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Research Article

Collisional Solitons Described by Two-Sided Beta Time Fractional Korteweg-de Vries Equations in Fluid-Filled Elastic Tubes

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This article deals with the basic features of collisional radial displacements in a prestressed thin elastic tube filled having inviscid fluid with the presence of nonlocal operator. By implementing the extended Poincare–Lighthill–Kuo method and a variational approach, the new two-sided beta time fractional Korteweg-de-Vries (BTF-KdV) equations are derived based on the concept of beta fractional derivative (BFD). Additionally, the BTF-KdV equations are suggested to observe the effect of related parameters on the local and nonlocal coherent head-on collision phenomena for the considered system. It is observed that the proposed equations along with their new solutions not only applicable with the presence of locality but also nonlocality to study the resonance wave phenomena in fluid-filled elastic tube. The outcomes reveal that the BFD and other physical parameters related to tube and fluid have a significant impact on the propagation of pressure wave structures.

1. Introduction

Nowadays, nonlinear partial differential equations of not only integer but also fractional order are widely applicable to describe the intricate phenomena in many environments, for example, biomechanics, plasma physics, quantum theories, water wave theories, signal processing, etc. As a result, a significant challenge to modern biological and medicinal science is to extract clinically trustworthy information about the disease because of the intricate dynamical interactions of hemodynamic waves (e.g., pressure and flow) described by such equations. Moreover, the analysis of nonlinear wave form in artery provides clinically valuable information about the local and global cardiovascular functions. Such functions help us to investigate the basic features of several cardiovascular diseases such as stenosis and predict how the blood flow can be disturbed by the local imperfections occurred in an artery. The arterial tree is stimulated by the pressure and pulse flow that produces from the intermittent ejection of blood. Taylor [1] has been demonstrated that the effect of elastic properties of the vessel wall is noteworthy on the velocity of blood in blood vessel based on the observed pressure and theoretical predictions. To depict the mechanism of action of blood wave through the use of weakly nonlinear theories, theoretical analysis have been done by several researchers [2–5]. They have found that the blood flow in human arteries can be studied by various types of evolution equations. A feasible explanation for the feature of the pulses, for example, "peaking" and "steepening" in arteries can be obtained from solitary wave model theories.

In contrast, the collisional solitary wave phenomenon is another enthralling feature and the phase shift is its observable effect. The overtaking, and head-on collisions are such types of solitary wave interactions [6]. Though, the multi-solution of KdV equation or any other evolution equations outline a way to investigate the overtaking collision of solitary waves. Whereas, the collision phenomena between two-counter propagating soliton can be studied by deriving two-sided evolution equations from the theoretical models via the extended Poincare–Lighthill–Kuo method [6, 7]. A huge number of researchers [6–11] have already reported the wave phenomena described by the evolution equations not only in the earlier mentioned environments but also in many branches of science and engineering without considering the nonlocality. For instance, Erbay et al. [7] and Tait et al. [9] have used the mathematical techniques to study propagating wave phenomena in fluid-filled elastic tube (FFET) with the presence of locality. Demiray [8, 10] have reported the collisional solitary waves in FFET with the consideration of local media only. Recently, Akter et al. [12] have reported the interactions of mulishocks in FFET with the presence of nonlocal operator. Ferdous and Hafez [13] have reported the collisional wave phenomena without considering the derivation of fractional wave equation in FFET. They have ignored how to derive the evolution equation with the presence of nonlocal operators. Very recently, Akter et al. [11] have reported the collisional soliton around the critical value only by formulating the coupled beta time fractional modified Korteweg-de-Vries (BTF-mKdV) equation in FFET.

However, the collisional wave phenomena between two-counter propagating soliton have not been previously reported by formulating the coupled BTF-KdV equation in the nonlocal dynamical systems to best of our knowledge. In fact, the nonlocal parameter gives a clear idea of what happen with the radial displacements in FFETs when the system supports either complexity or nonconservity due to certain time. In addition, what will happen with the physical issues for the presence of both past and future memories in the system? In such situations, the coupled evolution equations with fractional temporal evolution are only an arena to describe the collisional radial displacements in FFETs. As a result, one needs to appropriate definition along with their useful properties of fractional derivates. Many researchers [13–20] have already used various types of fractional operators (e.g. Coimbra, Riesz, Riemann-Liouville, Hadamard, Gr¨unwald-Letnikov, Liouville-Caputo, Caputo-Fabrizio, etc.) to study physical issues in various environments. Such derivatives have some limitation to fulfill all the fundamental characteristics of calculus. Later, Atangana and his research group [21] introduced the so called "beta-derivative" of fractional order or beta fractional derivative (BFD). The newly introduced derivative owned some characteristics that can defeat some limitation of fractional derivative and is used to model including of some physical problems. They have defined the useful definition of BFD as:

$$0^A D_\tau^\sigma\{f(\tau)\} = \lim_{\varepsilon\longrightarrow 0}\frac{f(\tau+\varepsilon(\tau+1/\Gamma(\sigma))^{1-\sigma})-f(\tau)}{\varepsilon}, \tag{1}$$

where σ is the beta fractional operator. Based on the above definition, they have also showed that BFD is satisfied all the fundamental properties of classical calculus. Some of the useful properties are given below:

$$0^A D_\tau^\sigma\{mf(\tau)+ng(\tau)\} = m0^A D_\tau^\sigma\{f(\tau)\} + n0^A D_\tau^\sigma\{g(\tau)\}, \tag{2}$$

$$0^A D_\tau^\sigma\{\mu\} = 0, \text{for any constant}\, \mu, \tag{3}$$

$$0^A D_\tau^\sigma\{f(\tau)\cdot g(\tau)\} = g(\tau)0^A D_\tau^\sigma\{f(\tau)\} + f(\tau)0^A D_\tau^\sigma\{g(\tau)\}, \tag{4}$$

$$\begin{aligned}0^A D_\tau^\sigma\{f(\tau)/g(\tau)\} = {}& [g(\tau)0^A D_\tau^\sigma\{f(\tau)\}\\ &-f(\tau)0^A D_\tau^\sigma\{g(\tau)\}]/g^2(\tau),\end{aligned} \tag{5}$$

here $m, n, \mu \in \Re, g \neq 0\sim$, and $\sim f$ are σ differentiable functions, $0<\sigma\leq 1$. Introducing $\varepsilon = (\tau + 1/\Gamma(\sigma))^{\sigma-1}h$, when $\sim\varepsilon \longrightarrow 0, h\longrightarrow 0$, ones obtain:

$$0^A D_\tau^\sigma\{f(\tau)\} = (\tau+1/\Gamma(\sigma))^{1-\sigma}\frac{df(\tau)}{d\tau}. \tag{6}$$

They have also defined the fractional integral operator as:

$$0^A I_\tau^\sigma\{f(\tau)\} = \int^\tau \left(t+\frac{1}{\Gamma(\sigma)}\right)^{\sigma-1} f(t)dt. \tag{7}$$

Thus, this work explores the collisional wave phenomena for the radial displacements (RDs) in FFET by deriving the coupled KdV equations involving of fractional order with the consideration of BFD from the previously proposed model as in [8]. With the variation of physical parameters, the collisional RDs and their corresponding phase shift are presented graphically with physical descriptions via the analytical solutions of these new equations.

2. Governing Model Equations

To study the collisional wave phenomena with the presence of nonlocality, the following normalized model equations are considered:

$$2\frac{\partial U}{\partial T} + (\lambda_{sr}+U)\frac{\partial W}{\partial Z} + 2W\frac{\partial W}{\partial Z} = 0, \tag{8}$$

$$\frac{\partial W}{\partial T} + W\frac{\partial W}{\partial Z} + \frac{\partial P}{\partial Z} = 0, \tag{9}$$

$$\begin{aligned}P = {}& \beta_{c1}U + \beta_{c2}U^2 - \alpha_{c0}\frac{\partial^2 U}{\partial Z^2} - \alpha_{c1}\left(\frac{\partial U}{\partial Z}\right)^2\\ &+\left(\frac{\alpha_{co}}{\lambda_{sr}} - 2\alpha_{c1}\right)U\frac{\partial^2 U}{\partial Z^2} + \frac{M}{\lambda_{sr}\lambda_{ar}}\frac{\partial^2 U}{\partial T^2} - \frac{M}{\lambda_{sr}^2\lambda_{ar}}U\frac{\partial^2 U}{\partial T^2}.\end{aligned} \tag{10}$$

The details derivation of the above model equations are given in [8]. The Equations (8–10) are normalized by introducing the nondimensionalized quantities $z = R_0 Z$, $t = \frac{R_0}{c_0}T, u = R_0 U, w = c_0 W, M = \frac{H\rho_0}{R_0\rho_f}, c_0^2 = \frac{\mu H}{\rho_f R_0}$, and $p = \rho_f c_0^2(p_0 + P)$. All the physical variables/parameters involved in the model equations are abbreviated in Table 1.

It is noted here that the constants β_{c1} and β_{c2} are calculated by the following density function [22, 23]:

Table 1: List of abbreviations and symbols.

Notation	Abbreviations
u	RD
w	Axial velocity
R_0	Radius of cylindrical long thin tube
p_0	Inner pressure
H	Initial thickness
ρ_0	Mass density of tube
ρ_f	Mass density
λ_{sr}	Initial stretch ratio in the circumferential direction
λ_{ar}	Axial stretch ratio
σ	Beta fractional parameter
M	Ratio between the tube and fluid body for mass density
z and t	Spatial and time coordinate

$$\sum = \frac{1}{2\alpha}\left[\exp\left\{\alpha\left(\lambda_{\mathrm{sr}}^2 + \lambda_{\mathrm{ar}}^2 + \frac{1}{\lambda_{\mathrm{sr}}^2\lambda_{\mathrm{ar}}^2}\right) - 3\right\} - 1\right]. \quad (11)$$

As a result, one can obtain the coefficients β_{c1} and β_{c2} by the following way:

$$f_0 = \frac{f}{\lambda_{sr}\lambda_{ar}}\left(\lambda_{sr} - \frac{1}{\lambda_{sr}^3\lambda_{ar}^2}\right), \quad (12)$$

$$f_1 = \frac{f}{\lambda_{sr}\lambda_{ar}}\left[\left(1 + \frac{1}{\lambda_{sr}^4\lambda_{ar}^2}\right) + 2\alpha\left(\lambda_{sr} - \frac{1}{\lambda_{sr}^3\lambda_{ar}^2}\right)^2\right], \quad (13)$$

$$f_2 = \frac{f}{2\lambda_{sr}\lambda_{ar}}\left[-\frac{12}{\lambda_{sr}^5\lambda_{ar}^2} + 6\alpha\left(1 + \frac{3}{\lambda_{sr}^4\lambda_{ar}^2}\right)\left(\lambda_{sr} - \frac{1}{\lambda_{sr}^3\lambda_{ar}^2}\right) + 4\alpha^2\left(\lambda_{sr} - \frac{1}{\lambda_{sr}^3\lambda_{ar}^2}\right)^3\right], \quad (14)$$

$$\beta_{c1} = f_1 - \frac{f_0}{\lambda_{sr}}, \quad (15)$$

and

$$\beta_{c2} = f_2 - \frac{\beta_{c1}}{\lambda_{sr}}, \quad (16)$$

where $f = \exp\left\{\alpha\left(\lambda_{\mathrm{sr}}^2 + \lambda_{\mathrm{ar}}^2 + \frac{1}{\lambda_{\mathrm{sr}}^2\lambda_{\mathrm{ar}}^2}\right) - 3\right\}$.

In [8], author has been considered the following stretched coordinates and perturb expansions:

$$\begin{pmatrix}\xi\\ \eta\\ \tau\end{pmatrix} = \begin{pmatrix}\epsilon^{\frac{1}{2}}(Z - c_{ps}T) + \epsilon F_0(\eta,\tau) + \epsilon^{\frac{4}{2}}F_1(\xi,\eta,\tau) + \cdots\\ \epsilon^{\frac{1}{2}}(Z + c_{ps}T) + \epsilon G_0(\xi,\tau) + \epsilon^{\frac{4}{2}}G_1(\xi,\eta,\tau) + \cdots\\ \epsilon^{\frac{4}{2}}T\end{pmatrix} \text{and} \begin{pmatrix}U(\xi,\eta,\tau)\\ W(\xi,\eta,\tau)\\ P(\xi,\eta,\tau)\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^{\infty}\epsilon^i U_i\\ \sum_{i=1}^{\infty}\epsilon^i W_i\\ \sum_{i=1}^{\infty}\epsilon^i P_i\end{pmatrix}, \quad (17)$$

where ξ and η are two counter-propagating trajectories. Inserting Equation (17) consecutively into the Equations (8–10), one obtains the set of partial differential equations by depending on the order of ϵ.

For $O(\epsilon)$:

$$\left.\begin{array}{c} 2c_{ps}\left(-\frac{\partial U_1}{\partial\xi} + \frac{\partial U_1}{\partial\eta}\right) + \lambda_\theta\left(\frac{\partial W_1}{\partial\xi} + \frac{\partial W_1}{\partial\eta}\right) = 0\\ c_{ps}\left(-\frac{\partial W_1}{\partial\xi} + \frac{\partial W_1}{\partial\eta}\right) + \left(\frac{\partial P_1}{\partial\xi} + \frac{\partial P_1}{\partial\eta}\right) = 0\\ P_1 = L_1 U_1 \end{array}\right\}, \quad (18)$$

For $O(\epsilon^2)$:

$$\begin{aligned} &2c_{ps}\left(-\frac{\partial U_2}{\partial\xi} + \frac{\partial U_2}{\partial\eta}\right) + \lambda_1\left(\frac{\partial W_2}{\partial\xi} + \frac{\partial W_2}{\partial\eta}\right)\\ &+2\left[\frac{\partial U_1}{\partial\tau} + c_{ps}\left\{\frac{\partial F_0}{\partial\eta}\frac{\partial U_1}{\partial\xi} - \frac{\partial G_0}{\partial\xi}\frac{\partial U_1}{\partial\eta}\right\}\right]\\ &+\lambda_1\left[\left\{\frac{\partial F_0}{\partial\eta}\frac{\partial W_1}{\partial\xi} + \frac{\partial G_0}{\partial\xi}\frac{\partial W_1}{\partial\eta}\right\}\right]\\ &+U_1\left(\frac{\partial W_1}{\partial\xi} + \frac{\partial W_1}{\partial\eta}\right) + 2W_1\left(\frac{\partial U_1}{\partial\xi} + \frac{\partial U_1}{\partial\eta}\right) = 0, \end{aligned} \quad (19)$$

$$\begin{aligned} &c_{ps}\left(-\frac{\partial W_2}{\partial\xi} + \frac{\partial W_2}{\partial\eta}\right) + W_1\left(\frac{\partial W_1}{\partial\xi} + \frac{\partial W_1}{\partial\eta}\right) + \left(\frac{\partial P_2}{\partial\xi} + \frac{\partial P_2}{\partial\eta}\right)\\ &+2\left[\frac{\partial W_1}{\partial\tau} + c\left\{\frac{\partial F_0}{\partial\eta}\frac{\partial W_1}{\partial\xi} - \frac{\partial G_0}{\partial\xi}\frac{\partial W_1}{\partial\eta}\right\}\right]\\ &+\left\{\frac{\partial F_0}{\partial\eta}\frac{\partial P_1}{\partial\xi} + \frac{\partial G_0}{\partial\xi}\frac{\partial P_1}{\partial\eta}\right\} = 0, \end{aligned} \quad (20)$$

$$\begin{aligned} P_4 = {} & L_1 U_4 + L_2 U_1^2 + \frac{Mc_{ps}^2}{\lambda_1\lambda_2}\left\{\frac{\partial^2 U_1}{\partial\xi^2} - 2\frac{\partial^2 U_1}{\partial\xi\partial\eta} + \frac{\partial^2 U_1}{\partial\eta^2}\right\}\\ & -K_1\left\{\frac{\partial^2 U_1}{\partial\xi^2} + 2\frac{\partial^2 U_1}{\partial\xi\partial\eta} + \frac{\partial^2 U_1}{\partial\eta^2}\right\}. \end{aligned} \quad (21)$$

By simplifying $O(\epsilon)$ and $O(\epsilon^2)$ equations, the following two-sided KdV equations have derived [8]:

$$U_\tau + \mu_1 U U_\xi + \mu_2 U_{\xi\xi\xi} = 0, \quad (22)$$

$$V_\tau - \mu_1 V V_\eta - \mu_2 V_{\eta\eta\eta} = 0, \tag{23}$$

where $U(\xi,\tau) \sim U_{1\xi}(\xi,\tau)$, $V(\eta,\tau) \sim U_{1\eta}(\eta,\tau)$, and $U_1 = U_{1\xi}(\xi,\tau) + U_{1\eta}(\eta,\tau)$ are left to right propagating, right to left propagating and resonance RDs, respectively (for simplicity). The coefficients μ_1 and μ_2 are defined by:

$$\mu_1 = \frac{5C_{ps}}{2\lambda_{sr}} + \frac{\lambda_{sr}\beta_{c2}}{2C_{ps}}, \mu_2 = \frac{\lambda_{sr}}{4C_{ps}}\left(\frac{MC_{ps}^2}{\lambda_{sr}\lambda_{ar}} - \alpha_{c0}\right), \tag{24}$$

where

$$C_{ps}^2 = \frac{\lambda_{sr}\beta_{c1}}{2}. \tag{25}$$

Additionally, the unfamiliar phase functions are represented [8] as:

$$\left.\begin{aligned} F_0(\eta,\tau) &= \frac{\lambda_{sr}}{8c_{ps}}\left(\frac{6c_{ps}}{\lambda_{sr}^2} - \frac{2\beta_{c2}}{c_{ps}}\right)\int_{-\infty}^{\eta} U_{1\eta}(\eta',\tau)d\eta' \\ G_0(\xi,\tau) &= \frac{\lambda_{sr}}{8c_{ps}}\left(\frac{6c_{ps}}{\lambda_{sr}^2} - \frac{2\beta_{c2}}{c_{ps}}\right)\int_{\infty}^{\xi} U_{1\xi}(\xi',\tau)d\xi' \end{aligned}\right\}. \tag{26}$$

However, the KdV equations in classical forms are not appropriate to describe the nonlocal behavior of collisional RD phenomena for this model. At this stage, one needs to require the fractional order evolution equations by assuming an appropriate fractional operator in investigating the features of nonlocal lucid structures. Currently, many kinds of nonlocal-operators are recently proposed by many researchers. One of the very effective fractional operators, so termed as the BFD has been proposed by Atangana et al. [21]. Such operator is perfectly full-filled all the useful properties of classical calculus. It is therefore motivated to derive new evolution equations, so called the two-sided BTF-KdV equations for reporting the nature of not only local but also nonlocal collisional wave phenomena in the considered system as mentioned earlier.

3. Formation of Two Sided BTF-KdV Equations

First of all, consider the potential functions $P(\xi,\tau)$ and $Q(\eta,\tau)$ defined by $U(\xi,\tau) \longrightarrow P_\xi(\xi,\tau)$ and $V(\eta,\tau) \longrightarrow Q_\eta(\eta,\tau)$ yields the potential Equations (22) and (23) in the following form:

$$\left.\begin{aligned} P_{\xi\tau}(\xi,\tau) + \mu_1 P_\xi(\xi,\tau)P_{\xi\xi}(\xi,\tau) + \mu_2 P_{\xi\xi\xi\xi}(\xi,\tau) = 0 \\ Q_{\eta\tau}(\eta,\tau) - \mu_1 Q_\eta(\eta,\tau)Q_{\eta\eta}(\eta,\tau) - \mu_2 Q_{\eta\eta\eta\eta}(\eta,\tau) = 0 \end{aligned}\right\}. \tag{27}$$

Applying the variational principle [24, 25], one can consider the functional of Equation (27) as:

$$\left.\begin{aligned} J(P) &= \iint_{\Omega T} P(\xi,\tau)\left(d_1 P_{\xi\tau}(\xi,\tau) + d_2\mu_1 P_\xi(\xi,\tau)P_{\xi\xi}(\xi,\tau) + d_3\mu_2 P_{\xi\xi\xi\xi}(\xi,\tau)\right)d\xi d\tau \\ J(Q) &= \iint_{\Omega T} Q(\eta,\tau)\left(d_1 Q_{\eta\tau}(\eta,\tau) - d_2\mu_1 Q_\eta(\eta,\tau)Q_{\eta\eta}(\eta,\tau) - d_3\mu_2 Q_{\eta\eta\eta\eta}(\eta,\tau)\right)d\eta d\tau \end{aligned}\right\}, \tag{28}$$

where Ω and T are stands for space and time unit, respectively, and d_i $(i = 1,2,3)$ are Lagrange's multipliers. Now, integrating Equation (28) by parts along with $P_\tau|_\Omega = P_\xi|_\Omega = P_\xi|_T = 0$, $Q_\tau|_\Omega = Q_\eta|_\Omega = Q_\eta|_T = 0$ and yields

$$\left.\begin{aligned} J(P) &= \iint_{\Omega T}\left[-d_1 P_\xi(\xi,\tau)P_\tau(\xi,\tau) - \frac{1}{2}d_2\mu_1 P_\xi^3(\xi,\tau) + d_3\mu_2 P_{\xi\xi}^2(\xi,\tau)\right]d\xi d\tau \\ J(Q) &= \iint_{\Omega T}\left[-d_1 Q_\eta(\eta,\tau)Q_\tau(\eta,\tau) + \frac{1}{2}d_2\mu_1 Q_\eta^3(\eta,\tau) - d_3\mu_2 Q_{\eta\eta}^2(\eta,\tau)\right]d\eta d\tau \end{aligned}\right\}. \tag{29}$$

Taking the variation of this functional with regard to $P(\xi,\tau)$, $Q(\eta,\tau)$, and integrating each term by parts and making the variation optimum, one can find the following expression:

$$\left.\begin{aligned} 2d_1 P_{\xi\tau}(\xi,\tau) + 3d_2\mu_1 P_\xi(\xi,\tau)P_{\xi\xi}(\xi,\tau) + 2d_3\mu_2 P_{\xi\xi\xi\xi}(\xi,\tau) = 0 \\ 2d_1 Q_{\eta\tau}(\eta,\tau) - 3d_2\mu_1 Q_\eta(\eta,\tau)Q_{\eta\eta}(\eta,\tau) - 2d_3\mu_2 Q_{\eta\eta\eta\eta}(\eta,\tau) = 0 \end{aligned}\right\}. \tag{30}$$

Comparing the expression (30) with the potential Equation (27), one obtains the unknown constants d_i $(i = 1,2,3)$ as:

$$d_1 = \frac{1}{2}, d_2 = \frac{1}{3}, d_3 = \frac{1}{2}. \tag{31}$$

Additionally, the Lagrangian form of the potential function for two sided KdV equation from the functional expression given by in Equation (29) becomes

$$\left.\begin{aligned}\mathscr{L}\left(P_\tau, P_\xi, P_{\xi\xi}\right) &= -\frac{1}{2}P_\xi(\xi,\tau)P_\tau(\xi,\tau) - \frac{1}{6}\mu_1 P_\xi^3(\xi,\tau) + \frac{1}{2}\mu_2 P_{\xi\xi}^2(\xi,\tau)\\ \mathscr{L}\left(Q_\tau, Q_\eta, Q_{\eta\eta}\right) &= -\frac{1}{2}Q_\eta(\eta,\tau)Q_\tau(\eta,\tau) + \frac{1}{6}\mu_1 Q_\eta^3(\eta,\tau) - \frac{1}{2}\mu_2 Q_{\eta\eta}^2(\eta,\tau)\end{aligned}\right\}. \tag{32}$$

By considering the following definition of BFD and beta fractional integral [21]:

$$\begin{aligned}0^A D_\tau^\sigma\{f(\tau)\} &= \lim_{\varepsilon\longrightarrow 0}\frac{f\left(\tau+\varepsilon\left(\tau+\frac{1}{\Gamma(\sigma)}\right)^{1-\sigma}\right)-f(\tau)}{\varepsilon},\\ 0^A I_\tau^\sigma\{f(\tau)\} &= \int^\tau\left(t+\frac{1}{\Gamma(\sigma)}\right)^{\sigma-1}f(t)dt,\end{aligned} \tag{33}$$

the time-fractional Lagrangian of the potential equations, like Equation (32) is defined as:

$$\left.\begin{aligned}\mathscr{L}_\sigma\left(0^A D_\tau^\sigma P, P_\xi, P_{\xi\xi}\right) &= -\frac{1}{2}0^A D_\tau^\sigma P(\xi,\tau)P_\xi(\xi,\tau) - \frac{1}{6}\mu_1 P_\xi^3(\xi,\tau) + \frac{1}{2}\mu_2 P_{\xi\xi}^2(\xi,\tau)\\ \mathscr{L}_\sigma\left(0^A D_\tau^\sigma Q, Q_\eta, Q_{\eta\eta}\right) &= -\frac{1}{2}0^A D_\tau^\sigma Q(\eta,\tau)Q_\eta(\eta,\tau) + \frac{1}{6}\mu_1 Q_\eta^3(\eta,\tau) - \frac{1}{2}\mu_2 Q_{\eta\eta}^2(\eta,\tau)\end{aligned}\right\}, \tag{34}$$

where $0^A D_\tau^\sigma$ is the BFD operator. Now, the functional of the potential equation in sense of BFD can be represented as:

$$\left.\begin{aligned}J_\sigma(P) &= \iint_{\Omega T}\mathscr{L}_\sigma\left(0^A D_\tau^\sigma P, P_\xi, P_{\xi\xi}\right)d\xi d\tau\\ J_\sigma(Q) &= \iint_{\Omega T}\mathscr{L}_\sigma\left(0^A D_\tau^\sigma Q, Q_\eta, Q_{\eta\eta}\right)d\eta d\tau\end{aligned}\right\}, \tag{35}$$

where the time-fractional Lagrangian of the potential equation is given by (33). Based on Agrawal's method [25], the variation of this functional as in Equation (35) with regard to $P(\xi,\tau)$ and $Q(\eta,\tau)$ yields

$$\left.\begin{aligned}\delta J_\sigma(P) &= \iint_{\Omega T}\left[\left(\frac{\partial\mathscr{L}_\sigma}{\partial 0^A D_\tau^\sigma P}\right)\delta 0^A D_\tau^\sigma P + \left(\frac{\partial\mathscr{L}_\sigma}{\partial P_\xi}\right)\delta P_\xi + \left(\frac{\partial\mathscr{L}_\sigma}{\partial P_{\xi\xi}}\right)\delta P_{\xi\xi}\right]d\xi d\tau\\ \delta J_\sigma(Q) &= \iint_{\Omega T}\left[\left(\frac{\partial\mathscr{L}_\sigma}{\partial 0^A D_\tau^\sigma Q}\right)\delta 0^A D_\tau^\sigma Q + \left(\frac{\partial\mathscr{L}_\sigma}{\partial Q_\eta}\right)\delta Q_\eta + \left(\frac{\partial\mathscr{L}_\sigma}{\partial Q_{\eta\eta}}\right)\delta Q_{\eta\eta}\right]d\eta d\tau\end{aligned}\right\}. \tag{36}$$

By simplifying Equation (36) with the assumptions $\delta P|_{\mathrm{T}} = \delta P|_\Omega = \delta P_\xi|_\Omega = 0$ and $\delta Q|_{\mathrm{T}} = \delta Q|_\Omega = \delta Q_\eta|_\Omega = 0$ leads to

$$\left.\begin{aligned}\delta J_\sigma(P) &= \iint_{\Omega T}\left[\left(-0^A D_\tau^\sigma\left(\frac{\partial\mathscr{L}_\sigma}{\partial 0^A D_\tau^\sigma P}\right) - \frac{\partial}{\partial\xi}\left(\frac{\partial\mathscr{L}_\sigma}{\partial P_\xi}\right) + \frac{\partial^2}{\partial\xi^2}\left(\frac{\partial\mathscr{L}_\sigma}{\partial P_{\xi\xi}}\right)\right)\delta P\right]d\xi d\tau\\ \delta J_\sigma(Q) &= \iint_{\Omega T}\left[\left(-0^A D_\tau^\sigma\left(\frac{\partial\mathscr{L}_\sigma}{\partial 0^A D_\tau^\sigma Q}\right) + \frac{\partial}{\partial\eta}\left(\frac{\partial\mathscr{L}_\sigma}{\partial Q_\eta}\right) - \frac{\partial^2}{\partial\eta^2}\left(\frac{\partial\mathscr{L}_\sigma}{\partial Q_{\eta\eta}}\right)\right)\delta Q\right]d\eta d\tau\end{aligned}\right\}. \tag{37}$$

To archive it, one can assume that let $f(t), g(t) : [a, b] \longrightarrow \mathbb{R}$ be two functions such that $f(t), g(t)$ is beta-differentiable, then

$$\int_a^b f(t) 0^A D_t^\sigma g(t) dt = f(t)g(t)|_a^b - \int_a^b g(t) 0^A D_t^\sigma f(t) dt. \tag{38}$$

Using $\delta J(P) = 0$ and $\delta J(Q) = 0$ leads to

$$\left. \begin{array}{l} -0^A D_\tau^\sigma \left(\dfrac{\partial \mathscr{L}_\sigma}{\partial 0^A D_\tau^\sigma P} \right) - \dfrac{\partial}{\partial \xi} \left(\dfrac{\partial \mathscr{L}_\sigma}{\partial P_\xi} \right) + \dfrac{\partial^2}{\partial \xi^2} \left(\dfrac{\partial \mathscr{L}_\sigma}{\partial P_{\xi\xi}} \right) = 0 \\ -0^A D_\tau^\sigma \left(\dfrac{\partial \mathscr{L}_\sigma}{\partial 0^A D_\tau^\sigma Q} \right) + \dfrac{\partial}{\partial \eta} \left(\dfrac{\partial \mathscr{L}_\sigma}{\partial Q_\eta} \right) - \dfrac{\partial^2}{\partial \eta^2} \left(\dfrac{\partial \mathscr{L}_\sigma}{\partial Q_{\eta\eta}} \right) = 0 \end{array} \right\}. \tag{39}$$

Substituting the time-fractional Lagrangian Equation (33) of the potential equation into this Euler–Lagrange Equation (39) yields

$$\left. \begin{array}{l} 0^A D_\tau^\sigma P_\xi(\xi, \tau) + \mu_1 P_\xi(\xi, \tau) P_{\xi\xi}(\xi, \tau) + \mu_2 P_{\xi\xi\xi\xi}(\xi, \tau) = 0 \\ 0^A D_\tau^\sigma Q_\eta(\eta, \tau) + \mu_1 Q_\eta(\eta, \tau) Q_{\eta\eta}(\eta, \tau) + \mu_2 Q_{\eta\eta\eta\eta}(\eta, \tau) = 0 \end{array} \right\}. \tag{40}$$

Now, replacing the potential function $P_\xi(\xi, \tau) = U(\xi, \tau)$ and $Q_\eta(\eta, \tau) = V(\eta, \tau)$ leads to the following two-sided BTF-KdV equations for the state function $U(\xi, \tau)$ and $V(\eta, \tau)$:

$$0^A D_\tau^\sigma U(\xi, \tau) + \mu_1 U(\xi, \tau) U_\xi(\xi, \tau) + \mu_2 U_{\xi\xi\xi}(\xi, \tau) = 0, \tag{41}$$

$$0^A D_\tau^\sigma V(\eta, \tau) - \mu_1 V(\eta, \tau) V_\eta(\eta, \tau) - \mu_2 V_{\eta\eta\eta}(\eta, \tau) = 0. \tag{42}$$

It is noted that many authors [26–31] have studied the wave phenomena by considering the fractional evolution equations. They have ignored how to form such equations from the evolution equations of integer orders. Very recently, Shahrina and Hafez [31] have studied the collisional soliton in plasmas without considering fractional evolution in plasmas. A few authors [11, 32–35] have only demonstrated how to obtain fractional evolution equations from the evolution equations of integer orders. Being motivated by these facts, Equations (41) and (42) are formulated for the first time with the presence of beta fractional operator to study the collisional wave phenomena in fluid filled elastic tube.

4. Solution of Two Sided BTF-KdV eEquations

Considering the properties of BFD and variable transformation, the solution of Equation (41) and Equation (42) is defined as

$$U(\xi, \tau) = \chi_L(\zeta_L) \text{ and } V(\eta, \tau) = \chi_R(\zeta_R), \tag{43}$$

where

$$\zeta_L = \xi - \frac{\nu}{\sigma} \left(\tau + \frac{1}{\Gamma(\sigma)} \right)^\sigma \text{ and } \zeta_R = \eta + \frac{\nu}{\sigma} \left(\tau + \frac{1}{\Gamma(\sigma)} \right)^\sigma. \tag{44}$$

Here, ν is the speed of the solitary wave. Equation (41) and Equation (42) can then be converted to the following ODE by plugging Equation (43) and Equation (44):

$$-\nu \chi_L'(\zeta_L) + \mu_1 \chi_L(\zeta_L) \chi_L'(\zeta_L) + \mu_2 \chi_L'(\zeta_L) = 0, \tag{45}$$

$$\nu \chi_R'(\zeta_R) - \mu_1 \chi_R(\zeta_R) \chi_R'(\zeta_R) - \mu_2 \chi_R'(\zeta_R) = 0. \tag{46}$$

By directly integrating Equation (45) and Equation (46), the analytical solutions of two sided BTF-KdV equations are attained as:

$$U(\xi, \tau) = \frac{3\nu}{\mu_1} \operatorname{sech}^2 \left(\frac{\xi - \dfrac{\nu}{\sigma} \left(\tau + \dfrac{1}{\Gamma(\sigma)} \right)^\sigma}{\sqrt{\dfrac{4\mu_2}{\nu}}} \right), \tag{47}$$

$$V(\eta, \tau) = \frac{3\nu}{\mu_1} \operatorname{sech}^2 \left(\frac{\eta + \dfrac{\nu}{\sigma} \left(\tau + \dfrac{1}{\Gamma(\sigma)} \right)^\sigma}{\sqrt{\dfrac{4\mu_2}{\nu}}} \right). \tag{48}$$

5. Phase Shifts

Implementing Equations (47) and (48) into Equation (26), the leading phase are obtained as

$$\left. \begin{array}{l} F_0(\eta, \tau) = \dfrac{\lambda_{sr}}{8c_{ps}} \left(\dfrac{6c_{ps}}{\lambda_{sr}^2} - \dfrac{2\beta_{c2}}{c_{ps}} \right) \displaystyle\int_{-\infty}^{\eta} U_a \operatorname{sech}^2 \left(\frac{\eta' + \dfrac{\nu}{\sigma} \left(\tau + \dfrac{1}{\Gamma(\sigma)} \right)^\sigma}{U_w} \right) d\eta' \\ G_0(\xi, \tau) = \dfrac{\lambda_{sr}}{8c_{ps}} \left(\dfrac{6c_{ps}}{\lambda_{sr}^2} - \dfrac{2\beta_{c2}}{c_{ps}} \right) \displaystyle\int_{\infty}^{\xi} U_a \operatorname{sech}^2 \left(\frac{\xi' - \dfrac{\nu}{\sigma} \left(\tau + \dfrac{1}{\Gamma(\sigma)} \right)^\sigma}{U_w} \right) d\xi' \end{array} \right\}. \tag{49}$$

Simplifying Equation (49), one obtains

$$\left.\begin{aligned} F_0(\eta,\tau) &= \frac{\lambda_{sr}}{8c_{ps}}\left(\frac{6c_{ps}}{\lambda_{sr}^2}-\frac{2\beta_{c2}}{c_{ps}}\right)U_aU_w\left[\tanh\left(\frac{\eta+\frac{\nu}{\sigma}\left(\tau+\frac{1}{\Gamma(\sigma)}\right)^{\sigma}}{U_w}\right)+1\right] \\ G_0(\xi,\tau) &= \frac{\lambda_{sr}}{8c_{ps}}\left(\frac{6c_{ps}}{\lambda_{sr}^2}-\frac{2\beta_{c2}}{c_{ps}}\right)U_aU_w\left[\tanh\left(\frac{\xi-\frac{\nu}{\sigma}\left(\tau+\frac{1}{\Gamma(\sigma)}\right)^{\sigma}}{U_w}\right)-1\right] \end{aligned}\right\}. \tag{50}$$

For the weak collisional solitons, the trajectories as defined in Equation (17) is converted to

$$\left.\begin{aligned} \xi &= \epsilon^{\frac{1}{2}}(Z-c_{ps}T)+\epsilon H\left[\tanh\left(\frac{\eta+\frac{\nu}{\sigma}\left(\tau+\frac{1}{\Gamma(\sigma)}\right)^{\sigma}}{U_w}\right)+1\right]+\cdots \\ \eta &= \epsilon^{\frac{1}{2}}(Z+c_{ps}T)+\epsilon H\left[\tanh\left(\frac{\xi-\frac{\nu}{\sigma}\left(\tau+\frac{1}{\Gamma(\sigma)}\right)^{\sigma}}{U_w}\right)-1\right]+\cdots \end{aligned}\right\}, \tag{51}$$

where

$$H = \frac{\lambda_{sr}}{8c_{ps}}\left(\frac{6c_{ps}}{\lambda_{sr}^2}-\frac{2\beta_{c2}}{c_{ps}}\right)U_aU_w. \tag{52}$$

Using the following relations as:

$$\left.\begin{aligned} \Delta F_0 &= \epsilon^{\frac{1}{2}}(Z-c_{ps}T)\big|_{\eta\longrightarrow-\infty,\xi=0}-\epsilon^{\frac{1}{2}}(Z-c_{ps}T)\big|_{\eta\longrightarrow\infty,\xi=0} \\ \Delta G_0 &= \epsilon^{\frac{1}{2}}(Z+c_{ps}T)\big|_{\xi\longrightarrow-\infty,\eta=0}-\epsilon^{\frac{1}{2}}(Z+c_{ps}T)\big|_{\xi\longrightarrow\infty,\eta=0} \end{aligned}\right\}, \tag{53}$$

the phase shifts due to collisional solitons are formulated as:

$$\left.\begin{aligned} \Delta F_0 &= -\epsilon\frac{\lambda_{sr}}{4c_{ps}}\left(\frac{6c_{ps}}{\lambda_{sr}^2}-\frac{2\beta_{c2}}{c_{ps}}\right)U_aU_w \\ \Delta G_0 &= \epsilon\frac{\lambda_{sr}}{4c_{ps}}\left(\frac{6c_{ps}}{\lambda_{sr}^2}-\frac{2\beta_{c2}}{c_{ps}}\right)U_aU_w \end{aligned}\right\}. \tag{54}$$

It is obviously found that the phase shift is independent of fractional parameter.

6. Results and Discussion

It is well confirmed that the wave–wave interaction is another fascinating feature of solitary wave phenomena because the collision of solitary waves exhibits many particles like features in the process of solitary wave propagation in arteries. It has commonly been assumed that a system is required to keep the solitary waves with striking colliding properties. As a result, the phase shift is a striking effect of such wave interactions. In the time of propagation solitary wave encounters, the collisional waves are actually formed. In this manuscript, the head-on collision between two solitary waves (i.e., the angle between two propagation directions of two solitary waves is equal to π) have been studied by employing a suitable asymptotic expansion. To do so, a coupled evolution equations involving fractional parameters have been derived. It is observed from the derived coupled BTF-KdV equations that the nonlinear (A_N) and dispersive (B_D) coefficients of such equations are strongly dependent on the initial stretch ratio in the circumferential direction, axial stretch ratio, mass density ratio between the tube and fluid body, and material constant etc. Besides, the experimentally founded average value of the parameters, that is, $R_0 = 0.38\,cm$, $H = 0.02\,cm$, $\rho_0 = 1.04\,g/cm^4$, $\rho_f = 1.05\,g/cm^4$, $\lambda_{sr} = 1.6$, and $\lambda_{ar} = 1.6$ in [3, 15], have applied to study the pressure waves in dogs' blood. That is why, the values of the parameters are considered by taking very close to the above experimental observational data, that is, $R_0 = 0.48\,cm$, $H = 0.02\,cm$,

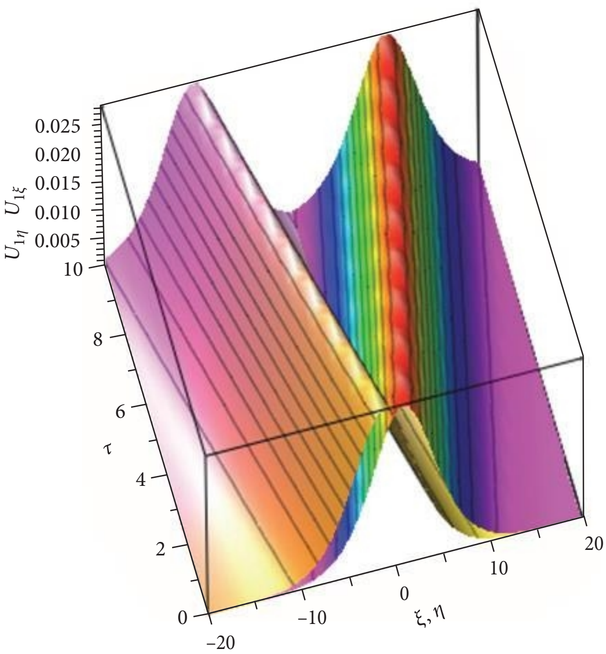

Figure 1: Left and right soliton propagation for $\sigma = 0.95$.

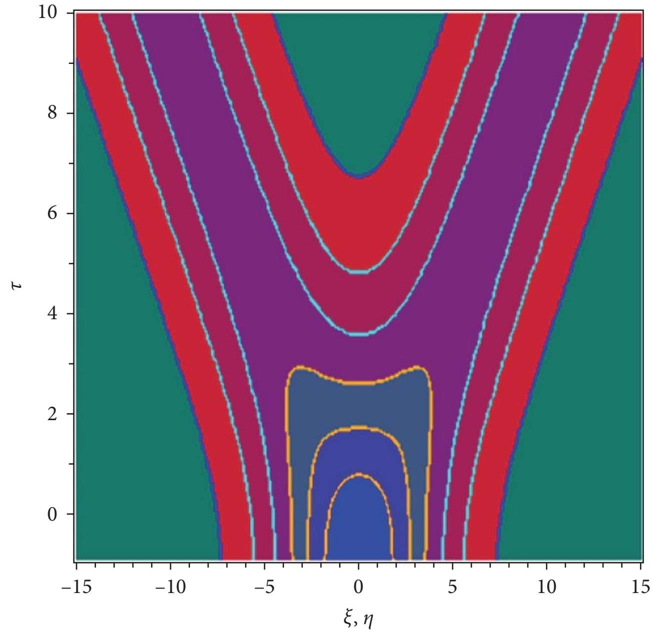

Figure 3: Contour plot of collisional radial displacement for $\sigma = 0.95$.

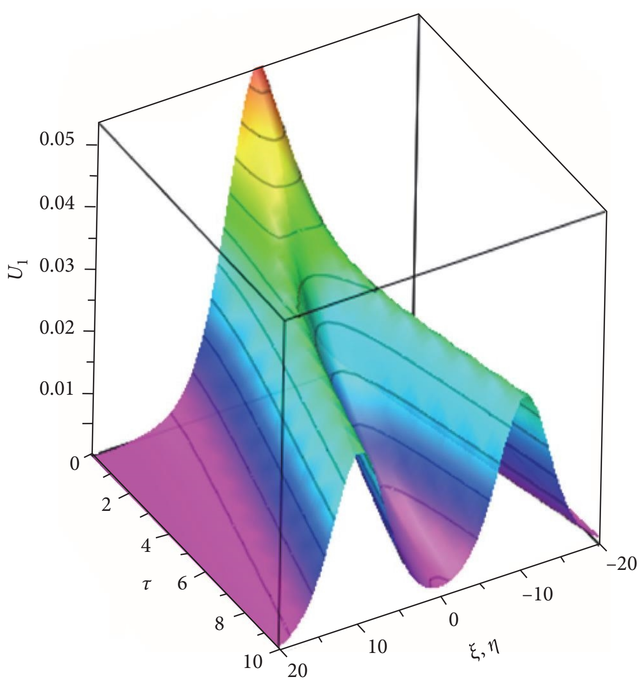

Figure 2: Collisional radial displacement for $\sigma = 0.95$.

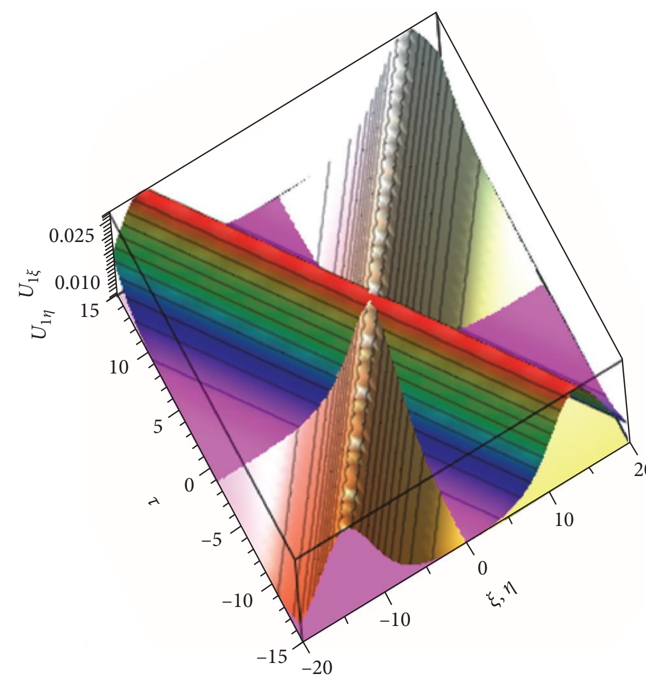

Figure 4: Left to right and right to left propagating radial displacement for $\sigma = 0.95$.

$\rho_0 = 1.04\, g/cm^4$, $\rho_f = 1.05\, g/cm^4$, $\lambda_{sr} = 1.6$, $\lambda_{ar} = 0.6$, and $\nu = 1\, cm/s$ in the presented analysis. In addition, the effect of fractional parameter and some other parameters on the radial displacement of pressure wave for the collisional soliton is presented graphically along with the physical interpretations. Using these experimental average data, the collisional RDs are displayed by 3D, 2D, and contour plots. Figures 1 and 2 show the 3D profile of left and right soliton propagation towards each other, whereas Figures 3 and 4 represent the 3D collisional radial displacement $U_1 = U_{1\xi}(\xi, \tau) + U_{1\eta}(\eta, \tau)$ profile described by two-sided BTF-KdV equations for the fractional parameter values $\sigma = 0.95$ and $\sigma = 1$, respectively. Also, the respective contour plots for $\sigma = 0.95$ and $\sigma = 1$ are shown in Figures 5 and 6. It is seen that an M-shaped solitary wave is formed after the head-on

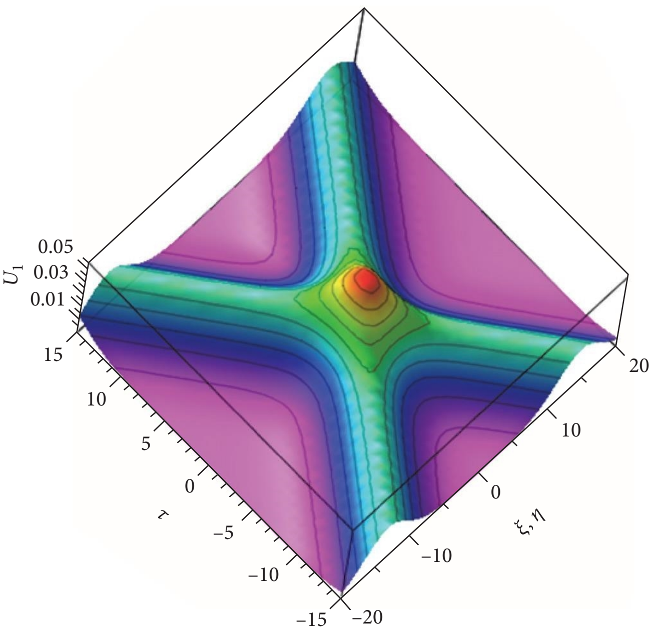

FIGURE 5: Collisional radial displacement for $\sigma = 1$.

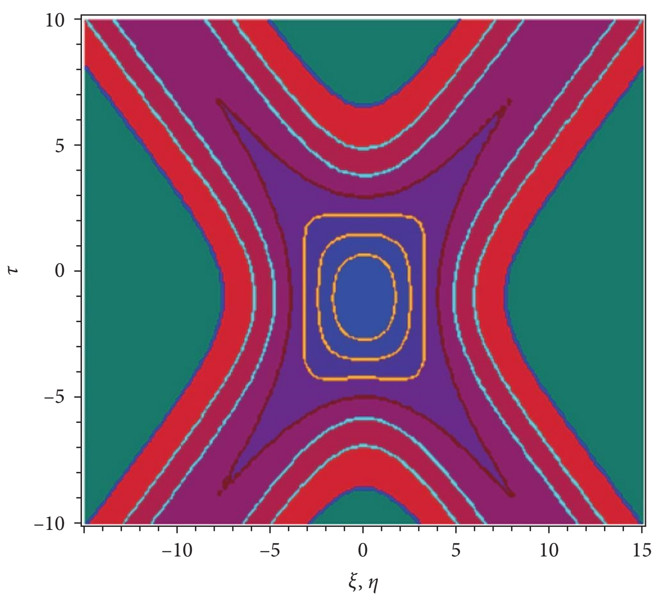

FIGURE 6: Contour plot of collisional radial displacement for $\sigma = 1$.

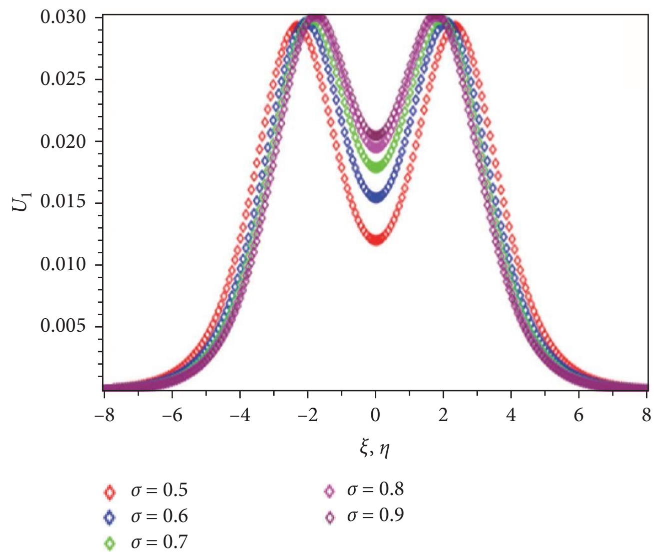

FIGURE 7: Effect of fractional parameter σ on collisional radial displacement.

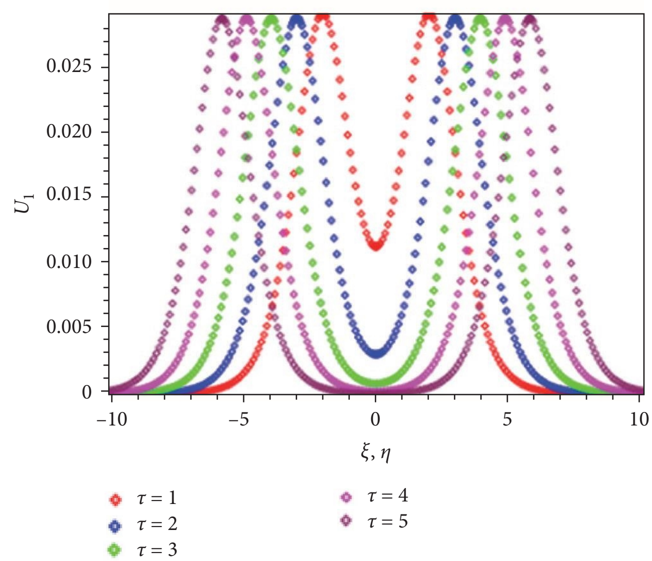

FIGURE 8: Effect of time on collisional radial displacement.

collision described by two-sided BTF-KdV equations for the considered model equations. Effect on the variation of fractional parameter σ for $\tau = 0.8$ and $M = 0.07$ to the radial displacement $U_1 = U_{1\xi}(\xi, \tau) + U_{1\eta}(\eta, \tau)$ for pressure waves is shown in Figure 7. From which it is seen that fractional parameter has a significant change on the head-on collision structures. The amplitude and width of the M-shaped solitary wave is slightly increased and decreased, respectively, due to the increase of fractional parameter. Figure 8 illustrates the effect of time variation for $\sigma = 0.95, M = 0.05$ to the radial displacement $U_1 = U_{1\xi}(\xi, \tau) + U_{1\eta}(\eta, \tau)$ which shows there is a minor modify in the wave peak, however the width is significantly increased for increasing values of time. Also, the effect of M for $\tau = 1$ and $\sigma = 0.95$ is shown in Figure 9. It is observed that the increase of M causes the enhance in both the M-shaped solition amplitude and width subsequent to the collision of solitons. Finally, the variation of phase shift (ΔF_0) with regards to the reference speed v for $M = 0.01$ (red color), $M = 0.04$ (green color), and $M = 0.05$ (orange color) is displayed in Figure 10. It is found that the time delayed is decreased with the increase of M. The collisional structures clearly indicated that the right propagating solitons is to begin with $\xi = 0, \eta \longrightarrow -\infty$, left propagating solitons is to begin with $\eta = 0, \xi \longrightarrow +\infty$, and afterward such waves asymptotically remote from each other. The completely overturn situations are obtained for $\tau \longrightarrow \pm\infty$, as it is expected. After that, the

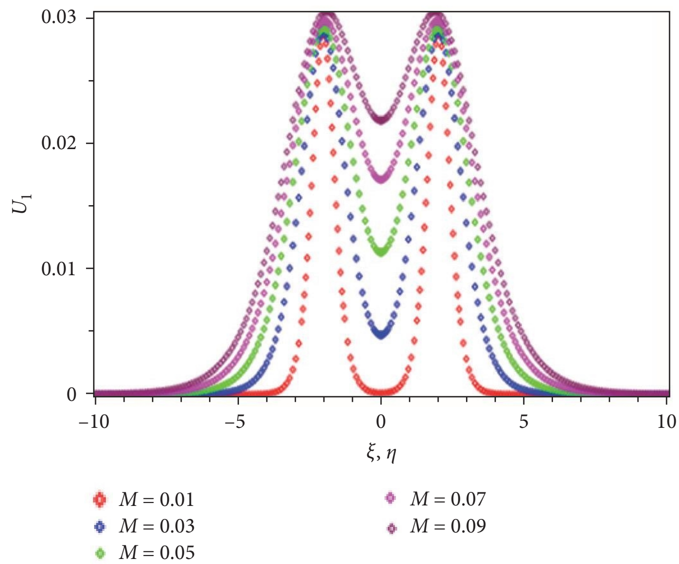

Figure 9: Effect of parameter M on collisional radial displacement.

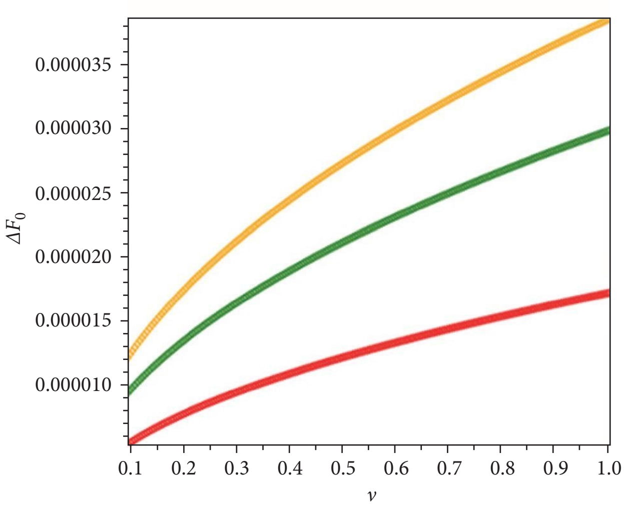

Figure 10: Variation of ΔF_0 with regards to the reference speed v for $M = 0.01$ (red color), $M = 0.04$ (green color), and $M = 0.05$ (orange color).

merged soliton profiles are produced by the composition processes of collisional solitons. Due to the deviation far from their initial position, the time delayed are obtained. The above discussion is concluded that the outcome attained in this study may be helpful for better understanding the collisional solitons described by not only the coupled BTF-KdV equations but also the coupled KdV equations in biomedical science, shallow water wave theories, plasma physics [36], etc.

7. Conclusion

A variational approach has been implemented to derive the new BTF-KdV equations with the presence of BFD. The exact solutions of BTF-KdV equations have been extracted by employing the suitable wave transformation based on the convenient properties of BFD. The nonlinear collisional wave structures along with the influence of various physical parameters involved in the system are determined by considering the physically relevant experimental data. It is observed that the proposed equations along with their new solutions are not only applicable with the presence of locality but also nonlocality to study the resonance wave phenomena in FFETs. In addition, the amplitude and width of the M-shaped solitary wave slightly gains and losses energy, respectively, due to the increase of fractional parameter. The mass density also has a significant impact on the collisional radial displacement in which the collisional radial displacements grow with the increase of mass density. From the physical point of view, it is observed that the nonlocal parameter gives a clear idea of what will happen with the radial displacements when the system has been arisen discontinuity due to a certain time. The idealized problem in this work predicts that the unlike values for initial stretch ratio to the circumferential direction and axial stretch ratio exhibit critical values and large amplitude single soliton. Consequently, the obtained negative potential creates stumbling block for the future mathematical analysis. We verify these predictions by performing numerical simulations. In such situation, one needs to search higher order correction via mKdV equations to study the interactions between couple single-soliton and their corresponding phase shifts around the critical values in fluid-filled elastic tube based on their earlier proposed model equations more accurately in the physical system. We will work with the experimental issue in near future. Hence, the findings of this study would be supportive of further theoretical and laboratory studies.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

All authors contributed equally to this work.

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