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*Review Article*

# New Multilinear Variable Separation Solutions of the $(3+1)$-Dimensional Burgers Hierarchy

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The multilinear variable separation (MLVS) approach has been proven to be very useful in solving $(2+1)$-dimensional integrable systems. Taking the $(3+1)$-dimensional Burgers hierarchy as an example, we extend the MLVS approach to a whole family of $(3+1)$-dimensional Burgers hierarchy. New exact solutions and universal formulas are obtained, which lead to abundant $(3+1)$-dimensional coherent structures. In particular, two ring-type soliton molecules and their interactions are shown in detail. We also generalize the MLVS results of the $(3+1)$-dimensional Jimbo–Miwa (JM) equation and modified JM equation.

## 1. Introduction

Recently, soliton theory has been widely applied in almost all branches of physics, like condensed matter physics, quantum field theory, fluid dynamics, and nonlinear optics. One of the most important research fields in soliton theory is to construct exact solutions of nonlinear evolution equations (NLEEs), which can be used to simulate natural phenomena. It is well known that the Fourier transform and the variable separation approach are the two most effective ways to find exact solutions for linear equations. The inverse scattering transformation serves as a nonlinear Fourier transform for integrable NLEEs. However, it is very difficult to extend the variable separation approach to nonlinear cases effectively. Fortunately, the multilinear variable separation (MLVS) approach has been proposed and proved to be a powerful method for finding exact solutions of many $(2+1)$-dimensional NLEEs [1–12] like the Davey–Stewartson equation, the Nizhnik–Novikov–Veselov equation, the Broer–Kaup–Kupershmidt equation and the long wave-short wave interaction equation. We call all these NLEEs the MLVS solvable models. We also can see that a $(2+1)$-dimensional universal formula is valid for suitable fields or potentials of MLVS solvable NLEEs, where at least one low-dimensional arbitrary function can be included. Thus, a large class of $(2+1)$-dimensional coherent structures, such as dromions, lumps, ring-solitons, breathers, instantons, and compactons, have been obtained. These studies are restricted to single-valued situations. For more complicated cases, multivalued functions also have been used to construct folded solitary waves and foldons. In [2, 3], it is also pointed out that the interactions among ring-solitons and some types of compactons are completely elastic, and the interactions among peakons or among some other types of compactons are not completely elastic because their shapes are changed during the interactions.

However, the MLVS approach has rarely been successfully generalized to higher dimensional NLEEs and hierarchies. As far as we know, there are only the $(3+1)$-dimensional Burgers equation [4, 5] and Jimbo–Miwa (JM) equation [8] that have been successfully solved by using this approach. Thus, our goal is primarily to apply the MLVS approach to a whole family of $(3+1)$-dimensional NLEEs, namely, the following $(3+1)$-dimensional Burgers hierarchy,

$$\begin{cases} u_t + \sum_{j=0}^{N} \beta_j \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} + u \right)^j u + \gamma_1 (2 v u_x + u_{xx}) + \gamma_2 (2 w u_z + u_{zz}) = 0, \\ v_y = u_x, \\ w_y = u_z, \end{cases} \tag{1}$$

where $N$ is a positive integer and $\beta_j, \gamma_1, \gamma_2$ are real constant parameters. This is a direct generalization of the $(2+1)$-dimensional Burgers hierarchy [12]. For $N=1$, $\gamma_2=0$, and $u \equiv u(x,y,t)$, Equation (1) corresponds to the $(2+1)$-dimensional Burgers equation and authors have obtained many new types of soliton structures of periodic waves [13]. For $\gamma_1=\gamma_2=0$ and $u \equiv u(y,t)$, Equation (1) degenerates to the $(1+1)$-dimensional Burgers hierarchy [14], which corresponds to the Burgers equation or the Sharma–Tasso–Olver equation when the positive integer $N=1$ or $N=2$.

The organization of this letter is as follows: Section 2 is devoted to describing the process of getting MLVS solutions of Equation (1) via the MLVS approach. New coherent structures, such as the two ring-type soliton molecules for the universal formula, are obtained in Section 3. A brief summary and discussions are given in Section 4.

## 2. MLVS Solutions of the Burgers Hierarchy

Similar to the usual steps in solving $(2+1)$-dimensional NLEEs via the MLVS approach [1–12], we construct the following auto-Bäcklund transformation first,

$$\begin{cases} u = (\ln f)_y + u_0, \\ v = (\ln f)_x + v_0, \\ w = (\ln f)_z + w_0, \end{cases} \tag{2}$$

through the truncated Laurent expansions. Here, $\{u_0 \equiv u_0(x, y, z, t), v_0 \equiv v_0(x, y, z, t), w_0 \equiv w_0(x, y, z, t)\}$ is a suitable seed solution of Equation (1) and the unknown function $f \equiv f(x, y, z, t)$ needs to be determined.

It is obvious that a simple seed solution of Equation (1) can be selected as $\{u_0 = 0, v_0 = v_0(x, z, t), w_0 = w_0(x, z, t)\}$. The substitution of Equation (2) with this seed solution into Equation (1) transforms Equation (1) into the following:

$$\begin{aligned} &\frac{f_{yt} f - f_y f_t}{f^2} + \sum_{j=0}^{N} \beta_j \frac{f_{(j+2)y} f - f_{(j+1)y} f_y}{f^2} + \gamma_1 \left[ 2\left(\frac{f_x}{f} + v_0\right)\left(\frac{f f_{xy} - f_x f_y}{f^2}\right) \right. \\ &\left. + \frac{f f_{xxy} - 2 f_x f_{xy} - f_{xx} f_y}{f^2} + 2\frac{f_x^2 f_y}{f^3} \right] + \gamma_2 \left[ 2\left(\frac{f_z}{f} + w_0\right)\left(\frac{f f_{zy} - f_z f_y}{f^2}\right) \right. \\ &\left. + \frac{f f_{zzy} - 2 f_z f_{zy} - f_{zz} f_y}{f^2} + 2\frac{f_z^2 f_y}{f^3} \right] = 0. \end{aligned} \tag{3}$$

To solve this equation, we change it to the form as follows:

$$\begin{aligned} &(f \partial_y - f_y) \\ &\left( f_t + 2\gamma_1 v_0 f_x + \gamma_1 f_{xx} + 2\gamma_2 w_0 f_z + \gamma_2 f_{zz} + \sum_{j=0}^{N} \beta_j f_{(j+1)y} \right) = 0. \end{aligned} \tag{4}$$

Thus, we have the following:

$$\begin{aligned} &f_t + 2\gamma_1 v_0 f_x + \gamma_1 f_{xx} + 2\gamma_2 w_0 f_z + \gamma_2 f_{zz} + \sum_{j=0}^{N} \beta_j f_{(j+1)y} \\ &= A(x, z, t) f. \end{aligned} \tag{5}$$

Next, the second step in the MLVS approach is to make an appropriate assumption for the expansion function $f(x, y, z, t)$. Essentially, the arbitrary functions in the MLVS solutions originate from the seed solution with respect to the same variables. Since the seed solution $\{u_0 = 0, v_0 = v_0(x, z, t), w_0 = w_0(x, z, t)\}$ and $A \equiv A(x, z, t)$ include three arbitrary functions of $\{x, z, t\}$, the unknown function $f(x, y, z, t)$ is taken as the following MLVS ansatz:

$$f \equiv \sum_{i=1}^{3} F_i G_i = \sum_{i=1}^{3} F_i(x, z, t) G_i(y, t). \tag{6}$$

Substituting this expansion, Equation (6) into Equation (5) arrives at the following:

$$\begin{aligned} &\sum_{i=1}^{3} G_i (2\gamma_1 v_0 F_{ix} + 2\gamma_2 w_0 F_{iz} - A F_i + F_{it} + \gamma_1 F_{ixx} + \gamma_2 F_{izz}) \\ &+ \sum_{i=1}^{3} F_i \left( G_{it} + \sum_{j=1}^{N} \beta_j (G_i)_{(j+1)y} \right) = 0. \end{aligned} \tag{7}$$

Obviously, Equation (7) can be divided into variable-separated equations, which are in the form of the following:

$$2\gamma_1 v_0 F_{ix} + 2\gamma_2 w_0 F_{iz} - AF_1 + F_{it} + \gamma_1 F_{ixx} + \gamma_2 F_{izz} = \sum_{k=1}^{3} c_{ik} F_k,$$
$$G_{it} + \sum_{j=1}^{N} \beta_j (G_i)_{(j+1)y} = -\sum_{k=1}^{3} c_{ki} G_k,\ i = 1, 2, 3, \tag{8}$$

where $c_{ik} \equiv c_{ik}(t), (i, k = 1, 2, 3)$ are arbitrary functions. In order to construct the explicit expression of the MLVS solution, we have the following:

$$2\gamma_1 F_{1x} v_0 + 2\gamma_2 F_{1z} w_0 - F_1 A + F_{1t} + \gamma_1 F_{1xx} + \gamma_2 F_{1zz} = 0, \tag{9}$$

$$2\gamma_1 F_{2x} v_0 + 2\gamma_2 F_{2z} w_0 - F_2 A + F_{2t} + \gamma_1 F_{2xx} + \gamma_2 F_{2zz} = 0, \tag{10}$$

$$2\gamma_1 F_{3x} v_0 + 2\gamma_2 F_{3z} w_0 - F_3 A + F_{3t} + \gamma_1 F_{3xx} + \gamma_2 F_{3zz} = 0, \tag{11}$$

$$G_{1t} + \sum_{j=1}^{N} \beta_j (G_1)_{(j+1)y} = 0, \tag{12}$$

$$G_{2t} + \sum_{j=1}^{N} \beta_j (G_2)_{(j+1)y} = 0, \tag{13}$$

$$G_{3t} + \sum_{j=1}^{N} \beta_j (G_3)_{(j+1)y} = 0, \tag{14}$$

by taking $c_{ik} = 0, (i, k = 1, 2, 3)$ without losing generality.

Because $v_0, w_0, A$ are arbitrary functions, we may treat this problem alternatively. We now consider $F_i,\ (i = 1, 2, 3)$ are arbitrary functions while $v_0, w_0, A$ can be determined by Equations (9)–(11). Namely, by using Cramer's rule in linear algebra, we can solve out $v_0, w_0$ and $A$ as follows:

$$v_0 = -\frac{\begin{vmatrix} F_{1t} + \gamma_1 F_{1xx} + \gamma_2 F_{1zz} & F_{1z} & F_1 \\ F_{2t} + \gamma_1 F_{2xx} + \gamma_2 F_{2zz} & F_{2z} & F_2 \\ F_{3t} + \gamma_1 F_{3xx} + \gamma_2 F_{3zz} & F_{3z} & F_3 \end{vmatrix}}{2\gamma_1 \begin{vmatrix} F_{1x} & F_{1z} & F_1 \\ F_{2x} & F_{2z} & F_2 \\ F_{3x} & F_{3z} & F_3 \end{vmatrix}}, \tag{15}$$

$$w_0 = -\frac{\begin{vmatrix} F_{1x} & F_{1t} + \gamma_1 F_{1xx} + \gamma_2 F_{1zz} & F_1 \\ F_{2x} & F_{2t} + \gamma_1 F_{2xx} + \gamma_2 F_{2zz} & F_2 \\ F_{3x} & F_{3t} + \gamma_1 F_{3xx} + \gamma_2 F_{3zz} & F_3 \end{vmatrix}}{2\gamma_2 \begin{vmatrix} F_{1x} & F_{1z} & F_1 \\ F_{2x} & F_{2z} & F_2 \\ F_{3x} & F_{3z} & F_3 \end{vmatrix}}, \tag{16}$$

$$A = \frac{\begin{vmatrix} F_{1x} & F_{1z} & F_{1t} + \gamma_1 F_{1xx} + \gamma_2 F_{1zz} \\ F_{2x} & F_{2z} & F_{2t} + \gamma_1 F_{2xx} + \gamma_2 F_{2zz} \\ F_{3x} & F_{3z} & F_{3t} + \gamma_1 F_{3xx} + \gamma_2 F_{3zz} \end{vmatrix}}{\begin{vmatrix} F_{1x} & F_{1z} & F_1 \\ F_{2x} & F_{2z} & F_2 \\ F_{3x} & F_{3z} & F_3 \end{vmatrix}}, \tag{17}$$

from Equations (9)– (11). In addition, for linear Equations (12)–(14), we have exact solutions of $G_i,\ (i = 1, 2, 3)$ such as

$$G_i = 1 + \sum_{k=1}^{M} \exp\left[b_{ik} y - \left(\sum_{j=0}^{N} \beta_j b_{ik}^{j+1}\right) t + d_{ik}\right],\ i = 1, 2, 3. \tag{18}$$

Thus, exact MLVS solutions of the $(3+1)$-dimensional Burgers hierarchy (Equation (1)) are obtained.

*Remark 1.* The $(3+1)$-dimensional JM equation [8, 15] in its potential form, which can be expressed as follows:

$$u_{yt} + \alpha u_{xxxy} + \alpha\beta\left(u_x u_y\right)_x + \gamma u_{xz} = 0. \tag{19}$$

By means of the following Bäcklund transformation:

$$u = \frac{6}{\beta} (\ln f)_x + u_1(x, t) + u_2(z, t), \tag{20}$$

and the MLVS ansatz

$$f = F(x, t) + G(\xi, t),\ \xi = y + c_0 z, \tag{21}$$

we have the following:

$$\left[-2 + \frac{F + G}{F_x} \partial_x\right] (F_t + \alpha F_{xxx} + \alpha\beta F_x u_{1x} + \gamma c_0 F_x)$$
$$+ \left[-2 + \frac{F + G}{G_\xi} \partial_\xi\right] (G_t) = 0. \tag{22}$$

Thus, this equation can be separated into the following:

$$F_t + \alpha F_{xxx} + \alpha\beta F_x u_{1x} + \gamma c_0 F_x = c_1 + c_2 F + c_3 F^2, \tag{23}$$

$$G_t = -c_1 + c_2 G - c_3 G^2, \tag{24}$$

where $c_1 \equiv c_1(t), c_2 \equiv c_2(t), c_3 \equiv c_3(t)$ are arbitrary functions. Because $u_1$ is an arbitrary function, we may treat Equation (23) alternatively. Now, we consider $F$ to be an arbitrary function while $u_1$ can be determined by Equation (23), that is

$$u_1 = \int \frac{c_1 + c_2 F + c_3 F^2 - F_t - \alpha F_{xxx} - \gamma c_0 F_x}{\alpha\beta F_x} \mathrm{d}x. \quad (25)$$

Considering the arbitrariness of the functions $c_1, c_2$, and $c_3$, it is quite straightforward to verify that the Riccati Equation (24) has an exact solution as follows:

$$G = \frac{d_1(t)}{d_2(t) + d_3(\xi)} + d_4(t), \quad (26)$$

here, we rewrite $c_1, c_2$, and $c_3$ as follows

$$c_1 = \frac{d_4 d_1' + d_4^2 d_2' - d_1 d_4'}{d_1}, \quad c_2 = \frac{d_1' + 2 d_4 d_2'}{d_1}, \quad c_3 = \frac{d_2'}{d_1}. \quad (27)$$

Thus, the MLVS solution, Equation (20) with Equations (21), (25), and (26) of Equation (19) is obtained.

*Remark 2.* The $(3+1)$-dimensional modified JM (mJM) equation in its potential form, which can be expressed as follows:

$$u_{yt} + \alpha u_{xxxy} + \alpha\left(-\frac{3}{2}\frac{u_{xy}u_{xxy}}{u_y} + \frac{3}{4}\frac{u_{xy}^3}{u_y^2}\right) + \frac{1}{2}\alpha\beta\left(u_x u_y\right)_x + \gamma u_{xz} = 0. \quad (28)$$

By using the Bäcklund transformation, Equation (20), and the MLVS ansatz, Equation (21), we have the following:

$$\left[-2 + \frac{F+G}{F_x}\partial_x\right]\left(F_t + \alpha F_{xxx} - \frac{3}{4}\alpha\frac{F_{xx}^2}{F_x} + \frac{1}{2}\alpha\beta F_x u_{1x} + \gamma c_0 F_x\right) + \left[-2 + \frac{F+G}{G_\xi}\partial_\xi\right](G_t) = 0. \quad (29)$$

So we have similar results.

## 3. Two Ring-Type Soliton Molecules

For $(2+1)$-dimensional MLVS solvable NLEEs, by setting the following MLVS ansatz:

$$f \equiv a_0 + a_1 F + a_2 G + a_3 FG = a_0 + a_1 F(x,t) + a_2 G(y,t) + a_3 F(x,t) G(y,t), \quad (30)$$

a quite universal formula,

$$U_1 = \lambda(\ln f)_{xy} = \frac{\lambda(a_0 a_3 - a_1 a_2) F_x G_y}{(a_0 + a_1 F + a_2 G + a_3 FG)^2}, \quad (31)$$

is derived to describe some special solutions for suitable physical quantities. In this formula, parameters $a_0$, $a_1$, $a_2$, and $a_3$ are arbitrary constants, and $F \equiv F(x,t)$ is an arbitrary function, and $G \equiv G(y,t)$ may be an arbitrary function or an arbitrary solution of a special equation such as Riccati-type equation. Setting $\widehat{F} = \frac{a_0 + a_1 F}{a_2 + a_3 F}$, we have $(\ln[\widehat{F} + G])_{xy} = (\ln[a_0 + a_1 F + a_2 G + a_3 FG])_{xy}$. Thus, $f = F + G$ can be viewed as a basic MLVS ansatz based on the corresponding Bäcklund transformation. For the $(3+1)$-dimensional JM Equation (19) and mJM Equation (28), we have extended the universal formula (Equation (31)) to the following:

$$U_2 = \lambda(\ln f)_{xy} = \frac{-\lambda F_x G_y}{(F+G)^2}, \quad (32)$$

where $F \equiv F(x,t)$ is an arbitrary function and $G \equiv G(y + c_0 z, t)$ is Equation (26). Further, for the $(3+1)$-dimensional Burger hierarchy (Equation (1)), we have two more generalized formulas as follows:

$$U_3 = \lambda\left(\ln\left[\sum_{i=1}^{3} F_i G_i\right]\right)_{xy}, \quad (33)$$

$$U_4 = \lambda\left(\ln\left[\sum_{i=1}^{3} F_i G_i\right]\right)_{zy}. \quad (34)$$

Because some arbitrary lower-dimensional functions have been included in Equations (32)–(34), we believe that abundant $(3+1)$-dimensional coherent structures can be constructed directly.

For $(3+1)$-dimensional case, it should be pointed out that we just can draw projective figures along some particular directions. So here we still give some $(2+1)$-dimensional coherent structures and their interactions. Of course, this also applies to the $(3+1)$-dimensional case, such as in Equation (32), because $\xi = y + c_0 z$. We will not discuss all the possible coherent structures but only list a new particular example, the ring-type soliton molecule. There are some types of elliptic, parabolic, and hyperbolic coherent structures, which are not equal to zero identically at some quadratic curves and decays exponentially apart from the curves [2]. By restricting the functions $F \equiv F(x,t), G \equiv G(y,t)$, as some summation forms of the exponential functions as follows:

$$F = \exp\left(-\frac{(x-4t)^2}{15} + 12\right) + \exp\left(-\frac{(x+4t)^2}{15} + 6\right), \quad G = \exp\left(\frac{y^2}{20} - 3\right), \quad (35)$$

and $\lambda = 30$, we can obtain a structure of two ring-type solitons for Equation (32) (see Figure 1). The interaction of two

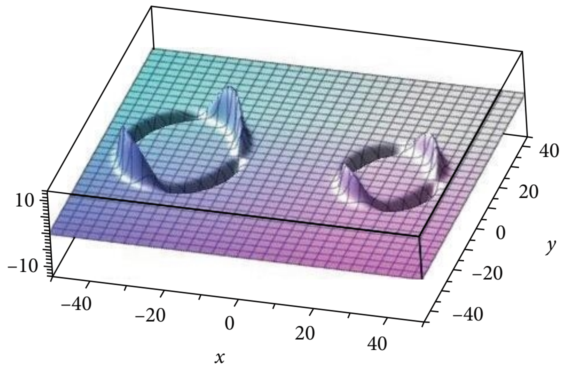


Figure 1: Two ring-type solitons of the universal formula (Equation (32)) with Equation (35), $t = -7$.

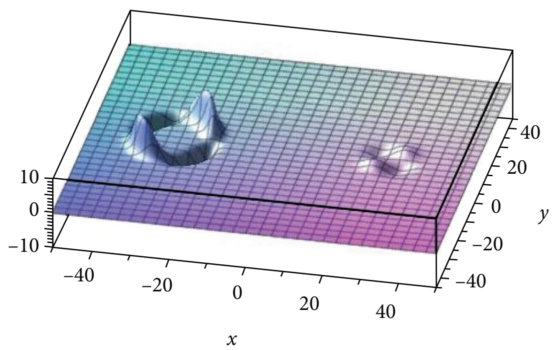


Figure 2: A ring-type soliton and a lump of the universal formula (Equation (32)) with Equation (36), $t = -7$.

ring-type solitons has been studied in [2]. If functions are changed to the following:

$$F = \exp\left(-\frac{(x-4t)^2}{15} + 12\right) + \exp\left(-\frac{(x+4t)^2}{15} + 6\right),$$
$$G = \exp\left(\frac{y^2}{20} + 6\right), \tag{36}$$

the ring-type soliton on the right degenerates into a lump-type structure (see Figure 2). This is different from the commonly defined lump structure, which is formed by rational functions.

Next, we investigate a new coherent structure, the ring-type soliton molecule, and its interaction. It is well-known that a soliton molecule is a bound state of solitons. The research on soliton molecules has attracted considerable attention and has focused on experimental and numerical fields. Recently, many novel forms of soliton molecules have been found in $(1+1)$-dimensional and $(2+1)$-dimensional integrable systems, such as the kink-type soliton molecules, dromion molecules, and breather molecules. Here, by means of the following:

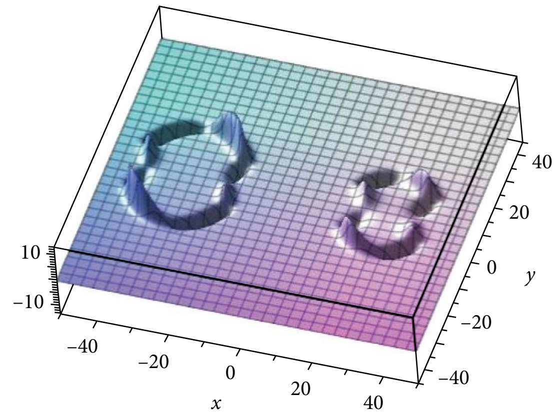


Figure 3: Two ring-type soliton molecules of the universal formula (Equation (32)) with Equation (37), $t = -7$.

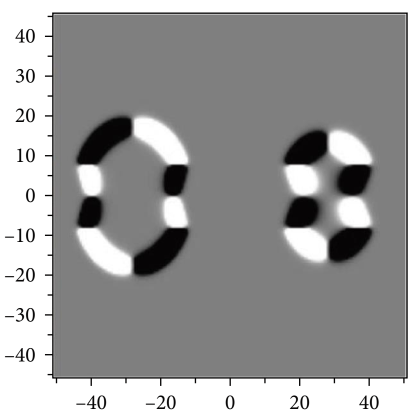


Figure 4: Denty plot of the two ring-type soliton molecules at $t = -7$.

$$F = \exp\left(-\frac{(x-4t)^2}{15} + 12\right) + \exp\left(-\frac{(x+4t)^2}{15} + 6\right),$$
$$G = \exp\left(\frac{y^2}{20} - 3\right) + \exp\left(\frac{-y^2}{20} + 3\right), \tag{37}$$

and $\lambda = 30$, we can obtain four ring-type solitons. Because of the special construction of the function $G$ in Equation (37), which is independent of the time variable $t$, we have actually obtained a structure of two ring-type soliton molecules for Equation (32) (see Figure 3). In density Figures 4–8, we plot the interaction property of the two ring-type soliton molecules for Equation (32) with the selection at the times $t = -7, -3, 0, 3, 7$, respectively. We can see that after the head-on collision of two ring-type soliton molecules, they preserve their shapes totally. In other words, the collision between the two ring-type soliton molecules is completely elastic.

## 4. Summary and Discussions

Exact solutions of NLEEs play a key role in understanding possible behaviors of natural phenomena [16–18]. There are

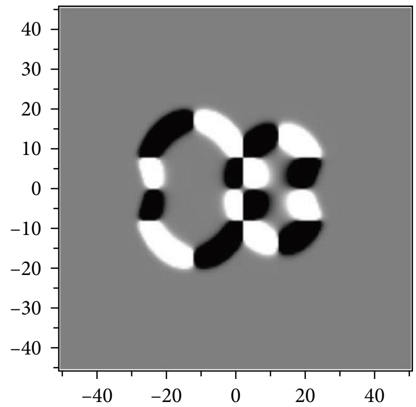


FIGURE 5: Denty plot of the interaction of the two ring-type soliton molecules at $t = -3$.

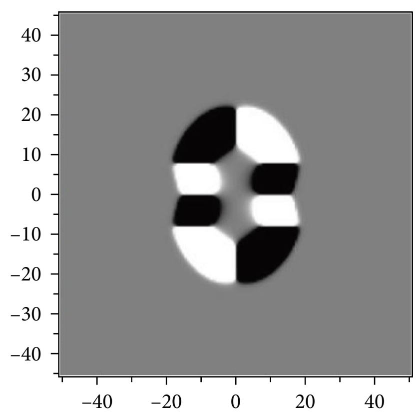


FIGURE 6: Denty plot of the interaction of the two ring-type soliton molecules at $t = 0$.

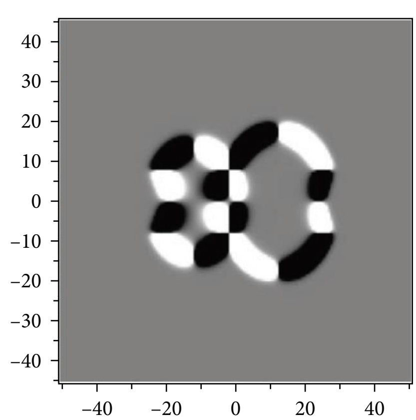


FIGURE 7: Denty plot of the interaction of the two ring-type soliton molecules at $t = 3$.

many powerful methods for constructing exact solutions of NLEEs, such as the inverse scattering transform method, the Darboux transformation, Hirota's bilinear method, the Lie group method, and so on. The MLSA approach is one of the most efficient methods because the constructed exact solution contains low-dimensional arbitrary functions, which

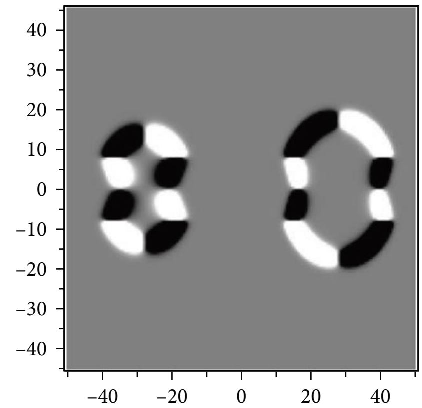


FIGURE 8: Denty plot of the two ring-type soliton molecules at $t = 7$.

allows us to construct abundant coherent structures. Since the MLVS approach has been widely applied to the $(2+1)$-dimensional NLEEs [1–12], how can it work for $(3+1)$-dimensional NLEEs or hierarchies? The $(3+1)$-dimensional Burgers equation and JM equation [8, 15] are the only two NLEEs that have been solved out their MLVS solutions via this approach. In this letter, we have considered the $(3+1)$-dimensional Burgers hierarchy (Equation (1)) by using the MLVS approach. Namely, starting from the auto-Bäcklund transformation (Equation (2)) and taking MLVS ansatz (Equation (5)), we can obtain much more general exact solutions of Equation (1). In Remarks 1 and 2, we improve the results of the $(3+1)$-dimensional JM equation and construct MLVS solutions of the $(3+1)$-dimensional mJM equation, respectively. We also have investigated a new coherent structure, the ring-type soliton molecule, and its interaction.

It is worth considering to construct other forms of variable separation solutions based on the Darboux transformation. We hope that the MLVS approach can be used to solve other high-dimensional equations or hierarchies [19–21]. How to construct $(3+1)$-dimensional coherent structures and study the properties of elastic or inelastic collisions is worthy of further study.

## Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## Ethical Approval

The authors state that this research complies with the ethical standards. This research does not involve either human participants or animals.

## Consent

All authors have given consent to participate in the study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

## Authors' Contributions

All authors contributed to the writing—original draft, methodology, software, formal analysis, and funding acquisition. All authors read and approved the final version of the manuscript for publication.

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## References

[1] S.-Y. Lou, "On the coherent structures of the Nizhnik–Novikov–Veselov equation," *Physics Letters A*, vol. 277, no. 2, pp. 94–100, 2000.

[2] X.-Y. Tang, S.-Y. Lou, and Y. Zhang, "Localized excitations in (2+1)-dimensional systems," *Physical Review E*, vol. 66, no. 4, Article ID 046601, 2002.

[3] X. Y. Tang and S. Y. Lou, "Extended multilinear variable separation approach and multivalued localized excitations for some (2+1)-dimensional integrable systems," *Journal of Mathematical Physics*, vol. 44, no. 9, pp. 4000–4025, 2003.

[4] J. P. Ying and S. Y. Lou, "Multilinear variable separation approach in (3+1)-dimensions: the Burgers equation," *Chinese Physics Letters*, vol. 20, no. 9, Article ID 1448, 2003.

[5] S. F. Shen, J. Zhang, and Z. L. Pan, "New exact solution to (3+1)-dimensional Burgers equation," *Communications in Theoretical Physics*, vol. 42, no. 1, pp. 49-50, 2004.

[6] X.-M. Qian, S.-Y. Lou, and X.-B. Hu, "Variable separation approach for a differential-difference asymmetric Nizhnik-Novikov-Veselov equation," *Zeitschrift für Naturforschung A*, vol. 59, no. 10, pp. 645–658, 2004.

[7] X.-M. Qian, S. Y. Lou, and X.-B. Hu, "Variable separation approach for a differential-difference system: special Toda equation," *Journal of Physics A: Mathematical and General*, vol. 37, no. 6, pp. 2401–2411, 2004.

[8] X. Y. Tang and Z. F. Liang, "Variable separation solutions for the -dimensional Jimbo–Miwa equation," *Physics Letters A*, vol. 351, no. 6, pp. 398–402, 2006.

[9] R. Zhang and S. Shen, "A new exact solution and corresponding localized excitations of the (2+1)-dimensional mKdV equation," *Physics Letters A*, vol. 370, no. 5-6, pp. 471–476, 2007.

[10] C. Z. Qu and S. F. Shen, "Nonlinear evolution equations admitting multilinear variable separable solutions," *Journal of Mathematical Physics*, vol. 50, no. 10, Article ID 103522, 2009.

[11] S. F. Shen, Y. Y. Jin, and J. Zhang, "Bäcklund Transformations and solutions of some generalized nonlinear evolution equations," *Reports on Mathematical Physics*, vol. 73, no. 2, pp. 225–279, 2014.

[12] K. Zhou, J.-D. Peng, G.-F. Wang, S.-J. Zhan, S. Shen, and Y. Jin, "New exact solutions of some (2+1)-dimensional Burgers-type systems and interactions," *Nonlinear Dynamics*, vol. 108, no. 4, pp. 4115–4122, 2022.

[13] C.-L. Bai and H. Zhao, "Interactions among periodic waves and solitary waves for a higher dimensional system," *Journal of Physics A: Mathematical and General*, vol. 39, no. 13, pp. 3283–3293, 2006.

[14] G. Qu, X. Hu, Z. Miao, S. Shen, and M. Wang, "Soliton molecules and abundant interaction solutions of a general high-order Burgers equation," *Results in Physics*, vol. 23, Article ID 104052, 2021.

[15] M. Singh, "New exact solutions for (3+1)-dimensional Jimbo–Miwa equation," *Nonlinear Dynamics*, vol. 84, no. 2, pp. 875–880, 2016.

[16] T. Mathanaranjan, M. S. Hashemi, H. Rezazadeh, L. Akinyemi, and A. Bekir, "Chirped optical solitons and stability analysis of the nonlinear Schrödinger equation with nonlinear chromatic dispersion," *Communications in Theoretical Physics*, vol. 75, no. 8, Article ID 085005, 2023.

[17] T. Mathanaranjan, "Optical solitons and stability analysis for the new (3+1)-dimensional nonlinear Schrödinger equation," *Journal of Nonlinear Optical Physics & Materials*, vol. 32, no. 2, Article ID 2350016, 2023.

[18] T. Mathanaranjan, D. Kumar, H. Rezazadeh, and L. Akinyemi, "Optical solitons in metamaterials with third and fourth order dispersions," *Optical and Quantum Electronics*, vol. 54, no. 5, Article ID 271, 2022.

[19] A. Silem, J. Lin, and N. Akhtar, "Nonautonomous dynamics of local and nonlocal Fokas–Lenells models," *Journal of Physics A: Mathematical and Theoretical*, vol. 56, no. 36, Article ID 365201, 2023.

[20] Q. Wang, D. Mihalache, M. R. Belic, L. W. Zeng, and J. Lin, "Spiraling Laguerre–Gaussian solitons and arrays in parabolic potential wells," *Optics Letters*, vol. 48, no. 16, Article ID 4233, 2023.

[21] Y.-J. Cai, J.-W. Wu, and J. Lin, "Nondegenerate N-soliton solutions for Manakov system," *Chaos, Solitons & Fractals*, vol. 164, Article ID 112657, 2022.