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Research Article

Extended Conformable *K*-Hypergeometric Function and Its Application

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The extended conformable k-hypergeometric function finds various applications in physics due to its ability to describe complex mathematical relationships arising in different physical scenarios. Here are a few instances of its uses in physics, including nuclear physics, fluid dynamics, quantum mechanics, and astronomy. The main objectives of this paper are to introduce the extended conformable k-hypergeometric and confluent hypergeometric functions by utilizing the new definition of the (α, k)-beta function and studying its important properties, like integral representation, summation formula, derivative formula, transform formula, and generating function. Also, introduce the extension of the Riemann–Liouville fractional derivative and establish some results related to the newly defined fractional operator, such as the Mellin transform and relations to extended (α, k)-hypergeometric functions.

1. Introduction

In the 20th century, there have been various waves of interest in special functions. A large number of special functions are defined in applied mathematics using improper integrals or infinite series. Special functions are essential tools for addressing particular problems in a wide range of domains, including scientific research, computational physics, chemistry, and statistical applications in technology [1–4]. Special functions are of great importance due to their extensive use in both pure and applied mathematics. One of the most significant special functions is the hypergeometric function [5, 6], which has many uses in the fields of evaluation of data, statistical theory, radio frequency field theory, quantum physics, and algebraic number theory [7–9]. The hypergeometric series is introduced by John Wallis in his book Arithmetica Infinitorum. Leonhard Euler studied the hypergeometric series, and Carl Friedrich Gauss (1813) presented the first complete standard method. The hypergeometric function is defined as follows:

$$F(\mu_1, \mu_2; \mu_3; w) = \sum_{p=0}^{\infty} \frac{(\mu_1)_p (\mu_2)_p}{(\mu_3)_p} \frac{(w)^p}{(p)!}, \tag{1}$$

where $w \in C$ and $\mathscr{R}(\mu_1)$, $\mathscr{R}(\mu_2)$, $\mathscr{R}(\mu_3) > 0$.

In order to extend the factorial to noninteger values, the Swiss mathematician Leonhard Euler (1707–1783) introduced the gamma function [10]. The definite integral defines the Gamma function as follows:

$$\Gamma(\mu_1) = \int_0^{\infty} e^{-\vartheta} (\vartheta)^{\mu_1 - 1} d\vartheta, \tag{2}$$

where Γ is the Gamma function and $\mathscr{R}(\mu_1) > 0$. The beta function [11] is a major and versatile special function that has many uses in a wide range of scientific and engineering fields. The beta function is used to express a variety of basic functions and unique polynomials. Legendre and Euler were the first mathematicians to discover the concept of the beta function by the name of Jacques Binet, using the symbol of the capital Latin word B or the capital Greek word β. $B(\mu_2, \mu_3)$ is a common form of beta function. Also, it has a symmetrical form such as $B(\mu_2, \mu_3) = B(\mu_3, \mu_2)$. To obtain the beta function integral representation as follows:

$$B(\mu_2, \mu_3) = \int_0^1 \vartheta^{\mu_2 - 1}(1 - \vartheta)^{\mu_3 - 1} d\vartheta, \tag{3}$$

where $\mathscr{R}(\mu_2), \mathscr{R}(\mu_3) > 0$. The given beta function can be written in the form of a gamma function as follows:

$$B(\mu_2, \mu_3) = \frac{\Gamma\mu_2 \Gamma\mu_3}{\Gamma\mu_2 + \mu_3}. \tag{4}$$

These functions often arise as solutions to differential equations or integral equations that cannot be expressed using elementary functions alone [2]. Special functions are defined by explicit formulas, power series expansions, and integral representations that allow for their computation and analysis. Many researchers recently examined the extensions of the beta function and hypergeometric functions [12, 13]. By utilizing the extended beta function, Chaudhary et al. [14] introduced extended hypergeometric and confluent hypergeometric functions $B_b(\mu_2, \mu_3)$.

$$F(\mu_1, \mu_2; \mu_3; w) = \sum_{p=0}^{\infty} (\mu_1)_p \frac{B_b(\mu_2 + p, \mu_3 - \mu_2)}{B(\mu_2, \mu_3 - \mu_2)} \frac{(w)^p}{(p)!}, \tag{5}$$

where $b > 0$, $|w| < 1$, and $\mathscr{R}(\mu_1), \mathscr{R}(\mu_2), \mathscr{R}(\mu_3) > 0$.

$$\Phi(\mu_2, \mu_3; w) = \sum_{p=0}^{\infty} \frac{B_b(\mu_2 + p, \mu_3 - \mu_2)}{B(\mu_2, \mu_3 - \mu_2)} \frac{(w)^p}{(p)!}, \tag{6}$$

where $b > 0$, $|w| < 1$, and $\mathscr{R}(\mu_3) > \mathscr{R}(\mu_2) > 0$.

Extended beta function introduced by Shadab et al. [15]. The definition of the extended beta function is defined as follows:

$$B_{b,\alpha}(\mu_2, \mu_3) = \int_0^1 (\vartheta)^{\mu_2 - 1}(1 - \vartheta)^{\mu_3 - 1} E_{(\alpha)}\left(\frac{-b}{(\vartheta(1 - \vartheta))}\right) d\vartheta, \tag{7}$$

where $b, \alpha > 0$ and $\mathscr{R}(\mu_3) > \mathscr{R}(\mu_2) > 0$.

$E_{(\alpha)}(w)$ is the Mittag–Leffler function [16] which is defined as follows:

$$E_\alpha(w) = \sum_{p=0}^{\infty} \frac{w^p}{\Gamma(p(\alpha) + 1)}, \tag{8}$$

where $\alpha, w \in C$. Recently, a novel concept known as conformable fractional calculus derivatives and integrals of fractional order, depending upon the fundamental limit explanation for derivatives [17]. The main point of the conformable fractional calculus principle is how to calculate the derivative and integral for either rational numbers or real numbers in fractional order. The conformable fractional calculus can be used to simulate complicated events in a variety of scientific and engineering fields.

The objectives of the manuscript are as follows: In Section 2, we list some basic definitions and terminologies that are needed in the paper. In Section 3, we introduce the extended conformable k-beta function and discuss its properties. In Section 4, we introduce the extended conformable k-Gauss and confluent hypergeometric functions and obtain integral and differentiation formulas. In addition, transformation, summation formulas, and generating functions are established. Extensions of the Riemann–Liouville fractional derivatives are presented in Section 5. Lastly, we highlight our observations and outlook in Section 6.

2. Preliminaries and Basic Concepts

In this section, we discuss some basic definitions and terminologies which are used further in this research.

Definition 1. Given a function $f : [0, \infty) \to \mathscr{R}$. Then the "conformable fractional derivative" of f of order α is defined by Khalil et al. [18] as follows:

$$D_\alpha(f)(\mu) = \lim_{\varepsilon \to 0} \frac{f(\mu + \varepsilon(\mu)^{1-\alpha}) - f(\mu)}{\varepsilon}, \tag{9}$$

where $\mu > 0$, $\alpha \in (0, 1)$.

Definition 2. Let $\alpha \in (0, 1)$, the conformable fractional integral [18] of the continuous $f : p, q \subseteq [0, \infty) \to \mathscr{R}$ of order α as follows:

$$I_\alpha(f)(\mu) = I_p^q(\mu^{\alpha-1} f) = \int_p^q f(\phi)(\phi)^{\alpha-1} d\phi, \tag{10}$$

where the integral is the usual Riemann improper integral.

Definition 3. Daiz introduced the k-gamma function and k-beta function [19]. Many scholars were inspired by this work and investigated the properties of the k-beta function and k-hypergeometric function [20–22].

Let $k > 0$, then the definition of the k-Gamma function is defined as follows [23]:

$$\Gamma_k(\mu_1)=\lim_{p\to\infty}\frac{p!k^p(pk)^{\frac{\mu_1}{k}-1}}{(\mu_1)_{p,k}}, \tag{11}$$

where $\mu_1\in C$, $k\in R^+$ and is $(\mu_1)_{p,k}$ Pochhammer k-symbol, the Pochhammer k-symbol defined as follows:

$$(\mu_1)_{p,k}=\begin{cases}(\mu_1)(\mu_1+k)(\mu_1+2k)\ldots\ldots(\mu_1+(p-1)k); & \text{if } p\in N\\ 1; & p=0\end{cases}. \tag{12}$$

The relationship between Pochhammer k-symbol and k-gamma function as follows:

$$(\mu_1)_{p,k}=\frac{\Gamma_k(\mu_1+pk)}{\Gamma_k(\mu_1)}, \tag{13}$$

where $\mu_1\in C$, $k\in R^+$, $p\in N$, and the integral form of Γ_k is expressed below:

$$\Gamma_k(\mu_1)=\int_0^\infty \vartheta^{\mu_1-1}e^{\frac{\vartheta^k}{k}}d\vartheta. \tag{14}$$

Note that $\Gamma_k(\mu_1)\to\Gamma(\mu_1)$ for $k\to1$ where $\Gamma(\mu_1)$ is the classical gamma function (2).

Definition 4. Let $k>0$, then the k-beta matrix function is defined as follows:

$$B_k(\mu_2,\mu_3)=\frac{1}{k}\int_0^1 \vartheta^{\frac{\mu_2}{k}-1}(1-\vartheta)^{\frac{\mu_3}{k}-1}d\vartheta. \tag{15}$$

The relationship between the k-beta function and k-gamma function is as follows:

$$B_k(\mu_2,\mu_3)=\Gamma_k(\mu_2)\Gamma_k(\mu_3)\Gamma_k^{-1}(\mu_2+\mu_3). \tag{16}$$

Also, the relationship between $B_k(\mu_2,\mu_3)$ and $B(\mu_2,\mu_3)$ is as follows:

$$B_k(\mu_2,\mu_3)=\frac{1}{k}B\left(\frac{\mu_2}{k},\frac{\mu_3}{k}\right). \tag{17}$$

Note that $B_k(\mu_2,\mu_3)\to B(\mu_2,\mu_3)$ for $k\to1$ where $B(\mu_2,\mu_3)$ is the classical beta function (3).

Definition 5. Mehmet Zeki Sarı kaya introduced the conformable k-gamma function [24]. It is denoted by the $\Gamma_k^\alpha(z)$. Conformable k-gamma functions are useful in the solution of specific integrals and differential equations with power functions and exponential terms.

Let $\alpha\in(0,1)\to\mathscr{R}$, for $0<\mu<\infty$, conformable gamma function Γ_k^α is given by the following:

$$\Gamma_k^\alpha(\mu)=\lim_{p\to\infty}\frac{p!\alpha^p k^p(pk\alpha)^{\frac{\mu+\alpha-1}{k\alpha}-1}}{(\mu)_{p,k}^\alpha}, \tag{18}$$

where is $(\mu)_{p,k}^\alpha$ Pochhammer (α,k)-symbol, then Pochhammer (α,k)-symbol defined as follows:

$$(\mu)_{p,k}^\alpha=(\mu+\alpha-1)(\mu+\alpha-1+k\alpha)(\mu+\alpha-1+2k\alpha)\ldots\ldots(\mu+\alpha-1+(p-1)k\alpha). \tag{19}$$

Integral form of (α,k)-gamma function is represent as follows:

$$\Gamma_k^\alpha(\mu_1)=\int_0^\infty(\vartheta)^{\mu_1-1}e^{-\frac{\vartheta^{k\alpha}}{k\alpha}}d_\alpha\vartheta. \tag{20}$$

Note that $\Gamma_{\alpha,k}(\mu_1)\to\Gamma_k(\mu_1)$ for $\alpha\to1$ where $\Gamma_k(\mu_1)$ is the k-gamma function (14).

Definition 6. Let $\alpha\in(0,1)$, the (α,k)-beta function [24] is given by the formula as follows:

$$B_k^\alpha(\mu_2,\mu_3)=\frac{1}{k\alpha}\int_0^1(\vartheta)^{\frac{\mu_2}{k\alpha}-1}(1-\vartheta)^{\frac{\mu_3}{k\alpha}-1}d_\alpha\vartheta. \tag{21}$$

Note that $B_{\alpha,k}(\mu_2,\mu_3)\to B_k(\mu_2,\mu_3)$ for $\alpha\to1$ where $B_k(\mu_2,\mu_3)$ is the k-beta function (15).

The (α,k)-beta function is an extensively studied mathematical function with applications in areas such as probability theory, statistics, mathematical physics, and engineering. It plays a fundamental role in various mathematical and statistical models.

Proposition 1. *Assume $\mu_1\in\mathscr{C}$, $k>0$, and $|w|<\frac{1}{k}$, the following identity holds [19]:*

$$\sum_{p=0}^\infty(\mu_1)_{p,k}\frac{w^p}{p!}=(1-kw)^{-\frac{\mu_1}{k}}. \tag{22}$$

3. Extended (α,k)-Beta Function $B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)$

Here, we introduce a new extension of the extended (α,k)-beta function and investigate various properties.

Definition 7. Let $k>0$ and $\alpha\in(0,1)$, then the extended (α,k) -beta function as follows:

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\frac{1}{k\alpha}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_3}{\alpha k}-1}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta, \tag{23}$$

where all the real $\mathscr{R}(\mu_2),\mathscr{R}(\mu_3)>0$, $\vartheta\in C$, $Q\geq 0$ and $E_{(k,q_1,q_2)}$ is mittag-laffler k-function.

Remark 1. If we consider $\alpha=1$, $k=1$, in Equation (23), we obtain $B_{q_1,q_2}^{Q}(\mu_2,\mu_3)$.

$$B_{1,q_1,q_2}^{1,Q}(\mu_2,\mu_3)=B_{q_1,q_2}^{Q}(\mu_2,\mu_3). \tag{24}$$

Now, we discover some interesting relations between summation formulas and integral representation for $B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)$.

Theorem 1. *Let $\alpha\in(0,1)$ and $k>0$, then following integral representations holds:*

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\frac{2}{k\alpha}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{2\mu_2}{\alpha k}-1}(\sin\theta)^{\frac{2\mu_3}{k\alpha}-1}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k(\cos^2\theta\sin^2\theta)}\right)d_\alpha\theta, \tag{25}$$

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\frac{1}{k\alpha}\int_0^{\infty}\frac{(\theta)^{\frac{\mu_2}{\alpha k}-1}}{(1+\theta)^{\frac{\mu_3}{\alpha k}+\frac{\mu_2}{\alpha k}}}E_{(k,q_1,q_2)}\left(\frac{-Q^k(1+\theta)^2}{k\theta}\right)d_\alpha\theta, \tag{26}$$

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\frac{2^{1-\frac{\mu_2}{k\alpha}-\frac{\mu_3}{k\alpha}}}{k\alpha}\int_{-1}^{1}(1+\theta)^{\frac{\mu_2}{\alpha k}-1}(1-\theta)^{\frac{\mu_3}{k\alpha}}E_{(k,q_1,q_2)}\left(\frac{-4Q^k}{k(1-\theta^2)}\right)d_\alpha\theta, \tag{27}$$

where $\mathscr{R}(\mu_2),\mathscr{R}(\mu_3)>0$.

Proof. Using the definition (23),

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\frac{1}{k\alpha}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_3}{\alpha k}-1}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta, \tag{28}$$

by substituting $\vartheta=cos^2\theta$ in Equation (28),

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\frac{1}{k\alpha}\int_0^1(\cos^2\theta)^{\frac{\mu_2}{\alpha k}-1}(1-\cos^2\theta)^{\frac{\mu_3}{\alpha k}-1}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\cos^2\theta(1-\cos^2\theta)}\right)2\cos(\theta)\sin(\theta)d_\alpha\theta. \tag{29}$$

After some algebraic manipulation, the last expression reads as follows:

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\frac{2}{k\alpha}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{2\mu_2}{\alpha k}-1}(\sin\theta)^{\frac{2\mu_3}{k\alpha}-1}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k(\cos^2\theta\sin^2\theta)}\right)d_\alpha\theta. \tag{30}$$

And $\vartheta=\frac{\theta}{1+\theta}$, $\vartheta=\frac{1+\theta}{2}$ put in Equation (23) and obtain Equations (26) and (27). □

Theorem 2. *The function $B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)$ has the following summation formula:*

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\sum_{p=0}^{q}B_{k,q_1,q_2}^{\alpha,Q}(\mu_2+pk\alpha,\mu_3+qk\alpha-pk\alpha), \tag{31}$$

where $k>0$ $\alpha\in(0,1)$ and $\mathscr{R}(\mu_2),\mathscr{R}(\mu_3)>0$.

Proof. From the definition of extended (α,k)-beta function (23), we have the following:

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\frac{1}{k\alpha}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_3}{\alpha k}-1}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta, \tag{32}$$

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=\frac{1}{k\alpha}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_3}{\alpha k}-1}[\vartheta+(1-\vartheta)]E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta, \tag{33}$$

$$B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3)=B_{k,q_1,q_2}^{\alpha,Q}(\mu_2+k\alpha,\mu_3)+B_{k,q_1,q_2}^{\alpha,Q}(\mu_2,\mu_3+k\alpha). \tag{34}$$

Repeating the same arguments to the above two terms in Equation (34) as follows:

$$B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3)=B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+2k\alpha,\mu_3)+2B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+k\alpha,\mu_3+k\alpha)+B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3+2k\alpha). \tag{35}$$

By continuing this process and using mathematical induction, the desired outcome is obtained.

$$B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3)=\sum_{p=0}^{q}B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3+qk\alpha-pk\alpha). \tag{36}$$

□

Theorem 3. *Let $\alpha\in(0,1)$ and $k>0$, then Mellin transform hold,*

$$M\left[B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3)\right]=\Gamma_k(s)B^{\alpha}_{k}(\mu_2+s\alpha,\mu_3+s\alpha), \tag{37}$$

where $\mathcal{R}(s)>0, \mathcal{R}(\mu_2+s)>0, \mathcal{R}(\mu_3+s)>0$.

Proof.

$$M\left[B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3)\right]=\int_0^\infty Q^{s-1}\left(B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3)\right)dQ. \tag{38}$$

Using the definition of extended (α,k) beta function as follows:

$$M\left[B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3)\right]=\int_0^\infty Q^{s-1}\left(\frac{1}{k\alpha}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_3}{\alpha k}-1}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta\right)dQ. \tag{39}$$

However, the integral in Equation (39) can be simplified in terms of k-gamma function by substituting, $\theta=\frac{Q}{\vartheta^{\frac{1}{k}}(1-\vartheta)^{\frac{1}{k}}}$, we have the following:

$$\int_0^\infty Q^{s-1}E\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)dQ=\int_0^\infty(\theta^{s-1})E\left(\frac{-\theta^k}{k}\right)d\theta=\Gamma_k(s), \tag{40}$$

$$M\left[B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3)\right]=\frac{\Gamma_k(s)}{k\alpha}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}+\frac{s}{k}-1}(1-\vartheta)^{\frac{\mu_3}{k\alpha}+\frac{s}{k}-1}d_\alpha\vartheta, \tag{41}$$

$$M\left[B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3)\right]=\Gamma_k(s)B^{\alpha}_{k}(\mu_2+s\alpha,\mu_3+s\alpha). \tag{42}$$

□

4. Extended (α,k) Hypergeometric Function and Confluent Hypergeometric Function

In this section, we introduce the extended conformable k-hypergeometric function and confluent hypergeometric function utilizing the $B^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3)$.

Hypergeometric functions are a function of special functions that are extensively used in many branches of mathematics, physics, and engineering. The extended conformable k-hypergeometric function is a specialized mathematical function that has its roots in this domain. This function can be applied to a wider variety of mathematical expressions and situations because of the extension and conformability features that it incorporates. These functions are well known for their ability to depict series expansions and solutions to a wide range of differential equations. Specifically, the extended conformable k-hypergeometric Function provides a parameter k that gives the function flexibility and enables it to handle a wider range of mathematical circumstances.

The extended conformable k-hypergeometric function is defined as follows:

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=\sum_{p=0}^{\infty}(\mu_1)_{p,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B^{\alpha}_{k}(\mu_2,\mu_3-\mu_2)}\frac{w^{p\alpha}}{p!}, \tag{43}$$

where $\alpha\in(0,1)$, $k>0$, $|w^\alpha|<1$ [25] and $\mathcal{R}(\mu_1),\mathcal{R}(\mu_2),\mathcal{R}(\mu_3)>0$.

The extended conformable k-confluent hypergeometric function is defined as follows:

$$\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3,w^\alpha)=\sum_{p=0}^{\infty}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B^{\alpha}_{k}(\mu_2,\mu_3-\mu_2)}\frac{w^{p\alpha}}{p!}, \tag{44}$$

where $k>0$, $|w^\alpha|<1$, $\alpha\in(0,1)$, and $\mathcal{R}(\mu_2),\mathcal{R}(\mu_3)>0$.

Remark 2. If we consider $k=1$ and $\alpha=1$, then extended (α,k) hypergeometric function (43) reduces to extended hypergeometric function (5) and extended (α,k) confluent hypergeometric function (44) reduces to an extended confluent hypergeometric function (6).

Theorem 4. *The following integral representations for the extended (α,k)-hypergeometric function $F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w)$*

and confluent hypergeometric function $\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3;w)$ *holds:*

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=\frac{1}{k\alpha B^\alpha_k(\mu_2,\mu_3-\mu_2)}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_3-\mu_2}{\alpha k}-1}(1-kw^\alpha\vartheta)^{\frac{-\mu_1}{k}}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta, \tag{45}$$

where $k>0$, $\mathcal{R}(\mu_1),\mathcal{R}(\mu_2),\mathcal{R}(\mu_3)>0$, $|w^\alpha|<1$, *and* $\alpha\in(0,1)$,

$$\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3;w^\alpha)=\frac{1}{k\alpha B^\alpha_k(\mu_2,\mu_3-\mu_2)}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_3-\mu_2}{\alpha k}-1}e^{w^\alpha\vartheta}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta. \tag{46}$$

Proof. From the definition (43),

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=\sum_{p=0}^{\infty}(\mu_1)_{p,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B^\alpha_k(\mu_2,\mu_3-\mu_2)}\frac{w^{p\alpha}}{p!}. \tag{47}$$

By using the definition (23) in Equation (47),

$$B^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2)=\frac{1}{k\alpha B^\alpha_k(\mu_2,\mu_3-\mu_2)}\int_0^1(\vartheta)^{\frac{\mu_1}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_2}{\alpha k}-1}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)\left\{\sum_{p=0}^{\infty}(\mu_1)_{p,k}\left(\frac{\vartheta w^\alpha)^p}{p!}\right)\right\}d_\alpha\vartheta. \tag{48}$$

Using the proposition (22) and after some calculations,

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=\frac{1}{k\alpha B^\alpha_k(\mu_2,\mu_3-\mu_2)}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_3-\mu_2}{\alpha k}-1}(1-kw^\alpha\vartheta)^{\frac{-\mu_1}{k}}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta. \tag{49}$$

By simply using the same procedure, Equation (46) yields the desired outcome. □

Theorem 5. *The following derivative formula for extended* (α,k)*-hypergeometric and extended* (α,k)*-confluent hypergeometric function holds:*

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=\alpha(\mu_1)_{1,k}\frac{\Gamma^\alpha_k(\mu_3)\Gamma^\alpha_k(\mu_2+k\alpha)}{\Gamma^\alpha_k(\mu_2)\Gamma^\alpha_k(\mu_3+k\alpha)}F^{\alpha,Q}_{k,q_1,q_2}(\mu_1+k,\mu_2+k\alpha,\mu_3+k\alpha,w^\alpha), \tag{50}$$

$$\begin{aligned}&\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3,w^\alpha)\\&=(\alpha)\frac{\Gamma^\alpha_k(\mu_3)\Gamma^\alpha_k(\mu_2+k\alpha)}{\Gamma^\alpha_k(\mu_2)\Gamma^\alpha_k(\mu_3+k\alpha)}\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2+k\alpha,\mu_3+k\alpha,w^\alpha),\end{aligned} \tag{51}$$

where $k>0$, $\mathcal{R}(\mu_1),\mathcal{R}(\mu_2),\mathcal{R}(\mu_3)>0$, $|w^\alpha|<1$, *and* $\alpha\in(0,1)$.

Proof. From the definition of $F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)$,

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=\sum_{p=0}^{\infty}(\mu_1)_{p,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B^\alpha_k(\mu_2,\mu_3-\mu_2)}\frac{w^{p\alpha}}{p!}. \tag{52}$$

Differentiating Equation (52) with respect to w,

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha) = \sum_{p=1}^{\infty}(\mu_1)_{p,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B^{\alpha}_{k}(\mu_2,\mu_3-\mu_2)}\frac{(\alpha p)w^{(p-1)\alpha}}{p!}, \tag{53}$$

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha) = \sum_{p=1}^{\infty}(\mu_1)_{p,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B^{\alpha}_{k}(\mu_2,\mu_3-\mu_2)}\frac{\alpha)w^{(p-1)\alpha}}{(p-1)!}. \tag{54}$$

Then replace $p \rightarrow p+1$ in Equation (54),

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha) = \sum_{p=0}^{\infty}(\mu_1)_{p+1,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+(p+1)k\alpha,\mu_3-\mu_2)}{B^{\alpha}_{k}(\mu_2,\mu_3-\mu_2)}\frac{\alpha w^{(p\alpha)}}{p!}, \tag{55}$$

using the following formula, we obtain the derivative formulas:

$$\sum_{p=0}^{\infty}(\mu_1)_{p+1,k} = (\mu_1)_{1,k}(\mu_1+k)_{p,k}, \tag{56}$$

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha) = \alpha(\mu_1)_{1,k}\frac{\Gamma^{\alpha}_{k}(\mu_3)\Gamma^{\alpha}_{k}(\mu_2+k\alpha)}{\Gamma^{\alpha}_{k}(\mu_2)\Gamma^{\alpha}_{k}(\mu_3+k\alpha)}F^{\alpha,Q}_{k,q_1,q_2}(\mu_1+k,\mu_2+k\alpha,\mu_3+k\alpha,w^\alpha). \tag{57}$$

We obtain Equation (51) by using the same derivative technique.

$$\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3,w^\alpha) = (\alpha)\frac{\Gamma^{\alpha}_{k}(\mu_3)\Gamma^{\alpha}_{k}(\mu_2+k\alpha)}{\Gamma^{\alpha}_{k}(\mu_2)\Gamma^{\alpha}_{k}(\mu_3+k\alpha)}\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2+k\alpha,\mu_3+k\alpha,w^\alpha). \tag{58}$$

□

Theorem 6. *The following derivative formula for extended (α, k)-hypergeometric and extended (α,k)-confluent hypergeometric function holds:*

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha) = (\alpha)^g(\mu_1)_{g,k}\frac{\Gamma^{\alpha}_{k}(\mu_3)\Gamma^{\alpha}_{k}(\mu_2+gk\alpha)}{\Gamma^{\alpha}_{k}(\mu_2)\Gamma^{\alpha}_{k}(\mu_3+gk\alpha)}F^{\alpha,Q}_{k,q_1,q_2}(\mu_1+gk,\mu_2+gk\alpha,\mu_3+gk\alpha,w^\alpha), \tag{59}$$

$$\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3,w^\alpha) = (\alpha)^g\frac{\Gamma^{\alpha}_{k}(\mu_3)\Gamma^{\alpha}_{k}(\mu_2+gk\alpha)}{\Gamma^{\alpha}_{k}(\mu_2)\Gamma^{\alpha}_{k}(\mu_3+gk\alpha)}\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2+gk\alpha,\mu_3+gk\alpha,w^\alpha), \tag{60}$$

where $k>0$, $(\mathscr{R}(\mu_2),\mathscr{R}(\mu_3)>0)$, $|w^\alpha|<1$, and $\alpha\in(0,1)$.

Proof. From the definition (43),

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha) = \sum_{p=0}^{\infty}(\mu_1)_{p,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B^{\alpha}_{k}(\mu_2,\mu_3-\mu_2)}\frac{w^{p\alpha}}{p!}. \tag{61}$$

Differentiating "g" time with respect to w,

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=\sum_{p=g}^{\infty}(\mu_1)_{p,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B^\alpha_k(\mu_2,\mu_3-\mu_2)}\frac{p\alpha(p\alpha-1)(p\alpha-2)\ldots\ldots(p\alpha-(g-1)\alpha)w^{(p-g)\alpha}}{p!}.$$
$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=\sum_{p=g}^{\infty}(\mu_1)_{p,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B^\alpha_k(\mu_2,\mu_3-\mu_2)}\frac{\alpha^g(p)!w^{(p-g)\alpha}}{p!(p-g)!}. \tag{62}$$

Then replace $p\to p+g$ in Equation (62) after some calculations,

$$\begin{aligned}&F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)\\&=\sum_{p=0}^{\infty}(\mu_1)_{p+g,k}\frac{B^{\alpha,Q}_{k,q_1,q_2}(\mu_2+gk\alpha+pk\alpha,\mu_3-\mu_2)}{B^\alpha_k(\mu_2,\mu_3-\mu_2)}\frac{(\alpha^g)w^{(p)\alpha}}{p!},\end{aligned} \tag{63}$$

using the following formula, we obtain the derivative formulas:

$$\sum_{p=0}^{\infty}(\mu_1)_{g+p,k}=(\mu_1)_{g,k}(\mu_1+gk)_{p,k}, \tag{64}$$

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=(\alpha)^g(\mu_1)_{g,k}\frac{\Gamma^\alpha_k(\mu_3)\Gamma^\alpha_k(\mu_2+gk\alpha)}{\Gamma^\alpha_k(\mu_2)\Gamma^\alpha_k(\mu_3+gk\alpha)}F^{\alpha,Q}_{k,q_1,q_2}(\mu_1+gk,\mu_2+gk\alpha,\mu_3+gk\alpha,w^\alpha). \tag{65}$$

Achieve the result, Equation (60), by using the same parallel line of explanation in the above term,

$$\begin{aligned}&\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3,w^\alpha)\\&=(\alpha)^g\frac{\Gamma^\alpha_k(\mu_3)\Gamma^\alpha_k(\mu_2+gk\alpha)}{\Gamma^\alpha_k(\mu_2)\Gamma^\alpha_k(\mu_3+gk\alpha)}\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2+gk\alpha,\mu_3+gk\alpha,w^\alpha).\end{aligned} \tag{66}$$

□

Theorem 7. *The following transformation and summation formulas hold:*

$$\begin{aligned}&F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3;w^\alpha)\\&=(1-kw^\alpha)^{\frac{-\mu_1}{k}}F^{\alpha,Q}_{k,q_1,q_2}\left(\mu_1,\mu_3-\mu_2,\mu_3;\frac{-kw^\alpha}{1-kw^\alpha}\right),\end{aligned} \tag{67}$$

$$\begin{aligned}&F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3;w^\alpha)\\&=(kw^\alpha)^{\frac{\mu_1}{k}}F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_3-\mu_2,\mu_3;1-kw^\alpha),\end{aligned} \tag{68}$$

$$\begin{aligned}&F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3;w^\alpha)\\&=(1+kw^\alpha)^{\frac{\mu_1}{k}}F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_3-\mu_2,\mu_3;-kw^\alpha),\end{aligned} \tag{69}$$

$$\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_2,\mu_3;w^\alpha)=e^{kw^\alpha}\Phi^{\alpha,Q}_{k,q_1,q_2}(\mu_3-\mu_2,\mu_3;-kw^\alpha), \tag{70}$$

where $k>0$, $\mathscr{R}(\mu_1),\mathscr{R}(\mu_2),\mathscr{R}(\mu_3)>0$, $|w^\alpha|<1$, *and* $\alpha\in(0,1)$.

Proof. Using the result of Theorem (4),

$$F^{\alpha,Q}_{k,q_1,q_2}(\mu_1,\mu_2,\mu_3,w^\alpha)=\frac{1}{k\alpha B^\alpha_k(\mu_2,\mu_3-\mu_2)}\int_0^1(\vartheta)^{\frac{\mu_2}{\alpha k}-1}(1-\vartheta)^{\frac{\mu_3-\mu_2}{\alpha k}-1}(1-kw^\alpha\vartheta)^{\frac{-\mu_1}{k}}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta. \tag{71}$$

Replacing ϑ by $1-\vartheta$ and substituting,

$$(1-kw^\alpha(1-\vartheta))^{\frac{-\mu_1}{k}}=(1-kw^\alpha)^{\frac{-\mu_1}{k}}\left(1+\frac{kw^\alpha}{1-kw^\alpha}\vartheta\right)^{\frac{-\mu_1}{k}}, \tag{72}$$

$$F_{k,q_1,q_2}^{\alpha,Q}(\mu_1,\mu_2,\mu_3,w^\alpha)=\frac{(1-kw^\alpha)^{\frac{-\mu_1}{k}}}{k\alpha B_k^\alpha(\mu_2,\mu_3-\mu_2)}\int_0^1(1-\vartheta)^{\frac{\mu_2}{\alpha k}-1}(\vartheta)^{\frac{\mu_3-\mu_2}{\alpha k}-1}\left(1+\frac{kw^\alpha}{1-kw^\alpha}\vartheta\right)^{\frac{-\mu_1}{k}}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta,\quad(73)$$

$$=\frac{(1-kw^\alpha)^{\frac{-\mu_1}{k}}}{k\alpha B_k^\alpha(\mu_2,\mu_3-\mu_2)}\int_0^1(1-\vartheta)^{\frac{\mu_2}{\alpha k}-1}(\vartheta)^{\frac{\mu_3-\mu_2}{\alpha k}-1}\left(1-\frac{-kw^\alpha}{1-kw^\alpha}\vartheta\right)^{\frac{-\mu_1}{k}}E_{(k,q_1,q_2)}\left(\frac{-Q^k}{k\vartheta(1-\vartheta)}\right)d_\alpha\vartheta.\quad(74)$$

In view of Equation (45), we get the desired result in Equation (67). Replacing $1-\frac{1}{kw^\alpha}$ and $\frac{kw^\alpha}{1+kw^\alpha}$ in Equation (67) yield Equations (68) and (69), respectively. Similarly, as Equation (67), we can establish Equation (70). □

Theorem 8. *The extended conformable* k*-hypergeometric function has the following summation formula:*

$$F_{k,q_1,q_2}^{\alpha,Q}(\mu_1,\mu_2,\mu_3;1)=\frac{B_{k,q_1,q_2}^{\alpha,Q}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B_k^\alpha(\mu_2,\mu_3-\mu_2)}.\quad(75)$$

Proof. Putting $w=1$ in Equation (43) and using the definition (23), we obtain the desired result. □

Theorem 9. *Let $\alpha\in(0,1)$ and $k>0$,then, the following generating function holds true:*

$$\sum_{r=0}^{\infty}(\mu_1)_{r,k}F_{k,q_1,q_2}^{\alpha,Q}(\mu_1+rk,\mu_2,\mu_3;w^\alpha)\frac{u^{\alpha r}}{r!}=(1-ku^\alpha)^{\frac{-\mu_1}{k}}F_{k,q_1,q_2}^{\alpha,Q}\left(\mu_1,\mu_3-\mu_2,\mu_3;\frac{w^\alpha}{1-ku^\alpha}\right),\quad(76)$$

where $|w^\alpha|<1$ and $|u^\alpha|<1$.

Proof.

$$\sum_{r=0}^{\infty}(\mu_1)_{r,k}\left\{F_{k,q_1,q_2}^{\alpha,Q}(\mu_1+rk,\mu_2,\mu_3;w^\alpha)\right\}\frac{u^{\alpha r}}{r!}=\sum_{r=0}^{\infty}(\mu_1)_{r,k}\left\{\sum_{p=0}^{\infty}(\mu_1+rk)_{p,k}\frac{B_{k,q_1,q_2}^{\alpha,Q}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B_k^\alpha(\mu_2,\mu_3-\mu_2)}\frac{w^{p\alpha}}{p!}\right\}\frac{u^{\alpha r}}{r!},\quad(77)$$

$$\sum_{r=0}^{\infty}(\mu_1+pk)_{r,k}\frac{u^{\alpha r}}{r!}\left[\sum_{p=0}^{\infty}(\mu_1)_{p,k}\frac{B_{k,q_1,q_2}^{\alpha,Q}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B_k^\alpha(\mu_2,\mu_3-\mu_2)}\frac{w^{p\alpha}}{p!}\right].\quad(78)$$

Using the proposition (22) and after some calculation,

$$=(1-ku^\alpha)^{\frac{-\mu_1+pk}{k}}\left[\sum_{p=0}^{\infty}(\mu_1)_{p,k}\frac{B_{k,q_1,q_2}^{\alpha,Q}(\mu_2+pk\alpha,\mu_3-\mu_2)}{B_k^\alpha(\mu_2,\mu_3-\mu_2)}\frac{w^{p\alpha}}{p!}\right],\quad(79)$$

$$\sum_{r=0}^{\infty}(\mu_1)_{r,k}F_{k,q_1,q_2}^{\alpha,Q}(\mu_1+rk,\mu_2,\mu_3;w^\alpha)\frac{u^{\alpha r}}{r!}=(1-ku^\alpha)^{\frac{-\mu_1}{k}}F_{k,q_1,q_2}^{\alpha,Q}\left(\mu_1,\mu_3-\mu_2,\mu_3;\frac{w^\alpha}{1-ku^\alpha}\right).\quad(80)$$

□

5. Application in Fractional Calculus

In this section, we examine the new extension of the Riemann–Liouville k-fractional derivative (RLKFD) for the extended conformable k-hypergeometric function. From the close relationship of the family of extended conformable k-hypergeometric functions with many special functions, we can easily construct various known and new fractional equations.

Fractional calculus and its applications [26] have been extensively studied by numerous scholars across a wide range of fields for many years, and interest in this subject has grown significantly. Fractional differential and integral equations are multidisciplinary and find application in a wide range of domains, including signal analysis, biomathematics, elasticity, electric motors, circuit systems, continuum mechanics, heat transport, quantum physics, and fluid mechanics.

Riemann–Liouville fractional derivative of order μ is given as follows [27]:

$$D^{R,\mu}[\vartheta^A]=\frac{1}{\Gamma(-\mu)}\int_0^\vartheta(\vartheta-t)^{-\mu-1}t^A dt,\quad(81)$$

where $\mathscr{R}(\mu)>0$. In particular, for case $p-1<\mathscr{R}(\mu)<p$, where $p=1,2,\ldots,$ is written by the following:

$$D^{R,\mu}[\vartheta^A]=\frac{d^p}{d\vartheta^p}D^{R,\mu-p}[\vartheta^A].\quad(82)$$

$$=\frac{d^p}{d\vartheta^p}\left\{\frac{1}{\Gamma(-\mu+p)}\int_0^\vartheta(\vartheta-t)^{-\mu+p-1}t^A dt\right\}.\quad(83)$$

Riemann–Liouville Fractional integral [28] of order μ is given as follows:

$$I^{R,\mu}[\vartheta^A]=\frac{1}{\Gamma(\mu)}\int_0^\vartheta(\vartheta-t)^{\mu-1}t^A dt,\quad(84)$$

where $\mathscr{R}(\mu)>0$. Recently, Rahman et al. [29] and Azam et al. [30] introduced RLKFD of order μ is defined as follows:

$$D_k^{R,\mu}\left[\vartheta^{\frac{A}{k}}\right]=\frac{1}{k\Gamma_k(-\mu)}\int_0^{\vartheta}(\vartheta-t)^{\frac{-\mu}{k}-1}t^{\frac{A}{k}}\,dt. \tag{85}$$

Definition 8. Assume $\alpha\in(0,1)$ and $k\in\mathscr{R}^+$, then new extension of conformable Riemann–Liouville k-fractional derivative as follows:

$$D_{k,q_1,q_2,Q}^{R,\mu,\alpha}\left[\vartheta^{\frac{A}{k\alpha}}\right]=\frac{1}{k\alpha\Gamma_k(-\mu)}\int_0^{\vartheta}(\vartheta^{\alpha}-t)^{\frac{-\mu}{k\alpha}-1}E_{k,q_1,q_2}\left(\frac{-Q^k\vartheta^{2\alpha}}{kt(\vartheta^{\alpha}-t)}\right)t^{\frac{A}{k\alpha}}\,dt, \tag{86}$$

where $\mathscr{R}(q_1),\mathscr{R}(q_2)>0$ and $\mathscr{R}(\mu)>0$.

Theorem 10. *The following result holds:*

$$D_{\alpha,q_1,q_2,Q}^{R,\mu,k,\alpha}\left[\vartheta^{\frac{A}{k\alpha}}\right]=\frac{\vartheta^{\frac{A-\mu}{k}}}{k\alpha\Gamma_k(-\mu)}B_{k,q_1,q_2}^{Q,\alpha}(A+k\alpha,-\mu), \tag{87}$$

where $k>0$, $\alpha\in(0,1)$, $\mathscr{R}(q_1)$, $\mathscr{R}(q_2)>0$, and $\mathscr{R}(\mu)>0$.

Proof. Using definition (8).

$$D_{k,q_1,q_2,S}^{R,\mu,\alpha}\left[\vartheta^{\frac{A}{k\alpha}}\right]=\frac{1}{k\alpha\Gamma_k(-\mu)}\int_0^{\vartheta}(\vartheta^{\alpha}-t)^{\frac{-\mu}{k\alpha}-1}E_{k,q_1,q_2}\left(\frac{-Q^k\vartheta^{2\alpha}}{kt(\vartheta^{\alpha}-t)}\right)t^{\frac{A}{k\alpha}}\,dt. \tag{88}$$

Then substitute $t=x\vartheta^{\alpha}$ in Equation (88), and after some calculation, we get the following:

$$D_{k,q_1,q_2,Q}^{R,\mu,\alpha}\left[\vartheta^{\frac{A}{k\alpha}}\right]=\frac{(\vartheta^{\alpha})^{\frac{A-\mu I}{k\alpha}}}{k\alpha\Gamma_k(-\mu)}\int_0^1(x)^{\frac{A}{k\alpha}}(1-x)^{\frac{-\mu}{k\alpha}-1}E_{k,q_1,q_2}\left(\frac{-Q^k}{kx(1-x)}\right)dx. \tag{89}$$

Using the definition (23), this is the desired result.

$$D_{k,q_1,q_2,Q}^{R,\mu,\alpha}\left[\vartheta^{\frac{A}{k\alpha}}\right]=\frac{\vartheta^{\frac{A-\mu I}{k}}}{k\alpha\Gamma_k(-\mu)}B_{k,q_1,q_2}^{Q}(A+k\alpha,-\mu). \tag{90}$$

□

Theorem 11. *Consider $k>0$, $\alpha\in(0,1)$, then the following result holds:*

$$D_{k,q_1,q_2,S}^{R,A-\mu,\alpha}\left[\vartheta^{\frac{A}{k\alpha}-1}(1-k\vartheta)^{\frac{-B}{k\alpha}}\right]=\frac{(\vartheta^{\alpha})^{\frac{\mu}{k\alpha}-1}}{\Gamma_k(\mu-A)}F_{k,q_1,q_2}^{\alpha,Q}(B,A,\mu;\vartheta^{\alpha}). \tag{91}$$

Proof. Using the definition of RL k-fractional derivative (8),

$$D_{k,q_1,q_2,S}^{R,A-\mu,\alpha}\left[\vartheta^{\frac{A}{k\alpha}-1}(1-k\vartheta)^{\frac{-B}{k\alpha}}\right]=\frac{1}{k\alpha\Gamma_k(\mu-A)}\int_0^{\vartheta}t^{\frac{A}{k\alpha}-1}(\vartheta^{\alpha}-t)^{\frac{\mu-A}{k\alpha}-1}(1-kt)^{\frac{-B}{k\alpha}}E_{k,q_1,q_2}\left(\frac{-Q^k\vartheta^{2\alpha}}{kt(\vartheta^{\alpha}-t)}\right)dt. \tag{92}$$

Then, put $t=\vartheta^{\alpha}u$ in Equation (92) and changing the limit from $t=0\rightarrow u=0$ to $t=\vartheta\rightarrow u=1$, after some rearranging of the terms,

$$=\frac{1}{k\alpha\Gamma_k(\mu-A)}\int_0^1(\vartheta^{\alpha}u)^{\frac{A}{k\alpha}-1}(\vartheta^{\alpha}-\vartheta^{\alpha}u)^{\frac{\mu-A}{k\alpha}-1}(1-k\vartheta^{\alpha}u)^{\frac{-B}{k\alpha}}E_{k,q_1,q_2}\left(\frac{-Q^k\vartheta^{2\alpha}}{k\vartheta^{\alpha}u(\vartheta^{\alpha}-\vartheta^{\alpha}u)}\right)\vartheta^{\alpha}du, \tag{93}$$

$$=\frac{(\vartheta)^{\frac{\mu}{k\alpha}-1}}{k\alpha\Gamma_k(\mu-A)}\int_0^1(u)^{\frac{A}{k\alpha}-1}(1-u)^{\frac{\mu-A}{k\alpha}-1}(1-k\vartheta^{\alpha}u)^{\frac{-B}{k\alpha}}E_{k,q_1,q_2}\left(\frac{-Q^k}{ku(1-u)}\right)du. \tag{94}$$

Now, using the integral representation of extended (α, k) hypergeometric function in Theorem (4), this is the desired result.

$$D^{R,A-\mu,\alpha}_{k,q_1,q_2,S}\left[\vartheta^{\frac{A}{k\alpha}-1}(1-k\vartheta)^{\frac{-B}{k\alpha}}\right]=\frac{(\vartheta^{\alpha})^{\frac{\mu}{k\alpha}-1}}{\Gamma_k(\mu-A)}F^{\alpha,Q}_{k,q_1,q_2}(B,A,\mu;\vartheta^{\alpha}). \tag{95}$$

□

Theorem 12. *The following result holds:*

$$M\left[e^{-\theta}D^{R,\mu,k,\alpha}_{\alpha,q_1,q_2,Q}\left\{\vartheta^{\frac{A}{k\alpha}}\right\}\right]=\Gamma(\theta)\frac{\vartheta^{\frac{A-\mu}{k}}}{k\alpha\Gamma_k(-\mu)}B^{Q,\alpha}_{k,q_1,q_2}(A+k\alpha,-\mu), \tag{96}$$

where $k>0,\alpha\in(0,1)$, $\mathscr{R}(q_1)$, $\mathscr{R}(q_2)>0$, and $\mathscr{R}(\mu)>0$.

Proof.

$$M\left[e^{-\theta}D^{R,\mu,k,\alpha}_{\alpha,q_1,q_2,Q}\left\{\vartheta^{\frac{A}{k\alpha}}\right\}\right]=\int_0^{\infty}(\theta^{s-1})\left\{e^{-\theta}D^{R,\mu,k,\alpha}_{\alpha,q_1,q_2,Q}\left\{\vartheta^{\frac{A}{k\alpha}}\right\}d\theta.\right. \tag{97}$$

Using definition (8).

$$M\left[e^{-\theta}D^{R,\mu,k,\alpha}_{\alpha,q_1,q_2,Q}\left\{\vartheta^{\frac{A}{k\alpha}}\right\}\right]=\frac{1}{k\alpha\Gamma_k(-\mu)}\int_0^{\infty}\theta^{s-1}e^{-\theta}\int_0^{\vartheta}(\vartheta^{\alpha}-t)^{\frac{-\mu}{k\alpha}-1}E_{k,q_1,q_2}\left(\frac{-Q^k\vartheta^{2\alpha}}{kt(\vartheta^{\alpha}-t)}\right)t^{\frac{A}{k\alpha}}dtd\theta, \tag{98}$$

$$M\left[e^{-\theta}D^{R,\mu,k,\alpha}_{\alpha,q_1,q_2,Q}\left\{\vartheta^{\frac{A}{k\alpha}}\right\}\right]=\frac{1}{k\alpha\Gamma_k(-\mu)}\int_0^{\infty}e^{-\theta}\theta^{s-1}d\theta\int_0^{\vartheta}(\vartheta^{\alpha}-t)^{\frac{-\mu}{k\alpha}-1}E_{k,q_1,q_2}\left(\frac{-Q^k\vartheta^{2\alpha}}{kt(\vartheta^{\alpha}-t)}\right)t^{\frac{A}{k\alpha}}dt. \tag{99}$$

Using Equation (2) and Theoren (10), this is the desired result.

$$M\left[e^{-\theta}D^{R,\mu,k,\alpha}_{\alpha,q_1,q_2,Q}\left\{\vartheta^{\frac{A}{k\alpha}}\right\}\right]=\Gamma(\theta)\frac{\vartheta^{\frac{A-\mu}{k}}}{k\alpha\Gamma_k(-\mu)}B^{Q,\alpha}_{k,q_1,q_2}(A+k\alpha,-\mu). \tag{100}$$

□

Theorem 13. *Consider $k>0$, $\alpha\in(0,1)$, then the following result holds:*

$$\begin{aligned}&M\left[e^{-\theta}D^{R,A-\mu,\alpha}_{k,q_1,q_2,S}\left\{\vartheta^{\frac{A}{k\alpha}-1}(1-k\vartheta)^{\frac{-B}{k\alpha}}\right\}\right]\\&\quad=\Gamma(\theta)\frac{(\vartheta^{\alpha})^{\frac{\mu}{k\alpha}-1}}{\Gamma_k(\mu-A)}F^{\alpha,Q}_{k,q_1,q_2}(B,A,\mu;\vartheta^{\alpha}).\end{aligned} \tag{101}$$

Proof.

$$\begin{aligned}&M\left[e^{-\theta}D^{R,A-\mu,\alpha}_{k,q_1,q_2,Q}\left\{\vartheta^{\frac{A}{k\alpha}-1}(1-k\vartheta)^{\frac{-B}{k\alpha}}\right\}\right]\\&\quad=\int_0^{\infty}\theta^{s-1}\left\{e^{-\theta}D^{R,A-\mu,\alpha}_{k,q_1,q_2,Q}\left\{(\vartheta)^{\frac{A}{k\alpha}-1}(1-k\vartheta)^{\frac{-B}{k\alpha}}\right\}\right\}d\theta.\end{aligned} \tag{102}$$

Using the definition of RL k-fractional derivative (8),

$$=\frac{1}{k\alpha\Gamma_k(\mu-A)}\int_0^{\infty}e^{-\theta}\theta^{s-1}\int_0^{\vartheta}t^{\frac{A}{k\alpha}-1}(\vartheta^{\alpha}-t)^{\frac{\mu-A}{k\alpha}-1}(1-kt)^{\frac{-B}{k\alpha}}E_{k,q_1,q_2}\left(\frac{-Q^k\vartheta^{2\alpha}}{kt(\vartheta^{\alpha}-t)}\right)dtd\vartheta. \tag{103}$$

Now, using Equation (2) and Theorem (11), this is the desired result.

$$\begin{aligned}&M\left[e^{-\theta}D^{R,A-\mu,\alpha}_{k,q_1,q_2,Q}\left\{\vartheta^{\frac{A}{k\alpha}-1}(1-k\vartheta)^{\frac{-B}{k\alpha}}\right\}\right]\\&\quad=\Gamma(\theta)\frac{(\vartheta^{\alpha})^{\frac{\mu}{k\alpha}-1}}{\Gamma_k(\mu-A)}F^{\alpha,Q}_{k,q_1,q_2}(B,A,\mu;\vartheta^{\alpha}).\end{aligned} \tag{104}$$

□

6. Conclusion

In this paper, we introduced a new extended conformable k-beta function in terms of the generalized Mittag–Leffler function, investigated its properties and its integral representations, also presented the extended conformable k-hypergeometric and extended conformable k-confluent hypergeometric functions. If consider $\alpha=1$, then all the results established in this paper will be true to the results related to the extended k-hypergeometric function. Some properties of these functions, such as integral representations, differentiation formulas, Mellin transformations, transformation and summation formulas, are also studied. The extended conformable k-hypergeometric function and conformable k-beta functions give the best solution for differential equations and integral order used in mathematics. Furthermore, established the new extended conformable Riemann–Liouville k-fractional derivative and derived some results containing extended conformable k-hypergeometric functions and extended conformable k-confluent hypergeometric functions. The extended conformable k-Riemann–Liouville definition of fractional derivatives plays an important role in the development of the theory of fractional calculus. It has numerous uses in the field of pure mathematics as well.

Data Availability

In this article, no data were utilized.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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