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Research Article

Monotone Iterative Technique for a Kind of Nonlinear Fourth-Order Integro-Differential Equations and Its Application

Yan Wang,[1] **Jinxiang Wang**,[2] **and Xiaobin Yao**[3]

[1]*School of Civil Engineering, Lanzhou University of Technology, Lanzhou 730050, China*
[2]*Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, China*
[3]*School of Mathematics and Statistics, Qinghai Nationalities University, Xi'ning, China*

Correspondence should be addressed to Jinxiang Wang; wjx19860420@163.com

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In this paper, we consider the existence and iterative approximation of solutions for a class of nonlinear fourth-order integro-differential equations (IDEs) with Navier boundary conditions. We first prove the existence and uniqueness of analytical solutions for a linear fourth-order IDE, which has rich applications in engineering and physics, and then we establish a maximum principle for the corresponding operator. Based upon the maximum principle, we develop a monotone iterative technique in the presence of lower and upper solutions to obtain iterative solutions for the nonlocal nonlinear problem under certain conditions. Some examples are presented to illustrate the main results.

1. Introduction

The aim of this paper is to develop a monotone iterative technique for the following nonlinear fourth-order Fredholm type integro-differential equation (IDE):

$$y^{(4)}(x) = f\left(x, y(x), \int_0^1 k(x,t)y(t)dt\right), x \in (0,1), \tag{1}$$

with the Navier boundary conditions as follows:

$$y(0) = y(1) = y''(0) = y''(1) = 0, \tag{2}$$

where $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R}), k \in C([0,1] \times [0,1], \mathbb{R})$.

Fourth-order differential equations were originally used in physics to describe the deformation of beams and plates (see [1], pp.1–2) and often take the form of linear equations. If the ends of the beam or the edges of the plate are simply supported, the equation may be completed by boundary conditions in the form of Equation (2). The first solution of the linear problem of bending of simply supported plates is due to Navier in the 1820s ([2], pp.105−109); for this reason, (2) has since been often referred to as Navier boundary condition. The nonlinear IDE boundary value problems (1) and (2), considered in this paper, can be seen as a generalization of the linear nonlocal fourth-order problem.

$$\begin{aligned} &y^{(4)}(x) + My(x) - N\int_0^1 k(x,t)y(t)dt = p(x), x \in (0,1) \\ &y(0) = y(1) = y''(0) = y''(1) = 0, \end{aligned} \tag{3}$$

where M, N are constants, $p \in C[0,1]$. Problem (3) arises from the models for suspension bridges [3, 4], quantum theory [5], and transient ultrasonic fields [6]. Since the nonlocal term under the integral sign will cause some mathematical difficulties, the analytical solutions for IDEs are usually not easy to obtain. For the linear fourth-order boundary value problems governed by IDEs like (3), only a few studies have been carried out by using numerical methods; see, e.g., [7–13] and the references therein.

The monotone iterative technique concerning upper and lower solutions is a powerful tool to solve nonlinear lower-order differential equations with various kinds of boundary conditions; see, e.g., [14–22] and the references therein. This technique has also been applied to the special case of (1) and (2) that f does not contain the integral term (see, e.g., [23–25]), namely simple local fourth-order boundary value problem as follows:

$$\begin{aligned} &y^{(4)}(x) = f(x, y(x)), x \in (0,1), \\ &y(0) = y(1) = y''(0) = y''(1) = 0. \end{aligned} \tag{4}$$

It is worth noticing that Cabada et al. [25] have pointed out that for general second-order differential equations with periodic, Neumann, or Dirichlet boundary conditions, it is well-known that the existence of a well-ordered pair of lower and upper solutions $\alpha \le \beta$ is sufficient to ensure the existence of a solution in the sector enclosed by them. However, this result is not true for fourth-order differential Equation (4); see the counterexample in [25, Remark 3.1]. Indeed, the application of the lower and upper solutions method in boundary value problems of the fourth order is heavily dependent on the conclusion of the maximum principle for the corresponding linear operators. For fourth-order local problems without integral terms in the equation but with lower-order derivative terms, related results on the lower and upper solutions method and monotone iterative technique, see [26–35] and the references therein.

We note that, on the one hand, the analytical solutions for fourth-order linear IDEs, such as Equation (3), have not been discussed in the aforementioned numerical solution works [7–13]. On the other hand, the existence and approximation of solutions for nonlinear nonlocal fourth-order Equation (1) with corresponding boundary value conditions have not been studied. Motivated by the above two factors, the main object of this paper is to discuss the analytical solution for the linear nonlocal problem (3) and then develop a monotone iterative technique in the presence of lower and upper solutions to solve the nonlinear nonlocal problems (1) and (2).

The rest paper is arranged as follows: In Section 2, we first prove a unique result of analytical solution for linear IDE (3) with inhomogeneous boundary value condition, and then we establish a maximum principle for the corresponding operator in (3). In Section 3, based upon the maximum principle, we develop a monotone iterative technique for nonlinear nonlocal problems (1) and (2) in the presence of lower and upper solutions under some monotonic condition on the nonlinearity f. In Section 4, we present two examples to illustrate the main results. The first one is a concrete nonlinear nonlocal fourth-order boundary value problems, and the second one is a general sixth-order boundary value problems, which can be transformed into fourth-order nonlocal problems like (1) and (2). Finally, Section 5 contains our conclusions.

2. The Linear Inhomogeneous Boundary Value Problem Governed by IDEs

In this section, we prove a unique result of solutions for Equation (3) with general inhomogeneous boundary value conditions and then establish the maximum principle for the corresponding operators.

As preliminaries, we first consider the following linear fourth-order inhomogeneous boundary value problem:

$$\begin{aligned} &y^{(4)}(x) + My(x) = p(x), \quad x \in (0,1), \\ &y(0) = A, y(1) = B, y''(0) = C, y''(1) = D, \end{aligned} \tag{5}$$

where $M > 0$ and A, B, C, D are constants, $p \in C[0,1]$. By Cabada [25, Lemma 2.1] or Ma et al. [28, Theorem 2.1], if $M \le c_0 \approx 950.8843$, then Problem (5) has a unique solution given by the following:

$$\begin{aligned} y(x) = &\int_0^1 G(x,s)p(s)ds + Aw(x) + Bw(1-x) + C\chi(x) \\ &+ D\chi(1-x), x \in [0,1], \end{aligned} \tag{6}$$

where

$$G(x,s) = \begin{cases} \dfrac{1}{2m^2}\left[\dfrac{\sin(mx)\sin(m(1-s))}{m\sin m} - \dfrac{\sinh(mx)\sinh(m(1-s))}{m\sinh m}\right], & 0 \le x \le s \le 1, \\ \dfrac{1}{2m^2}\left[\dfrac{\sin(ms)\sin(m(1-x))}{m\sin m} - \dfrac{\sinh(ms)\sinh(m(1-x))}{m\sinh m}\right], & 0 \le s \le x \le 1, \end{cases} \tag{7}$$

is the Green function for the boundary value problem as follows:

$$\begin{aligned} &y^{(4)}(x) + My(x) = 0, x \in (0,1), \\ &y(0) = y(1) = y''(0) = y''(1) = 0, \end{aligned} \tag{8}$$

with $M = m^4$; $w(x)$ is the unique solution of the inhomogeneous problem as follows:

$$y^{(4)}(x) + My(x) = 0, \quad y(0) = 1, y''(0) = y(1) = y''(1) = 0, \tag{9}$$

and $\chi(x)$ is the unique solution of the inhomogeneous problem as follows:

$$y^{(4)}(x) + My(x) = 0, \quad y(0) = 0, y''(0) = 1, y(1) = y''(1) = 0. \tag{10}$$

Moreover, by Cabada [25, Proposition 2.1.], when $0 < M = m^4 \le c_0 \approx 950.8843$, the Green function $G(x, s)$ given by (7) is nonnegative on $[0, 1] \times [0, 1]$.

Denote

$$h(x, A, B, C, D) = Aw(x) + Bw(1 - x) + C\chi(x) + D\chi(1 - x), \tag{11}$$

it is easy to see that h is the unique solution of the following:

$$y^{(4)}(x) + My(x) = 0, \quad x \in (0, 1), \tag{12}$$

$$y(0) = A, y(1) = B, y''(0) = C, y''(1) = D, \tag{13}$$

and the following conclusion holds:

Lemma 1. *Assume that $0 < M \le c_1 \approx 125.137$. Then if $A \ge 0$, $B \ge 0$, $C \le 0$, $D \le 0$, we have the following:*

$$h(x, A, B, C, D) = Aw(x) + Bw(1 - x) + C\chi(x) + D\chi(1 - x) \ge 0, \tag{14}$$

if $A \le 0$, $B \le 0$, $C \ge 0$, $D \ge 0$, we have the following:

$$h(x, A, B, C, D) = Aw(x) + Bw(1 - x) + C\chi(x) + D\chi(1 - x) \le 0. \tag{15}$$

Proof. By Cabada [25, Corollary 2.1.], if $0 < M \le c_1 \approx 125.137$, then $w(x) \ge 0, \forall x \in [0, 1]$ and $\chi(x) \le 0, \forall x \in [0, 1]$, and thus (14) and (15) are immediate consequences. □

Now, we give out the first main result of this section.

Theorem 1. *Assume that $0 < M \le c_0 \approx 950.8843$, and*

$$\|k\|_\infty = \max\{k(x, t) | (x, t) \in [0, 1] \times [0, 1]\} < \frac{1}{|N| \max_{x \in [0,1]} \int_0^1 G(x, s) ds}, \tag{16}$$

then for any $p \in C[0, 1]$ and constants A, B, C, D, the following nonlocal inhomogeneous boundary value problem:

$$\begin{aligned} &y^{(4)}(x) + My(x) - N \int_0^1 k(x, t) y(t) dt = p(x), x \in (0, 1), \\ &y(0) = A, y(1) = B, y''(0) = C, y''(1) = D, \end{aligned} \tag{17}$$

has a unique solution.

Proof. Observe that y is a solution of (17) if and only if y is a fixed point of the operator $K : C[0, 1] \to C[0, 1]$ given by the following: □

$$\begin{aligned} [Ky](x) = &\int_0^1 G(x, s) p(s) ds + N \int_0^1 \int_0^1 G(x, s) k(s, t) y(t) dt ds \\ &+ h(x, A, B, C, D), \end{aligned} \tag{18}$$

where $G(x, s)$ and h is as in (7) and (11), respectively.

For $u, v \in C[0, 1]$, we have the following:

$$\begin{aligned} \|Ku - Kv\|_\infty &= \left\| N \int_0^1 \int_0^1 G(x, s) k(s, t) [u(t) - v(t)] dt ds \right\|_\infty \\ &\le \|u - v\|_\infty |N| \left| \int_0^1 \int_0^1 G(x, s) k(s, t) dt ds \right| \\ &\le \|u - v\|_\infty |N| \|k\|_\infty \int_0^1 G(x, s) ds \\ &\le \|u - v\|_\infty |N| \|k\|_\infty \max_{x \in [0,1]} \int_0^1 G(x, s) ds, \end{aligned} \tag{19}$$

then, according to condition (16) and Banach fixed point theorem, there exists a unique fixed point for the operator K, which assures the existence and uniqueness of solution for (17).

Remark 1. When $A = B = C = D = 0$, the inhomogeneous problem (17) will degenerate into the homogeneous problem (3). Thus, we obtain a result of the existence and uniqueness of solutions for (3). As far as we know, this result is new.

In the sequel, we establish the maximum principle for the operator in (3) and (17).

We first derive an explicit expression of analytical solutions for (17). Using Picard's iterative method, we know that for any $y_0 \in C[0, 1]$, the sequence given by $y_n = Ky_{n-1}$, $n \ge 1$, converges to the unique solution given by Theorem 1. Taking

$$y_0 = \int_0^1 G(x, s) p(s) ds + h(x, A, B, C, D), \tag{20}$$

we get that

$$y_n(x) = y_0(x) + \int_0^1 Q_n(x,s)h(s,A,B,C,D)ds + \int_0^1 F_n(x,s)p(s)ds, \tag{21}$$

where

$$F_n(x,s) = \int_0^1 Q_n(x,t)G(t,s)dt, \tag{22}$$

$$Q_n(x,s) = \sum_{i=1}^{n} R^{(i)}(x,s). \tag{23}$$

Here

$$R^{(i)}(x,s) = \int_0^1 R^{(i-1)}(x,t)R(t,s)dt, i \geq 2, \tag{24}$$

and

$$R^{(1)}(x,s) = R(x,s) = N\int_0^1 G(x,t)k(t,s)dt. \tag{25}$$

By (16), we have the following:

$$N\int_0^1 G(x,s)k(s,t)dt \leq |N| \, \|k\|_\infty \max_{x\in[0,1]} \int_0^1 G(x,s)ds = d<1, \tag{26}$$

then

$$\left\|R^{(i)}\right\|_\infty \leq d^i, \tag{27}$$

and the series $\sum_{i=1}^{\infty} R^{(i)}(x,s)$ will converge to a function $Q \in C([0,1]\times[0,1])$. Meanwhile, $\{F_n(x,s)\}$ will converge to the function $F \in C([0,1]\times[0,1])$ given by the following:

$$F(x,s) = \int_0^1 Q(x,t)G(t,s)dt. \tag{28}$$

Now, by passing to the limit for the Picard's iterative $y_n = Ky_{n-1}$, we conclude that the unique analytical solution of (17) is given by the following:

$$y(x) = \int_0^1 G(x,s)p(s)ds + h(x,A,B,C,D) + \int_0^1 Q(x,s)h(s,A,B,C,D)ds + \int_0^1 F(x,s)p(s)ds. \tag{29}$$

Now, by Lemma 1, the expression (29), and the positivity of the Green function G, one can easily get the following maximum principle for problem (17).

Theorem 2. *Assume that* $0<M<c_1 \approx 125.137, Nk(x,t)\geq 0$ *and* $\|k\|_\infty < \frac{1}{|N| \max_{x\in[0,1]} \int_0^1 G(x,s)ds}$, *then*

(i) *If* $p(x)\geq 0, \forall x\in[0,1], A\geq 0, B\geq 0, C\leq 0, D\leq 0$, *then the unique solution* y *of (17) is nonnegative;*

(ii) *If* $p(x)\leq 0, \forall x\in[0,1], A\leq 0, B\leq 0, C\geq 0, D\geq 0$, *then the unique solution* y *of (17) is nonpositive.*

Remark 2. When $A=B=C=D=0, h(x,A,B,C,D)\equiv 0$, then the linear nonlocal fourth-order problem (3) has a unique analytical solution as follows:

$$y(x) = \int_0^1 G(x,s)p(s)ds + \int_0^1 F(x,s)p(s)ds, \tag{30}$$

according to (29). This explicit expression of analytical solutions can be seen as an improvement on the numerical solution work of (3) in [7–13].

Remark 3. In [25], the authors gave out the explicit expression of solutions for problem (17) when $N=0$, and then obtained similar maximum principle for linear local operator $Ly=y^{(4)}+My$, see [25, Lemma 2.1] and [25, Corollary 2.1] respectively. Thus, our results, (29), on the analytical solution of nonlocal problem (17) and the maximum principle Theorem 2 generalize corresponding results in [25].

Remark 4. The conclusion of Theorem 2 also holds for homogeneous problem (3). That is, we get a maximum principle for the fourth-order differential operator $Ly=y^{(4)}+My-N\int_0^1 k(x,t)y(t)dt$ in function space $D(L):=\{y\in C^4[0,1]: y(0)=y(1)=y''(0)=y''(1)=0\}$.

3. Main Results

We will use the following definition of lower and upper solutions:

Definition 1. The function $\alpha\in C^4[0,1]$ is said to be *a lower solution* for the BVP (1) and (2) if

$$\alpha^{(4)}(x) \leq f\left(x, \alpha(x), \int_0^1 k(x,t)\alpha(t)dt\right), x\in(0,1), \tag{31}$$

and

$$\alpha(0)\leq 0, \alpha(1)\leq 0, \alpha''(0)\geq 0, \alpha''(1)\geq 0. \tag{32}$$

An *upper solution* $\beta \in C^4[0,1]$ is defined analogously by reversing the inequalities in (31) and (32).

Theorem 3. *Assume* $f \in C([0,1]\times\mathbb{R}^2,\mathbb{R}), k \geq 0$ *and there exists a lower solution* α *and an upper solution* β *for problems (1) and (2) which satisfy the following:*

$$\alpha(x) \leq \beta(x) \quad \text{for } x \in [0,1]. \tag{33}$$

If there exist two constants $M, N > 0$ satisfying $M < c_1 \approx 125.137$ and $\|k\|_\infty < \frac{1}{|N| \max_{x\in[0,1]} \int_0^1 G(x,s)ds}$ such that

$$f(x,u_1,v_1) - f(x,u_2,v_2) \geq -M(u_1-u_2) + N(v_1-v_2), \tag{34}$$

for

$$\begin{gathered} \alpha(x) \leq u_2 \leq u_1 \leq \beta(x) \text{ and } \int_0^1 k(x,t)\alpha(t)dt \leq v_2 \\ \leq v_1 \leq \int_0^1 k(x,t)\beta(t)dt, x \in [0,1]. \end{gathered} \tag{35}$$

Then the iterative sequences $\{\alpha_n\}$ and $\{\beta_n\}$ produced by the iterative procedure as follows:

$$\begin{gathered} Ly_n(x) = f\left(x, y_{n-1}(x), \int_0^1 k(x,t)y_{n-1}(t)dt\right) + My_{n-1}(x) - N\int_0^1 k(x,t)y_{n-1}(t)dt, \\ y_n(0) = y_n(1) = y_n''(0) = y_n''(1) = 0, n = 1,2,\ldots, \end{gathered} \tag{36}$$

with the initial functions $y_0 = \alpha$ and $y_0 = \beta$, respectively, satisfy the following:

$$\alpha_{n-1} \leq \alpha_n \leq \beta_n \leq \beta_{n-1}, \tag{37}$$

and converge uniformly to the extremal solutions of BVP (1) and (2), in $[\alpha,\beta]$.

Proof. Define the mapping $E: C[0,1] \to C[0,1]$ by the following: □

$$\begin{aligned} E(\sigma)(x) = f&\left(x, \sigma(x), \int_0^1 k(x,t)\sigma(t)dt\right) + M\sigma(x) \\ &- N\int_0^1 k(x,t)\sigma(t)dt. \end{aligned} \tag{38}$$

Denote $\Phi = T \circ E$, where $T = L^{-1}: C[0,1] \to D(L)$. By Remark 1, it is easy to see that $T: C[0,1] \to C[0,1]$ is compact, then $\Phi: C[0,1] \to C[0,1]$ is completely continuous. Obviously, the solutions of (1) and (2) in $C[0,1]$ is equivalent to the fixed-points of the mapping Φ.

First, we show that

$$\alpha \leq u \leq \beta \Rightarrow \alpha \leq \Phi(u) \leq \beta. \tag{39}$$

Let $g = \Phi u - \alpha$, by the definition of the lower solutions and (34), we have the following:

$$\begin{aligned} &g^{(4)} + Mg - N\int_0 k(x,t)g(t)dt \\ &= \left[(\Phi u)^{(4)} + M(\Phi u) - N\int_0^1 k(x,t)(\Phi u)(t)dt\right] - \left[\alpha^{(4)} + M\alpha - N\int_0^1 k(x,t)\alpha(t)dt\right] \\ &= E(u)(x) - \left[\alpha^{(4)} + M\alpha - N\int_0^1 k(x,t)\alpha(t)dt\right] \\ &= \left[f\left(x, u, \int_0^1 k(x,t)u(t)dt\right) + Mu - N\int_0^1 k(x,t)u(t)dt\right] - \left[\alpha^{(4)} + M\alpha - N\int_0^1 k(x,t)\alpha(t)dt\right] \\ &\geq M(u-\alpha) - N\int_0^1 k(x,t)(u-\alpha)(t)dt + f\left(x,u,\int_0^1 k(x,t)u(t)dt\right) - f\left(x,\alpha,\int_0^1 k(x,t)\alpha(t)dt\right) \\ &\geq 0. \end{aligned} \tag{40}$$

On the other hand, since $\Phi: C[0,1] \to D(L)$, then

$$\begin{aligned} g(0) &= (\Phi u)(0) - \alpha(0) = -\alpha(0) \geq 0, \\ g(1) &= (\Phi u)(1) - \alpha(1) = -\alpha(1) \geq 0, \end{aligned} \tag{41}$$

and

$$\begin{aligned} g''(0) &= (\Phi u)''(0) - \alpha''(0) = -\alpha''(0) \leq 0, \\ g''(1) &= (\Phi u)''(1) - \alpha''(1) = -\alpha''(1) \leq 0. \end{aligned} \tag{42}$$

By maximum principle in Theorem 2. (i), (40) and (42) imply that $g \geq 0$ and then $\alpha \leq \Phi(u)$.

By a similar way, using the definition of the upper solutions, the maximum principle in Theorem 2, (ii) and (34), we can get that $\Phi(u) \leq \beta$, then (39) is proved.

Based upon Schauder fixed-point theorem, Φ has a fixed point in $[\alpha, \beta]$, which is a solution of (1) and (2).

Second, we show the following claim:

$$\beta \geq u_1 \geq u_2 \geq \alpha \Rightarrow \Phi u_1 \geq \Phi u_2. \tag{43}$$

In fact, let $\phi = \Phi u_1 - \Phi u_2$, by (34) again, we have the following:

$$\begin{aligned} &\phi^{(4)} + M\phi - N\int_0^1 k(x,t)\phi(t)dt \\ &= \left[(\Phi u_1)^{(4)} + M\Phi u_1 - N\int_0^1 k(x,t)(\Phi u_1)(t)dt\right] - \left[(\Phi u_2)^{(4)} + M\Phi u_2 - N\int_0^1 k(x,t)(\Phi u_2)(t)dt\right] \\ &= E(u_1)(x) - E(u_2)(x) = \left[f\left(x, u_1, \int_0^1 k(x,t)u_1(t)dt\right) + Mu_1 - N\int_0^1 k(x,t)u_1(t)dt\right] \\ &\quad - \left[f\left(x, u_2, \int_0^1 k(x,t)u_2(t)dt\right) + Mu_2 - N\int_0^1 k(x,t)u_2(t)dt\right] \\ &\geq 0. \end{aligned} \tag{44}$$

On the other hand,

$$\phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0. \tag{45}$$

Then, by Remark 2, we conclude that $\phi \geq 0$ and then the claim, (43), is proved.

By the definition of the mapping Φ, the iterative procedure, Equation (36), is equivalent to the iterative equation as follows:

$$y_n = \Phi y_{n-1}, n = 1, 2, \ldots \tag{46}$$

Define the iterative sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following:

$$\alpha_n = \Phi\alpha_{n-1} \quad \beta_n = \Phi\beta_{n-1}, \quad n = 1, 2, \ldots, \tag{47}$$

with $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Then, combining (39) with (43), it is easy to see that $\{\alpha_n\}$ and $\{\beta_n\}$ have the monotonicity (37). By the compactness of Φ and the monotonicity (37), it follows that $\{\alpha_n\}$ and $\{\beta_n\}$ are convergent in $C[0,1]$, that is, there exist $\underline{y}$ and $\overline{y} \in C[0,1]$ such that

$$\lim_{n\to\infty} \alpha_n(x) = \underline{y}(x), \qquad \lim_{n\to\infty} \beta_n(x) = \overline{y}(x). \tag{48}$$

On the other hand, it is easy to see that the operator Φ is continuous, then letting $n \to \infty$ in (47), we have the following:

$$\underline{y} = \Phi\left(\underline{y}\right), \quad \overline{y} = \Phi(\overline{y}), \tag{49}$$

thus, $\underline{y}$ and $\overline{y}$ are the solutions of (1) and (2).

Finally, we show that $\underline{y}$ and $\overline{y}$ are the extremal solutions of Equations (1) and (2) on $[\alpha, \beta]$.

Let $y \in [\alpha, \beta]$ be an arbitrary solution of problems (1) and (2), then combining (39) with (43) we have the following:

$$\Phi^n\alpha \leq \Phi^n y \leq \Phi^n\beta, \tag{50}$$

that is

$\alpha_n \leq y \leq \beta_n$.

Letting $n \to \infty$, we have the following:

$$\underline{y} \leq y \leq \overline{y}. \tag{51}$$

Hence, $\underline{y}$ and $\overline{y}$ are minimum and maximum solutions of (1) and (2) in $[\alpha, \beta]$, respectively.

Remark 5. In [25, Theorem 3.1 (I), Remark 3.1.3], the authors explored the method of lower and upper solutions for local problem (4) in order to prove the existence of solutions. This result can be seen as a particular case of our Theorem 3 when $f(x, y(x), \int_0^1 k(x,t)y(t)dt) = f(x, y(x))$, and thus Theorem 3 can be seen as an improvement on [25, Theorem 3.1 (I)].

4. Examples

We present two examples to illustrate the application of Theorem 3.

Example 1. Consider the nonlinear nonlocal fourth-order boundary value problem as follows:

$$
\begin{aligned}
&y^{(4)}(x) = \sin(\pi x)\left[\left(1+\frac{1}{2}\cos^2(\pi x)+y^2(x)\right)\int_0^1 y(t)dt + \frac{\sqrt{2}}{2}\left(\frac{\pi^5-3}{\pi}\right)\right], x\in(0,1),\\
&y(0)=y(1)=y''(0)=y''(1)=0.
\end{aligned}
\tag{52}
$$

It is easy to verify that problem (52) has an exact solution $y=\frac{\sqrt{2}}{2}\sin(\pi x)$.

Denote $k(x,t)=\sin(\pi x), (x,t)\in[0,1]\times[0,1]$ and $f(x, y(x), \int_0^1 k(x,t)y(t)dt) = (1+\frac{1}{2}\cos^2(\pi x)+y^2(x))\int_0^1 \sin(\pi x) y(t)dt + \sin(\pi x)\frac{\sqrt{2}}{2}(\frac{\pi^5-3}{\pi})$, then $k\ge 0, \|k\|_\infty = 1$ and the function as follows:

$$
f(x,u,v) = \left(1+\frac{1}{2}\cos^2(\pi x)+u^2\right)v + \sin(\pi x)\frac{\sqrt{2}}{2}\left(\frac{\pi^5-3}{\pi}\right),
\tag{53}
$$

is continuous on $[0,1]\times R^2$. Moreover, problem (52) is equivalent to the following:

$$
\begin{aligned}
&y^{(4)}(x) = f\left(x, y(x), \int_0^1 k(x,t)y(t)dt\right), x\in(0,1),\\
&y(0)=y(1)=y''(0)=y''(1)=0.
\end{aligned}
\tag{54}
$$

It is easy to see that $\alpha=0$ and $\beta=\sin(\pi x)$ are lower and upper solutions of (54), respectively, which satisfy (33).

Taking $M=\frac{2}{\pi}$, by (7) and simple computation, we have the following:

$$
\max_{x\in[0,1]}\int_0^1 G(x,s)ds = \frac{\cosh\sqrt[4]{\frac{2}{\pi}}}{\frac{2}{\pi}},
\tag{55}
$$

then, for any $0<N<\frac{\frac{2}{\pi}}{\cosh\sqrt[4]{\frac{2}{\pi}}}\approx 0.446$, it is not difficult to verify that (34) and other conditions in Theorem 3 are satisfied.

Observing that the linear problem corresponding to the example (52) is as follows:

$$
\begin{aligned}
&y^{(4)}(x) + \frac{2}{\pi}y(x) - N\int_0^1 \sin(\pi x)y(t)dt = p(x), x\in(0,1),\\
&y(0)=0, y(1)=0, y''(0)=0, y''(1)=0,
\end{aligned}
\tag{56}
$$

then, according to Theorem 1 and simple derivation, the analytical solution of (56) is as follows:

$$
y(x) = \int_0^1 G(x,s)p(s)ds + \frac{N\int_0^1 G(x,s)\sin(\pi s)ds}{1-N\int_0^1\int_0^1 G(x,s)\sin(\pi s)dsdx}\int_0^1\int_0^1 G(x,s)p(s)dsdx.
\tag{57}
$$

Now, we are ready to use the monotone iterative technique in Theorem 3. Taking $N=0.4$, substitute the lower solution $\alpha_0=0$ and the upper solution $\beta_0=\sin(\pi x)$ of problem (52) into right side of the iterative procedure (36), respectively, by using (57) we can calculate that

$$
\alpha_1 = 0.7066\ \sin(\pi x), \beta_1 = 0.7258\ \sin(\pi x),
\tag{58}
$$

consequently, the results of the second iteration are as follows:

$$
\alpha_2 = 0.7070\sin(\pi x), \beta_2 = 0.7206\sin(\pi x).
\tag{59}
$$

Compared to the exact solution $y=\frac{\sqrt{2}}{2}\sin(\pi x)$ of (52), the results show that the error is very small even after only one iteration, see Figure 1 below drawn with MATLAB:

Example 2. Consider the nonlinear sixth-order boundary value problem as follows:

$$
\begin{aligned}
&u^{(6)}(x) = f(x, u''(x), u(x)), x\in(0,1),\\
&u(0)=u(1)=u''(0)=u''(1)=u''''(0)=u''''(1)=0,
\end{aligned}
\tag{60}
$$

where $f\in C([0,1]\times\mathbb{R}^2,\mathbb{R})$. Let $u''(x)=y(x)$; then, (60) is equivalent to the following:

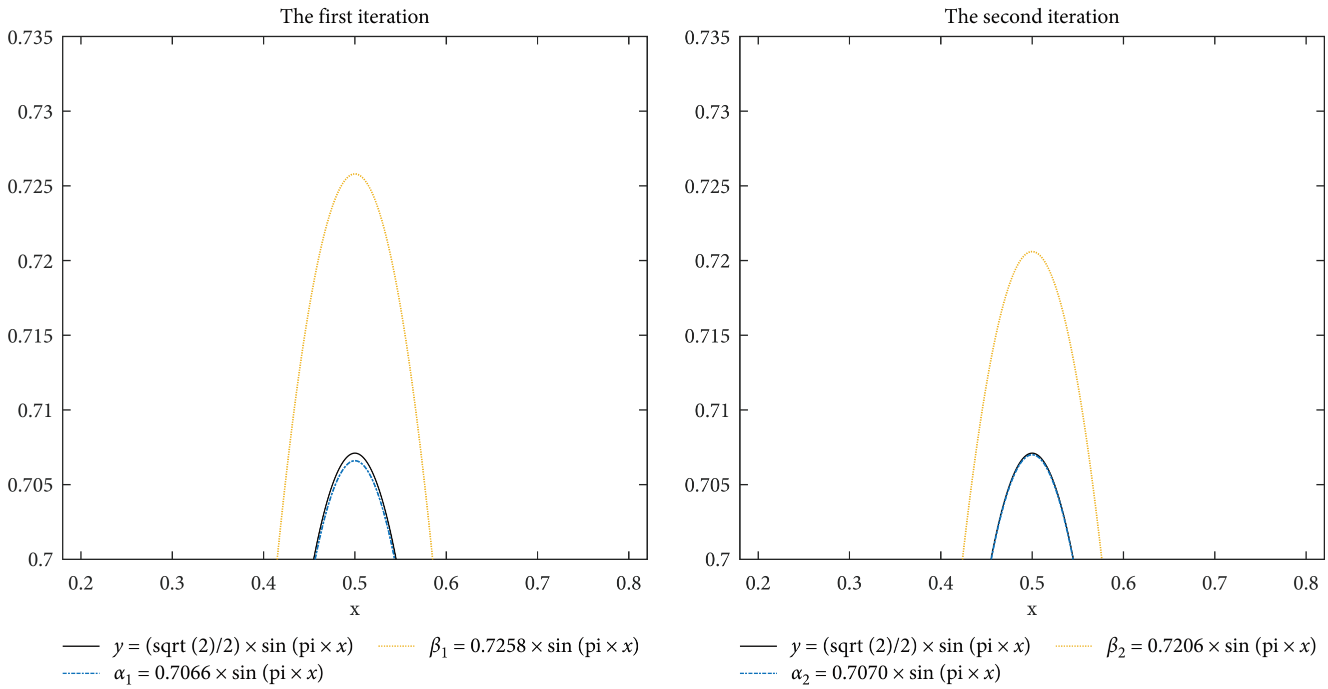

FIGURE 1: The first and second iterations.

$$y^{(4)}(x) = f\left(x, y(x), \int_0^1 k(x,t)y(t)dt\right), x \in (0,1),$$
$$y(0) = y(1) = y''(0) = y''(1) = 0. \tag{61}$$

in which

$$k(x,t) = \begin{cases} x(1-t), & 0 \le x \le t \le 1 \\ t(1-x), & 0 \le t \le x \le 1 \end{cases}, \tag{62}$$

is the Green function for the following:

$$u''(x) = 0, x \in (0,1), \tag{63}$$

$$u(0) = u(1) = 0. \tag{64}$$

Obviously, $k \in C([0,1]\times[0,1], \mathbb{R})$ and $k \ge 0, \|k\|_\infty = \frac{1}{4}$, then we can obtain the solutions $y = u''$ for problem (61) by the monotone iterative technique in Theorem 3 under certain conditions, and consequently, the nonlinear sixth order problem have solution $u = \int_0^1 k(x,t)y(t)dt$.

5. Conclusion

In this paper, we first derived an explicit expression of analytical solutions for the linear fourth-order IDE (3), which can be seen as an improvement on the numerical solution work [7–13], and then we proved the uniqueness of the analytical solutions and established a maximum principle for the corresponding integro-differential operator which can be regarded as a generalization of the maximum principles established in [23–33] without integral terms. Based upon the analytical solutions of the linear problem and the new maximum principle, we constructed two successively monotone iterative sequences which are monotonically convergent from above and from below, respectively, to the extremal solutions of the nonlinear nonlocal problems (1) and (2). The monotone iteration technique newly developed in this paper can not only prove the existence of the solutions to problems (1) and (2) but also provides an algorithm for the approximation of the solutions. We show numerically through Example 1 that the convergence of the iterative scheme requires only a few iterations; that is, the proposed method is very efficient to solve the nonlinear nonlocal problems (1) and (2). In addition, Example 2 shows that the monotone iteration technique we developed for Equations (1) and (2) can also be used to solve high-order local nonlinear problems.

Data Availability

Numerical data used to support the findings of this study are included within the article.

Disclosure

The initial draft of this work has been presented as arxiv in Cornell University to the following link: https://arxiv.org/abs/2003.04697, see [36].

Conflicts of Interest

All authors declare no conflicts of interest in this paper.

Authors' Contributions

Yan Wang's (first author) suggestion that the fourth-order nonlocal boundary value problems have important applications in engineering problems and physics inspired Jinxiang Wang (corresponding author) to think and write the initial draft of the work, see [36]. After reviewing the initial draft, Yan Wang (first author) rewrote Section 4 of the manuscript, in which a new example is given to demonstrate the main results of this paper perfectly and in detail. Yan Wang (first author) also modified the Abstract section and Section 1 of the manuscript and added three new important Remarks and the new Section 5 in the new version of the manuscript. Xiaobin Yao checked the proof process and verified the calculation for the new version of the paper. Moreover, all the authors read and approved the last version of the manuscript.

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