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Research Article

Near Crack Line Elastic-Plastic Field for Mode I Cracks under Plane Stress Condition in Rectangular Coordinates

Ya Li, Feng Huang, Min Wang, Chaohua Zhao, and Zhijian Yi

Department of Civil Engineering, Chongqing Jiaotong University, Chongqing 400074, China

Correspondence should be addressed to Ya Li; 554667539@qq.com

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By using the crack line analysis method, this paper carries out an elastic-plastic analysis for mode I cracks under plane stress condition in an elastic perfectly plastic solid and obtains the general form of matching equations of the elastic stress field and the plastic stress field near the crack line in rectangular coordinate form. The analysis in rectangular coordinates in this paper avoids the conversion from rectangular coordinates into polar coordinates in the existing analysis and greatly simplifies the power series forms of the elastic stress field and plastic stress field near the crack line during the solving process. Furthermore, by focusing on a new problem, i.e., the center-cracked plate with finite width under unidirectional uniform tension, this paper obtains the elastic stress field, plastic stress field, and the length of the elastic-plastic boundary near the crack line by using the general form of the solution. When the dimensions of the plate tend to be infinite, the results of this paper are consistent with those obtained for an infinite plate with a mode I crack. Furthermore, the variation curves of the length of the elastic-plastic boundary are also delineated in different sized center-cracked plates, and the results are compared with those obtained under the small-scale yielding conditions. The solving process and the results in this paper abandon the small-scale yielding conditions completely. The method used in this paper not only makes the solving process simpler during the elastic-plastic analysis near the crack line but also enriches the crack line analysis method.

1. Introduction

In linear elastic fracture mechanics [1–3], the stress intensity factor is the most critical parameter for the analysis of the elastic stress field near the crack tip [4–8]. However, in elastic-plastic fracture mechanics [1, 2, 9], the conventional crack tip asymptotic analysis method is widely used for the elastic-plastic analysis near the crack tip region [10–17]; such a method is confined by the small-scale yielding conditions. Nowadays, fracture mechanics has been further developed in analyzing different media with cracks and inclusions under complex conditions, and many important results have been obtained [18–24]. The crack line analysis method, which mainly focuses on the elastic and plastic fields near the crack line, is an effective way for elastic-plastic analysis of cracks. In 1984, the crack line analysis method was first proposed by Achenbach et al. [25, 26], but the results obtained were still under the small-scale yielding conditions. In 1994, Yi et al. developed the crack line analysis method by abandoning the small-scale yielding conditions [27–32].

The basic way of the crack line analysis method is as follows: to obtain the general solution of the plastic stress field near the crack line and then match it with the exact solution of the elastic stress field at the elastic-plastic boundary near the crack line. It abandons the small-scale yielding assumption, and thus, the related parameters such as the length of the plastic zone and the unit normal vector of the elastic-plastic boundary can be obtained. A series of problems have been solved by using the crack line analysis method [33–37]. Moreover, such a method can be extended for elastic-plastic analysis near the crack surface [38, 39]. However, as polar coordinates have to be used during the elastic stress field analysis or elastic-plastic stress field analysis near the crack tip, people have been accustomed to the use of polar coordinates in such analysis, so they naturally pick up polar coordinates when it comes to the analysis of the stress field near the crack line, without realizing that the solving process is quite complex as both expressions of the elastic and plastic stress fields in rectangular coordinates must be converted into polar coordinates. Thus, almost all the existing studies of the elastic-plastic analyses of the stress field near the crack line are carried out in polar coordinates.

By using polar coordinates, Yi and Yan have given the general steps of the elastic-plastic analysis near the crack line for the mode I crack under plane stress condition and have presented an instance of the solving process and the results [30].

In fact, the elastic and plastic fields near the crack line can be expressed in rectangular coordinates originally, without the necessity for further conversion into polar coordinates. In this paper, two aspects of the work are conducted: first, focusing on the general equations studied by Yi and Yan [30], this paper uses rectangular coordinates to carry out the elastic-plastic analysis near the crack line and obtains the general form of the matching equations of the elastic stress field and the plastic stress field. In our analysis, the expression of the elastic-plastic boundary near the crack line is given in rectangular coordinates, so are the expressions of the elastic and the plastic stress fields. The adoption of the rectangular coordinates reduces the complexity caused by the unnecessary conversion from rectangular coordinates into polar coordinates, especially the complexity in the expressions and conversion during the solving process. Second, by using the general form of the matching equations, a new problem, i.e., a center-cracked plate with finite width, is analyzed in this paper. For the new problem, the expression of the elastic field near the crack line is established, which satisfies not only the basic equations and the boundary conditions of the crack surface but also the boundary conditions obtained by the transformation of equilibrium conditions near the crack line. It is sufficiently precise near the crack line. Through matching the sufficiently precise elastic stress field and the general solution of the plastic stress field at the elastic-plastic boundary by using the general form of the matching equations, the matching results near the crack line are obtained for the center-cracked plate with finite width. When the dimensions of the plate with finite width tend to be infinite, the solution to a corresponding infinite plate can thus be obtained. This solution, if converted to polar coordinates, is consistent with the solution by Yi and Yan [30], but the solving process is substantially simplified. The solving process and results in this paper completely abandon the small-scale yielding conditions. In addition, the variation curves of the length of the elastic-plastic boundary are also delineated in different sized center-cracked plates, and the results are compared with those obtained under the small-scale yielding conditions.

2. The Plastic Stress Field near the Crack Line in Rectangular Coordinates

For a mode I crack under plane stress condition in an elastic perfectly plastic solid, the nonzero stress components are, respectively, σ_x, σ_y, and σ_{xy}, and the equilibrium equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \tag{1a}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \tag{1b}$$

The Tresca yielding condition is

$$\frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2} = 0, \tag{2}$$

where k is the yield stress in shear, which is related with the yield stress σ_s in tension as

$$k = \frac{\sigma_s}{2}. \tag{3}$$

Near the crack line (Figure 1), the stress components σ_x, σ_y, and σ_{xy} of the plastic stress field can be expressed by Taylor series near the crack line by the following forms (Yi and Yan [30]):

$$\sigma_x = p_0^{(p)}(x) + p_2^{(p)}(x)y^2 + p_4^{(p)}(x)y^4 + O\left(y^6\right), \tag{4a}$$

$$\sigma_y = q_0^{(p)}(x) + q_2^{(p)}(x)y^2 + q_4^{(p)}(x)y^4 + O\left(y^6\right), \tag{4b}$$

$$\sigma_{xy} = s_1^{(p)}(x)y + s_3^{(p)}(x)y^3 + O\left(y^5\right). \tag{4c}$$

In equations (4a)–(4c), superscript (p) is plasticity.

By substituting equations (4a)–(4c) into equilibrium differential equations (1a) and (1b) and Tresca yielding condition (2) and omitting y^4 and the higher-order infinitesimals, p_0, p_2, q_0, q_2, s_1 and s_3 can be solved exactly. By substituting them back into equations (4a)–(4c), the general solution of the plastic stress field near the crack line can be obtained in rectangular coordinates. Equations (5a)–(5c) are the results of the plastic stress field expressed in rectangular coordinates given by Yi and Yan [30]:

$$\sigma_x = \left(\frac{A}{x+L} + 2k\right) + \left[\frac{C}{(x+L)^3} + \frac{D}{(x+L)^4}\right]y^2 + O\left(y^4\right), \tag{5a}$$

$$\sigma_y = 2k + \left[\frac{A}{(x+L)^3}\right]y^2 + O\left(y^4\right), \tag{5b}$$

$$\sigma_{xy} = \left[\frac{A}{(x+L)^2}\right]y + \left[\frac{C}{(x+L)^4} + \frac{4}{3}\frac{D}{(x+L)^5}\right]y^3 + O\left(y^5\right), \tag{5c}$$

where A, C, D, and L are undetermined integral constants.

Equations (5a)–(5c) were expressed in rectangular coordinates in their first appearance. However, in order to match the solution of the elastic stress field with that of the plastic stress field in polar coordinates, through the relationship between rectangular coordinates and polar coordinates expressed in Taylor series, that is, $x = r\cos\theta = r[1 - (1/2)\theta^2 +$

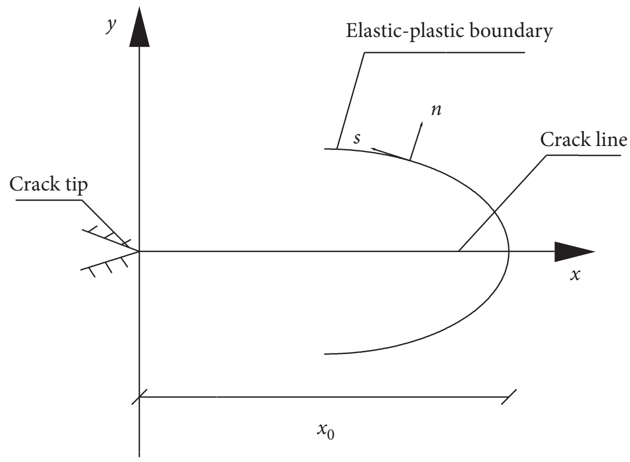

FIGURE 1: Elastic-plastic boundary near the crack line.

$O(\theta^4)]$ and $y = r\sin\theta = r[\theta - (1/6)\theta^3 + O(\theta^5)]$, Yi and Yan transformed equations (5a)–(5c) from rectangular coordinates into polar coordinates as shown in [30] as follows, among which lots of deduction and calculation were conducted:

$$\sigma_x = \left(\frac{A}{r+L} + 2k\right) + \left[\frac{Ar}{2(r+L)^2} + \frac{Cr^2}{(r+L)^3} + \frac{Dr^2}{(r+L)^4}\right]\theta^2 + O(\theta^4), \quad (6a)$$

$$\sigma_y = 2k + \left[\frac{Ar^2}{(r+L)^3}\right]\theta^2 + O(\theta^4), \quad (6b)$$

$$\sigma_{xy} = \left[\frac{Ar}{(r+L)^2}\right]\theta + \left[\frac{Ar^2}{(r+L)^3} - \frac{Ar}{6(r+L)^2} + \frac{Cr^3}{(r+L)^4} + \frac{4}{3}\frac{Dr^3}{(r+L)^5}\right]\theta^3 + O(\theta^5). \quad (6c)$$

By comparing the expressions of σ_x, σ_y, and σ_{xy} in equations (5a)–(6c), it can be found that the expression of σ_x in rectangular coordinates is one term less in quadratic terms than that in polar coordinates; the expression of σ_y rectangular coordinates is similar to that in polar coordinates; and the expression of σ_{xy} in rectangular coordinates is two terms less in cubic terms than that in polar coordinates.

The conversion of the expression of the plastic stress field near the crack line from rectangular coordinates into polar coordinates not only leads to more solving work but also the complexity of expressions. Moreover, the complex solution will further lead to the complex process of matching the expression of the plastic stress field at the elastic-plastic boundary near the crack line.

3. The Rectangular Form of the Elastic-Plastic Boundary near the Crack Line

The elastic-plastic boundary near the crack line can also be expressed directly in rectangular coordinates. By the premise that the elastic-plastic boundary is symmetric to and continuous near the crack line $y = 0$, the boundary x_P can be expanded by power series in rectangular coordinates near the crack line as follows:

$$x_p = x_0 + x_2 y^2 + O(y^4), \quad (7)$$

where x_0 is the length of the plastic zone along the crack line.

However, Yi and Yan [30] have given elastic-plastic boundary near the crack line in polar coordinates as $r_p = r_0 + r_2\theta^2 + O(\theta^4)$, which is not needed in the analysis in rectangular coordinates in this paper.

According to equation (7), the tangent vector of any point on the elastic-plastic boundary near the crack line can be expressed as $(1, (dy/dx))$ which equals to $(2x_2 y + O(y^3), 1)$. Therefore, the normal vector of that point is $(1, -2x_2 y + O(y^3))$. Then, the unit normal vector of any point on the elastic-plastic boundary near the crack line can be expressed as $\mathbf{n} = (n_x, n_y)$, where

$$n_x = 1 - 2x_2^2 y^2 + O(y^3), \quad (8a)$$

$$n_y = -2x_2 y + O(y^3). \quad (8b)$$

The deducing process shows that compared with the polar coordinates, the rectangular coordinates facilitate the work in obtaining the unit normal vector as it can be obtained directly through the definition of the tangent and normal vectors or through the implicit function. However, in polar coordinates, more solving work is involved to obtain the unit normal vector as it must project the function onto axis x and y separately. As a result, Yi and Yan [30] have given the expressions of n_x and n_y in polar coordinates as $n_x = 1 - (1/2)(1 - 2(r_2/r_0))^2\theta^2 + O(\theta^4)$ and $n_y = (1 - 2(r_2/r_0))\theta + O(\theta^3)$, which are quite complex. It can thus be seen that using rectangular coordinates to solve the vector obviously simplifies the matching process.

4. The Rectangular Form of the Plastic Stress Field at the Elastic-Plastic Boundary near the Crack Line

In the above analysis, the general solution of the plastic stress field near the crack line in rectangular coordinates (5a)–(5c) and the rectangular form of the elastic-plastic boundary near the crack line (7) have been obtained, respectively. By substituting equation (7) into equations (5a)–(5c), the rectangular form of the plastic stress field at the elastic-plastic boundary near the crack line can be obtained in this paper:

$$\sigma_x^{(p)} = \sigma_{x0}^{(p)} + \sigma_{x2}^{(p)} y^2 + O(y^4), \quad (9a)$$

$$\sigma_y^{(p)} = \sigma_{y0}^{(p)} + \sigma_{y2}^{(p)} y^2 + O(y^4), \quad (9b)$$

$$\sigma_{xy}^{(p)} = \sigma_{xy1}^{(p)} y + \sigma_{xy3}^{(p)} y^3 + O(y^5), \quad (9c)$$

where

$$\sigma_{x0}^{(p)} = \frac{A}{x_0 + L} + 2k, \quad (10a)$$

$$\sigma_{x2}^{(p)} = \frac{C}{(x_0 + L)^3} + \frac{D}{(x_0 + L)^4} - \frac{Ax_2}{(x_0 + L)^2}, \quad (10b)$$

$$\sigma_{y0}^{(p)} = 2k, \quad (10c)$$

$$\sigma_{y2}^{(p)} = \frac{A}{(x_0 + L)^3}, \quad (10d)$$

$$\sigma_{xy1}^{(p)} = \frac{A}{(x_0 + L)^2}, \quad (10e)$$

$$\sigma_{xy3}^{(p)} = \frac{C}{(x_0 + L)^4} + \frac{4}{3}\frac{D}{(x_0 + L)^5} - \frac{2Ax_2}{(x_0 + L)^3}. \quad (10f)$$

The expression of the plastic stress field in rectangular coordinates, i.e., equations (9a)–(9c), at the elastic-plastic boundary near the crack line is much more concise than that given by Yi and Yan [30] in polar coordinates, especially in the terms of $\sigma_{x2}^{(p)}$ and $\sigma_{xy3}^{(p)}$. The expressions of $\sigma_{x2}^{(p)}$ and $\sigma_{xy3}^{(p)}$ in polar coordinates by Yi and Yan [30] are as follows:

$$\sigma_{x2}^{(p)} = \frac{Cr_0^2}{(r_0 + L)^3} + \frac{Dr_0^2}{(r_0 + L)^4} - \frac{Ar_0}{(r_0 + L)^2}\left(\frac{r_2}{r_0} - \frac{1}{2}\right), \quad (11a)$$

$$\sigma_{xy3}^{(p)} = \frac{Cr_0^3}{(r_0 + L)^4} + \frac{4}{3}\frac{Dr_0^3}{(r_0 + L)^5} - \frac{2Ar_0^2}{(r_0 + L)^3}\left(\frac{r_2}{r_0} - \frac{1}{2}\right) + \frac{Ar_0}{(r_0 + L)^2}\left(\frac{r_2}{r_0} - \frac{1}{6}\right). \quad (11b)$$

It can be seen that the solutions in polar coordinates are more complex. Subsequent analysis will show that converting rectangular coordinates into polar coordinates is completely unnecessary.

5. The Power Series Form of the Elastic Stress Field near the Crack Line and Its Expression at the Elastic-Plastic Boundary in Rectangular Coordinates

As the stress components of the elastic field, namely, σ_x, σ_y, and σ_{xy}, are continuous near the crack line, they can be expanded by the Taylor series as follows:

$$\sigma_x = p_0^{(e)}(x) + p_2^{(e)}(x)y^2 + O(y^4), \quad (12a)$$

$$\sigma_y = q_0^{(e)}(x) + q_2^{(e)}(x)y^2 + O(y^4), \quad (12b)$$

$$\sigma_{xy} = s_1^{(e)}(x)y + s_3^{(e)}(x)y^3 + O(y^5). \quad (12c)$$

In equations (12a)–(12c), superscript (e) is elasticity.

The form (12a)–(12c) in a power series of the elastic stress field is a general form rather than a specific one as its specific form can only be obtained for a specific problem under its boundary conditions.

By substituting equation (7) into equations (12a)–(12c), the expression of the elastic stress field at the elastic-plastic boundary near the crack line can thus be obtained:

$$\sigma_x^{(e)} = \sigma_{x0}^{(e)} + \sigma_{x2}^{(e)}y^2 + O(y^4), \quad (13a)$$

$$\sigma_y^{(e)} = \sigma_{y0}^{(e)} + \sigma_{y2}^{(e)}y^2 + O(y^4), \quad (13b)$$

$$\sigma_{xy}^{(e)} = \sigma_{xy1}^{(e)} + \sigma_{xy3}^{(e)}y^3 + O(y^5). \quad (13c)$$

It should be noted that equations (13a)–(13c) are still a general form.

6. The Matching Results of the Elastic and Plastic Stress Fields at the Elastic-Plastic Boundary near the Crack Line in Rectangular Coordinates

The expressions of the elastic and plastic stress fields at the boundary have been obtained in rectangular coordinates as in equations (9a)–(9c) and equations (13a)–(13c), respectively. The matching condition of the stress fields is the stress is continuous at the boundary, that is, the tangent/normal stress components of the elastic and plastic fields are equal at any point of the elastic-plastic boundary. The matching condition can be expressed as $\sigma_{nn}^e = \sigma_{nn}^p$ and $\sigma_{ns}^e = \sigma_{ns}^p$, where $\sigma_{nn} = \sigma_x n_x^2 + \sigma_y n_y^2 + 2\sigma_{xy}n_x n_y$ and $\sigma_{ns} = (n_x^2 - n_y^2)\sigma_{xy} + (\sigma_y - \sigma_x)n_x n_y$. n_x and n_y have been obtained in equations (8a) and (8b) above.

Through matching, simplification, and decoupling of the two stress fields at the elastic-plastic boundary, the following equations can be obtained:

$$\sigma_{x0}^{(e)} = \sigma_{x0}^{(p)}, \quad (14a)$$

$$\sigma_{y0}^{(e)} = \sigma_{y0}^{(p)} = 2k, \quad (14b)$$

$$\sigma_{xy1}^{(e)} = \sigma_{xy1}^{(p)}, \quad (14c)$$

$$\sigma_{x2}^{(e)} = \sigma_{x2}^{(p)}, \quad (14d)$$

$$\sigma_{y2}^{(e)} = \sigma_{y2}^{(p)}, \quad (14e)$$

$$\sigma_{xy3}^{(e)} = \sigma_{xy3}^{(p)}. \quad (14f)$$

Equations (14a)–(14f) are the general form of the matching equation near the crack line by the crack line analysis method.

7. The Matching Results for a Center-Cracked Plate with Finite Width

Equations (12a)–(13c) are the general forms of the elastic field. Only for a specific problem, can the specific form of the stress field be obtained.

In Yi and Yan's study [30], after the obtainment of the general form of the matching equations in polar coordinates, a unidirectional tensile-cracked plate with infinite width was taken as an instance for the analysis.

This paper aims at a new problem, i.e., a center-cracked plate with finite width subjected to uniform stress σ along the y direction at infinity, as shown in Figure 2. Compared with an infinite plate, solving the elastic stress field of a finite plate is more complex due to the existence of specific boundary conditions. For instance, during solving of stress intensity factor in some classical crack problems, analytical solution can usually be obtained for an infinite plate but not for a finite plate, so the numerical method is often used for analysis of a finite plate.

For the new problem mentioned above, its plastic field still satisfies equations (5a)–(5c). However, as the boundary conditions of the plate with finite width are difficult to satisfy, no analytical solution for its elastic field has been obtained. This paper, based on the analytical exact solutions of the corresponding infinite plate, will establish the elastic field near the crack line of the finite plate and make the established stress field near the crack line precise enough.

When the width b of the plate trends to infinity, i.e., $b \longrightarrow \infty$, the finite plate turns into a corresponding infinite plate. For such an infinite plate, Westergaard's stress function can be used for the analysis to obtain analytical solutions [1–3, 9]. Specifically, the stress field for the problem can be expressed by using Westergaard's stress function $Z_I(z) = (\sigma z/\sqrt{z^2 - a^2})$ as follows:

$$\sigma_x = \mathrm{Re}Z_I(z) - y_1\mathrm{Im}Z_I'(z) - \sigma, \tag{15a}$$

$$\sigma_y = \mathrm{Re}Z_I(z) + y_1\mathrm{Im}Z_I'(z), \tag{15b}$$

$$\sigma_{xy} = -y_1\mathrm{Re}Z_I'(z), \tag{15c}$$

where Re and Im denote the real and imaginary parts of the complex function $Z_I(z)$ and $Z_I'(z)$, respectively; $Z_I'(z)$ denotes the first-order derivative of $Z_I(z)$ with respect to z. And complex variable $z = x_1 + iy_1$ (under the coordinate system of $x_1O_1y_1$ with the origin at the center of the crack).

Equations (15a)–(15c) expressed by Westergaard's stress function $Z_I(z)$ satisfy the basic equations in theory of elasticity (equilibrium differential equations and compatibility equation). Meanwhile, when $Z_I(z) = (\sigma z/\sqrt{z^2 - a^2})$, the far-field boundary condition $\sigma_x = 0$, $\sigma_y = \sigma$, and $\sigma_{xy} = 0$ of the cracked plate with infinite width, and the boundary condition on the crack surface, $\sigma_y = \sigma_{xy} = 0$, can be satisfied, so equations (15a)–(15c) are the exact solution of the cracked plate with infinite width corresponding to the center-cracked plate with finite width as shown in Figure 2.

Next, by using equations (15a)–(15c), a sufficiently precise elastic stress field near the crack line of the center-cracked plate with finite width can be established. After modified by constants M and N, equations (15a)–(15c) turn into

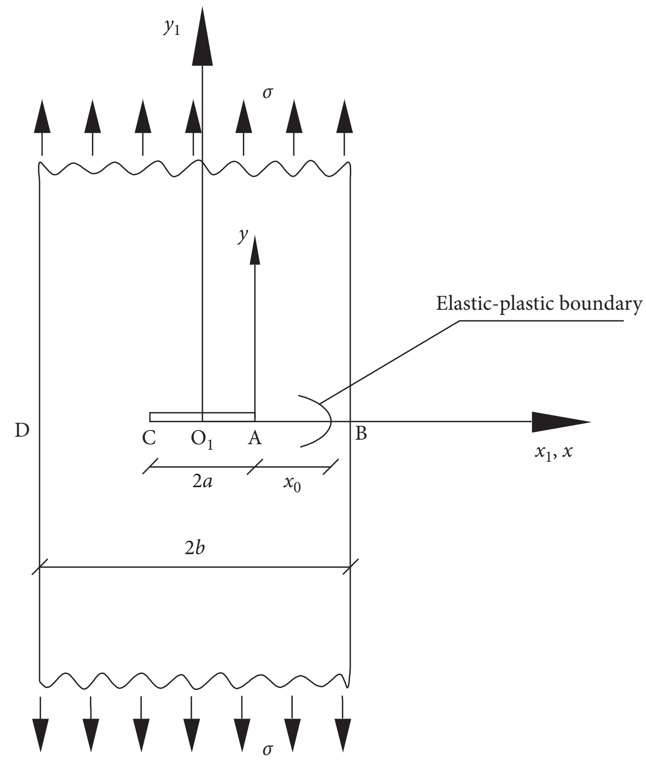

Figure 2: A center-cracked plate with finite width subjected to uniform stress σ along the y direction at infinity.

$$\sigma_x = M \cdot \left[\mathrm{Re}Z_I(z) - y_1\mathrm{Im}Z_I'(z)\right] - N \cdot \sigma, \tag{16a}$$

$$\sigma_y = M \cdot \left[\mathrm{Re}Z_I(z) + y_1\mathrm{Im}Z_I'(z)\right], \tag{16b}$$

$$\sigma_{xy} = M \cdot \left[-y_1\mathrm{Re}Z_I'(z)\right]. \tag{16c}$$

Obviously, equations (16a)–(16c) satisfy the basic equations of the theory of elasticity and satisfy the traction-free boundary condition on the crack surface also. The reasons lie in the following: first, as equations (15a)–(15c) satisfy equilibrium differential equations and the compatible equation of the theory of elasticity and equations (16a)–(16c) are modified by constants M and N from equations (15a)–(15c), thus the satisfaction of equilibrium differential equations, and the compatible equation by equations (16a)–(16c) is not affected by the modification; second, as equations (15a)–(15c) satisfy the traction-free boundary conditions $\sigma_y = 0$ and $\sigma_{xy} = 0$ on the crack surface and equations (16a)–(16c) are multiplied by constant M from equations (15a)–(15c), thus, the satisfaction of the boundary conditions $\sigma_y = 0$ and $\sigma_{xy} = 0$ by equations (16a)–(16c) is not compromised. Thus, as long as boundary conditions near the crack line are established, equations (16a)–(16c) will be sufficiently precise in the region near the crack line. And the boundary conditions can be built from the equilibrium conditions along the section of the crack line.

Through the shift of the coordinate system from $x_1O_1y_1$ to xAy with the origin at the crack tip, there will be $z = x + a + iy$ as $x_1 = x + a$ and $y_1 = y$. Then, by Westergaard's stress function with the complex function method, let $z = x + a + iy$ and expand equations (16a)–(16c) in Taylor series; the elastic stress field near the crack line of the finite plate can be expressed as follows:

$$\sigma_x^{(e)} = \frac{M\sigma(x+a)}{\sqrt{x^2+2ax}} - N\sigma - \frac{M\sigma}{\sqrt{x^2+2ax}}\frac{9a^2(x+a)}{2(x^2+2ax)^2}y^2 + O\left(y^4\right), \tag{17a}$$

$$\sigma_y^{(e)} = \frac{M\sigma(x+a)}{\sqrt{x^2+2ax}} + \frac{M\sigma}{\sqrt{x^2+2ax}}\frac{3a^2(x+a)}{2(x^2+2ax)^2}y^2 + O\left(y^4\right), \tag{17b}$$

$$\sigma_{xy}^{(e)} = \frac{M\sigma a^2}{\sqrt{\left(x_0{}^2+2ax_0\right)^3}}y\left[1 - \frac{3\left(4x^2+8ax+5a^2\right)}{2(x^2+2ax)^2}y^2\right] + O\left(y^4\right). \tag{17c}$$

Clearly, the above equations have been expressed in rectangular coordinates, so there is no need to further convert them into polar coordinates.

Suppose that the plate is divided into the upper and lower parts along the section of the crack lines AB and CD (Figure 2); then, the stress on the section of the crack line should be balanced by the external load. Thus, it can be obtained that

$$2\int_0^{x_0}\left(\sigma_y^{(p)}\right)_{y=0}\mathrm{d}x + 2\int_{x_0}^{b-a}\left(\sigma_y^{(e)}\right)_{y=0}\mathrm{d}x = 2b\sigma, \tag{18}$$

where $\sigma_y^{(p)}$ is presented in equation (9b) and $\sigma_y^{(e)}$ is presented in equation (17b). By substituting equations (9b) and (17b) into equation (18) and then integrating, it can be obtained that

$$M = \frac{b\sigma - 2kx_0}{\left[\sqrt{b^2-a^2} - \sqrt{x_0(x_0+2a)}\right]\sigma}. \tag{19}$$

Additionally, at the intersection point B (or D) between the crack line and the free boundaries, the stress should satisfy the boundary equation:

$$\left(\sigma_x^{(e)}\right)_{x=b-a,y=0} = 0. \tag{20}$$

By substituting equation (17a) into (18), it can be obtained that

$$N = \frac{Mb}{\sqrt{b^2-a^2}}. \tag{21}$$

The idea is to establish the elastic stress field near the crack line (i.e., equations (16a)–(16c)) of a cracked plate with finite width by modifying the precise solution of the elastic stress field (i.e., equations (15a)–(15c)) of a corresponding infinite plate originated from Yi [27]. Yi established the elastic stress field for a cracked plate with finite width during the analysis of a mode III crack. Before that, Yi proposed a method for solving stress intensity factor by establishing the precise elastic stress field of cracked plates with finite width [40, 41], which was called the crack-line stress field method for estimating stress intensity factor [42], and such a method has been widely used. In this paper, the sufficiently precise solution of the elastic stress field near the crack line was obtained in rectangular coordinates for the center-cracked plate with finite width and was then matched with the precise solution of the plastic stress field.

By substituting equation (7) into equations (17a)–(17c), the coefficients in equations (13a)–(13c) can all be obtained for a unidirectional tensile finite plate with the mode I center crack:

$$\sigma_{x0}^{(e)} = \frac{M\sigma(x_0+a)}{\sqrt{x_0^2+2ax_0}} - N\sigma, \tag{22a}$$

$$\sigma_{x2}^{(e)} = -\frac{M\sigma a^2}{\sqrt{x_0^2+2ax_0}}\left[\frac{9(x_0+a)}{2(x_0^2+2ax_0)^2} + \frac{x_2}{x_0^2+2ax_0}\right], \tag{22b}$$

$$\sigma_{y0}^{(e)} = \frac{M\sigma(x_0+a)}{\sqrt{x_0^2+2ax_0}}, \tag{22c}$$

$$\sigma_{y2}^{(e)} = \frac{M\sigma a^2}{\sqrt{x_0^2+2ax_0}}\left[\frac{3(x_0+a)}{2(x_0^2+2ax_0)^2} - \frac{x_2}{x_0^2+2ax_0}\right], \tag{22d}$$

$$\sigma_{xy1}^{(e)} = \frac{M\sigma a^2}{\sqrt{\left(x_0^2+2ax_0\right)^3}}, \tag{22e}$$

$$\sigma_{xy3}^{(e)} = -\frac{M\sigma a^2}{\sqrt{\left(x_0^2+2ax_0\right)^3}}\left[\frac{3\left(4x_0^2+8ax_0+5a^2\right)}{2(x_0^2+2ax_0)^2} + \frac{3x_2(x_0+a)}{x_0^2+2ax_0}\right]. \tag{22f}$$

By far, both left and right sides of equations (14a)–(14f) have obtained specific forms. By matching equations (10a)–(10c) and (22a)–(22f), it can be obtained that

$$x_0 = a\left[\sqrt{\frac{\sigma_s^2}{\sigma_s^2-(M\sigma)^2}} - 1\right], \tag{23}$$

$$L = -x_0\left[\frac{(x_0+2a)b}{a^2\sqrt{b^2-a^2}}\sqrt{x_0^2+2ax_0} + 1\right], \tag{24}$$

$$A = M\sigma x_0\frac{b^2(x_0+2a)}{(b^2-a^2)a^2}\sqrt{x_0^2+2ax_0}, \tag{25}$$

$$\frac{x_2}{x_0} = \frac{3(x_0+a)}{2x_0^2(x_0+2a)} + \frac{M\sqrt{b^2-a^2}}{b}\frac{\sigma a^2(x_0+a)}{\sigma_s x_0^3(x_0+2a)^2}, \tag{26}$$

where equation (3), i.e., $k = (\sigma_s/2)$, is used, in which k is the yield stress in shear and σ_s is the yield stress in tension.

Undetermined constant M can be solved from equations (19) and (23):

$$M = \frac{(\sigma_s a + b\sigma)\sqrt{b^2 - a^2}}{b^2\sigma} - \frac{a\sqrt{(\sigma_s b)^2 - (\sigma_s a + b\sigma)^2}}{b^2\sigma}. \quad (27)$$

In order to ensure that the radical $\sqrt{(\sigma_s b)^2 - (\sigma_s a + b\sigma)^2}$ in equation (27) is no less than zero, there is

$$\frac{\sigma}{\sigma_s} \le \frac{b-a}{b}. \quad (28)$$

Substitute equation (27) into equation (23), and by equation (28), it can be obtained that

$$x_0 \le b - a. \quad (29)$$

Equation (28) or (29) presents the applicable condition for obtaining the results of a finite plate, that is, the external load σ is less than $(\sigma_s (b-a)/b)$, or the length of the plastic zone is less than the length of the ligament $(b-a)$. By substituting equation (26) into equations (8a) and (8b), the unit normal vector of the elastic-plastic boundary near the crack line can also be obtained.

Assuming $b \longrightarrow \infty$, the solution of a corresponding infinite plate can be obtained. It can be seen from equation (19) or (27) that when $b \longrightarrow \infty$, there is $M \longrightarrow 1$. Furthermore, according to equation (21), when $b \longrightarrow \infty$ and $M \longrightarrow 1$, there is $N \longrightarrow 1$. If $M \longrightarrow 1$ and $N \longrightarrow 1$ are substituted into equations (17a)–(17c), the power series form of the elastic stress field near the crack line for the corresponding cracked plate with infinite width can thus be obtained. Similarly, if $M \longrightarrow 1$ and $N \longrightarrow 1$ are substituted into equations (23)–(26), the matching results x_0, L, and A and (x_2/x_0) can be obtained for the corresponding infinite plate:

$$\begin{aligned}
x_0 &= a\left(\sqrt{\frac{\sigma_s^2}{\sigma_s^2 - \sigma^2}} - 1\right),\\
L &= -x_0\left(\frac{x_0 + 2a}{a^2}\sqrt{x_0^2 + 2ax_0} + 1\right),\\
A &= \sigma x_0 \frac{(x_0 + 2a)}{a^2}\sqrt{x_0^2 + 2ax_0},\\
\frac{x_2}{x_0} &= \frac{3(x_0 + a)}{2x_0^2(x_0 + 2a)} + \frac{\sigma a^2 (x_0 + a)}{\sigma_s x_0^3 (x_0 + 2a)^2},
\end{aligned} \quad (30)$$

where x_0 is the length of the plastic zone along the crack line; (x_2/x_0) is related to the unit vector of the point at the elastic-plastic boundary near the crack line in rectangular coordinates; and L and A are the integral constants.

If x_0 in the results is replaced by r_0, it can be seen from the matching results that x_0, L, and A obtained in this paper are consistent with r_0, L, and A in Yi and Yan [30], except for (x_2/x_0) and (r_2/r_0) as they are related to the unit vector of the point at the elastic-plastic boundary near the crack line in rectangular coordinates and polar coordinates, respectively. When Yi and Yan [30] analyzed the cracked plate with infinite width (the cracked plate with finite width in this paper was not analyzed), they transformed all the expressions into polar coordinates, which made the solving process extremely complicated, although the final results are not so complicated.

For the center-cracked plate with the finite plate, Figure 3 shows the variations of (x_0/a) with (σ/σ_s) in several instances for $(b/a) = 1.5, 2, 5, 10$, and ∞. $(b/a) = \infty$ is the instance of a cracked plate with infinite dimensions, and the relevant results of which have also been shown in Figure 4.

Table 1 shows the variations of (x_0/a) with (σ/σ_s) in several instances for $(b/a) = 1.5, 2, 5, 10$, and ∞. From the table, it can be seen that if $(b/a) \ge 1.5$, when $(\sigma/\sigma_s) < 0.15$, the differences between the sizes of the plastic zones are not significant in different sized center-cracked plates; if $(b/a) \ge 5$, when $(\sigma/\sigma_s) < 0.3$, the sizes of the plastic zones are slightly different from those of the corresponding cracked plate with infinite width (when $(b/a) \longrightarrow \infty$), and the maximum difference is about 11%; and if $(b/a) \ge 10$, when $(\sigma/\sigma_s) < 0.50$, the sizes of the plastic zones are slightly different from those of the corresponding cracked plate, and the maximum difference is less than 8%. It can thus be seen that, for a cracked plate with finite width, if the dimensions of the plate are sufficiently large compared with those of the crack, the size of the plastic zone of the finite plate has only slight difference from that of the infinite plate with the crack if the stress level is relatively not high.

In the classical theory of fracture mechanics, for a cracked plate with infinite width (when $(b/a) \longrightarrow \infty$), Irwin has given the length of the plastic zone along the crack line, namely, $r_0 = (K_I^2/2\pi\sigma_s^2)$, and after modification by considering stress relaxation, the length of the plastic zone is $r_0' = (K_I^2/\pi\sigma_s^2)$ (where $K_I = \sigma\sqrt{\pi a}$) [43, 44]. Figure 4 shows a comparison of the variation of (x_0/a) with (σ/σ_s) between the results of this paper (when $b \longrightarrow \infty$) and those in [43, 44] for a cracked plate with infinite width. The compassion shows that the size of the plastic zone in this paper is in better agreement with r_0 in [43, 44] when (σ/σ_s) is not high enough. When $(\sigma/\sigma_s) < 0.4$, the difference between the two is no more than 10%.

It should be noted that the confining condition for the solutions given in this paper for a cracked plate with finite width, i.e., equation (28), is obtained by the satisfaction of equation (27), and equation (29) corresponding to equation (28) is $x_0 \le b - a$, which means that the largest length of the plastic zone is exactly the length of the ligament $(b-a)$. For an analytical solution, the reason why the confining condition (28) or (29) is so precise lies in that the plastic stress field, i.e., equations (5a)–(5c), is the general solution near the crack line, and the elastic stress field, i.e., equations (17a)–(17c), is sufficiently precise near the crack line. Therefore, by matching the elastic stress field with the solution of the plastic stress field near the crack line, the obtained results are sufficiently precise near the region of the crack line. The analysis in this paper abandoned the small-scale yielding condition. It is noteworthy that the crack line analysis is valid for the elastic-plastic analysis of the region near the crack line only, just as the conventional crack-tip asymptotic analysis method is valid for the analysis of the region near the crack tip only. By far, the analytical solution for the elastic-plastic crack problem on a larger range of cracked bodies can hardly be obtained.

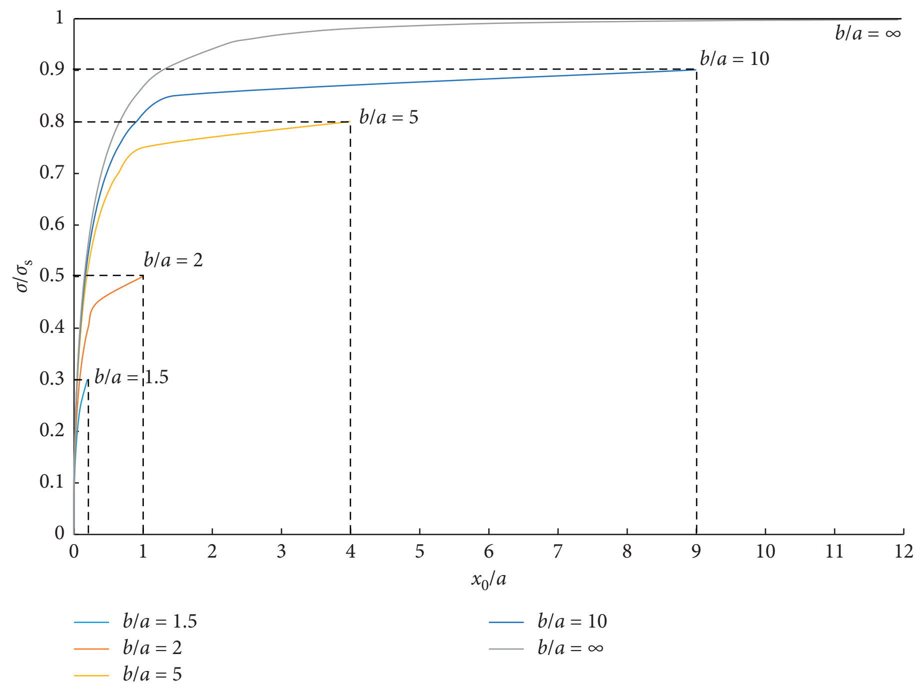

Figure 3: Comparison of variations of (x_0/a) with (σ/σ_s) in different sized center-cracked plates.

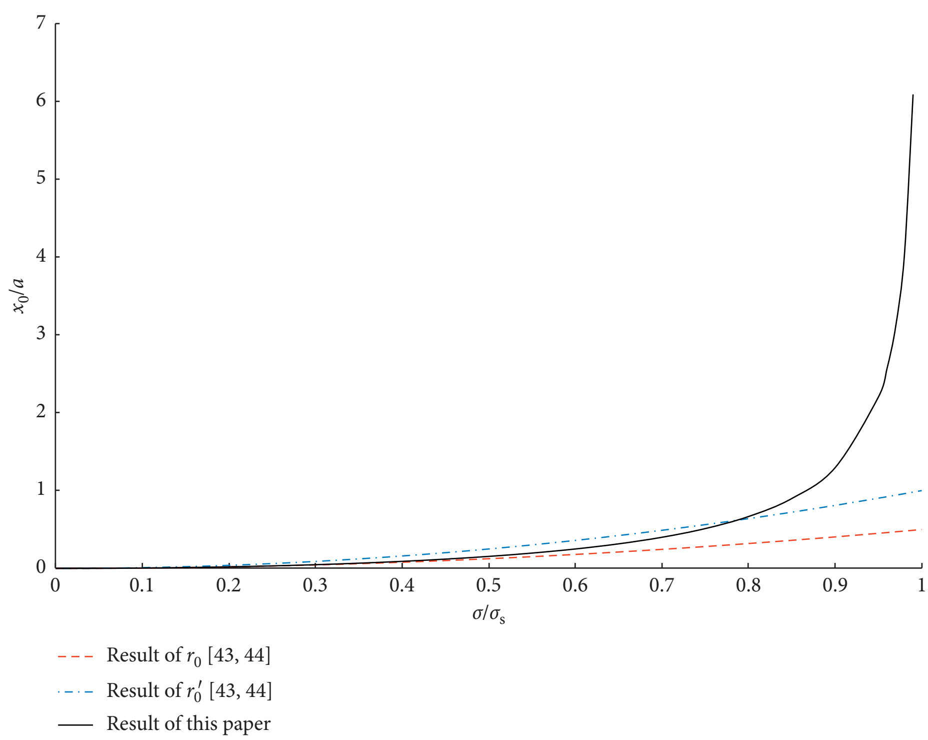

Figure 4: Comparison of the variation of (x_0/a) with (σ/σ_s).

Table 1: Variations of (x_0/a) with (σ/σ_s) in several instances for $(b/a) = 1.5, 2, 5, 10$, and ∞.

(σ/σ_s)		0.05	0.10	0.15	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
(x_0/a)	$(b/a) = 1.5$	0.002	0.010	0.026	0.053	0.100	0.197							
	$(b/a) = 2$	0.002	0.007	0.017	0.033	0.055	0.087	0.208	1.000					
	$(b/a) = 5$	0.001	0.005	0.012	0.022	0.036	0.054	0.106	0.191	0.337	0.647	4.000		
	$(b/a) = 10$	0.001	0.005	0.012	0.021	0.034	0.051	0.097	0.168	0.281	0.477	0.909	9.000	
	$(b/a) = \infty$	0.001	0.005	0.011	0.021	0.033	0.048	0.091	0.155	0.250	0.400	0.667	1.294	∞

8. Conclusions

Through the above analysis, the following conclusions can be made:

(1) By using the crack line analysis method, this paper obtains the general form of the matching equations in rectangular coordinates for the elastic and plastic stress fields near the crack line for mode I cracks under plane stress condition in an elastic perfectly plastic solid. In the analysis, the complexity caused by the unnecessary conversion from rectangular coordinates into polar coordinates is avoided, especially the complexity in the expressions and conversion during the solving process. Thus, by using rectangular coordinates, the analysis of near crack line elastic-plastic fields for mode I cracks is substantially simplified.

(2) For the center-cracked plate with finite width, the solution of its elastic field is established near the crack line by modifying the precise solution of the elastic field of a corresponding infinite plate, which is sufficiently precise as it satisfies both the basic equations in the theory of elasticity and the boundary conditions of the crack surface and the crack line. By matching the precise elastic stress field with the precise plastic stresses field at the elastic-plastic boundary, sufficiently precise matching results are obtained in this paper near the crack line, and it abandons the small-scale yielding conditions. The variation curves of the length of the elastic-plastic boundary are also delineated in different sized center-cracked plates. In the obtained results for a finite plate, by assuming $b \longrightarrow \infty$, the solution to a corresponding infinite plate can thus be obtained, and the results are compared with those obtained under the small-scale yielding conditions.

The analysis in this paper simplifies the matching process of the elastic and plastic fields near the crack line, and a new problem of a cracked plate with finite dimensions is solved. The adoption of rectangular coordinates enriches the crack line analysis method, and it can be used for the analysis of cracks under more complex conditions.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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