Methodology for Determining the Compliance Matrix and Analyzing the Skew Anisotropic Plate with an Arbitrarily Shaped Hole

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1. Introduction

Anisotropic materials are widely used in fields such as aerospace, machinery manufacturing, civil engineering, and automobile industry owing to the excellent mechanical properties. In order to meet the requirements of assembly and structure functions, various holes are performed in anisotropic plates, which leads to a reduction in the stability and safety of the structure. It is necessary to analyze the stress distributions to ensure the sufficient strength of structure and to predict the potential risk. Therefore, many scholars at home and abroad have conducted extensive research on the perforated anisotropic structures.

The complex variable method developed by Muskheilishvili [1] is an effective approach for studying the analytical solution of stress in elastic anisotropic mediums. Regarding the orthotropic plate with an elliptical hole, Lekhnitskiit [2, 3] firstly obtained accurate stress and displacement solutions for different external loadings at infinity, using the complex stress functions and conformal transformation method [4, 5]. Since then, the stress distributions in anisotropic mediums with circular [6, 7], triangular [8–10], rectangular [11–13], hexagonal [14, 15], or irregular-shaped holes [16] were investigated by scholars all over the world, subject to various external loadings. Sharma [17] even analyzed the stacking sequence of orthotropic lamina in an infinite laminated composite plate and proposed a general method to determine the stress field around a polygonal hole. Targeted for isotropic and anisotropic materials, Setiawan and Zimmerman [18] presented a unified methodology to compute the stresses around an arbitrarily shaped hole. Moreover, considering that three conformal mapping functions should be involved in solving the anisotropic problem, Manh et al. [19] derived an analytical solution for arbitrarily shaped tunnels excavated in transversely isotropic rock mass. Lu et al. [20] established three polar coordinate
systems and presented an accurate analytical method to analyze the stress distribution along an arbitrarily shaped tunnel excavated in orthotropic rock mass, using the power-series method. Based on this theoretical solution, Wang et al. [21] and Zhang et al. [22] performed the optimum design of hole shape, fiber angle, and hole orientation of orthotropic plates, respectively, in order to decrease the stress concentration along the hole boundary, using a Differential Evolution (DE) algorithm. Sun et al. [23] introduced the Boundary Element Method (BEM) into the analysis of stress solution (DE) algorithm. Sun et al. [23] introduced the Boundary Element Method (BEM) into the analysis of stress solution (DE) algorithm. Sun et al. [23] introduced the boundary element method into the analysis of stress solution (DE) algorithm. Sun et al. [23] introduced the Boundary Element Method (BEM) into the analysis of stress solution (DE) algorithm. Sun et al. [23] introduced the Boundary Element Method (BEM) into the analysis of stress solution (DE) algorithm. Sun et al. [23] introduced the Boundary Element Method (BEM) into the analysis of stress solution (DE) algorithm. Sun et al. [23] introduced the Boundary Element Method (BEM) into the analysis of stress solution (DE) algorithm.

In addition, many scholars have also carried out research on the buckling analysis or vibration response of plates with different shapes by using other solution methods. Civalek and Avcar [24] presented the free vibration and buckling analyses of functionally graded carbon nanotube-reinforced (CNT) laminated nonrectangular plates by utilizing a Discrete Singular Convolution (DSC) method, which was also used in the investigation of free vibration response of functionally graded cylindrical shells [25]. Ng et al. [26] were concerned with the comparison of Discrete Singular Convolution and generalized Differential Quadrature (DQ) for the vibration analysis of rectangular plates. For buckling analysis of plates with different shapes, the Differential Quadrature and Harmonic Differential Quadrature (HDQ) methods were proposed by Civalek [27]. Besides, Zhu et al. [28] utilized the Finite Element Method (FEM) to study static and free vibration modeling of CNT composites plates with the first-order shear deformation plate theory. Mishra and Barik [29] gave the nonuniform rational B-spline augmented Finite Element Method for stability analysis of arbitrary thin plates. However, the influence of hole performed in the plate has not been concerned in these researches.

Moreover, although extensive theoretical studies have been carried out to derive analytical solutions for anisotropic structures, the most complex conditions involved in the abovementioned studies have mostly been orthotropic or even transversely isotropic materials. General conditions of nonorthogonal materials have not been discussed. For orthotropic materials, there are three elastic symmetry planes. The elastic compliance matrix in the generalized Hooke’s law can easily be obtained when taking the directions of the axes along the principal material directions [30]. However, coupling exists between normal stress and shear strain, among others, for general conditions of nonorthogonal materials. Determination of the compliance matrix, especially the elements that reflect the coupling, becomes far more complicated.

At present, the inverse analysis method using numerical software is utilized to identify the elastic constants for general anisotropic materials. Using the Boundary Element Method (BEM) combined with a minimization algorithm, Comino and Gallego [31] presented an inverse approach for the plane problem and determined the six elastic constants. Hematiyan et al. [32] inversely analyzed the elastic constants according to the displacement measurements from more than one static experiment and using the multiloading BEM. Regarding a three-dimensional generally anisotropic solid with arbitrary geometry, Hematiyan et al. [33] proposed a new inverse approach for the identification of all elastic constants by means of measured strain data. However, inverse analysis method is carried out by inputting the displacement or strain data measured from experiments. These methods are very susceptible to small changes in the input data [34], which makes it difficult to guarantee the accuracy of the results. Moreover, further experiments are conducted in order to address the ill-posed nature of the inverse analysis, thereby increasing the computation amount. To overcome this difficulty, it is essential to establish a simple, efficient, and accurate method to determine the elastic constants for general anisotropic materials.

In this study, a methodology is provided for determining the compliance coefficient matrix of skew anisotropic materials, where only one elastic symmetry plane exists and two sets of fibers are not orthogonal but skew at any angle. On this basis, the infinite skew anisotropic plate with an arbitrarily shaped hole is considered, and the analytical stress solutions are derived using the conformal transformation method of complex variable function. The presented method is verified by comparison with the numerical simulation of ANSYS software. The influences of external loadings on stress distribution along the hole boundary are analyzed using the elliptical, hexagonal, and square holes as examples.

The methodology described in this study can be applied to obtain the material constants of anisotropic plates in practical engineering. It is based on theoretical derivation and the engineering constants, which can be determined by means of simple uniaxial tensile and pure shear tests. Compared with the previous numerical and inverse analysis methods, it has the advantages of simple calculation, high precision, and universality, and it can provide a theoretical basis for future research of the anisotropic material.

### 2. Problem Description and Assumption

The only difference in the fundamental equations of anisotropic and isotropic linear elastic mechanics is the constitutive equations, because geometrical and equilibrium conditions are independent of material physical properties. For general condition of anisotropic materials, there are 13 independent elastic constants when at least one elastic symmetry plane exists, to which the z-axis of the coordinate system is perpendicular. In this case, the generalized Hooke’s law in the Cartesian coordinate system \( xyz \) is expressed as follows:

\[
\begin{align*}
\epsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{16}\tau_{xy}, \\
\epsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{26}\tau_{xy}, \\
\epsilon_z &= a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z + a_{36}\tau_{xy}, \\
y_{xy} &= a_{44}\tau_{xy} + a_{45}\tau_{yz}, \\
y_{xz} &= a_{45}\tau_{yz} + a_{55}\tau_{xz}, \\
y_{yx} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{36}\sigma_z + a_{66}\tau_{xy},
\end{align*}
\]

where \( a_{ij} \) (\( i, j = 1, 2, 3, 4, 5, 6 \)) are the elastic constants that determine the material properties. Determination of these
values is a prerequisite to analyze the perforated anisotropic structures. When the material is orthotropic and the Cartesian coordinate system is established along the principal directions of the material, the number of independent elastic constants is reduced to nine. The values of $a_{11}$, $a_{22}$, $a_{33}$, and $a_{45}$ are equal to zero. The other elastic constants in (1) can be obtained in terms of engineering constants: Young’s modulus $E$, Poisson’s ratio $v$, and shear modulus $G$ [2]. However, several elastic constants, such as $a_{16}$ and $a_{26}$, are no longer equal to zero for nonorthogonal materials, regardless of the Cartesian coordinate system establishment. This study aims to determine the values of these elastic constants and obtain the compliance matrix through theoretical studies.

The anisotropic material discussed here is illustrated in Figure 1. The only elastic symmetry plane is the $xoy$-plane, to which the $z$-axis of the coordinate system is perpendicular. That is, the elastic properties along the $z$-direction are the same for each point in the material. Two sets of skew fibers exist in the $xoy$-plane and along the directions of $x_1$ and $x_2$, respectively. Similarly, each point in the material has the same elastic properties along the $x_1'$ or $x_2'$-direction, respectively. Taking the two fiber orientations as the $x_1'$ and $x_2'$-axes, respectively, the local Cartesian coordinate systems $x_1'oy_1'$ and $x_2'oy_2'$ are established. As can be seen from Figure 1, the global Cartesian coordinate system $xoy$ overlaps $x_1'oy_1'$ and $x_2'oy_2'$ under rotation by $\varphi_1$ and $\varphi_2$ degrees clockwise around the $z$-axis, respectively. It is an orthotropic problem for $\varphi_2 - \varphi_1 = \pi/2$, which means that the fibers are orthogonal. Only the general situation of $\varphi_2 - \varphi_1 \neq \pi/2$ is investigated in this study. In this case, neither the $x_1'oz$-plane nor the $x_2'oz$-plane is an elastic symmetry plane, that is, the skew anisotropic material, which appears frequently in the composite plates.

Based on the assumptions that the stress components $\sigma_{x}$, $\sigma_{z}$, and $\tau_{xz}$ are equal to zero at each point in the plane and the other stress components are functions of $x$ and $y$, the analysis of skew anisotropic plate is simplified to a plane stress problem. Thus, the generalized Hooke’s law in equation (1) can be reduced to the following equations:

\[
\begin{align*}
\sigma_{x} &= a_{11}\epsilon_{x} + a_{12}\epsilon_{y} + a_{16}\tau_{xy}, \\
\sigma_{y} &= a_{12}\epsilon_{x} + a_{22}\epsilon_{y} + a_{26}\tau_{xy}, \\
\tau_{xy} &= a_{16}\epsilon_{x} + a_{26}\epsilon_{y} + a_{66}\tau_{xy},
\end{align*}
\]

(2)

In the above relations, the strain components $\epsilon_{x}$ and $\epsilon_{y}$ are equal to zero, and $\epsilon_{z}$ is not involved in the calculation of the other components. The elastic constants in local coordinate systems $x_1'oy_1'$ and $x_2'oy_2'$ are represented by $b_{ij}$ and $c_{ij}$, respectively, and the generalized Hooke’s laws are expressed as follows:

\[
\begin{align*}
\epsilon_{x} &= b_{11}\sigma_{x} + b_{12}\sigma_{y} + b_{16}\tau_{xy}, \\
\epsilon_{y} &= b_{12}\sigma_{x} + b_{22}\sigma_{y} + b_{26}\tau_{xy}, \\
\gamma_{x,y} &= b_{16}\sigma_{x} + b_{26}\sigma_{y} + b_{66}\tau_{xy},
\end{align*}
\]

(3)

Replacing $b_{ij}$, $x_1$, and $y_1$ with $c_{ij}$, $x_2$, and $y_2$, respectively, the other set of linear equations regarding $c_{ij}$ can also be obtained. The elastic constants of anisotropic bodies are related to the direction of coordinates. Although $a_{ij}$, $b_{ij}$, and $c_{ij}$ take different values, they describe the same material. As shown in Figure 1, the three Cartesian coordinate systems are related to one another by certain angles, and there is also a transformation relationship between $a_{ij}$, $b_{ij}$, and $c_{ij}$.

3. Determination of Elastic Constants

Initially, the transformation relationship between the elastic constants is analyzed under rotation of the coordinate axes. The elastic constant in the original coordinate system $xyz$ is denoted as $d_{ij}$, and the constant in the new coordinate system $x'yz'$ is denoted as $d'_{ij}$. $d_{ij}$ and $d'_{ij}$ satisfy the following relationship:

\[
d'_{ij} = \sum_{m=1}^{6} \sum_{n=1}^{6} q_{im}q_{jn}d_{mn} = q_{im}q_{jn}d_{mn},
\]

(4)

where $i, j = 1, 2, 3, 4, 5, 6$; $m$ and $n$ are dummy indices indicating summation. $q_{ij}$ can be calculated according to Table 1.

\[
l_{ij} = \cos(x'_i x_j),
\]

representing the cosine value of the angle between $x'_i$-axis in the new coordinate system $\{x'_i\}$ and $x_j$ in the original coordinate system $\{x_i\}$. $x'_1, x'_2, x'_3$ represent $x', y'$, $z'$. $x_1, x_2, x_3$ represent $x, y, z$. The $z$-axis coincides with the $z'$-axis, and the new coordinate system is obtained by rotating the original coordinate system counterclockwise around the $z$-axis by an angle $\varphi$. Accordingly, the calculation rules of $l_{ij}$ can be reduced to Table 2.

Substituting Tables 1 and 2 into equation (4), the transformation formulae between elastic constants $d_{ij}$ and $d'_{ij}$ can be obtained. For the skew anisotropic plate discussed in this study, $b_{ij}$ and $c_{ij}$ correspond to the elastic constants in the new coordinate systems, and $a_{ij}$ corresponds to the
Table 1: Calculation rules of $a_{ij}$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$l_{11}$</td>
<td>$l_{12}$</td>
<td>$l_{13}$</td>
<td>$l_{21}$</td>
<td>$l_{22}$</td>
<td>$l_{23}$</td>
</tr>
<tr>
<td>2</td>
<td>$l_{21}$</td>
<td>$l_{22}$</td>
<td>$l_{23}$</td>
<td>$l_{31}$</td>
<td>$l_{32}$</td>
<td>$l_{33}$</td>
</tr>
<tr>
<td>3</td>
<td>$l_{31}$</td>
<td>$l_{32}$</td>
<td>$l_{33}$</td>
<td>$l_{11}$</td>
<td>$l_{12}$</td>
<td>$l_{13}$</td>
</tr>
<tr>
<td>4</td>
<td>$2l_{11}$</td>
<td>$2l_{12}$</td>
<td>$2l_{13}$</td>
<td>$l_{21} + l_{31}$</td>
<td>$l_{22} + l_{32}$</td>
<td>$l_{23} + l_{33}$</td>
</tr>
<tr>
<td>5</td>
<td>$2l_{21}$</td>
<td>$2l_{22}$</td>
<td>$2l_{23}$</td>
<td>$l_{31} + l_{11}$</td>
<td>$l_{32} + l_{12}$</td>
<td>$l_{33} + l_{13}$</td>
</tr>
<tr>
<td>6</td>
<td>$6l_{31}$</td>
<td>$6l_{32}$</td>
<td>$6l_{33}$</td>
<td>$l_{11} + l_{21}$</td>
<td>$l_{12} + l_{22}$</td>
<td>$l_{13} + l_{23}$</td>
</tr>
</tbody>
</table>

Table 2: Calculation rules of $l_{ij}$

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>x'</td>
<td>$\cosh$</td>
<td>$\sinh$</td>
<td>0</td>
</tr>
<tr>
<td>y'</td>
<td>$-\sinh$</td>
<td>$\cosh$</td>
<td>0</td>
</tr>
<tr>
<td>z'</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The equation (3) is reduced to the solution of the elastic constants, subsequently, according to the physical significance, $\sigma_{ij}$ indicates the applied stress direction and the second subscript $xy$ is Poisson’s ratio, in which the first subscript $x$ is the shear modulus. In summary, the elastic constants $c_{11}, c_{12}, c_{22}, c_{66}$ can also be obtained by analyzing the generalized Hooke’s law regarding $c_{ip}$ when the stresses $\sigma_{x1}, \sigma_{y1}$, and $\tau_{x1y1}$ are applied, respectively.

$$c_{11} = \frac{1}{E_{x1}}$$
$$c_{12} = \frac{\nu_{x1y1}}{E_{x1}}$$
$$c_{22} = \frac{1}{E_{y1}}$$
$$c_{66} = \frac{1}{G_{x1y1}}$$

where $E_{x1}$ and $E_{y1}$ are Young’s moduli in the $x_1$- and $y_1$-axes, respectively; $\nu_{x1y1}$ is Poisson’s ratio, and $G_{x1y1}$ is the shear modulus.
shear modulus. They can also be determined by means of simple uniaxial tensile and pure shear tests.

Analyzing the two sets of transformation formulae regarding \( a_{ij} \), \( b_{ij} \), and \( c_{ij} \) yields
\[
\begin{align*}
    b_{16} - 4b_{12} &= c_{66} - 4c_{12}, \\
    b_{11} + 2b_{22} + 2b_{12} &= c_{11} + c_{22} + 2c_{12},
\end{align*}
\]
where
\[
    \begin{align*}
    b_{16}, b_{26}, c_{16}, c_{26}, c_{12}, \text{ and } c_{66}.
\end{align*}
\]

Upon combining the engineering constants and the previous analysis, the unknown constants of \( b_{ij} \) and \( c_{ij} \) are

\[
\begin{align*}
    b_{16}, b_{26}, c_{16}, c_{26}, c_{12}, \text{ and } c_{66}. \quad (10)
\end{align*}
\]

Define
\[
\begin{align*}
    \sin \phi_1 \cos^3 \phi_1 &= g_1, \\
    \sin \phi_1 \cos \phi_1 &= g_2, \\
    \sin^2 \phi_1 \cos^2 \phi_1 &= g_3, \\
    \sin \phi_2 \cos^3 \phi_2 &= h_1, \\
    \sin \phi_2 \cos \phi_2 &= h_2, \\
    \sin^2 \phi_2 \cos^2 \phi_2 &= h_3.
\end{align*}
\]

The following relations containing \( b_{16}, b_{26}, c_{16}, c_{26}, c_{12}, \) and \( c_{66} \) are proposed based on equation (5).

Equation (12) can be expressed in the form of
\[
\begin{align*}
    AX &= B,
\end{align*}
\]
where
\[
\begin{align*}
    A &= \begin{bmatrix}
        -2g_1 & -2g_2 & 2h_1 & 2h_2 & -2h_3 & -h_3 \\
        g_1 - g_2 & g_2 - g_1 & h_2 - h_1 & h_1 - h_2 & 2h_3 - 1 & h_3 \\
        \cos^4 \phi_1 - 3g_3 & 3g_3 - \sin^4 \phi_1 & 3h_1 - \cos^4 \phi_2 & 2(\sin^3 \phi_2 - 3h_3) & 2(h_1 - h_2) & h_1 - h_2 \\
        2g_2 & 2g_1 & -2h_2 & -2h_1 & -2h_3 & -h_3 \\
        3g_3 - \sin^4 \phi_1 & \cos^3 \phi_1 - 3g_3 & \sin^2 \phi_2 - 3h_3 & 3h_3 - \cos^4 \phi_2 & 2(h_2 - h_1) & h_2 - h_1 \\
        4(g_1 - g_2) & 4(g_2 - g_1) & 4(h_2 - h_1) & 4(h_1 - h_2) & 8h_3 & 4h_3 - 1
\end{bmatrix},
\end{align*}
\]
\[
\begin{align*}
    B &= \begin{bmatrix}
        c_{11} \cos^2 \phi_1 - b_{11} \cos^4 \phi_1 - (2b_{12} + b_{66})g_3 - b_{22} \sin^4 \phi_1 + c_{22} \sin^2 \phi_2 \\
        c_{11} h_3 - (b_{11} + b_{22} - 2b_{12} - b_{66})g_3 - b_{12} + c_{22} h_3 \\
        2c_{11} h_1 + 2b_{22} g_3 - 2b_{11} g_1 + (2b_{12} + b_{66}) g_1 - g_2 & -2c_{22} h_2 \\
        c_{11} \sin^3 \phi_2 - b_{11} \sin^2 \phi_1 - (2b_{12} + b_{66}) g_3 - b_{22} \cos^4 \phi_1 + c_{22} \cos^2 \phi_2 \\
        2c_{11} h_1 + 2b_{22} g_3 - 2b_{11} g_1 - (2b_{12} + b_{66}) g_1 - g_2 & -2c_{22} h_1 \\
        4c_{11} h_3 - 4(b_{11} + b_{22} - 2b_{12} - b_{66}) g_3 - b_{66} + 4c_{22} h_3
\end{bmatrix}.
\end{align*}
\]

Theoretical method is also applicable for the plane strain problem.

**4. Derivation of Analytical Stress Solutions**

Since the material constants of the skew anisotropic plate are obtained, the stress distribution around the hole can be analyzed, as presented in this section. As illustrated in
Figure 2, an arbitrarily shaped hole exists in the plate of Figure 1, and uniform in-plane static loadings are applied to the edge at infinity and the application of loadings is completed instantaneously. When the size of the plate is much larger than that of the hole and the hole position is not near the edge, the research domain is considered to be infinite.

The conformal transformation of the complex variable function is an effective method for solving the stress problem of anisotropic plate with an irregular-shaped hole. Three complex variables \( z = x + iy, z_1 = x + \mu_1 y, \) and \( z_2 = x + \mu_2 y \) are involved in the solving process; thus, three polar coordinate systems \( \zeta = \rho e^{i\theta}, \zeta_1 = \rho_1 e^{i\theta_1}, \) and \( \zeta_2 = \rho_2 e^{i\theta_2} \) are established. With the analytic mapping functions \( z = \omega(\zeta), z_1 = \omega_1(\zeta_1), \) and \( z_2 = \omega_2(\zeta_2) \), the outer hole region in physical plane \( (z, z_1,\) and \( z_2\)-plane) is mapped onto the outer unit circle region in image plane \( (\zeta, \zeta_1,\) and \( \zeta_2\)-plane), respectively. The general expressions of the mapping functions are as follows \([20]\):

\[
\begin{align*}
z &= \omega(\zeta) = R \left( \zeta + \sum_{k=0}^{\infty} C_k \zeta^{-k} \right), \\
z_1 &= \omega_1(\zeta_1) = y_1 R \left( \zeta_1 + \sum_{k=1}^{n} C_k \zeta_1^{-k} \right) + \delta_1 R \left( \frac{1}{\zeta_1} + \sum_{k=1}^{n} C_k \zeta_1^{-k} \right), \\
z_2 &= \omega_2(\zeta_2) = y_2 R \left( \zeta_2 + \sum_{k=1}^{n} C_k \zeta_2^{-k} \right) + \delta_2 R \left( \frac{1}{\zeta_2} + \sum_{k=1}^{n} C_k \zeta_2^{-k} \right),
\end{align*}
\]

where \( C_k \) and \( R \) in equation (15) are coefficients denoting the hole geometry and size, respectively. Equation (15) can represent a variety of hole shapes, provided that \( k \) is sufficiently large. Herein, \( n \) is the maximum value of \( k \).

In equations (16) and (17),

\[
\begin{align*}
y_1 &= \frac{(1 - i\mu_1)}{2}, \\
y_2 &= \frac{(1 - i\mu_2)}{2}, \\
\delta_1 &= \frac{(1 + i\mu_1)}{2}, \\
\delta_2 &= \frac{(1 + i\mu_2)}{2}, \\
\mu_1 &= \alpha_1 + i\beta_1, \\
\mu_2 &= \alpha_2 + i\beta_2,
\end{align*}
\]

where \( \alpha_k \) and \( \beta_k \) \((k = 1, 2)\) are coefficients related to material properties, and \( \beta_1 > 0 \) and \( \beta_2 > 0 \).

The relational expressions between \( \zeta, \zeta_1, \) and \( \zeta_2 \) are established as follows:
Equations (19) and (20) are applicable for any point in the region \( \frac{r}{\mu} \geq 1 \), and \( \zeta = \xi = \sigma = e^{i\theta} \) in the unit circle, which corresponds to the hole boundary.

For the plane stress problem of an anisotropic plate with only one elastic symmetry plane, the body forces are neglected when the external loadings are substantially larger than the forces, and the compatibility equation for Airy’s stress function \( F = F(x, y) \) is expressed as [30]

\[
\begin{align*}
\gamma_1 R \left( \zeta + \sum_{k=1}^{n} C_k \zeta^{-k} \right) + \delta_1 R \left( \frac{1}{\zeta} + \sum_{k=1}^{n} C_k \zeta^{-k} \right) &= \gamma_1 R \left( \zeta + \sum_{k=1}^{n} C_k \zeta^{-k} \right), \\
\gamma_2 R \left( \zeta_2 + \sum_{k=1}^{n} C_k \zeta_2^{-k} \right) + \delta_2 R \left( \frac{1}{\zeta_2} + \sum_{k=1}^{n} C_k \zeta_2^{-k} \right) &= \gamma_2 R \left( \zeta + \sum_{k=1}^{n} C_k \zeta^{-k} \right).
\end{align*}
\] (19) (20)

The solutions to equation (21) are connected with the roots of the following characteristic equation:

\[
a_{11} \mu^4 - 2a_{16} \mu^3 + (2a_{12} + a_{66}) \mu^2 - 2a_{26} \mu + a_{22} = 0. \tag{22}
\]

As the elastic constants \( a_{ij} \) have been determined previously, the roots of equation (22) can be calculated. It is noteworthy that the roots are conjugate complex roots \( \mu_1, \mu_2, \mu_3, \) and \( \mu_4. \) In this study, we only investigate the situation of \( \mu_1 \neq \mu_2. \)

The stress boundary conditions are needed in the derivation process. By defining \( F = 2\text{Re}[F_1(z_1) + F_2(z_2)], \) \( \Phi_1(z_1) = dF_1(z_1)/dz_1, \) and \( \Phi_2(z_2) = dF_2(z_2)/dz_2, \) the stress boundary conditions are expressed by \( \Phi_1(z_1) \) and \( \Phi_2(z_2) \) as

\[
\frac{\partial^3 F}{\partial x^2 \partial y} - 2a_{26} \frac{\partial^2 F}{\partial x \partial y^2} + (2a_{12} + a_{66}) \frac{\partial F}{\partial y} - 2a_{26} \frac{\partial F}{\partial y} + a_{22} F = 0,
\]

where \( a_{ij} \) are elements of the compliance matrix analyzed in Section 3.

The stress components are reduced to solving the two analytic functions \( \Phi_1(z_1) \) and \( \Phi_2(z_2). \) The research domain is infinite, and no external loadings exist along the hole boundary. Accordingly, \( f_2 = f_2 = 0, \) and \( \Phi_1(z_1) \) and \( \Phi_2(z_2) \) are written as

\[
\begin{align*}
\Phi_1(z_1) &= B^* z_1 + \Phi_1^0(z_1), \\
\Phi_2(z_2) &= (B^* + iC^*) z_2 + \Phi_2^0(z_2),
\end{align*}
\] (23) (24)

where \( B^*, B^*, \) and \( C^* \) are real constants. By utilization of the external loadings at infinity (i.e., \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \)) and the constants \( \alpha_1, \alpha_2, \beta_1, \) and \( \beta_2, \) the constants can be determined as [30]

\[
B^* = \frac{\sigma_x^\infty + (\alpha_1 + \beta_1^2) \sigma_y^\infty + 2a_2 \tau_{xy}^\infty}{2[(a_2 - a_1)^2 + (\beta_1^2 - \beta_2^2)]},
\]

\[
B^* = \frac{(\alpha_1 - \beta_1^2 - 2a_1 \alpha_2) \sigma_y^\infty - \sigma_x^\infty - 2a_2 \tau_{xy}^\infty}{2[(a_2 - a_1)^2 + (\beta_1^2 - \beta_2^2)]},
\]

\[
C^* = \frac{(\alpha_1 - \alpha_2) \sigma_x^\infty + [\alpha_2 (a_1^2 - \beta_2^2) - \alpha_1 (a_2^2 - \beta_1^2)] \sigma_y^\infty + \left[[\alpha_2^2 - \beta_2^2] - (a_2^2 - \beta_2^2) \right] \tau_{xy}^\infty}{2 \beta_2 [(a_2 - a_1)^2 + (\beta_2^2 - \beta_1^2)]},
\]

Substitution of the expressions for \( z_1 \) and \( z_2, \) given by equations (16) and (17), into (25) and (26), respectively, gives the following forms of \( \Phi_1^0(z_1) \) and \( \Phi_2^0(z_2): \)
\[ \Phi_1^0(z_1) = \Phi_1^0(\zeta_1) = \Phi_1^0, (\xi_1) = \sum_{k=0}^{\infty} a_k \zeta_1^{-k}, \]

\[ \Phi_2^0(z_2) = \Phi_2^0(\zeta_2) = \Phi_2^0, (\xi_2) = \sum_{k=0}^{\infty} b_k \zeta_2^{-k}. \]

Thereafter, \( a_k \) and \( b_k \) \((k = 0, \ldots, \infty)\) are determined by substituting equations (23)–(31) into the stress boundary conditions and using the power-series method.

Finally, by means of equations (16), (17), and (23)–(31), we obtain

\[ \Phi_1^1(z_1) = B' - \sum_{k=1}^{n} k \alpha_k \left[ y_1 R \left( 1 - \sum_{n=1}^{k-1} kC_k \zeta_1^{-k} \right) + \delta_1 \left( 1 - \sum_{n=1}^{k-1} kC_k \zeta_1^{-k} \right) \right], \]

\[ \Phi_2^1(z_2) = (B' + iC' -) - \sum_{k=1}^{n} k \beta_k \left[ y_2 R \left( 1 - \sum_{n=1}^{k-1} kC_k \zeta_2^{-k} \right) + \delta_2 \left( 1 - \sum_{n=1}^{k-1} kC_k \zeta_2^{-k} \right) \right]. \]

In the Cartesian coordinate system, the stress components are determined by means of the following equations:

\[ \begin{align*}
\sigma_x &= 2 \text{Re} \left[ \mu_1' \Phi_1^1(z_1) + \ldots \right], \\
\sigma_y &= 2 \text{Re} \left[ \Phi_1^1(z_1) + \Phi_2^1(z_2) \right], \\
\tau_{xy} &= -2 \text{Re} \left[ \mu_1' \Phi_2^1(z_2) \right].
\end{align*} \]  

(34)

The stress components \( \sigma_\theta, \sigma_\rho, \) and \( \tau_\rho \) in the orthogonal curvilinear coordinate system are calculated by

\[ \begin{align*}
\sigma_\rho + \sigma_\theta &= \sigma_x + \sigma_y, \\
\sigma_\theta - \sigma_\rho + 2i \tau_\rho &= \frac{\zeta^2}{\rho^2} \omega' \left( \frac{\omega'}{\omega} \right) \left( \sigma_y - \sigma_x + 2i \tau_{xy} \right). 
\end{align*} \]  

(35)

Along the hole boundary, where \( \rho = 1 \) and \( \zeta_1 = \zeta_2 = \zeta = \sigma = e^{i\theta} \), the analytic solutions for stress components can easily be obtained by employing equations (32)–(35).

5. Examples and Comparisons

5.1. Comparison with Numerical Results Obtained by ANSYS Software. The Finite Element Method (FEM) is a particularly effective method for analyzing complex structural problems. In order to verify the correctness of the proposed method in this paper, an elliptical hole is used as an example, and the tangential stresses along the hole boundary are simulated by utilizing the ANSYS software in this section. Thereafter, the numerical results are compared to those obtained by means of the analytical method, using the same material parameters.

For the plate in Figure 2, the angles between two sets of skew fibers and the global coordinate system \( xoy \) are taken as \( \varphi_1 = 10^\circ \) and \( \varphi_2 = 150^\circ \), respectively. The engineering constants are \( E_{x_1} = 60 \text{ GPa}, \ E_{y_1} = 40 \text{ GPa}, \ E_{y_2} = 15 \text{ GPa}, \ E_{y_3} = 10 \text{ GPa}, \ y_{x_1 y_1} = 0.28, \) and \( G_{x_2 y_2} = 6.0 \text{ GPa}. \) On this basis, the calculation of the stress components along the hole boundary is conducted by implementing the presented theory in the FORTRAN program.

The results of the material constants \( a_{ij} \) in equation (2) are expressed as follows:

\[ a_{11} = 1.426 \times 10^{-2} \text{GPa}^{-1}, \]

\[ a_{12} = -1.226 \times 10^{-2} \text{GPa}^{-1}, \]

\[ a_{16} = -1.753 \times 10^{-3} \text{GPa}^{-1}, \]

\[ a_{22} = 8.427 \times 10^{-2} \text{GPa}^{-1}, \]

\[ a_{26} = 4.793 \times 10^{-2} \text{GPa}^{-1}, \]

\[ a_{66} = 0.136 \text{GPa}^{-1}. \]

The height of the elliptical hole is \( 2a \) in the \( x \)-axis direction, while the width is \( 2b \) in the \( y \)-axis direction. Taking the values of \( a/b = 0.5, 1.0, \) and \( 1.5 \), the hole shapes are shown in Figure 3. For \( a/b = 1.5 \), the ANSYS numerical model is established as illustrated in Figure 4. The model range is 10 times the size of the width \( 2a \), which is equivalent to the infinite domain problem in the analytical method. The values of the external loadings are taken as \( \sigma_{x, \infty} = 1.0 \text{ MPa} \) and \( \sigma_{y, \infty} = 0.5 \text{ MPa} \), applying to the four boundaries at infinity. The four-node quadrilateral element Plane 182 is adopted, and there are 252,720 nodes and 252,000 elements in the entire research domain after meshing the model by means of the mapped method. In order to improve the computational accuracy of the numerical solutions, the grid near the hole boundary is denser, while the grid away from the hole boundary is sparser. For the purpose of stress analysis, the constraints in the \( x \)- and \( y \)-directions are applied to point \( A \), and the constraint in the \( x \)-direction is applied to point \( B \), in order to limit the rigid body displacements of the structure.

The tangential stresses obtained by the two methods for different ratios of \( a/b \) are illustrated in Figure 5. The solid and dashed lines represent the analytical and numerical solutions, respectively. It is observed from Figure 5 that the results obtained by different methods are in good agreement, which verifies the theoretical method presented in this paper.
Figure 3: Shapes of the elliptical hole for different values of $a/b$.

Figure 4: Numerical calculation model of the ANSYS software.

Figure 5: Tangential stresses along the boundary of different elliptic holes obtained by analytical and numerical methods.
5.2. Stress Analysis for Different Hole Shapes and External Loadings. Elliptical, hexagonal, and square holes in the skew anisotropic plates are used as examples in this section. The values of external loadings are taken as $\sigma_x$, $\sigma_y$, and $\tau_{xy}$, in which the value of $\sigma$ remains unchanged and only $\lambda$ is taken for different values, namely, 0.5, 1.0, and 1.5. $\lambda$ is the biaxial loading factor, and $\lambda = 1$ represents the equibiaxial loading condition.

The size of the hole has no influence on the calculation of the stress field, because the research domain is infinite. In order not to lose the generality, take $R_1 = 1.0 + 0.0i$. In consideration that the hole shapes are all symmetric about the $x$-axis, the imaginary part of the coefficient $C_k$ is equal to zero.

The coefficients of the mapping functions for the three types of holes are given as follows [35].

For the elliptical hole, $a/b = 1.5$, $n = 1$, and $C_1 = (a - b)/(a + b) = 0.2$.

For the hexagonal hole, $n = 29$, $C_5 = 0.0667$, $C_{11} = 0.0101$, $C_{17} = 0.0036$, $C_{23} = 0.0018$, and $C_{29} = 0.0010$.

For the square hole, $n = 19$, $C_3 = -0.1667$, $C_7 = 0.0179$, $C_{11} = -0.0057$, $C_{15} = 0.0026$, and $C_{19} = -0.0014$.

All other $C_k$ are equal to zero.

The hole shapes illustrated in Figure 6 are obtained based on the aforementioned values of $C_k$ and $R$. By using the same values of skew angles and elastic constants as those in Section 5.1, the analytical solutions of the tangential stress along the hole boundary are illustrated in Figures 7–9 for different values of the biaxial loading factor, respectively.

With no external loading applied on the hole boundary, the radial stress $\sigma_r$, and shear stress $\tau_{r\theta}$ are equal to zero. Therefore, only the distribution of the normalized tangential stress $\sigma_{\theta}/\sigma$, which is the ratio of the tangential stress to the external loading in the $x$-direction, is provided in Figures 7–9. It should be noted that angle $\theta$ in Figures 7–9 is
the polar angle in the $\zeta$-plane, which begins from the positive $x$-axis and rotates anticlockwise. Sign convention is defined as positive for tension and negative for compression.

As can be seen from Figures 7–9 that, for the perforated skew anisotropic plate, the distributions of the tangential stress exhibit asymmetry, and mainly tensile stresses are presented for the biaxial loading factor $\lambda = 0.5, 1.0,$ and $1.5$. Only when $\lambda = 1.5$, a small region with compressive stresses exists. However, the magnitude of the compressive stress is much smaller than that of the tensile stress. Owing to the skew anisotropic properties of the material, the region with compressive stress deviates from the intersection area of the hole boundary and $y$-axis (i.e., $\theta = 90^\circ$ and $270^\circ$) and is mainly in the vicinity of $\theta = 80^\circ$ and $260^\circ$.

Moreover, when the external loading in the $x$-direction $\sigma_x^\infty$ remains unchanged, the tangential stresses near the intersection of the hole and $x$-axis gradually increase with increasing the value of $\sigma_y^\infty$. However, the tangential stresses near the intersection of the hole and $y$-axis decrease. It should be mentioned that there always exist four points in which the value of the tangential stress does not change with $\sigma_y^\infty$, such as point $J$ at $\theta = 52.40^\circ$ and point $K$ at $\theta = 110.50^\circ$ in Figure 7, where the stress values are always 1.2 and 2.3, respectively; the point near point $D$ at $\theta = 60.95^\circ$ and the point near point $E$ at $\theta = 117.75^\circ$ in Figure 8, where the stress values are always 7.3 and 9.0, respectively; and the point near point $G$ at $\theta = 47.05^\circ$ and the point near point $H$ at $\theta = 131.10^\circ$ in Figure 9, where the stress values are always 10.3 and 9.0, respectively. Considering that both the hole geometry and the external loadings are symmetric about the $x$-axis, the stress distribution at $\theta = 180^\circ$–$360^\circ$ is the repetition of that at $\theta = 0^\circ$–$180^\circ$. Therefore, only the points with unchanged stresses at $\theta = 0^\circ$–$180^\circ$ are given in the above discussion. Accordingly, we can conclude that the external loadings in the $y$-direction have no influence on the tangential stresses of the four points. Furthermore, we can infer that when the external loadings are uniaxial tension stresses acting at infinity, the tangential stresses of the four points at the hole boundary are equal to zero.

By comparing Figures 7–9, we can also obtain that the stress distribution along the elliptical hole boundary is relatively uniform and the stress value is smaller. However, the stress concentration is very significant near the corners of the hexagonal and square holes. Moreover, for the same external loadings, a greater curvature of the hole corner leads to a larger value of the maximum tangential stress.

The values of the maximum tangential stresses $(\sigma_{\theta}/\sigma)_{\text{max}}$ and their positions $\theta$ are illustrated in Table 3 for the hexagonal and square holes under different biaxial loading factors. As can be seen from the data in Table 3, for the two types of hole, the maximum tangential stress

![Figure 8: Normalized tangential stresses along boundary of the hexagonal hole for different external loadings ($\lambda = 0.5, 1.0,$ and $1.5$).](image-url)
does not exactly occur at the point of the corner but at the point near the corner. Besides, the position gets closer to the corner with increasing the value of the biaxial loading factor. Moreover, the position of the maximum stress even changes from the vicinity of point $H$ to the vicinity of point $G$ for the square hole under the loadings of $\lambda = 1.5$.

### 6. Conclusion

In this study, the skew anisotropic material, in which only one elastic symmetry plane exists and the fibers are in general nonorthogonal condition, is introduced into the problem of perforated anisotropic structure. From the view of mechanics, the stress analysis of skew anisotropic plate with an arbitrarily shaped hole is simplified to a plane stress problem in an infinite domain. A new methodology is provided for determining the compliance coefficient matrix by establishing the transformation formulae for the elastic constants under rotation of the coordinate axes and in combination with the engineering constants. On this basis, the analytical stress solutions for the skew anisotropic plate with an arbitrarily shaped hole are derived using the conformal transformation method of complex variable function.

Taking the elliptical, hexagonal, and square hole as an example, the tangential stresses along the hole boundary are analyzed for different external loadings, and the following conclusions are reached.

1. The distributions of the tangential stress in the perforated skew anisotropic plate show significant asymmetry owing to the anisotropic properties of the material.
2. Mainly tensile stresses are presented under the tensile external loadings for the biaxial loading factor $\lambda = 0.5, 1.0,$ and $1.5$, and the region with smaller compressive stresses deviates from the intersection area of the hole boundary and $y$-axis.
3. When the external loadings are uniaxial tension

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**Table 3: Values of the maximum tangential stresses $(\sigma_\theta/\sigma_\theta)^{\text{max}}$ and their positions $\theta$.**

<table>
<thead>
<tr>
<th>The biaxial loading factor $\lambda$</th>
<th>The hexagonal hole</th>
<th>The square hole</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$(\sigma_\theta/\sigma_\theta)^{\text{max}}$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>0.5</td>
<td>$118.50^\circ/298.50^\circ$</td>
<td>10.2</td>
</tr>
<tr>
<td>1.0</td>
<td>$118.70^\circ/298.70^\circ$</td>
<td>11.3</td>
</tr>
<tr>
<td>1.5</td>
<td>$118.80^\circ/298.80^\circ$</td>
<td>12.7</td>
</tr>
</tbody>
</table>
stresses applied at infinity, there always exist four points at the hole boundary where the tangential stresses are equal to zero. (4) For the same external loadings and material parameters, a greater curvature of the hole corner leads to a larger value of the maximum tangential stress. (5) The maximum tangential stress does not exactly occur at the point of the corner but at the point near the corner; besides, the position gets closer to the corner point with the increase of the value of the biaxial loading factor.

The ANSYS software is utilized to make a numerical verification of the presented method, and good agreement of stress distributions is found. The proposed solutions can provide insight into the mechanical behavior of skew anisotropic plates in practical engineering. Furthermore, they can also be applied in preliminary designs of future perforated anisotropic structures.

Data Availability

The data that support the findings of this study are available upon request from the corresponding author.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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