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Research Article

Modified Halfspace-Relaxation Projection Methods for Solving the Split Feasibility Problem

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This paper presents modified halfspace-relaxation projection (HRP) methods for solving the split feasibility problem (SFP). Incorporating with the techniques of identifying the optimal step length with positive lower bounds, the new methods improve the efficiencies of the HRP method (Qu and Xiu (2008)). Some numerical results are reported to verify the computational preference.

1. Introduction

Let C and Q be nonempty closed convex sets in R^n and R^m, respectively, and A an $m \times n$ real matrix. The problem, to find $x \in C$ with $Ax \in Q$ if such x exists, was called the split feasibility problem (SFP) by Censor and Elfving [1].

In this paper, we consider an equivalent reformulation [2] of the SFP:

$$\text{minimize } f(z) \quad \text{subject to } z = \begin{pmatrix} x \\ y \end{pmatrix} \in \Omega, \tag{1.1}$$

where

$$f(z) = \frac{1}{2}\|Bz\|^2 = \frac{1}{2}\|y - Ax\|^2, \qquad B = \begin{pmatrix} -A & I \end{pmatrix}, \qquad \Omega = C \times Q. \tag{1.2}$$

For convenience, we only consider the Euclidean norm. It is obvious that $f(z)$ is convex. If $z = (x^T, y^T)^T \in \Omega$ and $f(z) = 0$, then x solves the SFP. Throughout we assume that the solution set of the SFP is nonempty. And thus the solution set of (1.1), denoted by Ω^*, is nonempty. In addition, in this paper, we always assume that the set Ω is given by

$$\Omega = \{z \in R^{n+m} \mid c(z) \leq 0\}, \tag{1.3}$$

where $c : R^{n+m} \to R$ is a convex (not necessarily differentiable) function. This representation of Ω is general enough, because any system of inequalities $\{c_j(z) \leq 0,\ j \in J\}$, where $c_j(z)$ are convex and J is an arbitrary index set, can be reformulated as the single inequality $c(z) \leq 0$ with $c(z) = \sup\{c_j(z) \mid j \in J\}$. For any $z \in R^{n+m}$, at least one subgradient $\xi \in \partial c(z)$ can be calculated, where $\partial c(z)$ is a subgradient of $c(z)$ at z and is defined as follows:

$$\partial c(z) = \left\{\xi \in R^{n+m} \mid c(u) \geq c(z) + (u - z)^T \xi,\ \forall u \in R^{n+m}\right\}. \tag{1.4}$$

Qu and Xiu [2] proposed a halfspace-relaxation projection method to solve the convex optimization problem (1.1). Starting from any $z^0 \in R^n \times R^m$, the HRP method iteratively updates z^k according to the formulae:

$$\overline{z}^k = P_{\Omega_k}\left[z^k - \alpha_k \nabla f\left(z^k\right)\right], \tag{1.5}$$

$$z^{k+1} = z^k - \gamma_k \left[z^k - \overline{z}^k - \alpha_k\left(\nabla f\left(z^k\right) - \nabla f\left(\overline{z}^k\right)\right)\right], \tag{1.6}$$

where

$$\Omega_k = \left\{z \in R^{n+m} \mid c\left(z^k\right) + \left(z - z^k\right)^T \xi^k \leq 0\right\}, \tag{1.7}$$

ξ^k is an element in $\partial c(z^k)$, $\alpha_k = \gamma l^{m_k}$ and m_k is the smallest nonnegative integer m such that

$$\alpha_k \left(z^k - \overline{z}^k\right)^T \left(\nabla f\left(z^k\right) - \nabla f\left(\overline{z}^k\right)\right) \leq (1 - \rho)\left\|z^k - \overline{z}^k\right\|^2, \quad \rho \in (0, 1), \tag{1.8}$$

$$\gamma_k = \frac{\theta\rho\left\|z^k - \overline{z}^k\right\|^2}{\left\|z^k - \overline{z}^k - \alpha_k\left(\nabla f(z^k) - \nabla f\left(\overline{z}^k\right)\right)\right\|^2}, \quad \theta \in (0, 2). \tag{1.9}$$

The notation $P_{\Omega_k}(v)$ denotes the projection of v onto Ω_k under the Euclidean norm, that is,

$$P_{\Omega_k}(v) = \arg\min\{\|u - v\| \mid u \in \Omega_k\}. \tag{1.10}$$

Here the halfspace Ω_k contains the given closed convex set Ω and is related to the current iterative point z^k. From the expressions of Ω_k, the projection onto Ω_k is simple to be computed (for details, see Proposition 3.3). The idea to construct the halfspace Ω_k and replace P_Ω by P_{Ω_k} is from the halfspace-relaxation projection technique presented by Fukushima [3]. This technique is often used to design algorithms (see, e.g., [2, 4, 5]) to solve the SFP. The drawback of the HRP method in [2] is that the step length γ_k defined in (1.9) may be very small since $\lim_{k\to\infty}\|z^k - \bar{z}^k\| = 0$.

Note that the reformulation (1.1) is equivalent to a monotone variational inequality (VI):

$$z^* \in \Omega, \quad (z - z^*)^T \nabla f(z^*) \geq 0, \quad \forall z \in \Omega, \tag{1.11}$$

where

$$\nabla f(z) = B^T B z. \tag{1.12}$$

The forward-backward splitting method [6] and the extragradient method [7, 8] are considerably simple projection-type methods in the literature. They are applicable for solving monotone variational inequalities, especially for (1.11). For given z^k, let

$$\bar{z}^k = P_\Omega\left[z^k - \alpha_k \nabla f\left(z^k\right)\right]. \tag{1.13}$$

Under the assumption

$$\alpha_k \left\|\nabla f\left(z^k\right) - \nabla f\left(\bar{z}^k\right)\right\| \leq \nu \left\|z^k - \bar{z}^k\right\|, \quad \nu \in (0,1), \tag{1.14}$$

the forward-backward (FB) splitting method generates the new iterate via

$$z^{k+1} = P_\Omega\left[\bar{z}^k + \alpha_k\left(\nabla f\left(z^k\right) - \nabla f\left(\bar{z}^k\right)\right)\right], \tag{1.15}$$

while the extra-gradient (EG) method generates the new iterate by

$$z^{k+1} = P_\Omega\left[z^k - \alpha_k \nabla f\left(\bar{z}^k\right)\right]. \tag{1.16}$$

The forward-backward splitting method (1.15) can be rewritten as

$$z^{k+1} = P_\Omega\left[z^k - \gamma_k\left(z^k - \bar{z}^k - \alpha_k\left(\nabla f\left(z^k\right) - \nabla f\left(\bar{z}^k\right)\right)\right)\right], \tag{1.17}$$

where the descent direction $-(z^k - \bar{z}^k - \alpha_k(\nabla f(z^k) - \nabla f(\bar{z}^k)))$ is the same as (1.6) and the step length γ_k along this direction always equals to 1. He et al. [9] proposed the modified versions of the FB method and EG method by incorporating the optimal step length γ_k along the descent directions $-(z^k - \bar{z}^k - \alpha_k(\nabla f(z^k) - \nabla f(\bar{z}^k)))$ and $-\alpha_k \nabla f(\bar{z}^k)$, respectively. Here γ_k is defined by

$$\gamma_k = \theta\gamma_k^*, \quad \theta \in (0,2), \qquad \gamma_k^* = \frac{\left(z^k - \bar{z}^k\right)^T \left(z^k - \bar{z}^k - \alpha_k\left(\nabla f(z^k) - \nabla f\left(\bar{z}^k\right)\right)\right)}{\left\| z^k - \bar{z}^k - \alpha_k\left(\nabla f(z^k) - \nabla f\left(\bar{z}^k\right)\right)\right\|^2}. \tag{1.18}$$

Under the assumption (1.14), $\gamma_k^* \geq 1/2$ is lower bounded.

This paper is to develop two kinds of modified halfspace-relaxation projection methods for solving the SFP by improving the HRP method in [2]. One is an FB type HRP method, the other is an EG type HRP method. The numerical results reported in [9] show that efforts of identifying the optimal step length usually lead to attractive numerical improvements. This fact triggers us to investigate the selection of optimal step length with positive lower bounds in the new methods to accelerate convergence. The preferences to the HRP method are verified by numerical experiments for the test problems arising in [2].

The rest of this paper is organized as follows. In Section 2, we summarize some preliminaries of variational inequalities. In Section 3, we present the new methods and provide some remarks. The selection of optimal step length of the new methods is investigated in Section 4. Then, the global convergence of the new methods is proved in Section 5. Some preliminary numerical results are reported in Section 6 to show the efficiency of the new methods, and the numerical superiority to the HRP method in [2]. Finally, some conclusions are made in Section 7.

2. Preliminaries

In the following, we state some basic concepts for the variational inequality VI($\mathcal{S}, F$):

$$s^* \in \mathcal{S}, \quad (s - s^*)^T F(s^*) \geq 0, \quad \forall s \in \mathcal{S}, \tag{2.1}$$

where F is a mapping from R^N into R^N, and $\mathcal{S} \subseteq R^N$ is a nonempty closed convex set. The mapping F is said to be monotone on R^N if

$$(s - t)^T (F(s) - F(t)) \geq 0, \quad \forall s, t \in R^N. \tag{2.2}$$

Notice that the variational inequality VI($\mathcal{S}, F$) is invariant when we multiply F by some positive scalar α. Thus VI($\mathcal{S}, F$) is equivalent to the following projection equation (see [10]):

$$s = P_{\mathcal{S}}[s - \alpha F(s)], \tag{2.3}$$

that is, to solve VI($\mathcal{S}, F$) is equivalent to finding a zero point of the residue function

$$e(s, \alpha) := s - P_{\mathcal{S}}[s - \alpha F(s)]. \tag{2.4}$$

Note that $e(s, \alpha)$ is a continuous function of s because the projection mapping is nonexpansive. The following lemma states a useful property of $\|e(s, \alpha)\|$.

Lemma 2.1 ([4], Lemma 2.2). *Let F be a mapping from R^N into R^N. For any $s \in R^N$ and $\alpha > 0$, we have*

$$\min\{1,\alpha\}\|e(s,1)\| \leq \|e(s,\alpha)\| \leq \max\{1,\alpha\}\|e(s,1)\|. \tag{2.5}$$

Remark 2.2. Let $\tilde{\mathcal{S}} \supseteq \mathcal{S}$ be a nonempty closed convex set and let $e_{\tilde{\mathcal{S}}}(s,\alpha)$ be defined as follows:

$$e_{\tilde{\mathcal{S}}}(s,\alpha) = s - P_{\tilde{\mathcal{S}}}[s - \alpha F(s)]. \tag{2.6}$$

Inequalities (2.5) still hold for $e_{\tilde{\mathcal{S}}}(s,\alpha)$.

Some fundamental inequalities are listed below without proof, see, for example, [10].

Lemma 2.3. *Let $\tilde{\mathcal{S}}$ be a nonempty closed convex set. Then the following inequalities always hold*

$$\left(t - P_{\tilde{\mathcal{S}}}(t)\right)^T \left(P_{\tilde{\mathcal{S}}}(t) - s\right) \geq 0, \quad \forall t \in R^N,\ \forall s \in \tilde{\mathcal{S}}, \tag{2.7}$$

$$\left\|P_{\tilde{\mathcal{S}}}(t) - s\right\|^2 \leq \|t - s\|^2 - \left\|t - P_{\tilde{\mathcal{S}}}(t)\right\|^2, \quad \forall t \in R^N,\ s \in \tilde{\mathcal{S}}. \tag{2.8}$$

The next lemma lists some inequalities which will be useful for the following analysis.

Lemma 2.4. *Let $\tilde{\mathcal{S}} \supseteq \mathcal{S}$ be a nonempty closed convex set, s^* a solution of the monotone VI$(\mathcal{S}, F)$* (2.1) *and especially $F(s^*) = 0$. For any $s \in R^N$ and $\alpha > 0$, one has*

$$\alpha(s - s^*)^T F\left(P_{\tilde{\mathcal{S}}}[s - \alpha F(s)]\right) \geq \alpha e_{\tilde{\mathcal{S}}}(s,\alpha)^T F\left(P_{\tilde{\mathcal{S}}}[s - \alpha F(s)]\right), \tag{2.9}$$

$$\begin{aligned}(s - s^*)^T &\left\{e_{\tilde{\mathcal{S}}}(s,\alpha) - \alpha\left[F(s) - F\left(P_{\tilde{\mathcal{S}}}[s - \alpha F(s)]\right)\right]\right\} \\ &\geq e_{\tilde{\mathcal{S}}}(s,\alpha)^T \left\{e_{\tilde{\mathcal{S}}}(s,\alpha) - \alpha\left[F(s) - F\left(P_{\tilde{\mathcal{S}}}[s - \alpha F(s)]\right)\right]\right\}.\end{aligned} \tag{2.10}$$

Proof. Under the assumption that F is monotone we have

$$\left\{\alpha F\left(P_{\tilde{\mathcal{S}}}[s - \alpha F(s)]\right) - \alpha F(s^*)\right\}^T \left\{P_{\tilde{\mathcal{S}}}[s - \alpha F(s)] - s^*\right\} \geq 0, \quad \forall s \in R^N. \tag{2.11}$$

Using $F(s^*) = 0$ and the notation of $e_{\tilde{\mathcal{S}}}(s,\alpha)$, from (2.11) the assertion (2.9) is proved. Setting $t = s - \alpha F(s)$ and $s = s^*$ in the inequality (2.7) and using the notation of $e_{\tilde{\mathcal{S}}}(s,\alpha)$, we obtain

$$\left\{e_{\tilde{\mathcal{S}}}(s,\alpha) - \alpha F(s)\right\}^T \left\{P_{\tilde{\mathcal{S}}}[s - \alpha F(s)] - s^*\right\} \geq 0, \quad \forall s \in R^N. \tag{2.12}$$

Adding (2.11) and (2.12), and using $F(s^*) = 0$, we have (2.10). The proof is complete. □

Note that the assumption $F(s^*) = 0$ in Lemma 2.4 is reasonable. The following proposition and remark will explain this.

Proposition 2.5 ([2], Proposition 2.2). *For the optimization problem* (1.1), *the following two statements are equivalent:*

(i) $z^* \in \Omega$ *and* $f(z^*) = 0$,

(ii) $z^* \in \Omega$ *and* $\nabla f(z^*) = 0$.

Remark 2.6. Under the assumption that the solution set of the SFP is nonempty, if $z^* = ((x^*)^T, (y^*)^T)^T$ is a solution of (1.1), then we have

$$\nabla f(z^*) = B^T B z^* = B^T (y^* - Ax^*) = 0. \tag{2.13}$$

This point z^* is also the solution point of the VI (1.11).

The next lemma provides an important boundedness property of the subdifferential, see, for example, [11].

Lemma 2.7. *Suppose $h : R^N \to R$ is a convex function, then it is subdifferentiable everywhere and its subdifferentials are uniformly bounded on any bounded subset of R^N.*

3. Modified Halfspace-Relaxation Projection Methods

In this section, we will propose two kinds of modified halfspace-relaxation projection methods—Algorithms 1 and 2. Algorithm 1 is an FB type HRP method and Algorithm 2 is an EG type HRP method. The relationship of these two methods is that they use the same optimal step length along different descent directions. The detailed procedures are presented as below.

The Modified Halfspace-Relaxation Projection Methods

Step 1. Let $\alpha_0 > 0$, $0 < \mu < \nu < 1$, $z^0 \in R^{n+m}$, $\theta \in (0,2)$, $\varepsilon > 0$ and $k = 0$. (In practical computation, we suggest to take $\mu = 0.3$, $\nu = 0.9$ and $\theta = 1.8$).

Step 2. Set

$$\overline{z}^k = P_{\Omega_k}\left[z^k - \alpha_k \nabla f\left(z^k\right)\right], \tag{3.1}$$

where Ω_k is defined in (1.7). If $\|z^k - \overline{z}^k\| \le \varepsilon$, terminate the iteration with the iterate $z^k = ((x^k)^T, (y^k)^T)^T$, and then x^k is the approximate solution of the SFP. Otherwise, go to Step 3.

Step 3. If

$$r_k := \frac{\alpha_k \left\|\nabla f(z^k) - \nabla f\left(\overline{z}^k\right)\right\|}{\left\|z^k - \overline{z}^k\right\|} \le \nu, \tag{3.2}$$

then set

$$e_k\left(z^k,\alpha_k\right)=e_{\Omega_k}\left(z^k,\alpha_k\right)=z^k-\overline{z}^k,\qquad g_k\left(z^k,\alpha_k\right)=\alpha_k\nabla f\left(\overline{z}^k\right),\tag{3.3}$$

$$d_k\left(z^k,\alpha_k\right)=e_k\left(z^k,\alpha_k\right)-\alpha_k\nabla f\left(z^k\right)+g_k\left(z^k,\alpha_k\right),\tag{3.4}$$

$$\gamma_k^*=\frac{e_k\left(z^k,\alpha_k\right)^T d_k\left(z^k,\alpha_k\right)}{\left\|d_k\left(z^k,\alpha_k\right)\right\|^2},\quad \gamma_k=\theta\gamma_k^*,\tag{3.5}$$

$$z^{k+1}=P_{\Omega_k}\left[z^k-\gamma_k d_k\left(z^k,\alpha_k\right)\right],\ (\text{Algorithm 1}:\ \text{FB type HRP method})\tag{3.6}$$

or

$$z^{k+1}=P_{\Omega_k}\left[z^k-\gamma_k g_k\left(z^k,\alpha_k\right)\right],\ (\text{Algorithm 2}:\ \text{EG type HRP method})\tag{3.7}$$

$$\alpha_k:=\begin{cases}\dfrac{3}{2}\alpha_k & \text{if } r_k\le\mu,\\ \alpha_k & \text{otherwise },\end{cases}\tag{3.8}$$

$$\alpha_{k+1}=\alpha_k,\qquad k=k+1,\ \text{go to Step 2.}$$

Step 4. Reduce the value of α_k by $\alpha_k := (2/3)\alpha_k * \min\{1, 1/r_k\}$,

$$\text{set } \overline{z}^k=P_{\Omega_k}\left[z^k-\alpha_k\nabla f\left(z^k\right)\right] \text{ and go to Step 3.}\tag{3.9}$$

Remark 3.1. In Step 3, if the selected α_k satisfies $0<\alpha_k\le\nu/L$ (L is the largest eigenvalue of the matrix B^TB), then from (1.12), we have

$$\alpha_k\left\|\nabla f\left(z^k\right)-\nabla f\left(\overline{z}^k\right)\right\|\le\alpha_k L\left\|z^k-\overline{z}^k\right\|\le\nu\left\|z^k-\overline{z}^k\right\|,\tag{3.10}$$

and thus Condition (3.2) is satisfied. Without loss of generality, we can assume that $\inf_k\{\alpha_k\}=\alpha_{\min}>0$.

Remark 3.2. By the definition of subgradient, it is clear that the halfspace Ω_k contains Ω. From the expressions of Ω_k, the orthogonal projections onto Ω_k may be directly calculated and then we have the following proposition (see [3, 12]).

Proposition 3.3. *For any $z\in R^{n+m}$,*

$$P_{\Omega_k}(z)=\begin{cases} z-\dfrac{c\left(z^k\right)+\left(z-z^k\right)^T\xi^k}{\|\xi^k\|^2}\xi^k, & \textit{if } c\left(z^k\right)+\left(z-z^k\right)^T\xi^k>0;\\ z, & \textit{otherwise},\end{cases}\tag{3.11}$$

where Ω_k is defined in (1.7).

Remark 3.4. For the FB type HRP method, taking

$$z^{k+1} = z^k - \gamma_k d_k\left(z^k, \alpha_k\right) \tag{3.12}$$

as the new iterate instead of Formula (3.6) seems more applicable in practice. Since from Proposition 3.3 the projection onto Ω_k is easy to be computed, Formula (3.6) is still preferable to generate the new iterate z^{k+1}.

Remark 3.5. The proposed methods and the HRP method in [2] can be used to solve more general convex optimization problem

$$\text{minimize } f(z) \quad \text{subject to } z \in \Omega, \tag{3.13}$$

where $f(z)$ is a general convex function only with the property that $\nabla f(z^*) = 0$ for any solution point z^* of (3.13), and Ω is defined in (1.3). The corresponding theoretical analysis is similar as these methods to solve (1.1).

4. The Optimal Step Length

This section concentrates on investigating the optimal step length with positive lower bounds in order to accelerate convergence of the new methods. To justify the reason of choosing the optimal step length γ_k in the FB type HRP method (3.6), we start from the following general form of the FB type HRP method:

$$z_{\mathrm{FB}}^{k+1}(\gamma) = P_{\Omega_k}\left[z_{PC}^{k+1}(\gamma)\right], \tag{4.1}$$

where

$$z_{PC}^{k+1}(\gamma) = z^k - \gamma d_k\left(z^k, \alpha_k\right). \tag{4.2}$$

Let

$$\Theta_k(\mathrm{FB}_\gamma) := \left\|z^k - z^*\right\|^2 - \left\|z_{\mathrm{FB}}^{k+1}(\gamma) - z^*\right\|^2, \tag{4.3}$$

which measures the progress made by the FB type HRP method. Note that $\Theta_k(\mathrm{FB}_\gamma)$ is a function of the step length γ. It is natural to consider maximizing this function by choosing an optimal parameter γ. The solution z^* is not known, so we cannot maximize $\Theta_k(\mathrm{FB}_\gamma)$ directly. The following theorem gives an estimate of $\Theta_k(\mathrm{FB}_\gamma)$ which does not include the unknown solution z^*.

Theorem 4.1. *Let z^* be an arbitrary point in Ω^*. If the step length in the general FB type HRP method is taken $\gamma > 0$, then we have*

$$\Theta_k(FB_\gamma) := \left\|z^k - z^*\right\|^2 - \left\|z_{FB}^{k+1}(\gamma) - z^*\right\|^2 \geq \Upsilon_k(\gamma), \tag{4.4}$$

where

$$\Upsilon_k(\gamma) := 2\gamma e_k\left(z^k,\alpha_k\right)^T d_k\left(z^k,\alpha_k\right) - \gamma^2\left\|d_k\left(z^k,\alpha_k\right)\right\|^2. \tag{4.5}$$

Proof. Since $z_{\mathrm{FB}}^{k+1}(\gamma) = P_{\Omega_k}[z_{PC}^{k+1}(\gamma)]$ and $z^* \in \Omega \subseteq \Omega_k$, it follows from (2.8) that

$$\left\|z_{\mathrm{FB}}^{k+1}(\gamma) - z^*\right\|^2 \leq \left\|z_{PC}^{k+1}(\gamma) - z^*\right\|^2 - \left\|z_{PC}^{k+1}(\gamma) - z_{\mathrm{FB}}^{k+1}(\gamma)\right\|^2 \leq \left\|z_{PC}^{k+1}(\gamma) - z^*\right\|^2, \tag{4.6}$$

and consequently

$$\Theta_k(\mathrm{FB}_\gamma) \geq \left\|z^k - z^*\right\|^2 - \left\|z_{PC}^{k+1}(\gamma) - z^*\right\|^2. \tag{4.7}$$

Setting $\alpha = \alpha_k$, $s = z^k$, $s^* = z^*$ and $\widetilde{\mathcal{S}} = \Omega_k$ in the equality (2.10) and using the notation of $e_k(z^k,\alpha_k)$ (see (3.3)) and $d_k(z^k,\alpha_k)$ (see (3.4)), we have

$$\left(z^k - z^*\right)^T d_k\left(z^k,\alpha_k\right) \geq e_k\left(z^k,\alpha_k\right)^T d_k\left(z^k,\alpha_k\right). \tag{4.8}$$

Using this and (4.2), we get

$$\begin{aligned}\left\|z^k - z^*\right\|^2 - \left\|z_{PC}^{k+1}(\gamma) - z^*\right\|^2 &= \left\|z^k - z^*\right\|^2 - \left\|z^k - \gamma d_k\left(z^k,\alpha_k\right) - z^*\right\|^2 \\ &= 2\gamma\left(z^k - z^*\right)^T d_k\left(z^k,\alpha_k\right) - \gamma^2\left\|d_k\left(z^k,\alpha_k\right)\right\|^2 \\ &\geq 2\gamma e_k\left(z^k,\alpha_k\right)^T d_k\left(z^k,\alpha_k\right) - \gamma^2\left\|d_k\left(z^k,\alpha_k\right)\right\|^2,\end{aligned} \tag{4.9}$$

and then from (4.7) the theorem is proved. □

Similarly, we start from the general form of the EG type HRP method

$$z_{\mathrm{EG}}^{k+1}(\gamma) = P_{\Omega_k}\left[z^k - \gamma g_k\left(z^k,\alpha_k\right)\right]. \tag{4.10}$$

to analyze the optimal step length in the EG type HRP method (3.7). The following theorem estimates the "progress" in the sense of Euclidean distance made by the new iterate and thus motivates us to investigate the selection of the optimal length γ_k in the EG type HRP method (3.7).

Theorem 4.2. *Let z^* be an arbitrary point in Ω^*. If the step length in the general EG type HRP method is taken $\gamma > 0$, then one has*

$$\Theta_k(EG_\gamma) := \left\|z^k - z^*\right\|^2 - \left\|z_{EG}^{k+1}(\gamma) - z^*\right\|^2 \geq \Upsilon_k(\gamma), \tag{4.11}$$

where $\Upsilon_k(\gamma)$ is defined in (4.5) *and $z_{PC}^{k+1}(\gamma)$ is defined in* (4.2).

Proof. Since $z_{\mathrm{EG}}^{k+1}(\gamma) = P_{\Omega_k}[z^k - \gamma g_k(z^k, \alpha_k)]$ and $z^* \in \Omega \subseteq \Omega_k$, it follows from (2.8) that

$$\left\| z_{\mathrm{EG}}^{k+1}(\gamma) - z^* \right\|^2 \leq \left\| z^k - \gamma g_k\left(z^k, \alpha_k\right) - z^* \right\|^2 - \left\| z^k - \gamma g_k\left(z^k, \alpha_k\right) - z_{\mathrm{EG}}^{k+1}(\gamma) \right\|^2, \tag{4.12}$$

and consequently we get

$$\begin{aligned}\Theta_k(\mathrm{EG}_\gamma) &\geq \left\| z^k - z^* \right\|^2 - \left\| z^k - z^* - \gamma g_k\left(z^k, \alpha_k\right) \right\|^2 + \left\| z^k - z_{\mathrm{EG}}^{k+1}(\gamma) - \gamma g_k\left(z^k, \alpha_k\right) \right\|^2 \\ &= \left\| z^k - z_{\mathrm{EG}}^{k+1}(\gamma) \right\|^2 + 2\gamma\left(z^k - z^*\right)^T g_k\left(z^k, \alpha_k\right) - 2\gamma\left(z^k - z_{\mathrm{EG}}^{k+1}(\gamma)\right)^T g_k\left(z^k, \alpha_k\right).\end{aligned} \tag{4.13}$$

Setting $\alpha = \alpha_k$, $s = z^k$, $s^* = z^*$ and $\widetilde{\mathcal{S}} = \Omega_k$ in the equality (2.9) and using the notation of $e_k(z^k, \alpha_k)$ and $g_k(z^k, \alpha_k)$ (see (3.3)), we have

$$\left(z^k - z^*\right)^T g_k\left(z^k, \alpha_k\right) \geq e_k\left(z^k, \alpha_k\right)^T g_k\left(z^k, \alpha_k\right). \tag{4.14}$$

From the above inequality, we obtain

$$\Theta_k(\mathrm{EG}_\gamma) \geq \left\| z^k - z_{\mathrm{EG}}^{k+1}(\gamma) \right\|^2 + 2\gamma e_k\left(z^k, \alpha_k\right)^T g_k\left(z^k, \alpha_k\right) - 2\gamma\left(z^k - z_{\mathrm{EG}}^{k+1}(\gamma)\right)^T g_k\left(z^k, \alpha_k\right). \tag{4.15}$$

Using $g_k(z^k, \alpha_k) = d_k(z^k, \alpha_k) - [e_k(z^k, \alpha_k) - \alpha_k \nabla f(z^k)]$ (see (3.4)), it follows that

$$\begin{aligned}\Theta_k(\mathrm{EG}_\gamma) \geq{}& \left\| z^k - z_{\mathrm{EG}}^{k+1}(\gamma) \right\|^2 + 2\gamma e_k\left(z^k, \alpha_k\right)^T \left\{ d_k\left(z^k, \alpha_k\right) - \left[e_k\left(z^k, \alpha_k\right) - \alpha_k \nabla f\left(z^k\right) \right] \right\} \\ & - 2\gamma\left(z^k - z_{\mathrm{EG}}^{k+1}(\gamma)\right)^T \left\{ d_k\left(z^k, \alpha_k\right) - \left[e_k\left(z^k, \alpha_k\right) - \alpha_k \nabla f\left(z^k\right) \right] \right\},\end{aligned} \tag{4.16}$$

which can be rewritten as

$$\begin{aligned}\Theta_k(\mathrm{EG}_\gamma) \geq{}& \left\| z^k - z_{\mathrm{EG}}^{k+1}(\gamma) - \gamma d_k\left(z^k, \alpha_k\right) \right\|^2 - \gamma^2 \left\| d_k\left(z^k, \alpha_k\right) \right\|^2 + 2\gamma e_k\left(z^k, \alpha_k\right)^T d_k\left(z^k, \alpha_k\right) \\ & + 2\gamma\left(z^k - z_{\mathrm{EG}}^{k+1}(\gamma) - e_k\left(z^k, \alpha_k\right)\right)^T \left(e_k\left(z^k, \alpha_k\right) - \alpha_k \nabla f\left(z^k\right)\right) \\ \geq{}& \Upsilon_k(\gamma) + 2\gamma\left(z^k - z_{\mathrm{EG}}^{k+1}(\gamma) - e_k\left(z^k, \alpha_k\right)\right)^T \left(e_k\left(z^k, \alpha_k\right) - \alpha_k \nabla f\left(z^k\right)\right).\end{aligned} \tag{4.17}$$

Now we consider the last term in the right-hand side of (4.17). Notice that

$$z^k - z_{\mathrm{EG}}^{k+1}(\gamma) - e_k\left(z^k, \alpha_k\right) = P_{\Omega_k}\left[z^k - \alpha_k \nabla f\left(z^k\right)\right] - z_{\mathrm{EG}}^{k+1}(\gamma). \tag{4.18}$$

Setting $t := z^k - \alpha_k \nabla f(z^k)$, $s := z_{\mathrm{EG}}^{k+1}(\gamma)$ and $\tilde{\mathcal{S}} = \Omega_k$ in the basic inequality (2.7) of the projection mapping and using the notation of $e_k(z^k, \alpha_k)$, we get

$$\left\{P_{\Omega_k}\left[z^k - \alpha_k \nabla f\left(z^k\right)\right] - z_{\mathrm{EG}}^{k+1}(\gamma)\right\}^T \left\{e_k\left(z^k, \alpha_k\right) - \alpha_k \nabla f\left(z^k\right)\right\} \geq 0, \tag{4.19}$$

and therefore

$$\left\{z^k - z_{\mathrm{EG}}^{k+1}(\gamma) - e_k\left(z^k, \alpha_k\right)\right\}^T \left\{e_k\left(z^k, \alpha_k\right) - \alpha_k \nabla f\left(z^k\right)\right\} \geq 0. \tag{4.20}$$

Substituting (4.20) in (4.17), it follows that

$$\Theta_k(\mathrm{EG}_\gamma) \geq \Upsilon_k(\gamma) \tag{4.21}$$

and the theorem is proved. □

Theorems 4.1 and 4.2 provide the basis of the selection of the optimal step length of the new methods. Note that $\Upsilon_k(\gamma)$ is the profit-function since it is a lower-bound of the progress obtained by the new methods (both the FB type HRP method and EG type HRP method). This motivates us to maximize the profit-function $\Upsilon_k(\gamma)$ to accelerate convergence of the new methods. Since $\Upsilon_k(\gamma)$ a quadratic function of γ, it reaches its maximum at

$$\gamma_k^* := \frac{e_k(z^k, \alpha_k)^T d_k(z^k, \alpha_k)}{\|d_k(z^k, \alpha_k)\|^2}. \tag{4.22}$$

Note that under Condition (3.2), using the notation of $d_k(z^k, \alpha_k)$ we have

$$\begin{aligned} e_k\left(z^k, \alpha_k\right)^T d_k\left(z^k, \alpha_k\right) &= \left\|e_k\left(z^k, \alpha_k\right)\right\|^2 - \alpha_k e_k\left(z^k, \alpha_k\right)^T \left(\nabla f\left(z^k\right) - \nabla f\left(\overline{z}^k\right)\right) \\ &\geq (1-\nu)\left\|e_k\left(z^k, \alpha_k\right)\right\|^2. \end{aligned} \tag{4.23}$$

In addition, since

$$\begin{aligned} e_k\left(z^k, \alpha_k\right)^T d_k\left(z^k, \alpha_k\right) &= \left\|e_k\left(z^k, \alpha_k\right)\right\|^2 - \alpha_k e_k\left(z^k, \alpha_k\right)^T \left(\nabla f\left(z^k\right) - \nabla f\left(\overline{z}^k\right)\right) \\ &\geq \frac{1}{2}\left\|e_k\left(z^k, \alpha_k\right)\right\|^2 - \alpha_k e_k\left(z^k, \alpha_k\right)^T \left(\nabla f\left(z^k\right) - \nabla f\left(\overline{z}^k\right)\right) \\ &\quad + \frac{1}{2}\alpha_k \left\|\nabla f\left(z^k\right) - \nabla f\left(\overline{z}^k\right)\right\|^2 \\ &= \frac{1}{2}\|d_k\left(z^k, \alpha_k\right)\|^2, \end{aligned} \tag{4.24}$$

we have

$$\gamma_k^* := \frac{e_k(z^k, \alpha_k)^T d_k(z^k, \alpha_k)}{\|d_k(z^k, \alpha_k)\|^2} \geq \frac{1}{2}. \tag{4.25}$$

From numerical point of view, it is necessary to attach a relax factor to the optimal step length γ_k^* obtained theoretically to achieve faster convergence. The following theorem concerns how to choose the relax factor.

Theorem 4.3. *Let z^* be an arbitrary point in Ω^*, θ a positive constant and γ_k^* defined in* (4.22)*. For given $z^k \in \Omega_k$, α_k is chosen such that Condition* (3.2) *is satisfied. Whenever the new iterate $z^{k+1}(\theta\gamma_k^*)$ is generated by*

$$z^{k+1}(\theta\gamma_k^*) = P_{\Omega_k}\left[z^k - \theta\gamma_k^* d_k\left(z^k, \alpha_k\right)\right] \quad \text{or} \quad z^{k+1}(\theta\gamma_k^*) = P_{\Omega_k}\left[z^k - \theta\gamma_k^* g_k\left(z^k, \alpha_k\right)\right], \tag{4.26}$$

we have

$$\left\|z^{k+1}(\theta\gamma_k^*) - z^*\right\|^2 \leq \left\|z^k - z^*\right\|^2 - \frac{\theta(2-\theta)(1-\nu)}{2}\left\|e_k\left(z^k, \alpha_k\right)\right\|^2. \tag{4.27}$$

Proof. From Theorems 4.1 and 4.2 we have

$$\left\|z^k - z^*\right\|^2 - \left\|z^{k+1}(\theta\gamma_k^*) - z^*\right\|^2 \geq \Upsilon_k(\theta\gamma_k^*). \tag{4.28}$$

Using (4.5), (4.23), and (4.25), we obtain

$$\begin{aligned} \Upsilon_k(\theta\gamma_k^*) &= 2\theta\gamma_k^* e_k\left(z^k, \alpha_k\right)^T d_k\left(z^k, \alpha_k\right) - (\theta\gamma_k^*)^2\left\|d_k\left(z^k, \alpha_k\right)\right\|^2 \\ &= 2\theta\gamma_k^* e_k\left(z^k, \alpha_k\right)^T d_k\left(z^k, \alpha_k\right) - \theta^2\gamma_k^* e_k\left(z^k, \alpha_k\right)^T d_k\left(z^k, \alpha_k\right) \\ &= \gamma_k^*\theta(2-\theta) e_k\left(z^k, \alpha_k\right)^T d_k\left(z^k, \alpha_k\right) \\ &\geq \frac{\theta(2-\theta)(1-\nu)}{2}\left\|e_k\left(z^k, \alpha_k\right)\right\|^2, \end{aligned} \tag{4.29}$$

and the assertion is proved. □

Theorem 4.3 shows theoretically that any $\theta \in (0, 2)$ guarantees that the new iterate makes progress to a solution. Therefore, in practical computation, we choose $\gamma_k = \theta\gamma_k^*$ with $\theta \in (0, 2)$ as the step length in the new methods. We need to point out that from numerical experiments, $\theta \in [1, 2)$ is much preferable since it leads to better numerical performance.

5. Convergence

It follows from (4.27) that for both the FB type HRP method (3.6) and the EG type HRP method (3.7), there exists a constant $\tau > 0$, such that

$$\left\|z^{k+1} - z^*\right\|^2 \leq \left\|z^k - z^*\right\|^2 - \tau\left\|e_k\left(z^k, \alpha_k\right)\right\|^2, \quad \forall z^* \in \Omega^*. \tag{5.1}$$

The convergence result of the proposed methods in this paper is based on the following theorem.

Theorem 5.1. *Let* $\{z^k\}$ *be a sequence generated by the proposed method* (3.6) *or* (3.7)*. Then* $\{z^k\}$ *converges to a point* $\tilde{z}$*, which belongs to* Ω^**.*

Proof. First, from (5.1) we get

$$\lim_{k\to\infty}\left\|e_k\left(z^k, \alpha_k\right)\right\| = 0. \tag{5.2}$$

Note that

$$e_k\left(z^k, \alpha_k\right) = z^k - \overline{z}^k, \quad \text{(see (3.3))}. \tag{5.3}$$

We have

$$\lim_{k\to\infty}\left\|z^k - \overline{z}^k\right\| = 0. \tag{5.4}$$

Again, it follows from (5.1) that the sequence $\{z^k\}$ is bounded. Let $\tilde{z}$ be a cluster point of $\{z^k\}$ and the subsequence $\{z^{k_j}\}$ converges to $\tilde{z}$. We are ready to show that $\tilde{z}$ is a solution point of (1.1).

First, we show that $\tilde{z} \in \Omega$. Since $\overline{z}^{k_j} \in \Omega_{k_j}$, then by the definition of Ω_{k_j}, we have

$$c\left(z^{k_j}\right) + \left(\overline{z}^{k_j} - z^{k_j}\right)^T \xi^{k_j} \leq 0, \quad \forall j = 1, 2, \ldots. \tag{5.5}$$

Passing onto the limit in this inequality and taking into account (5.4) and Lemma 2.7, we obtain that

$$c(\tilde{z}) \leq 0. \tag{5.6}$$

Hence, we conclude $\tilde{z} \in \Omega$.

Next, we need to show $(z - \tilde{z})^T \nabla f(\tilde{z}) \geq 0, \forall z \in \Omega$. To do so, we first prove

$$\lim_{j\to\infty}\left\|e_{k_j}\left(z^{k_j}, 1\right)\right\| = 0. \tag{5.7}$$

It follows from Remark 3.1 in Section 3 that $\inf_j\{\alpha_{k_j}\} \geq \inf_k\{\alpha_k\} = \alpha_{\min} > 0$. Then from Lemma 2.1, we have

$$\left\| e_{k_j}\left(z^{k_j}, 1\right) \right\| \leq \frac{\left\| z^{k_j} - \overline{z}^{k_j} \right\|}{\min\left\{1, \alpha_{k_j}\right\}}, \tag{5.8}$$

which, together with (5.4), implies that

$$\lim_{j\to\infty} \left\| e_{k_j}\left(z^{k_j}, 1\right) \right\| \leq \lim_{j\to\infty} \frac{\left\| z^{k_j} - \overline{z}^{k_j} \right\|}{\min\left\{1, \alpha_{k_j}\right\}} \leq \lim_{j\to\infty} \frac{\left\| z^{k_j} - \overline{z}^{k_j} \right\|}{\min\{1, \alpha_{\min}\}} = 0. \tag{5.9}$$

Setting $t = z^{k_j} - \nabla f(z^{k_j})$, $\widetilde{S} = \Omega_{k_j}$ in the inequality (2.7), for any $z \in \Omega \subseteq \Omega_{k_j}$, we obtain

$$\left(z^{k_j} - \nabla f\left(z^{k_j}\right) - P_{\Omega_{k_j}}\left(z^{k_j} - \nabla f\left(z^{k_j}\right)\right)\right)^T \left(P_{\Omega_{k_j}}\left(z^{k_j} - \nabla f\left(z^{k_j}\right)\right) - z\right) \geq 0. \tag{5.10}$$

From the fact that $e_{k_j}(z^{k_j}, 1) = z^{k_j} - P_{\Omega_{k_j}}[z^{k_j} - \nabla f(z^{k_j})]$, we have

$$\left(e_{k_j}\left(z^{k_j}, 1\right) - \nabla f\left(z^{k_j}\right)\right)^T \left(z^{k_j} - e_{k_j}\left(z^{k_j}, 1\right) - z\right) \geq 0, \quad \forall z \in \Omega, \tag{5.11}$$

that is,

$$\left(z - z^{k_j}\right)^T \nabla f\left(z^{k_j}\right) + e_{k_j}\left(z^{k_j}, 1\right)^T \left(z^{k_j} - e_{k_j}\left(z^{k_j}, 1\right) - z + \nabla f\left(z^{k_j}\right)\right) \geq 0, \quad \forall z \in \Omega. \tag{5.12}$$

Letting $j \to \infty$, taking into account (5.7), we deduce

$$(z - \widetilde{z})^T \nabla f(\widetilde{z}) \geq 0, \quad \forall z \in \Omega, \tag{5.13}$$

which implies that $\widetilde{z} \in \Omega^*$. Then from (5.1), it follows that

$$\left\| z^{k+1} - \widetilde{z} \right\|^2 \leq \left\| z^k - \widetilde{z} \right\|^2 - \tau \left\| e_k\left(z^k, \alpha_k\right) \right\|^2. \tag{5.14}$$

Together with the fact that the subsequence $\{z^{k_j}\}$ converges to $\widetilde{z}$, we can conclude that $\{z^k\}$ converges to $\widetilde{z}$. The proof is complete. □

6. Numerical Results

In this section, we implement the proposed methods to solve some numerical examples arising in [2] and then report the results. To show the superiority of the new methods, we also compare them with the HRP method in [2]. The codes for implementing the

Table 1: Results for Example 6.1 using the HRP method in [2].

Starting points	Number of iterations	CPU (Sec.)	Approximate solution
$(1,2,3,0,0,0)^T$	43	0.0500	$(0.3213, 0.2815, 0.1425)^T$
$(1,1,1,1,1,1)^T$	67	0.0910	$(0.8577, 0.8577, 1.3097)^T$
$(1,2,3,4,5,6)^T$	85	0.1210	$(1.1548, 0.8518, 1.8095)^T$

Table 2: Results for Example 6.1 using FB type HRP method.

Starting points	Number of iterations	CPU (Sec.)	Approximate solution
$(1,2,3,0,0,0)^T$	15	<0.0001	$(0.7335, 0.9309, 1.2014)^T$
$(1,1,1,1,1,1)^T$	0	<0.0001	$(1.0000, 1.0000, 1.0000)^T$
$(1,2,3,4,5,6)^T$	36	<0.0001	$(2.5000, 0.9572, 1.0466)^T$

Table 3: Results for Example 6.1 using EG type HRP method.

Starting points	Number of iterations	CPU (Sec.)	Approximate solution
$(1,2,3,0,0,0)^T$	15	<0.0001	$(0.6505, 1.0000, 1.3744)^T$
$(1,1,1,1,1,1)^T$	0	<0.0001	$(1.0000, 1.0000, 1.0000)^T$
$(1,2,3,4,5,6)^T$	38	<0.0001	$(2.5000, 1.3024, 1.3927)^T$

proposed methods were written by Matlab 7.9.0 (R2009b) and run on an HP Compaq 6910p Notebook (2.00 GHz of Intel Core 2 Duo CPU and 2.00 GB of RAM). The stopping criterion is $\|e_k(z^k, \alpha_k)\| \leq \varepsilon$.

For the new methods, we take $\varepsilon = 10^{-10}$, $\alpha_0 = 1$, $\mu = 0.3$, $\nu = 0.9$ and $\theta = 1.8$. To compare with the HRP method and the new methods, we list the numbers of iterations, the computation times (CPU(Sec.)) and the approximate solutions in Tables 1, 2, 3, 4, 5, 6, 7, 8, and 9. For the HRP method in [2], we list the original numerical results in [2].

Example 6.1 (a convex feasibility problem). Let $C = \{x \in R^3 \mid x_2^2 + x_3^2 - 4 \leq 0\}$, $Q = \{x \in R^3 \mid x_3 - 1 - x_1^2 \leq 0\}$. Find some point x in $C \cap Q$.

Obviously this example can be regarded as an SFP with $A = I$.

For Example 6.1, it is easy to verify that the point $(1,1,1,1,1,1)^T$ is a solution of (1.1). Therefore, the FB type and EG type HRP method only use 0 iteration when we choose the starting point $z^0 = (1,1,1,1,1,1)^T$. While applying the HRP method in [2] to solve Example 6.1 and choosing the same starting point, the number of iterations is 67. This is the original numerical result listed in Table 1 of [2].

Example 6.2 (a split feasibility problem). Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 2 & 0 & 2 \end{pmatrix}$, $C = \{x \in R^3 \mid x_1 + x_2^2 + 2x_3 \leq 0\}$, $Q = \{x \in R^3 \mid x_1^2 + x_2 - x_3 \leq 0\}$. Find some point $x \in C$ with $Ax \in Q$.

Example 6.3 (a nonlinear programming problem). Consider the problem

$$\begin{aligned} &\text{Minimize} \quad f(z) = \sum_{i=1}^{n} z_i^2 \\ &\text{subject to} \quad \sum_{i \neq j} z_i^2 - z_j - j \leq 0, \quad j = 1, \ldots, n. \end{aligned} \tag{6.1}$$

Table 4: Results for Example 6.2 using the HRP method in [2].

Starting points	Number of iterations	CPU (Sec.)	Approximate solution
$(1,2,3,0,0,0)^T$	1890	2.7740	$(-0.1203, 0.0285, 0.0582)^T$
$(1,1,1,1,1,1)^T$	2978	4.2860	$(0.8603, -0.1658, -0.5073)^T$
$(1,2,3,4,5,6)^T$	3317	4.8570	$(3.6522, -0.1526, -2.3719)^T$

Table 5: Results for Example 6.2 using FB type HRP method.

Starting points	Number of iterations	CPU (Sec.)	Approximate solution
$(1,2,3,0,0,0)^T$	609	0.0620	$(-1.2024, 0.0724, 0.5986)^T$
$(1,1,1,1,1,1)^T$	630	0.0470	$(-1.2039, 0.0723, 0.5993)^T$
$(1,2,3,4,5,6)^T$	680	0.0620	$(-1.1284, 0.0759, 0.5613)^T$

Table 6: Results for Example 6.2 using EG type HRP method.

Starting points	Number of iterations	CPU (Sec.)	Approximate solution
$(1,2,3,0,0,0)^T$	757	0.0620	$(-1.1758, 0.0737, 0.5852)^T$
$(1,1,1,1,1,1)^T$	567	0.0470	$(-1.2161, 0.0716, 0.6055)^T$
$(1,2,3,4,5,6)^T$	711	0.0620	$(-1.1815, 0.0734, 0.5881)^T$

Table 7: Results for Example 6.3 using the HRP method in [2].

n (dimension)	Number of iterations	CPU (Sec.)
10	36	0.0160
100	38	0.2970
1000	40	15.5000
5000	41	416.7340

Table 8: Results for Example 6.3 using FB type HRP method.

n (dimension)	Number of iterations	CPU (Sec.)
10	15	<0.0001
100	16	0.0160
1000	17	0.1250
5000	17	0.3130

Table 9: Results for Example 6.3 using EG type HRP method.

n (dimension)	Number of iterations	CPU (Sec.)
10	15	<0.0001
100	16	0.0160
1000	17	0.1250
5000	17	0.3130

This example is a general nonlinear programming problem not the reformulation (1.1) for the SFP. Notice that it has a unique solution $z^* = (0, \ldots, 0)^T$ and $\nabla f(z^*) = 2z^* = (0, \ldots, 0)^T$. Then from Remark 3.5 in Section 3, the proposed methods and the HRP method in [2] can be used to find its solution.

The computational preferences to the HRP method (1.5)-(1.6) are revealed clearly in Tables 1–9. The numerical results demonstrate that the selection of optimal step length in both the FB type HRP method and the EG type HRP method reduces considerable computational load of the HRP method in [2].

7. Conclusions

In this paper we consider the split feasibility problem, which is a special case of the multiple-sets split feasibility problem [13–15]. With some new strategies for determining the optimal step length, this paper improves the HRP method in [2] and thus develops modified halfspace-relaxation projection methods for solving the split feasibility problem. Compared to the HRP method in [2], the new methods reduce the number of iterations moderately with little additional computation.

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