Hindawi Publishing Corporation
Advances in Operations Research
Volume 2012, Article ID 490983, 11 pages
doi:10.1155/2012/490983

Research Article

Adaptive Control of a Reverse Logistic Inventory Model with Uncertain Deteriorations and Disposal Rates

Ahmad Alshamrani

Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Ahmad Alshamrani, ahmadm@ksu.edu.sa

Received 31 October 2011; Revised 6 March 2012; Accepted 25 March 2012

Academic Editor: Chandal Nahak

An adaptive control of a reverse logistic inventory system with unknown deterioration and disposal rates is considered. An adaptive control approach with a feedback is applied to track the inventory levels toward their goal levels. Also, the updating rules of both deterioration and disposal rates are derived from the conditions of asymptotic stability of the reference model. Important characteristics of the adaptive inventory system are discussed. The adaptive controlled system is modeled by a nonlinear system of differential equations. Finally, the numerical solution of the controlled system is discussed and displayed graphically.

1. Introduction

Applications of optimal control theory to management science, and especially to production planning, are proving to be quite fruitful; see Sethi and Thompson [1] and El-Gohary et al. [2]. The optimal control problem characterizes the system optimal trajectory in time. The application of optimal control theory supplies nice information about the system optimal path and decision rules in time, as for the following examples.

(i) Tadj et al. [3] applied the optimal control theory to an inventory system with ameliorating and deteriorating items. They derived the optimal inventory level and optimal production rate for different cases of both amelioration and deterioration rates.

(ii) El-Gohary and Elsyed [4] discussed the problem of optimal control of multi-item inventory models with different types of deterioration. They derived inventory levels which minimize the total holding cost of the system.

(iii) Alshamrani and El-Gohary [5] applied Pontryagin principle to a two-item inventory model with different types of deteriorating items. They minimized the total cost which includes the sum of the holding costs of inventory items, the holding costs of one item due to the presence of the other, and the production cost.

(iv) Foul et al. [6] studied the adaptive control of a continuous-time model of a production inventory system in which a manufacturing firm produces a single product, selling some, and stocking the remaining. The model reference adaptive control with feedback is applied to track the output of the system toward the inventory goal level.

(v) El-Gohary and Yassen [7] used the adaptive control and synchronization procedures to the coupled dynamo system with unknown parameters. Based on the Liapunov stability technique, an adaptive control laws are derived so that the coupled dynamo system is asymptotically stable and the two identical dynamo systems are asymptotically synchronized. Also the updating rules of the unknown parameters are derived.

The reverse logistic model is a method for manufacturing of materials and remanufacturing of market returned reusable materials. In this model, the demand is satisfied with the new produced products and with the remanufactured used products. Therefore, there is no difference between manufactured and remanufactured items for satisfying the demand [8, 9].

The aim of this paper is to introduce the mathematical model of a reverse logistic system with two different types of natural deteriorations. We apply an adaptive control technique to derive the updating rules of both of deteriorations and disposal rates from the conditions of asymptotic stability of the reference model. Also, the manufacturing and remanufacturing rates are derived using a suitable feedback technique. Numerical examples for different sets of the system parameters and initial inventory levels are introduced.

The motivation of this paper is to extend and generalize the reverse logistic system and applying an adaptive control approach to stabilize the system about the steady states of this system.

2. The Reverse Logistic Model

In this section, we derive the mathematical model of a reverse logistic model with two different types of deterioration. Reverse logistic is a term for manufacturing new products and remanufacturing used products that is returned from market for future reuse. The demand is to be satisfied with new manufactured products and the remanufactured used products.

2.1. The Mathematical Model

In this subsection, we derive the mathematical model and the perturbed system of a reverse logistic inventory model with two different types of deteriorating items about the inventory goal levels. In what follows, we formulate a two inventory stores reverse logistic model with a continuous disposal rate. The manufactured items go to the first inventory store and then are used to meet the market demand. Returned used items from market go to the second inventory store. Some of the returned items will be disposed while the others will be remanufactured and then go to the first inventory store where they will be used to meet

the market demand. It is assumed that items might deteriorate in both inventory stores. The problem can be considered as an adaptive control problem with two state variables $I_i(t)$, $(i = 1,2)$ and two control variables $P_i(t)$, $(i = 1,2)$, and three unknown functions which are the two deterioration rates and the rate of disposal. The aim is to use the adaptive control technique to derive both of manufacturing and remanufacturing rates and the updating rules for the disposal and deterioration rates.

To derived the mathematical model we use the following notations:

$I_i(t)$: the inventory level in the ith store at time t, $(i = 1,2)$,

$P_1(t)$: the manufacturing rate at time t,

$P_2(t)$: the remanufacturing rate at time t,

$P_3(t)$: the disposal rate at time t,

$S(t)$: the continuous differentiable demand rate,

$R(t)$: the continuous differentiable return rate,

$\overline{I}_i$: the required inventory goal level in the ith store,

$\overline{P}_1$: the required manufacturing goal rate,

$\overline{P}_2$: the required remanufacturing goal rate,

$\overline{P}_3$: the required disposal goal rate,

$\overline{R}$: the return value at the inventory goal level,

$\overline{S}$: the demand value at the inventory goal level,

θ_i: the deterioration rate in the ith store.

The time evolution of the inventory levels at time t in both the first and second stores can be described by the following set of differential equations:

$$\begin{aligned} \dot{I}_1(t) &= P_1(t) + P_2(t) - S(t) - \theta_1 I_1(t), \\ \dot{I}_2(t) &= R(t) - P_2(t) - P_3(t) - \theta_2 I_2(t), \end{aligned} \tag{2.1}$$

with initial values $I_1(0) = I_{10}$, $I_2(0) = I_{20}$, where,

$$I_i(t) \geq 0, \quad (i = 1,2), \qquad P_i(t) \geq 0, \quad (i = 1,2,3). \tag{2.2}$$

The above inequalities represent the conditions of the nonnegativity of the inventory levels, the rates of manufacturing, remanufacturing, and disposal.

Next, we discuss the adaptive control of the reverse logistic inventory system with different types of deteriorating items using Liapunov technique.

3. Adaptive Control Procedure

In this section, we discuss, in details, the adaptive control problem of the reverse logistic inventory system. The rates of manufacturing and remanufacturing will be derived using a feedback control approach. Also, the updating rules of disposal and deterioration rates will be derived from the conditions of asymptotic stability of the perturbed system.

To study the adaptive control problem, we first start by considering the required inventory goal levels and the manufacturing and remanufacturing goal rates as a possible special solution of the modified system.

$$\begin{aligned} \dot{I}_1(t) &= P_1(t) + P_2(t) - S(t) - \widehat{\theta}_1(t) I_1(t), \\ \dot{I}_2(t) &= R(t) - P_2(t) - \widehat{P}_3(t) - \widehat{\theta}_2(t) I_2(t), \end{aligned} \tag{3.1}$$

where, $\widehat{\theta}_1(t)$, $\widehat{\theta}_2(t)$, and $\widehat{P}_3(t)$ are dynamic estimators of the unknown deterioration rates θ_i, $i = 1,2$ and disposal rate P_3. Following, we assume that $I_i(t) = \overline{I}_i$, $P_i(t) = \overline{P}_i$, $R(t) = \overline{R}$, $S(t) = \overline{S}$ and $\widehat{\theta}_i = \theta_i$, $\widehat{P}_3 = \overline{P}_3$ is a special solution for the system (3.1). Therefore, the equations of perturbed state about this solution can be derived by introducing the following new variables:

$$\begin{gathered} \xi_i(t) = I_i(t) - \overline{I}_i, \qquad \eta_i(t) = \widehat{\theta}_i(t) - \theta_i, \qquad v_i(t) = P_i(t) - \overline{P}_i, \qquad (i = 1,2), \\ d(t) = S(t) - \overline{S}, \qquad \nu(t) = \widehat{P}_3(t) - \overline{P}_3, \qquad r(t) = R(t) - \overline{R}. \end{gathered} \tag{3.2}$$

Substituting from (3.2) into (3.1), we get the following perturbed system:

$$\begin{aligned} \dot{\xi}_1(t) &= v_1(t) + v_2(t) - \theta_1 \xi_1(t) - \overline{I}_1 \eta_1(t) - \eta_1(t)\xi_1(t) - d(t), \\ \dot{\xi}_2(t) &= r(t) - v_2(t) - \nu(t) - \overline{I}_2 \eta_2(t) - \eta_2(t)\xi_2(t) - \theta_2 \xi_2(t). \end{aligned} \tag{3.3}$$

The system (3.3) will be used to study the adaptive control problem using the Liapunov technique. This technique uses the Liapunov function for the system (3.3).

The variables $\xi_i(t)$, $i = 1,2$ represent the deviations of the inventory levels $I_i(t)$ from the inventory goal levels $\overline{I}_i$, $i = 1,2$, while the variables $\eta_i(t)$, $i = 1,2$ represent the deviations of the estimators $\widehat{\theta}_i(t)$ of the deterioration rates from the real deterioration rates θ_i, $i = 1,2$.

The following theorem gives the manufacturing and remanufacturing rates and the updating rules of disposal rate and deterioration rates which ensure the asymptotic stability of the reverse logistic inventory model with uncertain deterioration and disposal rates.

Theorem 3.1. *If the manufacturing and remanufacturing rates are given by:*

$$\begin{gathered} v_1(t) = d(t) - r(t) - k_1 \xi_1(t) - k_2 \xi_2(t), \\ v_2(t) = r(t) - k_1 \xi_1(t), \end{gathered} \tag{3.4}$$

and the updating rules of deterioration rates and disposal rate are given by:

$$\begin{gathered} \dot{\eta}_1(t) = \overline{I}_1 \xi_1(t) + \eta_1(t)\xi_1(t) - l_1 \xi_1(t), \\ \dot{\eta}_2(t) = \overline{I}_2 \xi_2(t) + \eta_2(t)\xi_2(t) - l_2 \xi_2(t), \\ \dot{\nu}(t) = \xi_2(t) - \mu \nu(t), \end{gathered} \tag{3.5}$$

where, k_i, l_i, and μ are positive real control gain parameters that can be selected by the firm.

Then the special solution $\xi_i(t) = 0$, $\eta_i(t) = 0$, $(i = 1,2)$, $\nu(t) = 0$, $r(t) = 0$, $d(t) = 0$ *of the system composed of the two systems* (3.3) *and* (3.5) *are asymptotically stable in Liapunov sense.*

Proof. The proof of this theorem can be reached by choosing a suitable Liapunov function for the system consisting of (3.3) and (3.5).

We assume this function has the form:

$$\Phi(\xi_1, \xi_2, \eta_1, \eta_2, \nu) = \sum_{i=1}^{2}\left[\xi_i^2(t) + \eta_i^2(t)\right] + \nu^2(t). \tag{3.6}$$

This function is a positive definite function of the variables $\xi_i(t)$, $\eta_i(t)$, $(i = 1,2)$, and $\nu(t)$.The total time derivative of the Liapunov function (3.6) along the trajectory of the system composed of the systems (3.3) and (3.5) gives

$$\dot{\Phi} = -\left[(\theta_1 + k_1)\xi_1^2 + (\theta_2 + k_2)\xi_2^2 + l_1\eta_1^2 + l_2\eta_2^2 + \mu\nu^2\right]. \tag{3.7}$$

Since $\theta_1 + k_1 > 0$, $\theta_2 + k_2 > 0$, and $\mu > 0$, then $\dot{\Phi}$ is a negative definite function of the variables $\xi_i(t)$, $\eta_i(t)$, $(i = 1,2)$, and $\nu(t)$, so the special solution $\xi_i(t) = 0$, $\eta_i(t) = 0$, $(i = 1,2)$, $\nu(t) = 0$, $r(t) = 0$, and $d(t) = 0$ is asymptotically stable in the Liapunov sense. which completes the proof. □

Now, by substituting from (3.4) into (3.3) and considering the updating rules (3.5), we get the following controlled system:

$$\begin{aligned}
\dot{\xi}_1(t) &= -k_1\xi_1(t) - \overline{I}_1\eta_1(t) - \left(\theta_1 + \eta_1(t)\right)\xi_1(t),\\
\dot{\xi}_2(t) &= -k_1\xi_1 - \overline{I}_2\eta_2 - \left(\theta_2 + \eta_2\right)\xi_2 - \nu(t),\\
\dot{\eta}_1(t) &= \xi_1^2(t) + \overline{I}_1\xi_1(t) - l_1\eta_1(t),\\
\dot{\eta}_2(t) &= \xi_2^2(t) + \overline{I}_2\xi_2(t) - l_2\eta_2(t),\\
\dot{\nu}(t) &= \xi_2(t) - \mu\nu(t).
\end{aligned} \tag{3.8}$$

The above system is used to study the time evolution of inventory levels and dynamic estimators of deterioration rates and disposal rate. It appears from (3.8) that the analytical solution of the system is difficult to derive since it is nonlinear and therefore we solve it numerically in the Section 5. In the next section, we discuss some characteristics of the adaptive system.

4. Some Characteristics of the Model

In this section, we discuss some characteristics of the adaptive reverse logistic inventory system with unknown deterioration and disposal rates.

Lemma 4.1. *There are always manufacturing, remanufacturing, and disposal activities.*

Proof. Using the asymptotic stability conditions of the system (3.3), we find that the manufacturing and remanufacturing rates are:

$$\begin{aligned} P_1(t) &= \overline{P}_1 + d(t) - r(t) - k_1\xi_1(t) - k_2\xi_2(t), \\ P_2(t) &= \overline{P}_2 + r(t) - k_1\xi_1(t). \end{aligned} \tag{4.1}$$

Also, note that when the perturbations $\xi_i(t)$, $d(t)$, and $r(t)$ tend to zero, we find that the manufacturing and remanufacturing activity tend to their goal values $\overline{P}_1$ and $\overline{P}_2$, respectively.

The solution of the differential equation

$$\dot{\widehat{P}}_3(t) - \overline{P}_3 + \mu\left(\widehat{P}_3(t) - \overline{P}_3\right) = \xi_2(t) \tag{4.2}$$

is given by

$$\widehat{P}_3(t) = \overline{P}_3 + e^{\mu t}\left[\int e^{-\mu t}\xi_2(t)dt + c\right], \tag{4.3}$$

where c is a constant. This gives the estimator of the disposal activity. □

Lemma 4.1 can be interpreted as follows: the manufacturing rate is forced continuously to its goal rate and the dynamic estimator of the disposal rate is forced continuously to its goal value.

Lemma 4.2. *If the inventory size of goal levels $\overline{I}_1$ and $\overline{I}_2$ satisfy the system* (2.1). *That is*

$$\begin{aligned} \overline{P}_1 + \overline{P}_2 - \theta_1\overline{I}_1 - \overline{S} &= 0, \\ \overline{P}_2 + \overline{P}_3 - \theta_2\overline{I}_2 - \overline{R} &= 0, \end{aligned} \tag{4.4}$$

then the deviation of the inventory levels from the goal values is equal to zero and the inventory levels, the manufacturing, and remanufacturing rates and the disposal rates are equal to the goal values that is:

$$\lim_{t\to\infty} I_1(t) = \overline{I}_1, \qquad \lim_{t\to\infty} I_2(t) = \overline{I}_2. \tag{4.5}$$

Proof. Since $\Phi(\xi_1, \xi_2, \eta_1, \eta_2, \nu) > 0$ and $\dot{\Phi}(\xi_1, \xi_2, \eta_1, \eta_2, \nu) < 0$ along the trajectory of the system composed of the two systems (3.3) and (3.5), then, we have

$$\lim_{t\to\infty} \xi_i(t) = 0, \quad (i = 1, 2), \tag{4.6}$$

Table 1

par.	k_1	k_2	l_1	l_2	$\bar{I}_1$	$\bar{I}_2$	θ_1	θ_2	μ
val.	5	3	4	2	15	10	0.05	0.03	2

Table 2

par.	k_1	k_2	l_1	l_2	$\bar{I}_1$	$\bar{I}_2$	θ_1	θ_2	μ
val.	0.5	0.2	0.4	0.2	55	40	0.15	0.13	0.2

then we get

$$\lim_{t\to\infty} I_i(t) = \bar{I}_i, \quad (i = 1, 2). \tag{4.7}$$

□

5. Numerical Solution

The objective of this section is to provide numerical solutions of the system (3.8) for different sets of the system parameters and initial conditions. The numerical solution algorithm is based on numerical integration of the system using the Runge-Kutta method. This section displays graphically the numerical solution of the adaptive controlled system (3.8).

5.1. Example 1

This example considerers the case for which both of demand and disposal rates are constants. In this example, In Table 1, a set of parameter values is assumed.

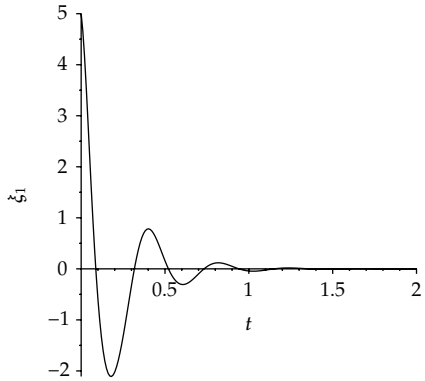
The numerical result of this example are displayed graphically in Figures 1(a) to 1(f). Figure 1 shows the time behavior of the perturbations of the inventory levels and the manufacturing and remanufacturing rates. It is depicted that after some time, the inventory levels and the manufacturing and remanufacturing rates track perfectly their goal levels and rates, respectively.

5.2. Example 2

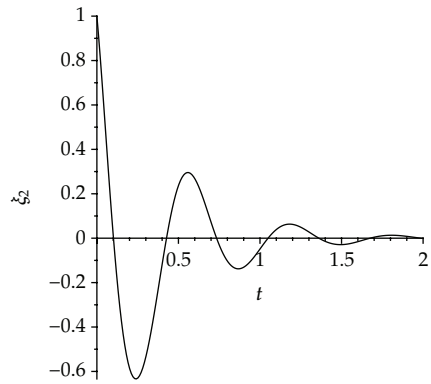
In this example, we discuss the case in which both of demand and disposal rates are time dependent. Namely, the demand rate is a sinusoidal function of time $d(t) = 1 + \sin(t)$ and the disposal rate is an exponential of time $r(t) = e^{-5t}$.

In Table 2, a set of parameter values is assumed.

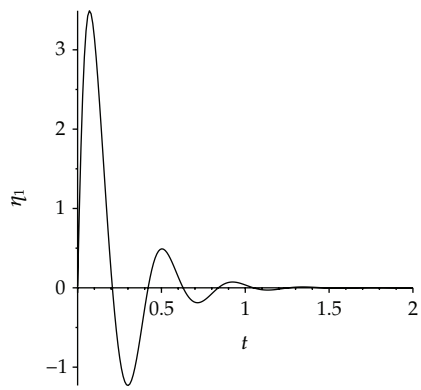
The numerical results of this example are illustrated in Figures 2(a) to 2(f). We conclude that both the perturbations of inventory levels and the manufacturing and remanufacturing rates extensively oscillate about their goal levels and rates, respectively, and finally tend to zero.



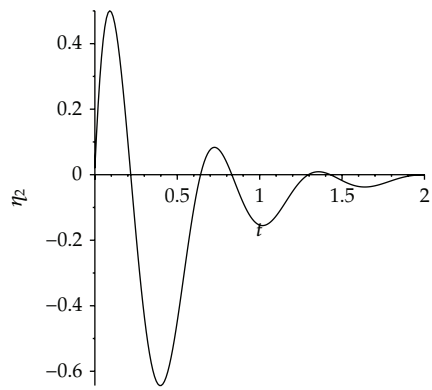
(a) Perturbation of the first inventory level



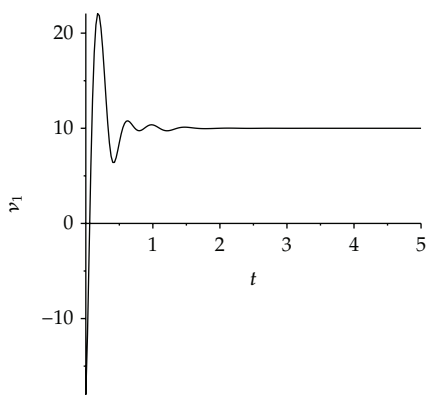
(b) Perturbation of the second inventory level



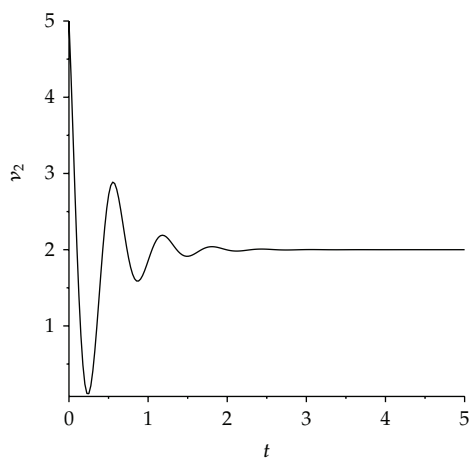
(c) Perturbation of the first deterioration rate



(d) Perturbation of the second deterioration rate

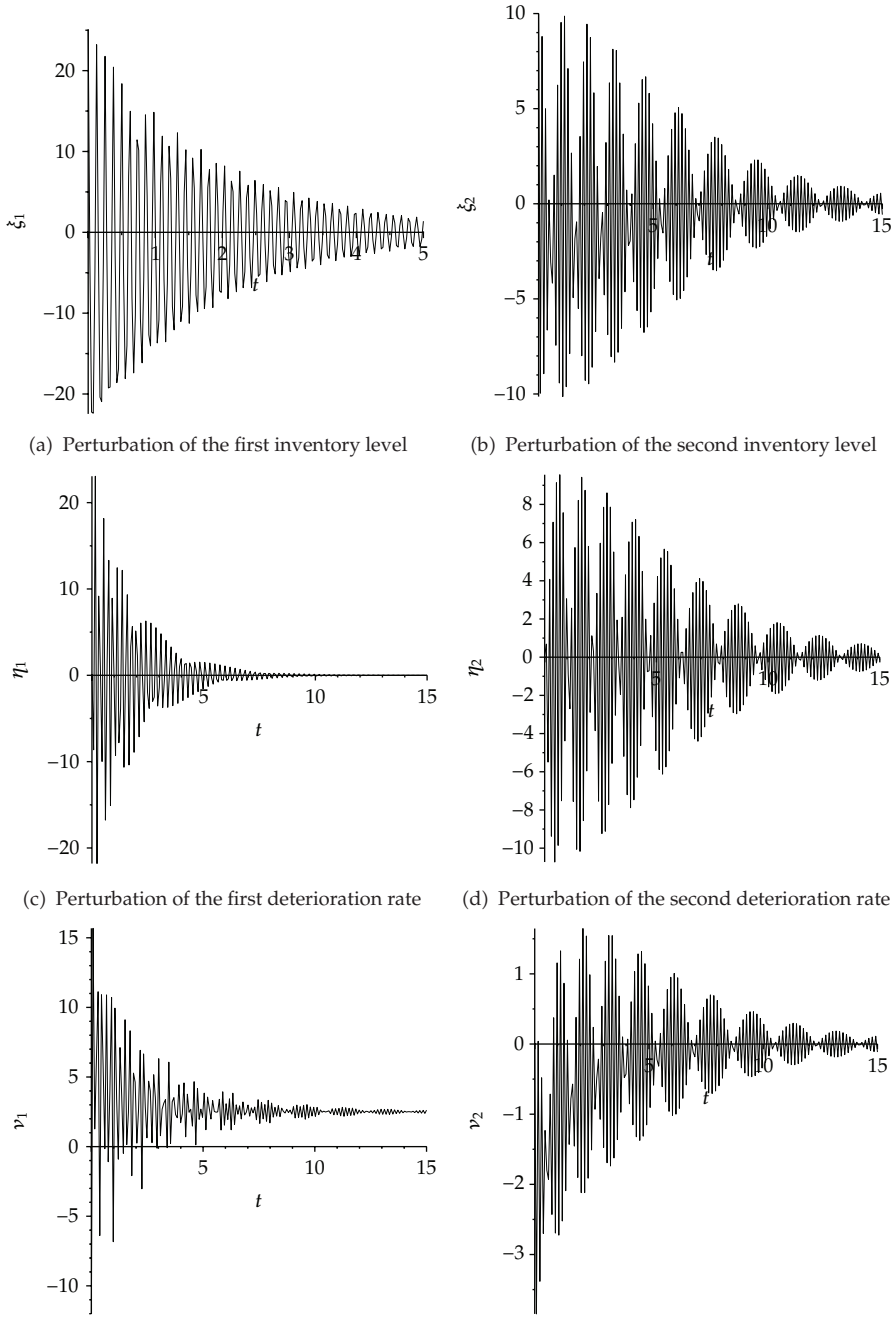


(e) Perturbation of the manufacturing rate



(f) Perturbation of the remanufacturing rate

Figure 1: (a): the perturbation of the first inventory level $\xi_1(t)$, (b): the perturbation of the second inventory level $\xi_2(t)$, (c): the perturbation of the first deterioration rate $\eta_1(t)$, (d): the perturbation of the second deterioration rate $\eta_2(t)$, (e): the perturbation of the manufacturing rate $\nu_1(t)$, and (f): the perturbation of the remanufacturing rate $\nu_2(t)$, with the initial perturbations: $\xi_1(0) = 5$, $\xi_2(0) = 1$; $\eta_1(0) = 0.02$, $\eta_2(0) = 0.02$; $\nu(0) = 5$, respectively.



(a) Perturbation of the first inventory level

(b) Perturbation of the second inventory level

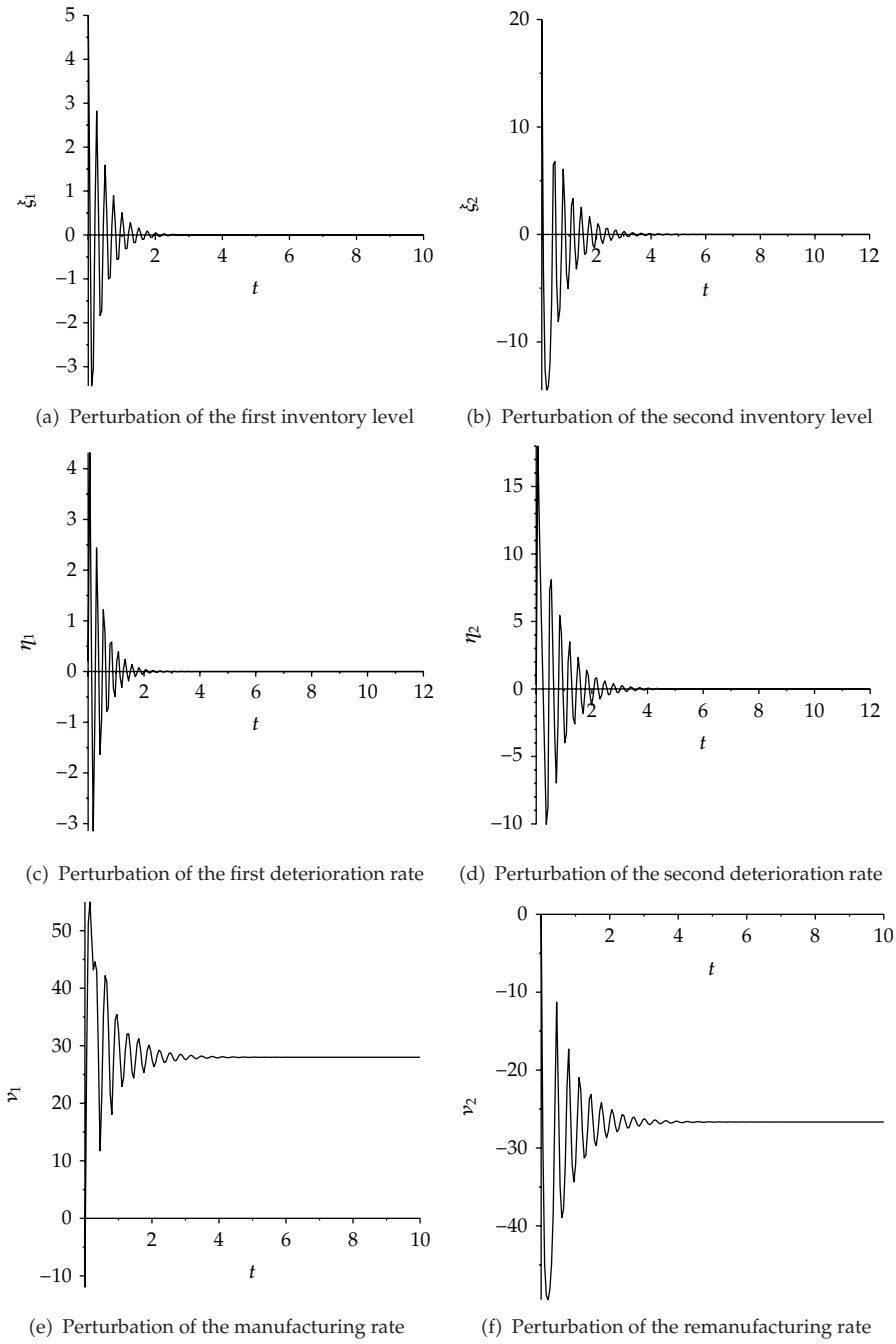
(c) Perturbation of the first deterioration rate

(d) Perturbation of the second deterioration rate

(e) Perturbation of the manufacturing rate

(f) Perturbation of the remanufacturing rate

Figure 2: (a): the perturbation of the first inventory level $\xi_1(t)$, (b): the perturbation of the second inventory level $\xi_2(t)$, (c): the perturbation of the first deterioration rate $\eta_1(t)$, (d): the perturbation of the second deterioration rate $\eta_2(t)$, (e): the perturbation of the manufacturing rate $\nu_1(t)$, and (f): the perturbation of the remanufacturing rate $\nu_2(t)$, with the initial perturbations: $\xi_1(0) = 25$, $\xi_2(0) = 10$; $\eta_1(0) = 0.02$, $\eta_2(0) = 0.02$; $\nu(0) = 25$, respectively.



(a) Perturbation of the first inventory level

(b) Perturbation of the second inventory level

(c) Perturbation of the first deterioration rate

(d) Perturbation of the second deterioration rate

(e) Perturbation of the manufacturing rate

(f) Perturbation of the remanufacturing rate

Figure 3: (a): the perturbation of the first inventory level $\xi_1(t)$, (b): the perturbation of the second inventory level $\xi_2(t)$, (c): the perturbation of the first deterioration rate $\eta_1(t)$, (d): the perturbation of the second deterioration rate $\eta_2(t)$, (e): the perturbation of the manufacturing rate $\nu_1(t)$, and (f): the perturbation of the remanufacturing rate $\nu_2(t)$, with the initial perturbations: $\xi_1(0) = 5$, $\xi_2(0) = 20$; $\eta_1(0) = 0.2$, $\eta_2(0) = 0.3$; $\nu(0) = 5$, respectively.

Table 3

par.	k_1	k_2	l_1	l_2	$\bar{I}_1$	$\bar{I}_2$	θ_1	θ_2	μ
val.	2	1.5	2.5	1.25	25	20	0.05	0.03	20

5.3. Example 3

In this example, we discuss the case when both of demand rate and disposal rate are solenoidal functions of time, where $r(t) = 1 + \cos(5t)$, $d(t) = 1 + \sin(5t)$.

In Table 3, a set of parameter values is assumed.

The numerical results are illustrated in Figures 3(a) to 3(f). We conclude that both of the perturbations of inventory levels and the manufacturing and remanufacturing rates are oscillating about their goal levels and rates, respectively, and tend to zero.

6. Conclusion

In this paper, the mathematical model of a reverse logistic inventory system with deterioration rates has been studied. We have shown how to use the Liapunov technique to study an adaptive control with feedback to solve a reverse logistic inventory model. Some characteristics of the adaptive control system have been discussed. Numerical simulations have been conducted to validate the results obtained. The updating rules of dynamic estimators of deteriorations and disposal rates have been derived from the conditions of the asymptotic stability.

References

[1] S. P. Sethi and G. L. Thompson, *Optimal Control Theory: Applications to Management Science and Economics*, Kluwer Academic Publishers, Boston, Mass, USA, 2nd edition, 2000.

[2] A. El-Gohary, L. Tadj, and A. Al-Rahmah, "Optimal control of a stochastic production planning model with different demand rates," *International Journal of Applied Mathematics*, vol. 20, no. 3, pp. 333–349, 2007.

[3] L. Tadj, A. M. Sarhan, and A. El-Gohary, "Optimal control of an inventory system with ameliorating and deteriorating items," *Jounal of Applied Sciences*, vol. 10, pp. 243–255, 2008.

[4] A. El-Gohary and A. Elsayed, "Optimal control of a multi-item inventory model," *International Mathematical Forum*, vol. 3, no. 25–28, pp. 1295–1312, 2008.

[5] A. Alshamrani and A. El-Gohary, "Optimal control of a two-item inventory system with different types of item deterioration," *Economic Quality Control*, vol. 26, pp. 201–213, 2011.

[6] A. Foul, L. Tadj, and R. Hedjar, "Adaptive control of inventory systems with unknowndeterioration rate," *Journal of King Saud University*. In press.

[7] A. El-Gohary and R. Yassen, "Adaptive control and synchronization of a coupled dynamo system with uncertain parameters," *Chaos, Solitons and Fractals*, vol. 29, no. 5, pp. 1085–1094, 2006.

[8] I. Dobos and K. Kistner, "Production-inventory control in a reverse logistics system," in *Proceedings of the 11th International Working Seminar On Production Economics*, vol. 2, pp. 67–86, Igls/Innsbruck, Austria, 2000, Pre-Prints.

[9] I. Dobos, "Optimal production-inventory strategies for a HMMS-type reverse logistics system," *International Journal of Production Economics*, vol. 81-82, pp. 351–360, 2003.