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Research Article

Development of a Bounded Two-Stage Data Envelopment Analysis Model in the Intuitionistic Fuzzy Environment

Shabnam Mohammadi Ardakani, Hamid Babaei Meybodi, and Hossein Sayyadi Tooranloo

Department of Management, Meybod University, Meybod, Iran

Correspondence should be addressed to Hamid Babaei Meybodi; babaei@meybod.ac.ir

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Data Envelopment Analysis is a powerful tool for evaluating the efficiency of decision-making units for the purpose of ranking, comparing, and differentiating efficient and inefficient units. Classical Data Envelopment Analysis methods operate by measuring the efficiency of each DMU compared to similar units without considering their internal workings and structures, which make them unsuitable for cases where DMUs are multistaged processes with intermediate products or when inputs and outputs are ambiguous or nonconfigurable. In problems that involve uncertainty, intuitionistic fuzzy sets can offer a better representation and interpretation of information than classic sets. In this paper, the noncooperative network data envelopment analysis model of Liang et al. (2008), which is based on Stackelberg game theory and efficiency decomposition, is expanded using the concepts of best and worst relative returns Data Envelopment Analysis model of Azizi et al. (2013) into an interval efficiency estimation model with α-β cuts for two-stage DMUs with trapezoidal intuitionistic fuzzy data. Furthermore, the method of Yue (2011) is used to rank these DMUs in terms of their intuitionistic fuzzy interval efficiency. A numerical example is also provided to illustrate the application of the proposed bounded two-stage intuitionistic Data Envelopment Analysis model.

1. Introduction

With significant developments in management science over the past decades, performance evaluation has become an essential part of this science. One of the prominent tools in the field of performance measurement and evaluation is Data Envelopment Analysis (DEA) [1]. First introduced by Charnes et al. [2] DEA is a nonparametric data-driven method for measuring the efficiency of homogeneous decision-making units [3]. In general, the foundation of nonparametric efficiency evaluation methods was laid by Farrell [4], who used a purely mathematical approach to develop a new performance measurement method opposite to the existing parametric methods. Lotfi et al. [5] used an interactive decision-making technique that incorporates both DEA linear programming and multiobjective linear programming (MOLP) to combine preferential information without prior judgment. Lotfi et al. [6] established an equivalence model between DEA and MOLP and showed how a DEA problem could be solved interactively by transforming it into an MOLP formulation. They used the Zionts–Wallenius (Z–W) method to reflect the DM's preferences in the process of assessing efficiency. Maddahi et al. [7] proposed the proportional weight assignment technique that makes a selection of weight proportional to its corresponding input or output as a secondary goal in DEA cross-efficiency evaluation. Ebrahimnejad et al. [8] proposed an integrated DEA and simulation method for group consensus ranking. The ranking method proposed in this study has several unique features. In contrast to most voting methods that assume equal voting power to voters, the proposed method classifies voters into different groups and allows for assigning a different voting power to each group. Tavana et al. [9] in their study, established an equivalence relationship between MOLP problems and combined-oriented DEA models using a direction distance function designed to account for desirable and undesirable inputs and outputs together with uncontrollable variables. Ebrahimnejad and Tavana [10] obtained a new link between a BCC model and the weighted minimax reference point of the MOLP formulation that simultaneously and interactively considers the increase in the total desirable outputs and the decrease in the total undesirable outputs. Ebrahimnejad et al. [11] established an equivalence model between the general combined-oriented CCR model and multiple objective linear programming and used Zionts–Wallenius's method to integrate combined-oriented CCR performance assessment and target setting such that the DM's preference can be taken into account in an interactive fashion.

The standard DEA takes a black-box approach to the analysis of DMUs, meaning that it simply ignores their internal workings and structures. However, this approach makes it impossible to trace the source of inefficiency in inefficient DMUs [12]. To overcome this problem, over the last two decades, further research has been done on the DEA of DMUs with internal structures, which has led to the development of network data envelopment analysis 2 models [13]. In NDEA, each DMU is assumed to be a network of interconnected subunits, where the outputs of the first stage are transformed to inputs for the second stage and the outputs of the second stage are transformed to inputs for the third stage, and so on [14]. NDEA is a powerful and practical approach for evaluating the efficiency of DMUs with multistage, series, parallel, hierarchical, and hybrid network structures [15]. In a study by Zhou et al. [16] DEA models for the efficiency evaluation of two-stage systems were divided into three categories based on the relationship between the two stages. In the first category, the two stages are considered to be unrelated and a standard DEA model is used independently for each stage [17]. In this category, potential conflicts between the two stages due to their intermediate products are ignored [18]. In the second category, the two stages are considered to have an equal cooperative relationship, a relationship that can be formulated using the relational models of Kao and Hwang [19]. In this approach, the efficiency of the entire DMU is decomposed into the product of the efficiency of its two stages. Liang et al. [20] proposed a centralized model from the perspective of cooperative game theory for these DMUs [21, 22]. Later, Cook et al. [23] proved that the relational model of Kao and Hwang [19] and the centralized model of Liang et al. [20] are equivalent to the NDEA model of Färe and Grosskopf [24]. In the third category, it is assumed that the two stages have an unequal relationship. This relationship can be formulated using the leader-follower model developed by Liang et al. [20] for noncooperative games. This approach requires an assumption about which of the two stages would be the leader and which would be the follower [25]. Li et al. [25] developed the model of Despotis et al. [26] for determining the leader stage in the noncooperative two-stage DEA approach. The noncooperative or leader-follower model is highly practical in representing the many cases where one stage dominates the process and has the power to control other stages, which become its followers [27]. This leader makes the first move, which then determines the behavior of the followers. This is relatively common in traditional manufacturer-retailer supply chains, where manufacturers act as leaders with the power to manipulate the chain and retailers act as followers [28].

While the existing noncooperative DEA models assume that the data will be definitive and precise [16, 29], this is not true for many real-world problems, where the vague and fuzzy nature of the data makes it impossible to use classical mathematics for model formulation. This problem can be resolved by merging fuzzy logic into classical logic. The fuzzy set theory is a useful tool for dealing with vagueness and uncertainty in real-world problems [30]. Over the years, many researchers in the field of DEA have proposed fuzzy methods for working with imprecise and vague data [31–35]. Hatami-Marbini et al. [36] proposed a novel fully fuzzified DEA approach where, in addition to input and output data, all the variables are considered fuzzy, including the resulting efficiency scores. A lexicographic multiobjective linear programming approach is suggested to solve the fuzzy models proposed in this study. Nasseri et al. [37] proposed a fuzzy stochastic DEA model with undesirable outputs. Kachouei et al. [38] proposed a novel approach for finding the common set of weights to compute efficiencies in the DEA model with undesirable outputs when the data are represented by fuzzy numbers. Ebrahimnejad and Amani [39] proposed an approach for solving DEA model in the presence of undesirable outputs in which all input/output data are represented by triangular fuzzy numbers. To this end, two virtual fuzzy DMUs called fuzzy ideal DMU and fuzzy anti-ideal DMU are introduced into the proposed fuzzy DEA framework.

Furthermore, many real-world problems require dealing with data that are verbal or subjective and cannot be accurately measured. One effective way to model this type of data is to avoid Zimmermann's fuzzy set theory [40] and instead use the intuitionistic fuzzy set theory proposed by Atanassov [41]. The intuitionistic fuzzy set theory is a suitable tool for describing vague and imprecise information and dealing with uncertainties and ambiguities in the decision-making process [42]. Since the introduction of intuitionistic fuzzy sets, some researchers have used this approach in NDEA models [43]. For example, Ameri et al. [44] developed an NDEA self-assessment model for measuring the efficiency of a parallel system with intuitionistic fuzzy inputs and outputs. Shakouri et al. [45] proposed a new parametric method for ranking intuitionistic fuzzy numbers in a general form by introducing a hesitation function. Javaherian et al. [4] developed a two-stage DEA model for efficiency measurement with intuitionistic fuzzy data for determining the minimum and maximum input and output levels of DMUs based on the efficiency of each stage. In another study, Javaherian et al. [46] expanded the model of Chen et al. [47] to develop a two-stage model based on intuitionistic fuzzy numbers with the assumption of variable returns to scale. Roy et al. [48] analyzed multiobjective transportation problem under an intuitionistic fuzzy environment. Considering the specific cut interval, the IF transportation cost matrix is converted to the interval cost matrix in their proposed problem. Two approaches, namely, intuitionistic fuzzy programming and goal programming, are used to derive the optimal solutions to the proposed problem. Ebrahimnejad and Verdegay [49] proposed an efficient computational solution approach to solve intuitionistic fuzzy transport problems based on classical transport algorithms. Given the high applicability of the noncooperative game theory in real-world problems, the present study aims to expand the two-stage DEA model of Liang et al. [20] based on Stackelberg game theory and efficiency decomposition in the intuitionistic fuzzy environment. Considering the use of α-β cuts in the proposed model, which requires converting intuitionistic fuzzy data into interval numbers, the article presents a new approach based on the bounded DEA model of Azizi et al. [50] for working with interval data. In the conventional DEA, performances of DMUs are measured from the optimistic point of view; namely, each DMU seeks a set of weights that is the most favorable to itself to maximize its efficiency. If the best relative efficiency of a DMU is equal to one, the DMU is said to be DEA efficient or optimistic efficient; otherwise, it is said to be DEA nonefficient. On the other hand, the performances of DMUs can also be measured from a pessimistic point of view. That is, each DMU seeks a set of weights that is the most unfavorable to itself to minimize its efficiency. It is realized that the overall performance of DMUs should consider both optimistic and pessimistic efficiencies at the same time. In this approach, optimistic and pessimistic relative efficiencies of Azizi's model are turned into upper and lower efficiency bounds, which are then transformed into an overall interval efficiency. Azizi et al. [51] argued that optimistic and pessimistic relative efficiencies should be considered simultaneously, and any approach that ignores one of them would be biased. They proposed merging the two efficiencies into an interval to which the overall efficiency would belong, arguing that this efficiency interval would provide more information than either individual efficiency score. A review of the literature will show that, so far, no study has been conducted on the subject of bounded two-stage NDEA using the concepts of best and worst relative returns in the intuitionistic fuzzy environment. The models developed in this study provide a realistic and practical assessment of efficiency under conditions of uncertainty and ambiguity. The rest of this research is organized as follows. In Section 2, the relevant preliminary concepts are described. In Section 3, the proposed model is presented. In Section 4, the proposed method is used to solve a numerical example. Finally, Section 5 discusses the results and concludes the article.

2. Preliminaries

2.1. Intuitionistic Fuzzy Set 3. The notion of intuitionistic fuzzy sets was first introduced by Atanassov [52] as an extension of classic fuzzy sets. An intuitionistic fuzzy set in the reference set X is defined as $\widetilde{A}^I = \{\prec x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\succ, \forall x \in X\}$, where $\mu_{\widetilde{A}^I}(x)\colon X \longrightarrow [0,1], \nu_{\widetilde{A}^I}(x)\colon X \longrightarrow [0,1]$ subject to $0 \le \mu_{\widetilde{A}^I}(x) + \nu_{\widetilde{A}^I}(x) \le 1, \forall x \in X$, where $\mu_{\widetilde{A}^I}(x), \nu_{\widetilde{A}^I}(x)$ represent the degree of membership and the degree of nonmembership of the element $x \in X$ to the set $\widetilde{A}^I$. In addition to the degree of membership and the degree of nonmembership, a hesitation index in the form of $\pi_X = 1 - \mu_{\widetilde{A}^I}(x) - \nu_{\widetilde{A}^I}(x), \forall x \in X$ is also defined for the intuitionistic fuzzy set. If $\pi_{\widetilde{A}^I} = 0$, then X is a fuzzy number.

2.2. Intuitionistic Fuzzy Number 4

Definition 1. The intuitionistic fuzzy set $\widetilde{A}^I$ is to be an intuitionistic fuzzy number if

(1) $\widetilde{A}^I$ is normalized such that $\forall x_0 \in R\colon \mu_{\widetilde{A}^I}(x_0) = 1, \nu_{\widetilde{A}^I}(x_0) = 0$.

(2) $\mu_{\widetilde{A}^I}(x)$ is convex; i.e.,

$$\mu_{\widetilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \ge \min\left(\mu_{\widetilde{A}^I}(x_1), \mu_{\widetilde{A}^I}(x_2)\right), \quad \forall x_1, x_2 \in X, \lambda \in [0,1]. \tag{1}$$

(3) $\nu_{\widetilde{A}^I}(x)$ is concave; i.e.,

$$\nu_{\widetilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \le \max\left(\nu_{\widetilde{A}^I}(x_1), \nu_{\widetilde{A}^I}(x_2)\right), \quad \forall x_1, x_2 \in X, \lambda \in [0,1]. \tag{2}$$

Definition 2. A trapezoidal intuitionistic fuzzy number is an intuitionistic fuzzy number in the form of $\widetilde{A}^I = \langle(a,b,c,d;w_{\widetilde{A}^I}), (a',b',c',d';\gamma_{\widetilde{A}^I})\rangle$ (Figure 1) with the degrees of membership and nonmembership, $\mu_{\widetilde{A}^I}(x), \nu_{\widetilde{A}^I}(x)$, defined as follows [53]:

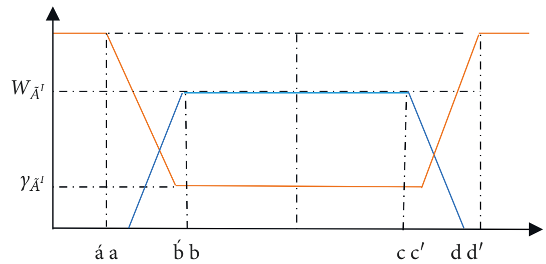

FIGURE 1: Trapezoidal intuitionistic fuzzy number.

$$\mu_{\tilde{A}^I}(x)=\begin{cases}\dfrac{w(x-a)}{b-a}, & a\le x\le b,\\ w & b\le x\le c,\\ \dfrac{w(d-x)}{d-c}, & c\le x\le d,\\ 0, & \text{otherwise},\end{cases},\quad \nu_{\tilde{A}^I}(x)=\begin{cases}\dfrac{(\gamma-1)(x-a')}{b'-a'}+1, & a'\le x\le b',\\ \gamma, & b'\le x\le c',\\ \dfrac{(\gamma-1)(x-d')}{c'-d'}+1, & c'\le x\le d',\\ 1, & \text{otherwise},\end{cases} \tag{3}$$

where $a'\le a\le b'\le b\le c\le c'\le d\le d'$ and $0\le \mu_{\tilde{A}^I}(x)+\nu_{\tilde{A}^I}(x)\le 1$. In this definition, if $w=1,\gamma=0$, then the number is said to be a normal intuitionistic fuzzy number.

2.3. α, β-Cut Sets of Intuitionistic Fuzzy Numbers. Atanassov [52] has provided the following definition for α, β-cuts for the membership and nonmembership of intuitionistic fuzzy sets [45].

The α, β-cut set of the trapezoidal intuitionistic fuzzy number $\tilde{A}^I=(a,b,c,d;w_{\tilde{A}^I}),(a',b',c',d';\gamma_{\tilde{A}^I})$ is a crisp subset of real numbers in the following form:

$$\tilde{A}_{\alpha,\beta}=\left\{x|\mu_{\tilde{A}^I}(x)\ge\alpha,\nu_{\tilde{A}^I}(x)\le\beta\right\},\ 0\le\alpha\le w,\ \gamma_{\tilde{A}}\le\beta\le 1,\ 0\le\alpha+\beta\le 1. \tag{4}$$

where $\alpha\in[0,1]$.

Obviously, α, β and α-β cuts each define an interval as follows:

The α-cut $\tilde{A}_\alpha$ and the β-cut $\tilde{A}_\beta$ for the set X are defined as follows:

$$\begin{aligned}\tilde{A}_\alpha&=\left\{x|\mu_{\tilde{A}}(x)\ge\alpha,x\in X\right\},\ \alpha\in[0,1],\\ \tilde{A}_\beta&=\left\{x|\nu_{\tilde{A}}(x)\le\beta,x\in X\right\}\ \beta\in[0,1],\end{aligned} \tag{5}$$

$$\begin{aligned}\tilde{A}^I{}_\alpha&=\left[L^\mu_{\tilde{A}}(\alpha),R^\mu_{\tilde{A}}(\alpha)\right]=\left[a+\frac{\alpha}{\omega}(b-a),c+\frac{\alpha}{\omega}(b-c)\right],\\ \tilde{A}^I{}_\beta&=\left[L^\nu_{\tilde{A}}(\beta),R^\nu_{\tilde{A}}(\beta)\right]=\left[a'+\frac{1-\beta}{1-\gamma}(b'-a'),c'+\frac{1-\beta}{1-\gamma}(b'-c')\right],\\ \tilde{A}^I{}_{\alpha,\beta}&=\left[L^{\mu,\nu}_{\tilde{A}}(\alpha,\beta),R^{\mu,\nu}_{\tilde{A}}(\alpha,\beta)\right]=\left[a+\frac{\alpha}{\omega}(b-a),d+\frac{\alpha}{\omega}(c-d)\right]\cap\left[a'+\frac{1-\beta}{1-\gamma}(b'-a'),d'+\frac{1-\beta}{1-\gamma}(c'-d')\right].\end{aligned} \tag{6}$$

The α, β-cut set of the intuitionistic fuzzy number is determined from the intersection of intervals that are obtained from α-cut and β-cut.

2.4. Archimedean Operators of Intuitionistic Fuzzy Numbers. For any two intuitionistic fuzzy numbers like [54]

$$\begin{aligned}
&\widetilde{A}_1{}^I = (a_1, b_1, c_1, d_1; w_1), (a_1', b_1', c_1', d_1'; \gamma_1), \widetilde{A}_2{}^I = (a_2, b_2, c_2, d_2; w_2), (a_2', b_2', c_2', d_2'; \gamma_2),\\
&\widetilde{A}_1 \oplus \widetilde{A}_2 = \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2;\ \min\{w_1, w_2\}), (a_1' + a_2', b_1' + b_2', c_1' + c_2', d_1' + d_2';\ \max\{\gamma_1, \gamma_2\})\rangle,\\
&\widetilde{A}_1 \Theta \widetilde{A}_2 = \langle (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2;\ \min\{w_1, w_2\}), (a_1' - d_2', b_1' - c_2', c_1' - b_2', d_1' - a_2';\ \max\{\gamma_1, \gamma_2\})\rangle,\\
&\lambda \widetilde{A}_1{}^I = \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1; w_1), (\lambda a_1', \lambda b_1', \lambda c_1', \lambda d_1'; \gamma_1), & \lambda \geq 0,\\ (\lambda d_1, \lambda c_1, \lambda b_1, \lambda a_1; w_1), (\lambda d_1', \lambda c_1', \lambda b_1', \lambda a_1'; \gamma_1), & \lambda < 0.\end{cases}
\end{aligned} \tag{7}$$

2.5. Defuzzification Method of Trapezoidal Intuitionistic Fuzzy Numbers. There are several ways to defuzzify intuitionistic fuzzy numbers. Since intuitionistic fuzzy numbers are more complex than fuzzy numbers, they are more commonly defuzzified. A defuzzification method of intuitionistic fuzzy number, like $\widetilde{A}^I = \langle (a, b, c, d; w_{\widetilde{A}^I}), (a', b', c', d'; \gamma_{\widetilde{A}^I})\rangle$, is given as follows [55]:

$$DM_{IFS} = \frac{1}{2} * \left(\left[\frac{(d-a) + ((1-\gamma) * b - a) + ((1-\gamma) * c - a)}{4} + a \right] + \left[\frac{(d-a) + (w * b - a) + (w * c - a)}{4} + a \right] \right). \tag{8}$$

3. Proposed Model: Intuitionistic Fuzzy Bounded Two-Stage Noncooperative DEA

The two-stage structure is a highly popular network data envelopment configuration and is widely discussed in the NDEA literature [13, 56]. Figure 2 shows the general layout of a two-stage structure. In this structure, we have n homogeneous DMUS ($DMU_j\,(j = 1, \ldots, n)$), each with m inputs for the first stage in the form of $x_{ij}\,(i = 1, \ldots, m)$, D intermediate products in the form of Z_{dj}, $(d = 1, \ldots, D)$, which are the outputs of the first stage and the inputs of the second stage, and R outputs for the second stage in the form of $Y_{rj}\,(r = 1, \ldots, s)$. Also, $x_{ij}\,(i = 1, \ldots, m)$, $z_{dj}\,(d = 1, \ldots, D)$, and $y_{rj}\,(r = 1, \ldots, s)$ are given nonnegative weights $v_i\,(i = 1, \ldots, m), w_d\,(d = 1, \ldots, D), u_r\,(r = 1, \ldots, s)$ respectively. The weights considered for intermediate products as the output of the first stage and the input of the second stage are the same. For DMUj, we denote the efficiency for the first stage as E_j^1 and the second as E_j^2. On the basis of the radial (CRS) DEA model of Charnes et al. [2] we define the following

$$E_j^1 = \frac{\sum_{d=1}^{D} w_d z_{dj}}{\sum_{i=1}^{m} v_i x_{ij}}, E_j^2 = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{d=1}^{D} w_d z_{dj}}. \tag{9}$$

It is noted that w_d can be set equal to $\widetilde{w}_d$, and in many if not most situations, this would be an appropriate course of action. In the case examined herein, we make the assumption that the "worth" or value according to the intermediate variables is the same regardless of whether they are being viewed as inputs or outputs. Clearly, one can apply two separate DEA analyses to the two stages as in Seiford and Zhu [17]. One criticism of such an approach is the inherent conflict that arises between these two analyses. For example, suppose the first stage is DEA efficient and the second stage is not. When the second stage improves its performance (by reducing the inputs Zdj via an input-oriented DEA model), the reduced Zdj may render the first stage inefficient. This indicates a need for a DEA approach that provides for coordination between the two stages. Before presenting our models, it is useful to point out that given the individual efficiency measures E_j^1 and E_j^2, it is reasonable to define the efficiency of the overall two-stage process either as $(1/2\,(E_j^1 + E_j^2))$ or as $E_j^1 \cdot E_j^2$. If the input-oriented model is used, then we should have $E_j^1 \leq 1$ and $E_j^2 \leq 1$. The above definition ensures that the two-stage process is efficient if and only if $E_j^1 = E_j^2 = 1$. Finally, if we define $E_j = (\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij})$ as the two-stage overall efficiency, our model imply $E_j = E_j^1 = E_j^2$ at optimality. In a similar manner, if assumed that the first stage is the leader, then the first stage performance is more important, and the efficiency of the second stage (follower) is computed, subject to the requirement that the leader's efficiency stays fixed. Adopting the convention that the first stage is the leader, and the second stage, the follower, the efficiency for the first stage can be calculated using the CCR model [2]. That is, we solve for a specific DMUo the linear programming model (10) [20]:

$$\begin{aligned}
&E_o^{1*} = \text{Max} \sum_{d=1}^{D} w_d z_{do}\\
&\text{s.t.} \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0; \quad j = 1, \ldots, n\\
&\sum_{i=1}^{m} v_i x_{io} = 1;\\
&w_d \geq 0,\ d = 1, \ldots, D;\ v_i \geq 0,\ i = 1, \ldots, m.
\end{aligned} \tag{10}$$

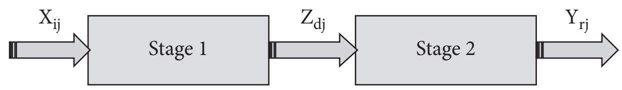

FIGURE 2: General layout of the two-stage structure.

The second stage now treats $\sum_{d=1}^{D} w_d z_{dj}$ as the "single" input subject to the restriction that the efficiency score of the first stage remains at $E_o^{1\,*}$. The model for computing E_o^2, the second stage's efficiency, can be expressed as the following model (11)

$$\begin{aligned}
&E_o^{2\,*} = \mathrm{Max}\frac{\sum_{r=1}^{s} u_r y_{ro}}{E_o^{1\,*}}\\
&\text{s.t.} \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \le 0; \quad j = 1, \ldots, n\\
&\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0; \quad j = 1, \ldots, n\\
&\sum_{i=1}^{m} v_i x_{io} = 1;\\
&\sum_{d=1}^{D} w_d z_{do} = E_o^{1\,*};\\
&w_d \ge 0,\ d = 1, \ldots, D;\ v_i \ge 0,\ i = 1, \ldots, m;\ u_r \ge 0,\ r = 1, \ldots, s.
\end{aligned} \tag{11}$$

In a similar manner, if we assumed the second stage to be the leader, we first calculate the regular DEA efficiency (E_o^2) for that stage, using the appropriate CCR model. Then, one solves the first stage (follower) model, with the restriction that the second stage score, having already been determined, cannot be decreased from that value. Finally, note that in model (11), $E_O^{1\,*} \bullet E_O^{2\,*} = \sum_{r=1}^{S} u_r^* y_{ro}$ at optimality, with $\sum_{i=1}^{m} v_i x_{io} = 1$. That is, $E_o^{1\,*} \cdot E_o^{2\,*} = \sum_{r=1}^{s} u_r y_{ro} / \sum_{i=1}^{m} v_i x_{io}$. This indicates that the noncooperative approach implies an efficiency decomposition for the two-stage DEA analysis. That is, the overall efficiency is equal to the product of the efficiencies of individual stages. The present study aims to expand this two-stage DEA model based on intuitionistic trapezoidal fuzzy numbers as follows.

Step 1. If the first stage is considered as the leader, model (10) is rewritten into model (12) based on trapezoidal intuitionistic fuzzy numbers as follows:

$$\begin{aligned}
&\widetilde{E}_o^{1I*} = \mathrm{Max}\sum_{d=1}^{D} w_d \otimes \langle (a_{do}, b_{do}, c_{do}, d_{do}; \omega_{do}), (a'_{do}, b'_{do}, c'_{do}, d'_{do}; \gamma_{do}) \rangle\\
&\text{s.t.} \sum_{d=1}^{D} w_d \otimes \langle (a_{dj}, b_{dj}, c_{dj}, d_{dj}; \omega_{dj}), (a'_{dj}, b'_{dj}, c'_{dj}, d'_{dj}; \gamma_{dj}) \rangle \Theta\\
&\sum_{i=1}^{m} v_i \otimes \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}; \omega_{ij}), (a'_{ij}, b'_{ij}, c'_{ij}, d'_{ij}; \gamma_{ij}) \rangle \le \widetilde{0}^I\\
&\sum_{i=1}^{m} v_i \otimes \langle (a_{io}, b_{io}, c_{io}, d_{io}; \omega_{io}), (a'_{io}, b'_{io}, c'_{io}, d'_{io}; \gamma_{io}) \rangle = \widetilde{1}^I; w_d, v_i \ge 0.
\end{aligned} \tag{12}$$

Step 2. Applying α, β cuts: upon substitution of input, intermediate, and output variables using their corresponding α, β cuts in the following form,

$$\widetilde{A}^I{}_{\alpha,\beta} = \left[L_{\widetilde{A}}^{\mu,\nu}(\alpha,\beta), R_{\widetilde{A}}^{\mu,\nu}(\alpha,\beta)\right] = \left[a + \frac{\alpha}{\omega}(b - a), d + \frac{\alpha}{\omega}(c - d)\right] \cap \left[a' + \frac{1-\beta}{1-\gamma}(b' - a'), d' + \frac{1-\beta}{1-\gamma}(c' - d')\right], \tag{13}$$

then model (12) can be rewritten into model (14) as follows:

$$\begin{aligned}
(\widetilde{E}_O)^{1I*}_{\alpha,\beta} = \operatorname{Max} \sum_{d=1}^{D} w_d \otimes \left(\left[a_{do} + \frac{\alpha}{\omega}(b_{do} - a_{do}), d_{do} + \frac{\alpha}{\omega}(c_{do} - d_{do}) \right] \cap \left[a'_{do} + \frac{1-\beta}{1-\gamma}(b'_{do} - a'_{do}), d'_{do} + \frac{1-\beta}{1-\gamma}(c'_{do} - d'_{do}) \right] \right), \\
\text{s.t.} \sum_{d=1}^{D} w_d \otimes \left(\left[a_{dj} + \frac{\alpha}{\omega}(b_{dj} - a_{dj}), d_{dj} + \frac{\alpha}{\omega}(c_{dj} - d_{dj}) \right] \cap \left[a'_{dj} + \frac{1-\beta}{1-\gamma}(b'_{dj} - a'_{dj}), d'_{dj} + \frac{1-\beta}{1-\gamma}(c'_{dj} - d'_{dj}) \right] \right) \Theta, \\
\cdot \sum_{i=1}^{m} v_i \otimes \left(\left[a_{ij} + \frac{\alpha}{\omega}(b_{ij} - a_{ij}), d_{ij} + \frac{\alpha}{\omega}(c_{ij} - d_{ij}) \right] \cap \left[a'_{ij} + \frac{1-\beta}{1-\gamma}(b'_{ij} - a'_{ij}), d'_{ij} + \frac{1-\beta}{1-\gamma}(c'_{ij} - d'_{ij}) \right] \right) \leq [0,0], \\
\cdot \sum_{i=1}^{m} v_i \otimes \left(\left[a_{io} + \frac{\alpha}{\omega}(b_{io} - a_{io}), d_{io} + \frac{\alpha}{\omega}(c_{io} - d_{io}) \right] \cap \left[a'_{io} + \frac{1-\beta}{1-\gamma}(b'_{io} - a'_{io}), d'_{io} + \frac{1-\beta}{1-\gamma}(c'_{io} - d'_{io}) \right] \right) = [1,1], \\
\cdot\, w_d \geq 0,\ d = 1, \ldots, D;\ v_i \geq 0,\ i = 1, \ldots, m.
\end{aligned} \tag{14}$$

Step 3. Determining the upper and lower bounds of intervals: after applying α, β cuts, the values of input, intermediate, and output variables must be turned into intervals. The lower bound of the interval is the Max of the lower bound of α, β cut, and the upper bound of the interval is the Min of the upper bound of α, β cut. With this transformation, then model (14) can be rewritten into model (15) as follows:

$$\begin{aligned}
\left[(\widetilde{E}_o)^{1LI*}_{\alpha,\beta}, (\widetilde{E}_o)^{1UI*}_{\alpha,\beta} \right] = \operatorname{Max} \sum_{d=1}^{D} \Bigg(w_d \otimes \operatorname{Max} \left[\left(a_{do} + \frac{\alpha}{\omega}(b_{do} - a_{do}), a'_{do} + \frac{1-\beta}{1-\gamma}(b'_{do} - a'_{do}) \right) \right], \\
\operatorname{Min} \left[d_{do} + \frac{\alpha}{\omega}(c_{do} - d_{do}), d'_{do} + \frac{1-\beta}{1-\gamma}(c'_{do} - d'_{do}) \right], \\
\text{s.t.} \sum_{d=1}^{D} \Bigg(w_d \otimes \operatorname{Max} \left[a_{dj} + \frac{\alpha}{\omega}(b_{dj} - a_{dj}), a'_{dj} + \frac{1-\beta}{1-\gamma}(b'_{dj} - a'_{dj}) \right], \\
\operatorname{Min} \left[d_{dj} + \frac{\alpha}{\omega}(c_{dj} - d_{dj}), d'_{dj} + \frac{1-\beta}{1-\gamma}(c'_{dj} - d'_{dj}) \right] \Bigg) \Theta, \\
\sum_{i=1}^{m} \Bigg(v_i \otimes \operatorname{Max} \left[a_{ij} + \frac{\alpha}{\omega}(b_{ij} - a_{ij}), a'_{ij} + \frac{1-\beta}{1-\gamma}(b'_{ij} - a'_{ij}) \right], \\
\operatorname{Min} \left[d_{ij} + \frac{\alpha}{\omega}(c_{ij} - d_{ij}), d'_{ij} + \frac{1-\beta}{1-\gamma}(c'_{ij} - d'_{ij}) \right] \Bigg) \leq [0,0], \\
\sum_{i=1}^{m} \Bigg(v_i \otimes \operatorname{Max} \left[a_{io} + \frac{\alpha}{\omega}(b_{io} - a_{io}), a'_{io} + \frac{1-\beta}{1-\gamma}(b'_{io} - a'_{io}) \right], \\
\operatorname{Min} \left[d_{io} + \frac{\alpha}{\omega}(c_{io} - d_{io}), d'_{io} + \frac{1-\beta}{1-\gamma}(c'_{io} - d'_{io}) \right] \Bigg) = [1,1], \\
w_d \geq 0,\ d = 1, \ldots, D;\ v_i \geq 0,\ i = 1, \ldots, m.
\end{aligned} \tag{15}$$

Step 4. Integrating optimistic and pessimistic efficiency intervals: in the model proposed by Azizi et al. [50] interval efficiency is determined through the simultaneous use of both optimistic and pessimistic efficiencies. In this approach, the optimistic and pessimistic efficiency intervals of DMUs are combined to reach a new interval called the overall efficiency interval. By considering both upper and lower bounds of efficiency intervals from two different perspectives, this approach gives an overall efficiency interval in the form of $[\psi_o^{L*}, \theta_o^{U*}]$ for each DMUj. Wang et al. [57] proposed the following model (16) to calculate the best relative efficiency of DMUs or the upper and lower bounds of optimistic efficiency:

$$
\begin{aligned}
&\max \theta_o^U = \sum_{r=1}^{s} u_r y_{ro}^U, \\
&\text{s.t.} \sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L \le 0, \\
&\sum_{i=1}^{m} v_i x_{io}^L = 1, \quad u_r, v_i \ge 0, \\
&\max \theta_o^L = \sum_{r=1}^{s} u_r y_{ro}^L, \\
&\text{s.t.} \sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L \le 0, \\
&\sum_{i=1}^{m} v_i x_{io}^U = 1, \quad u_r, v_i \ge 0.
\end{aligned} \tag{16}
$$

Considering the use of α, β-cut, which converts the values of variables into intervals, in this step, interval efficiency is calculated using the bounded model of Azizi et al. [50]. To calculate the optimistic interval efficiency for a given DMU such as DMUo, the optimistic intuitionistic fuzzy relative efficiency interval is the first model (16) rewritten in the form of upper and lower bounds as Models (17) and (18) shown below as follows:

$$
\begin{aligned}
&\left(\widetilde{\theta}_O\right)_{\alpha,\beta}^{1*IL} = \text{Max} \sum_{d=1}^{D} \left(w_d \otimes \text{Max}\left[a_{do} + \frac{\alpha}{\omega}\left(b_{do} - a_{do}\right), a_{do}' + \frac{1-\beta}{1-\gamma}\left(b_{do}' - a_{do}'\right) \right] \right), \\
&\text{s.t.} \sum_{d=1}^{D} \left(w_d \otimes \text{Max}\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right) \right] \right) \Theta, \\
&\sum_{i=1}^{m} \left(v_i \otimes \text{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right) \right] \right) \le 0, \\
&\sum_{i=1}^{m} \left(v_i \otimes \text{Min}\left[d_{io} + \frac{\alpha}{\omega}\left(c_{io} - d_{io}\right), d_{io}' + \frac{1-\beta}{1-\gamma}\left(c_{io}' - d_{io}'\right) \right] \right) = 1 \quad w_d, v_i \ge 0,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\left(\widetilde{\theta}_O\right)^{1*IU}_{\alpha,\beta} &= \operatorname{Max} \sum_{d=1}^{D} \left(w_d \otimes \operatorname{Min}\left[d_{do} + \frac{\alpha}{\omega}\left(c_{do} - d_{do}\right), d_{do}{}' + \frac{1-\beta}{1-\gamma}\left(c_{do}{}' + (-d)_{do}{}'\right) \right] \right), \\
\text{s.t.} & \sum_{d=1}^{D} \left(w_d \otimes \operatorname{Min}\left[d_{dj} + \frac{\alpha}{\omega}\left(c_{dj} - d_{dj}\right), d_{dj}{}' + \frac{1-\beta}{1-\gamma}\left(c_{dj}{}' - d_{dj}{}'\right) \right] \right) \Theta, \\
& \sum_{i=1}^{m} \left(v_i \otimes \operatorname{Max}\left[a_{ij} + \frac{\alpha}{\omega}\left(b_{ij} - a_{ij}\right), a_{ij}' + \frac{1-\beta}{1-\gamma}\left(b_{ij}' - a_{ij}'\right) \right] \right) \leq 0, \\
& \sum_{i=1}^{m} \left(v_i \otimes \operatorname{Max}\left[a_{io} + \frac{\alpha}{\omega}\left(b_{io} - a_{io}\right), a_{io}' + \frac{1-\beta}{1-\gamma}\left(b_{io}' - a_{io}'\right) \right] \right) = 1; \quad w_d, v_i \geq 0.
\end{aligned} \tag{18}
$$

Here, $\widetilde{\theta}_O^U$ is the best relative efficiency under the most favorable conditions and $\widetilde{\theta}_O^L$ is the best relative efficiency under the most unfavorable conditions. These values form the optimistic efficiency interval $[\widetilde{\theta}_O^L, \widetilde{\theta}_O^U]$. Similarly, the worst relative efficiency of DMUs is determined using the following pessimistic model (19) [58]:

$$
\begin{aligned}
\min \phi_o^L &= \sum_{r=1}^{s} u_r y_{ro}^L, \\
\text{s.t.} & \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U \geq 0, \\
& \sum_{i=1}^{m} v_i x_{io}^U = 1, \quad u_r, v_i \geq 0, \\
\min \phi_o^U &= \sum_{r=1}^{s} u_r y_{ro}^U, \\
\text{s.t.} & \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U \geq 0, \\
& \sum_{i=1}^{m} v_i x_{io}^L = 1, \quad u_r, v_i \geq 0.
\end{aligned} \tag{19}
$$

To calculate the pessimistic interval efficiency for a given DMU such as DMUo, the relative pessimistic intuitive fuzzy efficiency interval is transformed by rewriting model (19) into the following models (20) and (21) (upper and lower boundaries):

$$
\begin{aligned}
\left(\widetilde{\phi}_O\right)^{1*IU}_{\alpha,\beta} &= \operatorname{Min} \sum_{d=1}^{D} \left(w_d \otimes \operatorname{Min}\left[d_{do} + \frac{\alpha}{\omega}\left(c_{do} - d_{do}\right), d_{do}{}' + \frac{1-\beta}{1-\gamma}\left(c_{do}{}' - d_{do}{}'\right) \right] \right), \\
\text{s.t.} & \sum_{d=1}^{D} \left(w_d \otimes \operatorname{Max}\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right) \right] \right) \Theta, \\
& \sum_{i=1}^{m} \left(v_i \otimes \operatorname{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right) \right] \right) \geq 0, \\
& \sum_{i=1}^{m} \left(v_i \otimes \operatorname{Max}\left[a_{ij} + \frac{\alpha}{\omega}\left(b_{ij} - a_{ij}\right), a_{ij}' + \frac{1-\beta}{1-\gamma}\left(b_{ij}' - a_{ij}'\right) \right] \right) = 1; w_d, \quad v_i \geq 0,
\end{aligned} \tag{20}
$$

$$\begin{aligned}
&\left(\widetilde{\phi}_{O}\right)_{\alpha,\beta}^{1*IL} = \operatorname{Min}\sum_{d=1}^{D} w_d \otimes \operatorname{Max}\left[a_{do} + \frac{\alpha}{\omega}\left(b_{do} - a_{do}\right), a_{do}' + \frac{1-\beta}{1-\gamma}\left(b_{do}' - a_{do}'\right)\right],\\
&\text{s.t.} \sum_{d=1}^{D}\left(w_d \otimes \operatorname{Max}\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right)\right]\right)\Theta,\\
&\sum_{i=1}^{m}\left(v_i \otimes \operatorname{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right) \geq 0,\\
&\sum_{i=1}^{m}\left(v_i \otimes \operatorname{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right) = 1; \quad w_d, v_i \geq 0,
\end{aligned} \tag{21}$$

where $\widetilde{\phi}_o^L$ is the worst relative efficiency under the most unfavorable conditions and $\widetilde{\phi}_o^U$ is the worst relative efficiency under the most favorable conditions. These values form the pessimistic efficiency interval $[\widetilde{\phi}_o^L, \widetilde{\phi}_o^U]$.

Step 5. Adjusting the lower bound of interval efficiency: since the best and worst relative efficiencies are placed in an interval, the worst relative efficiency determined by pessimistic models needs to be adjusted [50]. Supposing that β $(0 < \beta \leq 1)$ is the adjustment coefficient, the adjusted worst relative efficiency can be calculated as follows:

$$\beta\phi_j^* = \beta\left[\phi_j^{L*}, \phi_j^{U*}\right] = \widehat{\phi}_j^* = \left[\widehat{\phi}_j^{L*}, \widehat{\phi}_j^{U*}\right], \tag{22}$$

and thus $\widehat{\phi}_j^* = \beta\phi_j^* = \beta[\phi_j^{L*}, \phi_j^{U*}] \leq \theta_j^* = [\theta_j^{L*}, \theta_j^{U*}]$ or $\beta \leq \min_{j=1,\ldots,n}(\theta_j^{L*}/\phi_j^{U*})$.

Assuming that $\phi_{\max}^{U*} = \max_{j=1,\ldots,n}\{\phi_j^{U*}\}, \theta_{\min}^{L*} = \min_{j=1,\ldots,n}\{\theta_j^{L*}\}$, by substituting $\beta = (\theta_{\min}^{L*}/\phi_{\max}^{U*})$, where $\beta \leq \min_{j=1,\ldots,n}\{\theta_j^{L*}/\phi_j^{U*}\}$, the worst efficiency of the units in the interval $[\beta, 1]$ can be obtained from the following model (23) [59]:

$$\begin{aligned}
&\min \psi_o^L = \sum_{r=1}^{s} u_r y_{ro}^L,\\
&\text{s.t.} \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i\left(\beta x_{ij}^U\right) \geq 0,\\
&\sum_{i=1}^{m} v_i x_{io}^U = 1, \quad u_r, v_i \geq 0,\\
&\min \psi_o^U = \sum_{r=1}^{s} u_r y_{ro}^U,\\
&\text{s.t.} \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i\left(\beta x_{ij}^U\right) \geq 0,\\
&\sum_{i=1}^{m} v_i x_{io}^L = 1, \quad u_r, v_i \geq 0.
\end{aligned} \tag{23}$$

Therefore, the best and worst adjusted relative pessimistic intuitionistic fuzzy relative efficiencies can be obtained by rewriting model (23) into the following models:

$$\begin{aligned}
&\left(\widetilde{\psi}_o\right)_{\alpha,\beta}^{1*IU} = \operatorname{Min}\sum_{d=1}^{D}\left(w_d \otimes \operatorname{Min}\left[d_{do} + \frac{\alpha}{\omega}\left(c_{do} - d_{do}\right), d_{do}' + \frac{1-\beta}{1-\gamma}\left(c_{do}' - d_{do}'\right)\right]\right),\\
&\text{s.t.} \sum_{d=1}^{D}\left(w_d \otimes \operatorname{Max}\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right)\right]\right)\Theta,\\
&\sum_{i=1}^{m}\left(v_i \otimes \beta \otimes \operatorname{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\nu}\left(c_{ij}' - d_{ij}'\right)\right]\right) \geq 0,\\
&\sum_{i=1}^{m}\left(v_i \otimes \operatorname{Max}\left[a_{ij} + \frac{\alpha}{\omega}\left(b_{ij} - a_{ij}\right), a_{ij}' + \frac{1-\beta}{1-\nu}\left(b_{ij}' - a_{ij}'\right)\right]\right) = 1; \quad w_d, v_i \geq 0,
\end{aligned} \tag{24}$$

$$
\begin{aligned}
&(\tilde{\psi}_o)^{1*IL}_{\alpha,\beta} = \mathrm{Min}\sum_{d=1}^{D}\left(w_d\otimes \mathrm{Max}\left[a_{do}+\frac{\alpha}{\omega}\left(b_{do}-a_{do}\right), a_{do}{}'+\frac{1-\beta}{1-\gamma}\left(b_{do}'-a_{do}'\right)\right]\right),\\
&\text{s.t.}\sum_{d=1}^{D}\left(w_d\otimes \mathrm{Max}\left[a_{dj}+\frac{\alpha}{\omega}\left(b_{dj}-a_{dj}\right), a_{dj}'+\frac{1-\beta}{1-\gamma}\left(b_{dj}'-a_{dj}'\right)\right]\right)\Theta,\\
&\sum_{i=1}^{m}\left(v_i\otimes \mathrm{Min}\left[d_{ij}+\frac{\alpha}{\omega}\left(c_{ij}-d_{ij}\right), d_{ij}'+\frac{1-\beta}{1-\nu}\left(c_{ij}'-d_{ij}'\right)\right]\right)\geq 0,\\
&\sum_{i=1}^{m}\left(v_i\otimes \mathrm{Min}\left[d_{ij}+\frac{\alpha}{\omega}\left(c_{ij}-d_{ij}\right), d_{ij}'+\frac{1-\beta}{1-\nu}\left(c_{ij}'-d_{ij}'\right)\right]\right)=1;\quad w_d, v_i\geq 0.
\end{aligned}
\tag{25}
$$

Step 6. Determining the efficiency interval of the first stage: in this step, the optimistic and pessimistic efficiency intervals for DMUj are combined to reach an overall efficiency interval for the first stage of this DMU in the form of $[E_o^{1*IL}, E_o^{1*IU}] = [\psi_o^{1*IL}, \theta_o^{1*IU}]$ [59].

Step 7. Developing the efficiency calculation model for the second stage: having the efficiency calculation model for the first stage, intuitionistic fuzzy DEA models with interval values can be similarly developed for the second stage by rewriting model (11) into models (26)–(34) as follows:

$$
\begin{aligned}
&\tilde{E}_o^{2I*} = \mathrm{Max}\frac{\left(\sum_{r=1}^{s} u_r\otimes\langle(a_{ro}, b_{ro}, c_{ro}, d_{ro};\omega_{ro}), (a_{ro}', b_{ro}', c_{ro}', d_{ro}';\gamma_{ro})\rangle\right)}{\tilde{E}_o^{1I*}},\\
&\text{s.t.}\sum_{r=1}^{s} u_r\otimes\langle(a_{rj}, b_{rj}, c_{rj}, d_{rj};\omega_{rj}), (a_{rj}', b_{rj}', c_{rj}', d_{rj}';\gamma_{rj})\rangle\Theta\\
&\cdot\sum_{d=1}^{D} w_d\otimes\langle(a_{dj}, b_{dj}, c_{dj}, d_{dj};\omega_{dj}), (a_{dj}', b_{dj}', c_{dj}', d_{dj}';\gamma_{dj})\rangle\leq\tilde{0}^{I},\\
&\cdot\sum_{d=1}^{D} w_d\otimes\langle(a_{dj}, b_{dj}, c_{dj}, d_{dj};\omega_{dj}), (a_{dj}', b_{dj}', c_{dj}', d_{dj}';\gamma_{dj})\rangle\Theta\\
&\cdot\sum_{i=1}^{m} v_i\otimes\langle(a_{ij}, b_{ij}, c_{ij}, d_{ij};\omega_{ij}), (a_{ij}', b_{ij}', c_{ij}', d_{ij}';\gamma_{ij})\rangle\leq\tilde{0}^{I},\\
&\cdot\sum_{i=1}^{m} v_i\otimes\langle(a_{io}, b_{io}, c_{io}, d_{io};\omega_{io}), (a_{io}', b_{io}', c_{io}', d_{io}';\gamma_{io})\rangle=\tilde{1}^{I},\\
&\cdot\sum_{d=1}^{D} w_d\otimes\langle(a_{do}, b_{do}, c_{do}, d_{do};\omega_{do}), (a_{do}', b_{do}', c_{do}', d_{do}';\gamma_{do})\rangle=\tilde{E}_o^{1I*};\quad w_d, v_i, u_r\geq 0,
\end{aligned}
\tag{26}
$$

$$\begin{aligned}
&\left(\widetilde{E}_O\right)^{2I*}_{\alpha,\beta} = \text{Max}\sum_{r=1}^{s} \frac{\left(u_r \otimes \left[a_{ro} + (\alpha/\omega)(b_{ro} - a_{ro}), d_{ro} + (\alpha/\omega)(c_{ro} - d_{ro})\right] \cap \left[a_{ro}' + ((1-\beta)/(1-\gamma))(b_{ro}' - a_{ro}'), d_{ro}' + ((1-\beta)/(1-\gamma))(c_{ro}' - d_{ro}')\right]\right)}{\left(\widetilde{E}_O\right)^{1I*}_{\alpha,\beta}},\\
&\text{s.t.} \sum_{r=1}^{s} u_r \otimes \left(\left[a_{rj} + \frac{\alpha}{\omega}\left(b_{rj} - a_{rj}\right), d_{rj} + \frac{\alpha}{\omega}\left(c_{rj} - d_{rj}\right)\right] \cap \left[a_{rj}' + \frac{1-\beta}{1-\gamma}\left(b_{rj}' - a_{rj}'\right), d_{rj}' + \frac{1-\beta}{1-\gamma}\left(c_{rj}' - d_{rj}'\right)\right]\right)\Theta,\\
&\sum_{d=1}^{D} w_d \otimes \left(\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), d_{dj} + \frac{\alpha}{\omega}\left(c_{dj} - d_{dj}\right)\right] \cap \left[a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right), d_{dj}' + \frac{1-\beta}{1-\gamma}\left(c_{dj}' - d_{dj}'\right)\right]\right) \leq [0,0],\\
&\sum_{d=1}^{D} w_d \otimes \left(\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), d_{dj} + \frac{\alpha}{\omega}\left(c_{dj} - d_{dj}\right)\right] \cap \left[a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right), d_{dj}' + \frac{1-\beta}{1-\gamma}\left(c_{dj}' - d_{dj}'\right)\right]\right)\Theta,\\
&\sum_{i=1}^{m} v_i \otimes \left(\left[a_{ij} + \frac{\alpha}{\omega}\left(b_{ij} - a_{ij}\right), d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right)\right] \cap \left[a_{ij}' + \frac{1-\beta}{1-\gamma}\left(b_{ij}' - a_{ij}'\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right) \leq [0,0],\\
&\sum_{i=1}^{m} v_i \otimes \left(\left[a_{io} + \frac{\alpha}{\omega}\left(b_{io} - a_{io}\right), d_{io} + \frac{\alpha}{\omega}\left(c_{io} - d_{io}\right)\right] \cap \left[a_{io}' + \frac{1-\beta}{1-\gamma}\left(b_{io}' - a_{io}'\right), d_{io}' + \frac{1-\beta}{1-\gamma}\left(c_{io}' - d_{io}'\right)\right]\right) = [1,1],\\
&\sum_{d=1}^{D} w_d \otimes \left(\left[a_{do} + \frac{\alpha}{\omega}\left(b_{do} - a_{do}\right), d_{do} + \frac{\alpha}{\omega}\left(c_{do} - d_{do}\right)\right] \cap \left[a_{do}' + \frac{1-\beta}{1-\gamma}\left(b_{do}' - a_{do}'\right), d_{do}' + \frac{1-\beta}{1-\gamma}\left(c_{do}' - d_{do}'\right)\right]\right) = \widetilde{E}_o^{1I*};\\
&w_d, v_i, u_r \geq 0,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\left[\left(\widetilde{E}_o\right)^{2*IL}_{\alpha,\beta}, \left(\widetilde{E}_o\right)^{2*IU}_{\alpha,\beta}\right] = \text{Max}\sum_{r=1}^{s} \frac{\left(u_r \otimes \left[\text{Max}\left[a_{ro} + (\alpha/\omega)(b_{ro} - a_{ro}), a_{ro}' + ((1-\beta)/(1-\gamma))(b_{ro}' - a_{ro}')\right], \text{Min}\left[d_{ro} + (\alpha/\omega)(c_{ro} - d_{ro}), d_{ro}' + ((1-\beta)/(1-\gamma))(c_{ro}' - d_{ro}')\right]\right]\right)}{\left[\left(\widetilde{E}_o\right)^{1*IL}_{\alpha,\beta}, \left(\widetilde{E}_o\right)^{1*IU}_{\alpha,\beta}\right]},\\
&\text{s.t.} \sum_{r=1}^{s}\left(u_r \otimes \left[\text{Max}\left[a_{rj} + \frac{\alpha}{\omega}\left(b_{rj} - a_{rj}\right), a_{rj}' + \frac{1-\beta}{1-\gamma}\left(b_{rj}' - a_{rj}'\right)\right], \text{Min}\left[d_{rj} + \frac{\alpha}{\omega}\left(c_{rj} - d_{rj}\right), d_{rj}' + \frac{1-\beta}{1-\gamma}\left(c_{rj}' - d_{rj}'\right)\right]\right]\right)\Theta,\\
&\sum_{d=1}^{D}\left(w_d \otimes \left[\text{Max}\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right)\right], \text{Min}\left[d_{dj} + \frac{\alpha}{\omega}\left(c_{dj} - d_{dj}\right), d_{dj}' + \frac{1-\beta}{1-\gamma}\left(c_{dj}' - d_{dj}'\right)\right]\right]\right) \leq [0,0],\\
&\sum_{d=1}^{D}\left(w_d \otimes \left[\text{Max}\left[a_3 + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right)\right], \text{Min}\left[d_{dj} + \frac{\alpha}{\omega}\left(c_{dj} - d_{dj}\right), d_{dj}' + \frac{1-\beta}{1-\gamma}\left(c_{dj}' - d_{dj}'\right)\right]\right]\right)\Theta,\\
&\sum_{i=1}^{m}\left(v_i \otimes \left[\text{Max}\left[a_{ij} + \frac{\alpha}{\omega}\left(b_{ij} - a_{ij}\right), a_{ij}' + \frac{1-\beta}{1-\gamma}\left(b_{ij}' - a_{ij}'\right)\right], \text{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right]\right) \leq [0,0],\\
&\sum_{i=1}^{m}\left(v_i \otimes \left[\text{Max}\left[a_{io} + \frac{\alpha}{\omega}\left(b_{io} - a_{io}\right), a_{io}' + \frac{1-\beta}{1-\gamma}\left(b_{io}' - a_{io}'\right)\right], \text{Min}\left[d_{io} + \frac{\alpha}{\omega}\left(c_{io} - d_{io}\right), d_{io}' + \frac{1-\beta}{1-\gamma}\left(c_{io}' - d_{io}'\right)\right]\right]\right) = [1,1],\\
&\sum_{d=1}^{D}\left(w_d \otimes \left[\text{Max}\left[a_{do} + \frac{\alpha}{\omega}\left(b_{do} - a_{do}\right), a_{do}' + \frac{1-\beta}{1-\gamma}\left(b_{do}' - a_{do}'\right)\right], \text{Min}\left[d_{do} + \frac{\alpha}{\omega}\left(c_{do} - d_{do}\right), d_{do}' + \frac{1-\beta}{1-\gamma}\left(c_{do}' - d_{do}'\right)\right]\right]\right)\\
&\quad = \left[\left(\widetilde{E}_O\right)^{1*IL}_{\alpha,\beta}, \left(\widetilde{E}_O\right)^{1*IU}_{\alpha,\beta}\right] \quad w_d, v_i, u_r \geq 0.
\end{aligned} \tag{28}$$

To calculate the optimistic intuitionistic fuzzy interval relative efficiency for the second stage, model (28) is rewritten into the following models in the form of upper and lower bounds as follows:

$$\begin{aligned}
\left(\widetilde{\theta}_O\right)_{\alpha,\beta}^{2*IU} &= \operatorname{Max}\sum_{r=1}^{s}\frac{\left(u_r\otimes \operatorname{Min}\left[d_{ro}+(\alpha/\omega)\left(c_{ro}-d_{ro}\right),d_{ro}'+\left((1-\beta)/(1-\gamma)\right)\left(c_{ro}'-d_{ro}'\right)\right]\right)}{\left(\widetilde{E}_O\right)_{\alpha,\beta}^{1*IL}},\\
\text{s.t. }&\sum_{r=1}^{s}\left(u_r\otimes \operatorname{Min}\left[d_{rj}+\frac{\alpha}{\omega}\left(c_{rj}-d_{rj}\right),d_{rj}'+\frac{1-\beta}{1-\gamma}\left(c_{rj}'-d_{rj}'\right)\right]\right)\Theta\sum_{d=1}^{D}\left(w_d\otimes \operatorname{Max}\left[a_{dj}+\frac{\alpha}{\omega}\left(b_{dj}-a_{dj}\right),a_{rj}'+\frac{1-\beta}{1-\gamma}\left(b_{rj}'-a_{rj}'\right)\right]\right)\leq 0,\\
&\sum_{d=1}^{D}\left(w_d\otimes \operatorname{Min}\left[d_{dj}+\frac{\alpha}{\omega}\left(c_{dj}-d_{dj}\right),d_{dj}'+\frac{1-\beta}{1-\gamma}\left(c_{dj}'-d_{dj}'\right)\right]\right)\Theta\sum_{i=1}^{m}\left(v_i\otimes \operatorname{Max}\left[a_{ij}+\frac{\alpha}{\omega}\left(b_{ij}-a_{ij}\right),a_{dj}'+\frac{1-\beta}{1-\gamma}\left(b_{dj}'-a_{dj}'\right)\right]\right)\leq 0,\\
&\sum_{i=1}^{m}\left(v_i\otimes \operatorname{Max}\left[a_{io}+\frac{\alpha}{\omega}\left(b_{io}-a_{io}\right),a_{io}'+\frac{1-\beta}{1-\gamma}\left(b_{io}'-a_{io}'\right)\right]\right)=1,\\
&\sum_{d=1}^{D}\left(w_d\otimes \operatorname{Max}\left[a_{do}+\frac{\alpha}{\omega}\left(b_{do}-a_{do}\right),a_{do}'+\frac{1-\beta}{1-\gamma}\left(b_{do}'-a_{do}'\right)\right]\right)=\left(\widetilde{E}_O\right)_{\alpha,\beta}^{1*IL};\quad w_d,v_i,u_r\geq 0,
\end{aligned}\tag{29}$$

$$\begin{aligned}
\left(\widetilde{\theta}_O\right)_{\alpha,\beta}^{2*IL} &= \operatorname{Max}\sum_{r=1}^{s}\frac{\left(u_r\otimes \operatorname{Max}\left[a_{ro}+(\alpha/\omega)\left(b_{ro}-a_{ro}\right),a_{ro}'+\left((1-\beta)/(1-\gamma)\right)\left(b_{ro}'-a_{ro}'\right)\right]\right)}{\left(\widetilde{E}_O\right)_{\alpha,\beta}^{1*IU}},\\
\text{s.t. }&\sum_{r=1}^{s}\left(u_r\otimes \operatorname{Min}\left[d_{rj}+\frac{\alpha}{\omega}\left(c_{rj}-d_{rj}\right),d_{rj}'+\frac{1-\beta}{1-\gamma}\left(c_{rj}'-d_{rj}'\right)\right]\right)\Theta\sum_{d=1}^{D}\left(w_d\otimes \operatorname{Max}\left[a_{dj}+\frac{\alpha}{\omega}\left(b_{dj}-a_{dj}\right),a_{dj}'+\frac{1-\beta}{1-\gamma}\left(b_{dj}'-a_{dj}'\right)\right]\right)\leq 0,\\
&\sum_{d=1}^{D}\left(w_d\otimes \operatorname{Min}\left[d_{dj}+\frac{\alpha}{\omega}\left(c_{dj}-d_{dj}\right),d_{dj}'+\frac{1-\beta}{1-\gamma}\left(c_{dj}'-d_{dj}'\right)\right]\right)\Theta\sum_{i=1}^{m}\left(v_i\otimes \operatorname{Max}\left[a_{ij}+\frac{\alpha}{\omega}\left(b_{ij}-a_{ij}\right),a_{dj}'+\frac{1-\beta}{1-\gamma}\left(b_{dj}'-a_{dj}'\right)\right]\right)\leq 0,\\
&\sum_{i=1}^{m}\left(v_i\otimes \operatorname{Min}\left[d_{io}+\frac{\alpha}{\omega}\left(c_{io}-d_{io}\right),d_{io}'+\frac{1-\beta}{1-\gamma}\left(c_{io}'-d_{io}'\right)\right]\right)=1,\\
&\sum_{d=1}^{D}w_d\otimes\left[\operatorname{Min}\left[d_{do}+\frac{\alpha}{\omega}\left(c_{do}-d_{do}\right),d_{do}'+\frac{1-\beta}{1-\gamma}\left(c_{do}'-d_{do}'\right)\right]\right]=\left(\widetilde{E}_O\right)_{\alpha,\beta}^{1*IU}\quad w_d,v_i,u_r\geq 0.
\end{aligned}\tag{30}$$

Then, to calculate the interval-valued intuitionistic fuzzy DEA model for measuring the worst relative efficiency of DMUs in the second stage, models (29) and (30) is rewritten into the following models:

$$
\begin{aligned}
&\left(\widetilde{\phi}_o\right)_{\alpha,\beta}^{2*IU} = \operatorname{Min} \sum_{r=1}^{s} \frac{\left(u_r \otimes \operatorname{Max}\left[a_{ro} + (\alpha/\omega)\left(b_{ro} - a_{ro}\right), a_{ro}' + \left((1-\beta)/(1-\gamma)\right)\left(b_{ro}' - a_{ro}'\right)\right]\right)}{\left(\widetilde{E}_O\right)_{\alpha,\beta}^{1*IL}},\\
&\text{s.t.} \sum_{r=1}^{s}\left(u_r \otimes \operatorname{Max}\left[a_{rj} + \frac{\alpha}{\omega}\left(b_{rj} - a_{rj}\right)a_{rj}' + \frac{1-\beta}{1-\gamma}\left(b_{rj}' - a_{rj}'\right)\right]\right)\Theta \sum_{d=1}^{D}\left(w_d \otimes \operatorname{Min}\left[d_{dj} + \frac{\alpha}{\omega}\left(c_{dj} - d_{dj}\right), d_{dj}' + \frac{1-\beta}{1-\gamma}\left(c_{dj}' - d_{dj}'\right)\right]\right) \geq 0,\\
&\sum_{d=1}^{D}\left(w_d \otimes \operatorname{Max}\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right)\right]\right)\Theta \sum_{i=1}^{m}\left(v_i \otimes \operatorname{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right) \geq 0,\\
&\sum_{i=1}^{m}\left(v_i \otimes \operatorname{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right) = 1,\\
&\sum_{d=1}^{D}\left(w_d \otimes \operatorname{Max}\left[a_{do} + \frac{\alpha}{\omega}\left(b_{do} - a_{do}\right), a_{do}' + \frac{1-\beta}{1-\gamma}\left(b_{do}' - a_{do}'\right)\right]\right) = \left(\widetilde{E}_O\right)_{\alpha,\beta}^{1*IL}; \quad w_d, v_i, u_r \geq 0,
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&\left(\widetilde{\phi}_O\right)_{\alpha,\beta}^{2IL*} = \operatorname{Min} \sum_{r=1}^{s} \frac{\left(u_r \otimes \operatorname{Max}\left[a_{ro} + (\alpha/\omega)\left(b_{ro} - a_{ro}\right), a_{ro}' + \left((1-\beta)/(1-\gamma)\right)\left(b_{ro}' - a_{ro}'\right)\right]\right)}{\left(\widetilde{E}_O\right)_{\alpha,\beta}^{1*IU}},\\
&\text{s.t.} \sum_{r=1}^{s}\left(u_r \otimes \operatorname{Max}\left[a_{rj} + \frac{\alpha}{\omega}\left(b_{rj} - a_{rj}\right), a_{rj}' + \frac{1-\beta}{1-\gamma}\left(b_{rj}' - a_{rj}'\right)\right]\right)\Theta \sum_{d=1}^{D}\left(w_d \otimes \operatorname{Min}\left[d_{dj} + \frac{\alpha}{\omega}\left(c_{dj} - d_{dj}\right), d_{dj}' + \frac{1-\beta}{1-\gamma}\left(c_{dj}' - d_{dj}'\right)\right]\right) \geq 0,\\
&\sum_{d=1}^{D}\left(w_d \otimes \operatorname{Max}\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right)\right]\right)\Theta \sum_{i=1}^{m}\left(v_i \otimes \operatorname{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right) \geq 0,\\
&\sum_{i=1}^{m}\left(v_i \otimes \operatorname{Max}\left[a_{io} + \frac{\alpha}{\omega}\left(b_{io} - a_{io}\right), a_{io}' + \frac{1-\beta}{1-\gamma}\left(b_{io}' - a_{io}'\right)\right]\right) = 1,\\
&\sum_{d=1}^{D}\left(w_d \otimes \operatorname{Min}\left[c_{do} + \frac{\alpha}{\omega}\left(b_{do} - c_{do}\right), c_{do}' + \frac{1-\beta}{1-\gamma}\left(b_{do}' - c_{do}'\right)\right]\right) = \left(\widetilde{E}_O\right)_{\alpha,\beta}^{1IU*} \quad w_d, v_i, u_r \geq 0.
\end{aligned}
\tag{32}
$$

Here, the adjusted pessimistic efficiency ($\widetilde{\psi}$), the lower bound of which is used as the final efficiency, is determined by introducing the adjustment coefficient to models (31) and (32) and is rewritten as the following models:

$$(\widetilde{\psi}_o)^{2*IU}_{\alpha,\beta} = \text{Min} \sum_{r=1}^{s} \frac{(u_r \otimes \text{Min}[d_{ro} + (\alpha/\omega)(c_{ro} - d_{ro}), d_{ro}' + ((1-\beta)/(1-\gamma))(c_{ro}' - d_{ro}')])}{(\widetilde{E}_O)^{1*IL}_{\alpha,\beta}},$$

$$\begin{aligned}
&\text{s.t.} \sum_{r=1}^{s}\left(u_r \otimes \text{Max}\left[a_{rj} + \frac{\alpha}{\omega}\left(b_{rj} - a_{rj}\right), a_{rj}' + \frac{1-\beta}{1-\gamma}\left(b_{rj}' - a_{rj}'\right)\right]\right)\Theta,\\
&\sum_{d=1}^{D}\left(w_d \otimes \beta_2 \otimes \text{Min}\left[d_{dj} + \frac{\alpha}{\omega}\left(c_{dj} - d_{dj}\right), d_{dj}' + \frac{1-\beta}{1-\gamma}\left(c_{dj}' - d_{dj}'\right)\right]\right) \geq 0,\\
&\sum_{d=1}^{D}\left(w_d \otimes \text{Max}\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right)\right]\right)\Theta,\\
&\sum_{i=1}^{m}\left(v_i \otimes \beta_1 \otimes \text{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right) \geq 0,\\
&\sum_{i=1}^{m}\left(v_i \otimes \text{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right) = 1,\\
&\sum_{d=1}^{D}\left(w_d \otimes \text{Max}\left[a_{do} + \frac{\alpha}{\omega}(b_{do} - a_{do}), a_{do}' + \frac{1-\beta}{1-\gamma}(b_{do}' - a_{do}')\right]\right) = (\widetilde{E}_O)^{1*IL}_{\alpha,\beta}; \quad w_d, v_i, u_r \geq 0,
\end{aligned} \tag{33}$$

$$(\widetilde{\psi}_O)^{2IL*}_{\alpha,\beta} = \text{Min} \sum_{r=1}^{s} \frac{(u_r \otimes \text{Max}[a_{ro} + (\alpha/\omega)(b_{ro} - a_{ro}), a_{ro}' + ((1-\beta)/(1-\gamma))(b_{ro}' - a_{ro}')])}{(\widetilde{E}_O)^{1*IU}_{\alpha,\beta}},$$

$$\begin{aligned}
&\text{s.t} \sum_{r=1}^{s} \Theta \sum_{d=1}^{D}\left(w_d \otimes \beta_2 \otimes \text{Min}\left[d_{dj} + \frac{\alpha}{\omega}\left(c_{dj} - d_{dj}\right), d_{dj}' + \frac{1-\beta}{1-\gamma}\left(c_{dj}' - d_{dj}'\right)\right]\right) \geq 0\\
&\left(u_r \otimes \text{Max}\left[a_{rj} + \frac{\alpha}{\omega}\left(b_{rj} - a_{rj}\right), a_{rj}' + \frac{1-\beta}{1-\gamma}\left(b_{rj}' - a_{rj}'\right)\right]\right),\\
&\sum_{d=1}^{D}\left(w_d \otimes \text{Max}\left[a_{dj} + \frac{\alpha}{\omega}\left(b_{dj} - a_{dj}\right), a_{dj}' + \frac{1-\beta}{1-\gamma}\left(b_{dj}' - a_{dj}'\right)\right]\right)\Theta\\
&\sum_{i=1}^{m}\left(v_i \otimes \beta_1 \otimes \text{Min}\left[d_{ij} + \frac{\alpha}{\omega}\left(c_{ij} - d_{ij}\right), d_{ij}' + \frac{1-\beta}{1-\gamma}\left(c_{ij}' - d_{ij}'\right)\right]\right) \geq 0,\\
&\sum_{i=1}^{m}\left(v_i \otimes \text{Max}\left[a_{io} + \frac{\alpha}{\omega}(b_{io} - a_{io}), a_{io}' + \frac{1-\beta}{1-\gamma}(b_{io}' - a_{io}')\right]\right) = 1,\\
&\sum_{d=1}^{D}\left(w_d \otimes \text{Min}\left[c_{do} + \frac{\alpha}{\omega}(b_{do} - c_{do}), c_{do}' + \frac{1-\beta}{1-\gamma}(b_{do}' - c_{do}')\right]\right) = (\widetilde{E}_O)^{1*IU}_{\alpha,\beta}; \quad w_d, v_i, u_r \geq 0.
\end{aligned} \tag{34}$$

Next, to determine the efficiency of the second stage, optimistic and pessimistic efficiencies are combined to reach an interval efficiency. This gives an overall efficiency interval in the form of $[E_o^{2*IL}, E_o^{2*IU}] = [\psi_o^{2*IL}, \theta_o^{2*IU}]$ for each DMUj.

Step 8. Determining the overall interval efficiency: since the efficiency scores of the first and second stages are positive values between [0, 1], the overall efficiency is determined from their Cartesian product:

$$\left[E_o^{1*IL}, E_o^{1*UL}\right] \otimes \left[E_o^{2*IL}, E_o^{2*IU}\right] = \left[\psi_o^{1*IL}, \theta_o^{1*IU}\right] \otimes \left[\psi_o^{2*IL}, \theta_o^{2*IU}\right] = \left[\psi_o^{1*IL} \times \psi_o^{2*IL}, \theta_o^{1*IU} \times \theta_o^{2*IU}\right]. \tag{35}$$

Step 9. Ranking DMUs: since this method computes the efficiency of each DMU as an interval, an appropriate method is needed to rank the DMUs. In this study, the method proposed by Yue [60] for the ranking of interval numbers is used for this purpose. This method is based on computing the probability that an interval number is larger than the other interval number. For two interval numbers $a = [a^l, a^u], b = [b^l, b^u]$, where $l_a = a^u - a^l$ and $l_b = b^u - b^l$, this probability is defined as follows:

$$p(a \geq b) = \max\left\{1 - \max\left(\frac{b^u - a^l}{l_a + l_b}, 0\right), 0\right\}. \tag{36}$$

To rank a series of interval numbers like $a_j = [a_j^L, a_j^U]$, $j = 1, \ldots, n$, first each $a_i = [a_i^L, a_i^U]$ must be compared with all $a_j = [a_j^l, a_j^u]$, $(j = 1, \ldots, n)$ using the above formula. For the sake of convenience, p_{ij} is defined as the probability that a_i is larger than a_j; i.e., $p_{ij} = p(a_i \geq a_j)$. Next, the matrix $p = (p_{ij})_{n\times n}$ is formed such that $p_{ii} + p_{ij} = 1$ and $p_{ii} = (1/2)p_{ij}$, $p_{ij} \geq 0$, $i, j = 1, \ldots, n$. Then, the sum of the elements of each row of the matrix p is calculated using the formula $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, \ldots, n$. Finally, the interval numbers are sorted in descending order.

4. Numerical Example

In this section, a numerical example is solved to provide a better understanding of the proposed model. In this example, there are 12 DMUs, each with two inputs, two intermediate products, and three outputs in the second stage. The general layout of the model in this example is shown in Figure 3. The trapezoidal intuitionistic fuzzy data of this example are given in Tables 1–3.

In this example, α, β cuts ($\alpha, \beta \in [0, 1]$) were applied at five levels of($\alpha = 0, \beta = 1$; $\alpha = 0.25, \beta = 0.75$; $\alpha = 0.5$, $\beta = 0.5$; $\alpha = 0.75, \beta = 0.25$; $\alpha = 1, \beta = 0$) and their intersections for two inputs, two intermediate products, and two outputs were calculated. The results obtained in the form of interval numbers were used to calculate lower and upper bounds of efficiency. To determine the efficiency of the first stage, first, the pessimistic and optimistic efficiency scores had to be calculated. To calculate the β coefficient, the minimum value of the lower bound of optimistic efficiency was divided by the maximum value of the upper bound of the pessimistic efficiency and then the product was put into the model to calculate the lower bound of overall efficiency. For example, for $\alpha = 0.5$ and $\beta = 0.5$, the lower bound of optimistic efficiency was determined as follows:

$$\widetilde{\theta}^{IL} = 1, 0.5, 1, 0.302, 0.582, 1, 1, 0.721, 0.31, 0.419, 0.28, 0.162 \quad \mathrm{Min}\left(\widetilde{\theta}^{IL}\right) = 0.162. \tag{37}$$

The upper bound of pessimistic efficiency for this level of α and β cut was determined as follows:

$$\widetilde{\phi}^{IU} = 4.174, 1.127, 1.956, 1, 1.529, 3.858, 3.025, 3.224, 1, 1.398, 2.359, 1 \quad \mathrm{Max}\left(\widetilde{\phi}^{IU}\right) = 4.174. \tag{38}$$

Dividing these two values by each other gave the following:

$$\beta \frac{\min \widetilde{\theta}^{IL}}{\max \widetilde{\phi}^{IU}} = \frac{0.162}{4.174} = 0.0388. \tag{39}$$

By inserting this coefficient in the model, we calculated the adjusted pessimistic efficiency, which is the lower bound of interval efficiency. The adjusted pessimistic efficiency (the lower bound of efficiency interval) was calculated as follows:

$$\widetilde{\psi}^{IL} = 0.2007, 0.089, 0.108, 0.0388, 0.0698, 0.249, 0.195, 0.166, 0.0499, 0.0595, 0.0723, 0.0574. \tag{40}$$

The upper bound of optimistic efficiency was used as the upper bound of overall efficiency. The results of efficiency calculations for the defined α and β cuts through the integration of optimistic and pessimistic efficiencies are provided in Table 4.

The optimistic and pessimistic efficiency values of the second stage for different levels of α and β cut were also calculated in the same way. The results of efficiency calculations for the second stage for the defined α and β cuts through the integration of optimistic and pessimistic efficiencies are presented in Table 5.

Since the efficiency of the first and second stages was a positive value between 0 and 1, the overall efficiency was determined by the Cartesian multiplication of the lower and upper bounds, as shown in Table 6.

According to a study by Hatami-Marbini et al. [61] on bounded DEA, when efficiency is measured in the form of an interval like $[E_O^{L*}, E_O^{U*}]$, a DMU is efficient only if its upper efficiency bound (E_O^{U*}) is 1. Here, efficient and inefficient DMUs at different levels of α-cut and β-cut were identified according to this rule. For example, for $\alpha = 0.5$

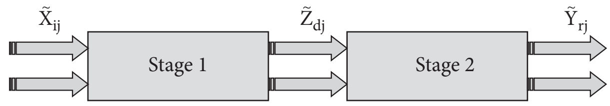

Figure 3: Two-stage model of the numerical example.

Table 1: Trapezoidal intuitionistic fuzzy inputs for the first stage of the 12 DMUs.

DMU	Input1	Input2
1	(7,9,11,16;0.9), (3,9,11,17;0.1)	(3,6,6,7;1), (1,6,6,8;0)
2	(3,5,5,8;1), (2,5,5,11;0)	(7,9,9,12;0.9), (6,9,9,23;0.1)
3	(5,7,7,10;0.9), (4,7,7,13;0.1)	(9,11,11,14; 1), (8,11,15,45;0)
4	(22,22,22,41;0.8), (21,22,22,42;0.1)	(3,10,10,10;1), (3,10,10,11;0)
5	(8,10,12,17;0.9), (4,10,12,18;0.1)	(6,6,6,31;1), (5,6,6,33;0)
6	(5,7,7,10;1), (4,7,13,1;0)	(4,7,7,8;1), (2,7,7,9;0)
7	(10,10,10,29;0.8), (10,10,10,30;0.1)	(8,10,10,13;0.9), (7,10,10,24;0.1)
8	(10,12,13,19;0.9), (6,12,14,21;0.1)	(4,6,8,13;1), (1,6,8,15;0)
9	(5,7,9,14;0.9), (1,7,9,15;0.1)	(6,13,13,13;1), (6,13,13,14;0)
10	(10,12,14,19;0.9), (6,12,14,20;0.1)	(7,7,7,32;1), (6,7,7,34;0)
11	(11,13,20,30;0.9), (7,13,20,32;0.1)	(6,8,20,25;0.9), (3,8,20,27;0.1)
12	(10,15,30,35;0.8), (7,15,30,38;0.1)	(16,20,36,40;0.8), (13,20,36,41;0.1)

Table 2: Trapezoidal intuitionistic fuzzy intermediate products of the 12 DMUs.

DMU	Intermediate 1	Intermediate 2
1	(52,53,53,54;0.9), (31,53,53,56;0.1)	(51,52,52,53;0.9), (21,52,52,56;0.1)
2	(15,17,17,18;0.9), (15,20,17,20;0.1)	(18,20,20,21;0.9), (18,23,20,23;0.1)
3	(55,56,60,65;0.9), (50,51,60,69;0.1)	(14,14,20,30;1), (14,14,20,30,0)
4	(25,27,27,28;0.9), (25,27,27,30;0.1)	(8,8,8,14;0.8), (7,8,8,15;0.2)
5	(53,54,54,55;0.9), (32,54,54,58;0.1)	(6,9,12,13;0.9), (6,9,12,29;0.1)
6	(45,47,47,48;1),(42,47,47,51;0)	(50,51,51,52;0.9),(20,51,51,55;0.1)
7	(51,51,58,64;1), (30,51,58,64;0)	(96,96,100,105;0.9), (89,96,100,106;0.1)
8	(50,51,51,52;0.9), (29,51,51,54;0)	(55,56,56,58;0.9),(22,56,56,61;0.1)
9	(20,22,22,23;0.9), (20,22,22,25;0.1)	(10,10,10,16;0.8), (9,10,10,17;0.2)
10	(42,44,44,45;1), (39,44,44,48;0)	(7,10,13,14;0.9), (7,10,13,30;0.1)
11	(35,37,45,48;0.9), (30,35,45,49;0.1)	(30,31,38,42;0.9), (29,31,38,45;0.1)
12	(27,28,35,36;0.9), (21,28,35,40;0.1)	(23,24,34,40;0.9), (22,25,34,41;0.1)

Table 3: Trapezoidal intuitionistic fuzzy outputs of the second stage of the 12 DMUs.

DMU	Output1	Output2
1	(96,97,115,120;1), (95,97,115,121;0)	(77,79,90,91;0.8), (76,79,90,95;0.1)
2	(65,68,80,82;0.8), (64,69,80,86;0.1)	(5,6,12,18;0.9), (4,5,12,20;0.1)
3	(148,148,165,170;0.9), (147,147,165,175;0.1)	(107,108,115,120;1), (80,88,115,120;0)
4	(7,8,9,10;1), (3,8,9,11;0)	(141,142,143,144;0.9), (113,142,143,145;0.1)
5	(9,11,16,18;0.9), (9,11,16,21;0.1)	(131,133,135,137;0.9), (117,133,135,148;0)
6	(225,226,227,229;0.9), (167,226,227,230;0.1)	(226,227,228,230;0.9), (165,227,228,231;0.1)
7	(100,105,110,117;0.8), (98,101,110,120;0.1)	(106,110,114,119;0.8), (85,90,114,120;0.1)
8	(100,101,110,117;0.9), (43,101,110,119;0.1)	(207,208,216,221;0.9), (180,188,216,225;0.1)
9	(10,13,20,25;0.9),(8,13,20,26;0.1)	(80,82,85,89;0.8),(79,82,85,90;0.1)
10	(55,56,88,89;0.9), (54,57,88,90;0.1)	(41,42,83,95;0.9), (40,43,83,90;0.1)
11	(65,68,95,98;1), (41,48,95,99;0)	(75,76,93,94;0.9), (75,77,93,100;0.1)
12	(60,61,78,80;0.8), (59,60,78,80;0.1)	(55,60,95,105;0.9), (50,55,95,110;0.1)

and $\beta = 0.5$, DMUs 1, 3, 6, and 7 were found to be efficient in their first stage and DMUs 2, 3, 4, 6, and 10 were found to be efficient in their second stage. According to Liang [20] a DMU is fully efficient only if it is efficient in both of its stages. At this α and β cut level, only DMUs 3 and 6 were found to be fulfilling this requirement. The results of the efficiency analysis for different cut levels are presented in Table 7.

Since efficiency values were obtained in the form of intervals, we ranked them using the method of Yue [60],

Table 4: Optimistic-pessimistic efficiency calculations of the first stage for different levels of α-β cuts.

DMU	$\alpha = 0, \beta = 1$	$\alpha = 0.25, \beta = 0.75$	$\alpha = 0.5, \beta = 0.5$	$\alpha = 0.75, \beta = 0.25$	$\alpha = 1, \beta = 0$
1	[0.20945,1]	[0.2044,1]	[0.20033,1]	[0.1971,1]	[0.1427,1]
2	[0.0723,0.669]	[0.0772,0.6144]	[0.08186,0.579]	[0.0858,0.5399]	[0.07145,0.5071]
3	[0.1185,1]	[0.1064,1]	[0.10749,1]	[0.11503,1]	[0.09569,1]
4	[0.03614,0.497]	[0.03685,0.3872]	[0.0387,0.3352]	[0.04228,0.3042]	[0.03547,0.282]
5	[0.059,0.689]	[0.06258,0.7564]	[0.06976,0.8146]	[0.08286,0.8651]	[0.08622,0.9735]
6	[0.248,1]	[0.2488,1]	[0.2484,1]	[0.2466,1]	[0.171,1]
7	[0.1706,0.976]	[0.17882,1]	[0.1951,1]	[0.2353,1]	[0.2467,1]
8	[0.1621,0.813]	[0.16395,0.899]	[0.16524,0.9676]	[0.1667,1]	[0.1234,1]
9	[0.0556,0.4176]	[0.05172,0.3922]	[0.04985,0.3823]]	[0.04339,0.3766]	[0.035,0.354]
10	[0.049,0.461]	[0.052226,0.5094]	[0.05946,0.5521]	[0.070525,0.5939]	[0.0639,0.683]
11	[0.065,0.5184]	[0.06706,0.5441]	[0.07233,0.5642]	[0.079528,0.58]	[0.08566,0.5899]
12	[0.05,0.3806]	[0.05261,0.35198]	[0.05744,0.3337]	[0.064667,0.3196]	[0.07189,0.3067]

Table 5: Optimistic-pessimistic efficiency calculations of the second stage for different levels of α-β cuts.

DMU	$\alpha = 0, \beta = 1$	$\alpha = 0.25, \beta = 0.75$	$\alpha = 0.5, \beta = 0.5$	$\alpha = 0.75, \beta = 0.25$	$\alpha = 1, \beta = 0$
1	[0.173,0.4782]	[0.1738,0.48082]	[0.17475,0.47929]	[0.1756,0.47903]	[0.1765,0.46944]
2	[0.10589,1]	[0.11645,1]	[0.12587,1]	[0.1382,0.98335]	[0.13866,0.9567]
3	[0.255,1]	[0.25525,1]	[0.2555,1]	[0.2558,1]	[0.256,1]
4	[0.29778,1]	[0.38355,1]	[0.44468,1]	[0.49179,1]	[0.49353,1]
5	[0.20319,1]	[0.18656,1]	[0.17459,0.8673]	[0.1657,0.7398]	[0.16697,0.6633]
6	[0.451,1]	[0.4516,1]	[0.4521,1]	[0.45267,1]	[0.4532,1]
7	[0.21107,0.4558]	[0.2088,0.45572]	[0.211625,0.45552]	[0.2144,0.45523]	[0.21725,0.399]
8	[0.3776,0.81574]	[0.3421,0.82067]	[0.31843,0.8255]	[0.3087,0.8302]	[0.30922,0.8348]
9	[0.2155,0.8069]	[0.2332,0.78739]	[0.24305,0.7679]	[0.25059,0.7484]	[0.25447,0.73621]
10	[0.20824,1]	[0.18955,1]	[0.17589,1]	[0.16444,0.91818]	[0.16539,0.85957]
11	[0.27006,0.6273]	[0.2592,0.6119]	[0.25178,0.54249]	[0.2467,0.53415]	[0.24847,0.5205]
12	[0.30215,0.7761]	[0.33069,0.727024]	[0.35482,0.680127]	[0.37579,0.5923]	[0.38112,0.5392]

Table 6: Calculation of overall efficiency in the form of interval numbers.

DMU	$\alpha = 0, \beta = 1$	$\alpha = 0.25, \beta = 0.75$	$\alpha = 0.5, \beta = 0.5$	$\alpha = 0.75, \beta = 0.25$	$\alpha = 1, \beta = 0$
1	[0.036235,0.4782]	[0.03553,0.48082]	[0.03501,0.4793]	[0.0346,0.47903]	[0.02519,0.469]
2	[0.00766,0.6686]	[0.00899,0.6144]	[0.0103,0.5792]	[0.01186,0.5309]	[0.0099,0.4694]
3	[0.03022,1]	[0.027159,1]	[0.027464,1]	[0.029425,1]	[0.02449,1]
4	[0.1076,0.497]	[0.014134,0.3872]	[0.01721,0.3352]	[0.02079,0.3042]	[0.01751,0.282]
5	[0.011988,0.689]	[0.011675,0.7564]	[0.01218,0.7065]	[0.0137,0.6399]	[0.01439,0.646]
6	[0.011848,1]	[0.11236,1]	[0.1123,1]	[0.11165,1]	[0.77497,1]
7	[0.03601,0.4448]	[0.03734,0.4557]	[0.041284,0.455]	[0.05044,0.4552]	[0.05359,0.399]
8	[0.61209,0.6632]	[0.056089,0.7378]	[0.05262,0.7987]	[0.05146,0.8302]	[0.0382,0.8348]
9	[0.01198,0.3369]	[0.01206,0.3088]	[0.01212,0.2936]	[0.01087,0.2819]	[0.0089,0.2606]
10	[0.0102,0.461]	[0.00989,0.5094]	[0.10458,0.5521]	[0.11597,0.5454]	[0.0106,0.6117]
11	[0.01756,0.3252]	[0.01738,0.3329]	[0.0182,0.3061]	[0.0196,0.3098]	[0.0213,0.3071]
12	[0.01511,0.2954]	[0.01739,0.2559]	[0.02038,0.227]	[0.0243,0.1893]	[0.0274,0.1654]

which estimates the probability that an interval number is larger than another interval number. A total of five 12×12 matrices were formed to determine this probability (Holsten et al.) for each α and β cut level. Table 8 shows the results of the ranking of DMUs based on their overall efficiency and their efficiency in the first and second stages.

In addition to differentiating efficient and inefficient DMUs, the proposed model can also rank the DMUs that are determined to be efficient. In this example, the ranking of units in the order of their overall efficiency was determined to be DMU6 > DMU8 > DMU3 > DMU5 > DMU10 > DMU2 > DMU1 > DMU7 > DMU4 > DMU11 > DMU9 > DMU12. The efficient units DMU3 and DMU6 were placed third and first in this ranking, respectively. The ranking obtained based on efficiency in each individual stage is also provided in the table above. Finally, the results of the proposed approach (IFBNDEA) were compared with the results of the traditional NDEA with crisp data. For this comparison, all trapezoidal intuitionistic fuzzy data had to be converted to crisp numbers. Thus, all of the intuitionistic fuzzy numbers given in Tables 3–5 were defuzzified using the formulas provided in Sections 2-6. After defuzzifying input data, the traditional NDEA was implemented using models 1, 2, and 3 for crisp data. The results and rankings obtained

Table 7: Classification of efficient units at different α-β cut levels.

	Stage 1	Stage 2	Overall efficiency
$\alpha=0,\ \beta=1$	1-3-6	2-3-4-5-6-10	3–6
$\alpha=0.25,\ \beta=0.75$	1-3-6-7	2-3-4-5-6-10	3–6
$\alpha=0.5,\ \beta=0.5$	1-3-6-7	2-3-4-6-10	3–6
$\alpha=0.75,\ \beta=0.25$	3-6-7-8	3-4-6	3–6
$\alpha=1,\ \beta=0$	1-3-6-7-8	3-4-6	3–6

Table 8: Ranking of DMUs based on overall interval efficiency and interval efficiency of stages 1 and 2.

DMU	Stage 1		Stage 2		Overall efficiency	
	$\sum p_i$	Rank	$\sum p_i$	Rank	$\sum p_i$	Rank
1	39.14258082	3	16.15182302	11	28.96102	7
2	27.17203834	7	31.75176534	6	30.83417	6
3	36.85854836	5	35.422537	3	39.59082	3
4	17.85001624	11	41.26247523	2	24.50996	9
5	32.63482202	6	29.64642532	8	33.73091	4
6	40.42232604	1	42.27404503	1	41.31146	1
7	39.52082274	2	15.19922567	12	28.20581	8
8	37.09956699	4	34.05678769	4	40.62764	2
9	19.42494098	10	29.53995879	9	21.13303	11
10	25.89864635	9	32.38234671	5	31.33729	5
11	26.43903311	8	22.57220575	10	22.29064	10
12	17.536658	12	29.74040445	7	17.46726	12

Table 9: Comparison of the ranking of DMUs with the proposed model and Liang et al. [20] model.

DMU	Liang et al. [20]		Proposed model		Liang et al. [20]		Proposed model		Liang et al. [20]		Proposed model	
	Stage 1	Rank	Pi (stage 1)	Rank	Stage 2	Rank	Pi (stage 2)	Rank	Overall	Rank	Pi (overall)	Rank
1	1	1	39.143	3	0.482	8	16.1518	11	0.482	5	28.961	7
2	0.553	4	27.172	7	0.8902	4	31.7517	6	0.492	4	30.834	6
3	1	1	36.859	5	1	1	35.4225	3	1	1	39.591	3
4	0.346	6	17.850	11	1	1	41.2625	2	0.346	8	24.509	9
5	0.649	3	32.635	6	0.917	3	29.6464	8	0.5953	3	33.731	4
6	1	1	40.422	1	1	1	42.274	1	1	1	41.311	1
7	1	1	39.521	2	0.376	10	15.199	12	0.376	7	28.206	8
8	0.777	2	37.099	4	0.834	5	34.057	4	0.648	2	40.628	2
9	0.331	7	19.425	10	0.714	6	29.539	9	0.236	9	21.133	11
10	0.494	4	25.899	9	0.952	2	32.382	5	0.471	6	31.337	5
11	0.406	5	26.439	8	0.472	9	22.5722	10	0.197	10	22.291	10
12	0.249	8	17.537	12	0.489	7	29.7404	7	0.122	11	17.467	12

from the proposed method and from the NDEA with crisp data are compared in Table 9.

These results clearly demonstrate the higher discriminatory power of IFBNDEA in the ranking of DMUs in terms of their efficiency. In other words, the proposed method can fully rank all DMUs based on their overall efficiency and efficiency in the first and second stages, whereas the classical NDEA cannot rank efficient DMUs and identifies multiple units as equally efficient. This is while the proposed approach is able to differentiate different levels of efficiency in different units.

5. Discussion and Conclusion

The assessment of the efficiency of DMUs is a complex but important decision-making issue that involves multiple quantitative and qualitative selection criteria. The classic DEA models were designed to work with deterministic data and cannot deal with uncertainties in their inputs.

Over the years, many studies in the field of Data Envelopment Analysis have proposed using fuzzy methods to deal with imprecise and ambiguous data. This is crucial for analyzing real-world problems where data are vague verbal or subjective and cannot be accurately measured. A highly effective way to model this type of data is to use intuitionistic fuzzy numbers. In addition to degrees of membership, intuitionistic fuzzy sets also consider a degree of non-membership and a degree of hesitation for entries, which results in reaching a more accurate decision matrix, thus more accurate and reliable evaluations, and consequently more efficient and effective decision-making. By expanding on the notion of fuzzy sets, the intuitionistic fuzzy set theory provides a more effective and efficient tool for dealing with uncertainty in a wide variety of situations. Despite the high applicability of the noncooperative or leader-follower game theory in real-world problems (e.g., traditional manufacturer-retailer supply chains), a review of the literature revealed that this theory has never been applied to the two-stage DEA in an intuitionistic fuzzy environment. Therefore, in this study, the intuitionistic fuzzy set theory was used to develop the two-stage DEA model of Liang et al. [20] based on Stackelberg's game theory and efficiency decomposition using α and β cuts. Considering that applying α and β cuts convert the data to interval numbers, the article presented a new approach based on the bounded DEA model of Azizi et al. [50] for working with interval data, which involves combining the optimistic and pessimistic relative efficiencies and turning them into upper and lower efficiency bounds and then obtaining an overall interval efficiency from these bounds. With this overall interval efficiency, it is possible to estimate the imprecise efficiency of DMUs and rank them in terms of efficiency accordingly. The overall efficiency interval obtained by the proposed approach uses fixed and unified production possibility frontiers (boundaries of efficiency and inefficiency) as a measure of efficiency for all DMUs. The overall interval efficiency provides a more detailed and realistic description of the condition of DMUs. Interval efficiency can better characterize the overall performance of a DMU than the traditional DEA efficiency. Furthermore, because of the development of the model in the intuitionistic fuzzy environment, the proposed DEA model is more effective in real-world applications in the sense that it also accounts for hesitations in decisions. The article also provided a numerical example to demonstrate the simplicity and applicability of the proposed method in measuring the efficiency of DMUs. By comparing the results of the proposed method (IFBNDEA) with the traditional NDEA results with clear data, the results clearly show the higher discriminatory power of IFBNDEA in ranking DMUs in terms of their efficiency. In other words, the proposed method can completely rank all DMUs based on their performance and overall efficiency in the first and second stages, while the classic NDEA cannot rank efficient DMUs and assign multiple units to recognize the same efficiency title. However, the proposed approach is able to differentiate different levels of performance in different units. Given the intuitionistic fuzzy nature of data in the proposed model, it can be expected to provide more accurate results than crisp models. The proposed two-stage model compared to that of Liang et al. (2008) has more power and application in the real world due to all intuitionistic fuzzy data and due to the intuitionistic fuzzy environment and two stages of the model, and the overall structure and each of the stages are separable and rankable, which has made the model a unique advantage over other models. Also, in our proposed model, optimistic and pessimistic efficiencies under the assumption of constant returns to scale and ranking of decision-making units for the noncooperative model of DEA in an intuitionistic fuzzy environment are expressed.

For the future research directions, the present work assumed that all outputs of the first stage enter the second stage as input; a similar model can be developed for cases where only part of these outputs is given to the second stage as input. In other words, a similar model can be developed for a more general two-stage network structure where each stage has its own inputs and outputs. Also, since the current models are based on the assumption of a fixed returns-to-scale (CRS), future studies might be able to modify these models for analyzing the efficiency of network structures under the variable returns-to-scale (VRS) assumption. Furthermore, the proposed approach and ranking method can be expanded for other network structures, including parallel, hybrid, and dynamic structures. In this research, a noncooperative model has been developed. A cooperative model can be developed in an intuitionistic fuzzy environment as a suggestion for future research. A real case study can also be used to determine the limitations of the proposed model. Moreover, the uncertain network DEA models for dealing with uncertainty could be proposed based on other approaches to uncertain programming in literature, such as robust optimization.

However, how to measure the interval efficiencies of a group of DMUs is yet a problem that has not been well resolved and needs further investigation. One of the limitations of the proposed model is when we are faced with crisp data, in which case the numbers can be intuitionistic fuzzy with existing methods. It is expected that the improved DEA models could find more applications in performance assessment in the near future.

Data Availability

Data to support the findings of this study are available on reasonable request from the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] J. E. Holsten, J. A. Deatrick, S. Kumanyika, J. Pinto-Martin, and C. W. Compher, "Children's food choice process in the home environment. a qualitative descriptive study," *Appetite*, vol. 58, no. 1, pp. 64–73, 2012.

[2] A. Charnes, W. W. Cooper, and E. Rhodes, "Measuring the efficiency of decision making units," *European Journal of Operational Research*, vol. 2, no. 6, pp. 429–444, 1978.

[3] S. Lozano and S. Khezri, "Network DEA smallest improvement approach," *Omega*, vol. 98, Article ID 102140, 2021.

[4] N. Javaherian, A. Hamzehee, and H. Sayyadi Tooranloo, "Designing an intuitionistic fuzzy network data envelopment analysis model for efficiency evaluation of decision-making units with two-stage structures," *Advances in Fuzzy Systems*, vol. 2021, Article ID 8860634, 15 pages, 2021.

[5] F. Hosseinzadeh Lotfi, G. R. Jahanshahloo, A. Ebrahimnejad, M. Soltanifar, and S. M. Mansourzadeh, "Target setting in the general combined-oriented CCR model using an interactive MOLP method," *Journal of Computational and Applied Mathematics*, vol. 234, no. 1, pp. 1–9, 2010.

[6] F. H. Lotfi, G. R. Jahanshahloo, M. Soltanifar, A. Ebrahimnejad, and S. M. Mansourzadeh, "Relationship

between MOLP and DEA based on output-orientated CCR dual model," *Expert Systems with Applications*, vol. 37, no. 6, pp. 4331–4336, 2010.

[7] R. Maddahi, G. R. Jahanshahloo, F. H. Lotfi, and A. Ebrahimnejad, "Optimising proportional weights as a secondary goal in DEA cross-efficiency evaluation," *International Journal of Operational Research*, vol. 19, no. 2, pp. 234–245, 2014.

[8] A. Ebrahimnejad, M. Tavana, and F. J. Santos-Arteaga, "An integrated data envelopment analysis and simulation method for group consensus ranking," *Mathematics and Computers in Simulation*, vol. 119, pp. 1–17, 2016.

[9] M. Tavana, A. Ebrahimnejad, F. J. Santos-Arteaga, S. M. Mansourzadeh, and R. K. Matin, "A hybrid DEA-MOLP model for public school assessment and closure decision in the city of Philadelphia," *Socio-Economic Planning Sciences*, vol. 61, pp. 70–89, 2018.

[10] A. Ebrahimnejad and M. Tavana, "An interactive MOLP method for identifying target units in output-oriented DEA models: the NATO enlargement problem," *Measurement*, vol. 52, pp. 124–134, 2014.

[11] A. Ebrahimnejad and F. Hosseinzadeh Lotfi, "Equivalence relationship between the general combined-oriented CCR model and the weighted minimax MOLP formulation," *Journal of King Saud University Science*, vol. 24, no. 1, pp. 47–54, 2012.

[12] Y. Chen, W. D. Cook, and J. Zhu, "Deriving the DEA Frontier for two-stage processes," *European Journal of Operational Research*, vol. 202, no. 1, pp. 138–142, 2010.

[13] P. Peykani, E. Mohammadi, and A. Emrouznejad, "An adjustable fuzzy chance-constrained network DEA approach with application to ranking investment firms," *Expert Systems with Applications*, vol. 166, Article ID 113938, 2021.

[14] W. Zhu, Z. Miao, and X. Pu, "Serial network DEA models with a single intermediate measure," *Journal of the Operational Research Society*, vol. 72, no. 5, pp. 1168–1173, 2021.

[15] C. Kao, "Network data envelopment analysis: a review," *European Journal of Operational Research*, vol. 239, no. 1, pp. 1–16, 2014.

[16] Z. Zhou, W. Sun, H. Xiao, Q. Jin, and W. Liu, "Stochastic leader-follower DEA models for two-stage systems," *Journal of Management Science and Engineering*, vol. 6, no. 4, pp. 413–434, 2021.

[17] L. M. Seiford and J. Zhu, "Profitability and marketability of the top 55 U.S. Commercial banks," *Management Science*, vol. 45, no. 9, pp. 1270–1288, 1999.

[18] W. Liu, Z. Zhou, C. Ma, D. Liu, and W. Shen, "Two-stage DEA models with undesirable input-intermediate-outputs," *Omega*, vol. 56, pp. 74–87, 2015.

[19] C. Kao and S.-N. Hwang, "Efficiency decomposition in two-stage data envelopment analysis: an application to non-life insurance companies in Taiwan," *European Journal of Operational Research*, vol. 185, no. 1, pp. 418–429, 2008.

[20] L. Liang, W. D. Cook, and J. Zhu, "DEA models for two-stage processes: game approach and efficiency decomposition," *Naval Research Logistics*, vol. 55, no. 7, pp. 643–653, 2008.

[21] H.-C. Huang and C.-F. Hu, "Performance measurement for the recycling production system using cooperative game network data envelopment analysis," *Sustainability*, vol. 13, no. 19, Article ID 11060, 2021.

[22] Y. Wen, Q. An, J. Hu, and X. Chen, "DEA game for internal cooperation between an upper-level process and multiple lower-level processes," *Journal of the Operational Research Society*, vol. 72, pp. 1–12, 2021.

[23] W. D. Cook, L. Liang, and J. Zhu, "Measuring performance of two-stage network structures by DEA: a review and future perspective," *Omega*, vol. 38, no. 6, pp. 423–430, 2010.

[24] R. Färe and S. Grosskopf, "Productivity and intermediate products: a Frontier approach," *Economics Letters*, vol. 50, no. 1, pp. 65–70, 1996.

[25] H. Li, C. Chen, W. D. Cook, J. Zhang, and J. Zhu, "Two-stage network DEA: who is the leader?" *Omega*, vol. 74, pp. 15–19, 2018.

[26] D. K. Despotis, D. Sotiros, and G. Koronakos, "A network DEA approach for series multi-stage processes," *Omega*, vol. 61, pp. 35–48, 2016.

[27] X. Zhou, R. Luo, Y. Tu, B. Lev, and W. Pedrycz, "Data envelopment analysis for bi-level systems with multiple followers," *Omega*, vol. 77, pp. 180–188, 2018.

[28] C. Kao and S.-T. Liu, "Efficiencies of two-stage systems with fuzzy data," *Fuzzy Sets and Systems*, vol. 176, no. 1, pp. 20–35, 2011.

[29] Q. An, P. Wang, H. Yang, and Z. Wang, "Fixed cost allocation in two-stage system using DEA from a noncooperative view," *Spectrum*, vol. 43, no. 4, pp. 1077–1102, 2021.

[30] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[31] E. Bernal-Jurado, A. Mozas-Moral, D. Fernández-Uclés, and M. J. Medina-Viruel, "Online popularity as a development factor for cooperatives in the winegrowing sector," *Journal of Business Research*, vol. 123, pp. 79–85, 2021.

[32] C. Dai, X. S. Qin, and W. T. Lu, "A fuzzy fractional programming model for optimizing water footprint of crop planting and trading in the Hai River Basin, China," *Journal of Cleaner Production*, vol. 278, Article ID 123196, 2021.

[33] R. Krishankumar, S. S. Nimmagadda, P. Rani, A. R. Mishra, K. S. Ravichandran, and A. H. Gandomi, "Solving renewable energy source selection problems using a q-rung orthopair fuzzy-based integrated decision-making approach," *Journal of Cleaner Production*, vol. 279, Article ID 123329, 2021.

[34] R. Pelissari, M. C. Oliveira, A. J. Abackerli, S. Ben-Amor, and M. R. P. Assumpção, "Techniques to model uncertain input data of multi-criteria decision-making problems: a literature review," *International Transactions in Operational Research*, vol. 28, no. 2, pp. 523–559, 2021.

[35] L. Xu, Z. Wang, Y. Liu, and L. Xing, "Energy allocation strategy based on fuzzy control considering optimal decision boundaries of standalone hybrid energy systems," *Journal of Cleaner Production*, vol. 279, Article ID 123810, 2021.

[36] A. Hatami-Marbini, A. Ebrahimnejad, and S. Lozano, "Fuzzy efficiency measures in data envelopment analysis using lexicographic multiobjective approach," *Computers & Industrial Engineering*, vol. 105, pp. 362–376, 2017.

[37] S. H. Nasseri, A. Ebrahimnejad, and O. Gholami, "Fuzzy stochastic data envelopment analysis with undesirable outputs and its application to banking industry," *International Journal of Fuzzy Systems*, vol. 20, no. 2, pp. 534–548, 2018.

[38] M. Kachouei, A. Ebrahimnejad, and H. Bagherzadeh-Valami, "A common-weights approach for efficiency evaluation in fuzzy data envelopment analysis with undesirable outputs: application in banking industry," *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 5, pp. 7705–7722, 2020.

[39] A. Ebrahimnejad and N. Amani, "Fuzzy data envelopment analysis in the presence of undesirable outputs with ideal points," *Complex & Intelligent Systems*, vol. 7, no. 1, pp. 379–400, 2021.

[40] H.-J. Zimmermann, "Fuzzy programming and linear programming with several objective functions," *Fuzzy Sets and Systems*, vol. 1, no. 1, pp. 45–55, 1978.

[41] K. T. Atanassov, "More on intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 37–45, 1989.

[42] A. Arya and S. P. Yadav, "Development of intuitionistic fuzzy super-efficiency slack based measure with an application to health sector," *Computers & Industrial Engineering*, vol. 115, pp. 368–380, 2018.

[43] I. Zhelyazkov, E. Benova, and V. Atanassov, "Axial structure of a plasma column produced by a large-amplitude electromagnetic surface wave," *Journal of Applied Physics*, vol. 59, no. 5, pp. 1466–1472, 1986.

[44] Z. Ameri, S. S. Sana, and R. Sheikh, "Self-assessment of parallel network systems with intuitionistic fuzzy data: a case study," *Soft Computing*, vol. 23, no. 2, pp. 12821–12832, 2019.

[45] B. Shakouri, R. Abbasi Shureshjani, B. Daneshian, and F. Hosseinzadeh Lotfi, "A parametric method for ranking intuitionistic fuzzy numbers and its application to solve intuitionistic fuzzy network data envelopment analysis models," *Complexity*, vol. 2020, Article ID 6408613, 25 pages, 2020.

[46] N. Javaherian, A. Hamzehee, and H. S. Tooranloo, "A compositional approach to two-stage data envelopment analysis in intuitionistic fuzzy environment," *Operations Research and Decisions*, vol. 31, no. 1, pp. 21–39, 2021.

[47] Y. Chen, W. D. Cook, N. Li, and J. Zhu, "Additive efficiency decomposition in two-stage DEA," *European Journal of Operational Research*, vol. 196, no. 3, pp. 1170–1176, 2009.

[48] S. K. Roy, A. Ebrahimnejad, J. L. Verdegay, and S. Das, "New approach for solving intuitionistic fuzzy multi-objective transportation problem," *Sādhanā*, vol. 43, no. 1, pp. 1–12, 2018.

[49] A. Ebrahimnejad and J. L. Verdegay, "An efficient computational approach for solving type-2 intuitionistic fuzzy numbers based transportation problems," *International Journal of Computational Intelligence Systems*, vol. 9, no. 6, pp. 1154–1173, 2016.

[50] H. Azizi, "A note on "a decision model for ranking suppliers in the presence of cardinal and ordinal data, weight restrictions, and nondiscriminatory factors"," *Annals of Operations Research*, vol. 211, no. 1, pp. 49–54, 2013.

[51] H. Azizi, "DEA efficiency analysis: a DEA approach with double frontiers," *International Journal of Systems Science*, vol. 45, no. 11, pp. 2289–2300, 2014.

[52] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Springer, Berlin, Germany, pp. 1–137, 1999.

[53] G. Mahapatra and T. Roy, "Reliability evaluation using triangular intuitionistic fuzzy numbers arithmetic operations," *World Academy of Science, Engineering and Technology*, vol. 50, pp. 574–581, 2009.

[54] A. Nagoorgani and K. Ponnalagu, "A new approach on solving intuitionistic fuzzy linear programming problem," *Applied Mathematical Sciences*, vol. 6, no. 70, pp. 3467–3474, 2012.

[55] H. Ercan-Teksen and A. S. Anagün, "Intuitionistic fuzzy c-control charts using defuzzification and likelihood methods," *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 5, pp. 6465–6473, 2020.

[56] G. E. Halkos, N. G. Tzeremes, and S. A. Kourtzidis, "A unified classification of two-stage DEA models," *Surveys in Operations Research and Management Science*, vol. 19, no. 1, pp. 1–16, 2014.

[57] Y.-M. Wang, R. Greatbanks, and J.-B. Yang, "Interval efficiency assessment using data envelopment analysis," *Fuzzy Sets and Systems*, vol. 153, no. 3, pp. 347–370, 2005.

[58] H. Azizi and H. G. Ajirlu, "Measurement of the worst practice of decision-making units in the presence of non-discretionary factors and imprecise data," *Applied Mathematical Modelling*, vol. 35, no. 9, pp. 4149–4156, 2011.

[59] H. Azizi and Y.-M. Wang, "Improved DEA models for measuring interval efficiencies of decision-making units," *Measurement*, vol. 46, no. 3, pp. 1325–1332, 2013.

[60] Z. Yue, "An extended TOPSIS for determining weights of decision makers with interval numbers," *Knowledge-Based Systems*, vol. 24, no. 1, pp. 146–153, 2011.

[61] A. Hatami-Marbini, M. Tavana, K. Gholami, and Z. G. Beigi, "A bounded data envelopment analysis model in a fuzzy environment with an application to safety in the semiconductor industry," *Journal of Optimization Theory and Applications*, vol. 164, no. 2, pp. 679–701, 2015.