

## Research Article

# Optimal HARA Investments with Terminal VaR Constraints

Marcos Escobar-Anel <sup>1</sup>, Maximilian Keller,<sup>2</sup> and Rudi Zagst<sup>1</sup>

<sup>1</sup>Department of Statistical and Actuarial Sciences, University of Western Ontario, London, ON, Canada

<sup>2</sup>Technical University of Munich, Department of Mathematics, Munich, Germany

Correspondence should be addressed to Marcos Escobar-Anel; [marcos.escobar@uwo.ca](mailto:marcos.escobar@uwo.ca)

Received 15 November 2021; Accepted 11 March 2022; Published 9 May 2022

Academic Editor: Egidio D'Amato

Copyright © 2022 Marcos Escobar-Anel et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper studies the impact of Value at Risk (VaR) constraints on investors with hyperbolic absolute risk aversion (HARA) risk preferences. We derive closed-form representations for the “triplet”: optimal investment, terminal wealth, and value function, via extending the Bellman-based methodology from constant relative risk aversion (CRRA) utilities to HARA utilities. In the numerical part, we compare our solution (HARA-VaR) to three critical embedded cases, namely, CRRA, CRRA-VaR, and HARA, assessing the influence of key parameters like the VaR probability and floor on the optimal wealth distribution and allocations. The comparison highlights a stronger impact of VaR on a CRRA-VaR investor compared to a HARA-VaR (HV). This is in terms of not only lower Sharpe ratios but also higher tail risk and lower returns on wealth. The HV analysis demonstrates that combining both, capital guarantee and VaR, may lead to a correction of the partially adverse effects of the VaR constraint on the risk appetite. Moreover, the HV portfolio strategy also does not show the high kurtosis observed for the PV strategy. A wealth-equivalent loss (WEL) analysis is also implemented demonstrating that, for a HV investor, losses would be more serious if adopting a CRRA-VaR strategy as compared to a HARA strategy.

## 1. Introduction

More than fifty years ago, [1] derived the famous Merton approach to unconstrained portfolio optimization by using the Bellman principle in the context of a Geometric Brownian motion (GBM) and a hyperbolic absolute risk aversion (HARA) utility. By setting up the Hamilton-Jacobi Bellman (HJB) equation, Merton derived the investment strategy given the investor's risk preferences in closed-form. However, the global evolution of financial markets and the professionalisation of the asset management industry within the last decades have brought up a multitude of additional requirements to the process of portfolio optimization.

Evidence of this is the increasingly quantitative regulatory environment for financial institutions, which are usually enforced via risk measures. For instance, as early as 1998, the Basel I Capital Accord Market Risk Amendment was implemented by the U.S. financial supervision authorities and required commercial banks to measure their market risk using the Value at Risk (VaR) risk measure (e.g., [2]). Other examples are the European insurance sector regulatory system Solvency II (see [3]) and banking regulation Basel III (e.g.,

[4]). Both require market risk measurements by VaR, whereas the Swiss banking regulation requires the use of Expected Shortfall (ES) as risk measure (e.g., [5]).

In parallel, the emergence of new financial products, guaranteeing the investor a minimum level of wealth during or at the end of the investment period, has called for academic contributions capable of reflecting the financial engineering of these product features. Dynamic investment strategies with protection and guarantee features such as Constant Proportion Portfolio Insurance (CPPI), Dynamic Proportion Portfolio Insurance (DPPI), Option-Based Portfolio Insurance (OBPI) (see [6]), and others have been proposed, as they capture the guarantee element of such product innovations.

Our research work refines the well-known CPPI investment strategy introduced by [7, 8] by incorporating risk measure (RM) constraints such as VaR. These multiconstraints, i.e., a risk measure constraint combined with a minimum capital guarantee, are implemented in a continuous Black-Scholes (BS) financial market using stochastic control methods. We extend the work of [9] to HARA utilities obtaining closed-form representations for the optimal investment strategy and wealth. In addition, we perform a

thorough comparative statistical, suboptimality, and empirical analysis of the derived investment strategies to compare the different strategies with each other.

Whereas the CPPI approach has been analyzed extensively with regard to dominance criteria (e.g., [6, 10] examine stochastic dominance criteria of CPPI vs. OBPI strategies) and risk measures [11] and against other portfolio insurance strategies [12, 13], the literature has largely ignored the joint constraints due to risk measures and portfolio insurance. The exception is the work of [14], which investigates the potential effect of a VaR constraint in the context of an OBPI setting, i.e., VaR constraint in combination with a minimum capital requirement (MCR) at maturity. They provide evidence that a combination of VaR and portfolio insurance constraint limits the size of losses which the standalone VaR is by nature “blind” against. In this case, the floor constraint  $F_T$  represents a comprehensive and not costly insurance against losses, excluding gambling strategies that would occur in the standalone VaR case (see the VaR gambling behavior suggested by [15]). In addition, [14] suggests that the criticism of limited upside potential of a portfolio insurance strategy resulting from a reduced exposure regarding risky assets compared to the Merton solution is less severe: in fact, the proportion invested in the risky asset is larger than under a standalone MCR regulation. They conclude that an investment under the combined constraints can be seen as “the best from both worlds,” hence complementing both standalone constraints.

Although [14] investigates the combination of the VaR and general portfolio protection criteria, there is yet no examination of the incorporation of risk measures such as VaR into the heavily used CPPI framework. A CPPI limits the portfolio value from below due to the dynamic capital guarantee with the value (floor)  $e^{-r(T-t)}F_T$  throughout the investment horizon  $[0, T]$  but may lead to a used-up cushion in volatile markets and hence has less room for risky investments thereafter. Also, in bullish markets, the capital guarantee translates to a less risky behavior, which means that the investor must give up a certain share of performance. The idea behind the implementation of VaR constraints for a HARA investor is to limit the frequency of the portfolio value to fall short of a certain portfolio value  $K$  (soft-floor) greater than the hard-floor  $F_T$ . There are multiple reasons for this: The HARA investor might want to limit his risk to deplete his cushion  $C_t$ , i.e., the distance of the portfolio value to the floor, too fast in bad market cycles. Also, the investors might not like their wealth falling below the “soft-floor”  $K$  too frequently. HARA investors are typically long-term oriented pension investors that undergo bullish as well as bearish cycles. In the latter HARA investors may become pure cash account investors for the rest of their (long-term) investment period. As previous contributions suggest that a CPPI strategy underperforms especially in market cycles which are first bearish and later bullish, the VaR could improve this problem of CPPI strategies by limiting the probability of a diminished cushion. Additionally, an investor might not only be interested in limiting the absolute losses but also want to control the volatility of his wealth or VaR capital provisions due to regulatory requirements.

Summarizing, the contributions of this paper are as follows:

- (i) We use a combination of HJB and financial derivatives to find a mathematical representation for the optimal CPPI strategy and its value function under VaR constraints. We call it HARA-VaR solution. This allows us to investigate the impact of VaR on capital guarantee strategies popular among financial institutions.
- (ii) We examine the performance of the newly found HARA-VaR solution in a comparison to a plain HARA solution (CPPI strategy) as well as to CRRA-VaR solutions, highlighting the benefits and pitfalls of VaR on CPPI investors.
- (iii) Our analysis demonstrates that combining both, capital guarantee and VaR, leads to a correction of the adverse effects of VaR constraint on risk appetite. Moreover, the HARA-VaR portfolio shows lower kurtosis than the CRRA-VaR portfolio.
- (iv) A wealth-equivalent loss (WEL) analysis demonstrates that losses would be more serious for a HARA-VaR investor if taking a CRRA-VaR rather than a plain HARA strategy.

The paper is structured as follows: In Section 2, we describe and solve various problems of interest combining VaR and minimum guarantee constraints. Section 3 compares the commonly applied portfolio constraints with the unconstrained investment strategy as well as the individual minimum guarantee and VaR constraint. We examine the expected performance of all derived investment strategies from an ex-ante view and derive the sensitivities to major input parameters (i.e., financial market and portfolio constraint parameters) using the four (centralized) moments of the terminal return distributions as well as the Sharpe ratio (SR) as measures for the performance of the strategies. In Section 4, we investigate the adverse effects that trading strategy restrictions such as risk or minimum guarantee constraints have on a previously unconstrained investor. We derive the equivalent percentage of wealth that an investor would sacrifice in order to be able to follow the unconstrained (or less constrained) portfolio strategy compared to the otherwise constrained investment strategy and show the relationship between the Wealth-Equivalent Loss (WEL) and major input parameters.

## 2. Dynamic Investment Strategies under Combined VaR and Minimum Guarantee Constraints

This section summarizes the optimal dynamic investment, terminal wealth, and value function for the well-known cases CRRA (P) and CRRA-VaR (PV), both based on a Power utility and HARA (H), and also derives this optimal triplet for a HARA investor with an additional VaR constraint (HV). In contrast to the HARA case, the HV constraints on terminal wealth cannot be simply implemented into the utility function  $u(v)$  itself. Hence, we follow the approach of the VaR in [9] to derive the optimal triplet: value function  $\Phi_t^{HV}$ , investment strategy  $\pi_t^{HV}$ , and investor wealth  $V_t^{HV}$  for the HV case, all to be introduced next.

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  represent a complete filtered probability space with a standard  $\mathcal{F}_t$ -adapted one-dimensional Brownian motion  $W = (W(t))_{t \in [0, T]}$ .  $\mathcal{F}$  denotes the augmented filtration, where  $\mathcal{F}_t := \sigma(W(s): 0 \leq s \leq t) \forall t \in [0, T]$  is the filtration generated by  $W$ . Under these conditions, the equation for the bank account (risk-free asset)  $B_t$  for  $t \in [0, T]$  is given by

$$dB_t = rB_t dt, \quad B_t = b_0 > 0, \quad (1)$$

where  $r$  is the risk-free interest rate. Furthermore, the stock (risky asset) with the corresponding price process  $S_t$  evolves according to the following SDE:

$$dS_t = S_t (\mu dt + \sigma dW_t), \quad S_0 = s > 0, \quad (2)$$

where the drift rate is  $\mu > r$  and the volatility rate is  $\sigma > 0$ . Now, consider an investor with an initial investment amount  $v \in \mathbb{R}$ . Based on our financial market assumptions, we suppose that both securities  $B$  and  $S$  can be traded continuously and without transaction costs by any small investor. The investor can participate in the market by allocating his funds  $v$  towards the risky and riskless asset. The process  $\pi = (\pi_B, \pi_S) = (\pi_B(t), \pi_S(t))_{t \in [0, T]}$  with  $\pi(t) := (\pi_B(t), \pi_S(t))' \in \mathbb{R}^2$  is assumed to be a self-financing relative portfolio process. It describes the fraction of wealth invested in the single assets. Then, the fraction invested in the bank account is given by  $\pi_B(t) = 1 - \pi_S(t)$ . For  $t \in [0, T]$ , the wealth process  $V_t := V_t(\pi)$  with initial wealth  $V_0(\pi) = v$  evolves according to the following SDE:

$$dV_t = V_t [r + \pi_S(t)(\mu - r)dt + \pi_S(t)\sigma dW_t], \quad V_0 = v. \quad (3)$$

**2.1. CRRA, CRRA-VaR, and HARA.** The CRRA unconstrained maximization problem was examined by [1]; it is given by

$$\begin{aligned} \Phi^M(t, v) &= \sup_{\pi \in \Lambda'(t, v)} \mathbb{E}[u(V_T) | V_t = v] \\ &= \sup_{\pi} \mathbb{E}\left[\frac{1}{\gamma} V_T^\gamma | V_t = v\right], \end{aligned} \quad (4)$$

where  $\Lambda'(t, v)$  is the set of admissible strategies and  $\gamma \leq 1$  and  $\gamma \neq 0$  capture the level of risk aversion of the investor. The solution is given next.

**Proposition 1.** *The triplet optimal value function  $\Phi_t^M$ , investment strategy  $\pi_t^M$ , and investors wealth  $V_t^M$  are given by*

$$\begin{aligned} \Phi^M(t, v) &= \frac{1}{\gamma} v^\gamma \cdot g(t), \quad g(t) = e^{\gamma((\theta^2/2(1-\gamma))+r)(T-t)}, \\ \pi_t^M &= \frac{\mu - r}{\sigma^2(1-\gamma)} = \frac{\theta}{\sigma(1-\gamma)}, \quad \theta = \frac{\mu - r}{\sigma}, \\ V_t^M &= V_0^M \exp\left[\left(r + \frac{\theta^2}{1-\gamma} - \frac{1}{2} \frac{\theta^2}{(1-\gamma)^2}\right)t + \frac{\theta}{1-\gamma} W_t\right]. \end{aligned} \quad (5)$$

A HARA utility function can be interpreted in the way that the investor requires his terminal wealth at the end of the investment horizon  $T$  to exceed a certain threshold  $F_T$  with certainty; i.e., the investor must serve financial obligations at the end of the investment period and thus measures his satisfaction in terms of the surplus of  $C_T = V_T - F_T$ , also called the cushion. The optimization problem for a HARA investor was investigated in [16] and states

$$\begin{aligned} \Phi^H(t, v) &= \sup_{\pi \in \Lambda'(t, v)} \mathbb{E}[u(V_T)] \\ &= \sup_{\pi} \mathbb{E}\left[\frac{(V_T - F_T)^\gamma}{\gamma}\right]. \end{aligned} \quad (6)$$

**Proposition 2.** *The optimal value function  $\Phi_t^H$ , investment strategy  $\pi_t^H$ , and investors wealth  $V_t^H$  for the (unconstrained) HARA investor are given by*

$$\begin{aligned} \Phi^H(t, v) &= \frac{1}{\gamma} (v - F_t)^\gamma \cdot g(t), \\ \pi_t^H(t, v) &= \pi_t^M \frac{v - F_t}{v}, \\ V_t^H &= V_0^M \exp\left[\left(r + \frac{\theta^2}{1-\gamma} - \frac{1}{2} \frac{\theta^2}{(1-\gamma)^2}\right)t + \frac{\theta}{1-\gamma} W_t\right] + F_t \\ &= V_t^M + F_t, \end{aligned} \quad (7)$$

with  $F_t = F_T e^{-r(T-t)}$ ,  $\theta$ ,  $g$ , and  $\pi_t^M$  as in proposition (1), and  $V_t^M$  denoting the optimal Merton wealth given by Proposition 1 and determined via the relation  $V_0^M = V_0^H - F_0$ .

Lastly, we present the CRRA-VaR utility portfolio optimization problem:

TABLE 1: Sensitivities for comparative analysis.

Sensitivity	Description	Minimum	Maximum	Step size	Settings j
$\epsilon_{\text{VaR}, T, \gamma}$	VaR probability	0.5%	50%	1.2%	42
$\mu_{1y}$	BS risky asset drift	4%	20%	1.4%	12
$\sigma_{1y}$	BS risky asset volatility	10%	60%	5%	11
$T$	Investment horizon	1Y	25Y	1Y	13
$K$	VaR barrier	10% of $v_0$	90% of $v_0$	5% of $v_0$	18
$F_T$	Minimum guarantee at T	25% of $v_0$	75% of $v_0$	3.2% of $v_0$	16

$$\begin{aligned}\Phi^{\text{PV}}(t, v) &= \sup_{\pi \in \Lambda'(t, v)} \mathbb{E}[u(V_T) | V_t = v] \\ &= \sup_{\pi} \mathbb{E} \left[ \frac{1}{\gamma} V_T^\gamma | V_t = v \right]\end{aligned}$$

Subject to  $\{\mathbb{P}(V_T < K) \leq \epsilon\}$ .

(8)

First, let us denote by  $P(t, v^M, K, r, \sigma)$  the price of an European Put Option with interest rate  $r$ , volatility  $\sigma$ , and strike  $K$ , given that the price of its underlying is  $v^M$  at time  $t$ . Furthermore, let  $N(\cdot)$  denote the cumulative distribution function of the normal distribution, and

$$d_2^{\text{Q}}(t, v^M, k_\epsilon) = \frac{\log(v_t^M/k_\epsilon) + (r - (\theta^2/(2(1-\gamma)^2)))(T-t)}{(\theta/(1-\gamma))\sqrt{T-t}},$$

$$d_1^{\text{Q}}(\cdot) = d_2^{\text{Q}}(\cdot) + \frac{\theta}{1-\gamma} \sqrt{T-t},$$

(9)

$$d_2^{\text{P}}(t, v^M, k_\epsilon) = \frac{\log(v_t^M/k_\epsilon) + (r + (\theta^2/(1-\gamma)) - (\theta^2/2(1-\gamma^2)))(T-t)}{(\theta/(1-\gamma))\sqrt{T-t}},$$

$$d_1^{\text{P}}(\cdot) = d_2^{\text{P}}(\cdot) + \frac{\theta}{1-\gamma} \sqrt{T-t}.$$

The next proposition presents the solutions as described in [9], Proposition 2; this result follows from the main theorem in the aforementioned paper which we reproduce in (A.1) for completeness; see also [15] for an alternative representation.

**Proposition 3.** *The optimal value function  $\Phi_t^{\text{PV}}$ , investment strategy  $\pi_t^{\text{PV}}$ , and investor wealth  $V_t^{\text{PV}}$  are given by*

$$\begin{aligned}\Phi^{\text{PV}}(t, v^M) &= \frac{1}{\gamma} \tilde{r}^{(T-t)} \left( (v^M)^\gamma + P\left(t, (v^M)^\gamma, K^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) - P\left(t, (v^M)^\gamma, k_\epsilon^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) \right) - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma + \gamma\lambda_\epsilon) N(-d_2^{\text{P}}(t, v^M, k_\epsilon)), \\ \pi^{\text{PV}}(t, v^M) &= \pi_t^M \cdot \frac{v^M (1 + N(d_1^{\text{Q}}(t, v^M, K)) - N(d_1^{\text{Q}}(t, v^M, k_\epsilon))) + (K - k_\epsilon) e^{-r(T-t)} ((1-\gamma)/\theta) (N'(d_2^{\text{Q}}(t, v^M, k_\epsilon)) / (v^M \sqrt{T-t}))}{v^M + P(t, v^M, K, r, (\theta/(1-\gamma))) - P(t, v^M, k_\epsilon, r, (\theta/(1-\gamma))) - (K - k_\epsilon) e^{-r(T-t)} N(-d_2^{\text{Q}}(t, v^M, k_\epsilon))},\end{aligned}\quad (10)$$

$$V_t^{\text{PV}} = v^M + P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) - (K - k_\epsilon) e^{-r(T-t)} N(-d_2^{\text{Q}}(t, v^M, k_\epsilon)),$$

with  $\theta, \pi_t^M$  as in (3) and the investor's wealth at  $t = T$  given by

$$V_T^{\text{PV}} = V_T^M + (K - V_T^M) \mathbf{1}_{\{k_\epsilon < V_T^M < K\}}, \quad (11)$$

where the required unconstrained wealth  $v_t^M$  as well as the lower strike  $k_\epsilon$  and the penalty term  $\lambda_\epsilon$  throughout the whole investment horizon  $[0, T]$  are given as the solution to the equation system:

TABLE 2: Default parameters for the sensitivity analysis.

$\epsilon_{\text{VaR},1y}$	$\mu_{1y}$	$\sigma_{1y}$	$r_{1y}$	$\gamma$	$T$	$K$	$F_T$
5%	7%	20%	2%	0.45	1y	85% of $v_0$	75% of $v_0$

$$\begin{aligned}
& N\left(\frac{\ln(k_\epsilon) - (\ln v_t^M + (r + (\theta^2/(1-\gamma)) - (1/2)(\theta^2/(1-\gamma)^2))(T-t))}{(\theta/(1-\gamma))\sqrt{T-t}}\right) - \epsilon = 0, \\
& v_t^M + P\left(t, v_t^M, K, r, \frac{\theta}{1-\gamma}\right) - P\left(t, v_t^M, r, k_\epsilon, \frac{\theta}{1-\gamma}\right) - (K - k_\epsilon)e^{-r(T-t)}\Phi\left(\frac{\ln(k_\epsilon) - (\ln v_t^M + (r - (1/2)(\theta^2/(1-\gamma)^2))(T-t))}{(\theta/(1-\gamma))\sqrt{T-t}}\right) - v = 0, \\
& ((K - k_\epsilon)\gamma k_\epsilon^{\gamma-1} - (K^\gamma - k_\epsilon^\gamma))\frac{1}{\gamma} - \lambda_\epsilon = 0,
\end{aligned} \tag{12}$$

and  $\tilde{r} = \gamma(r + (1/2)(\theta^2/(1-\gamma)))$ ,  $\gamma \in (0, 1)$ .

**2.2. The HARA-VaR Solution.** Here we study the following Value at Risk HARA utility portfolio optimization problem:

$$\Phi^{\text{HV}}(t, v) = \sup_{\pi \in \Lambda'(t, v): \mathbb{P}(V_T < K) \leq \epsilon} \mathbb{E}[u(V_T)] = \sup_{s.t. \pi: \mathbb{P}(V_T < K) \leq \epsilon} \mathbb{E}\left[\frac{(V_T - F_T)^\gamma}{\gamma}\right]. \tag{13}$$

The main proposition in this work, yielding the solution to this optimization problem, is provided next. The solution is built along the lines of [9]; see Appendixes A.1. The VaR constraint is satisfied given the conjecture  $f^{\text{HV}}(V_T^M)$ . The investor wealth  $v_0$  is linked to the present value  $\Pi(0, v_0^M)$  of a contingent claim with payoff  $f^{\text{HV}}(V_T^M)$ . The process underlying this claim,  $V_t^M$ , is the solution to the unconstrained expected CRRA utility problem, and its price must be equal to the initial investor wealth  $v_0$ .

The intuition here is that we find a derivative with terminal payoff  $f(V_T^M)$  and price  $\Pi(t, v_t^M)$  at time  $t$  such that if a condition holds (see equation (A.3))  $\Pi(t, v_t^M)$  is the optimal wealth in the constraint optimization problem and  $\pi_t$  is the optimal investment strategy.

**Proposition 4.** *The optimal value function  $\Phi_t^{\text{HV}}$ , investment strategy  $\pi^{\text{HV}}$ , and investors wealth  $V_t^{\text{HV}}$  for the HV case are given by*

$$\begin{aligned}
\Phi^{\text{HV}}(t, v^M) &= \frac{e^{\tilde{r}(T-t)}}{\gamma} \left[ (v^M)^\gamma + P\left(t, (v^M)^\gamma, K^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) - P\left(t, (v^M)^\gamma, k_\epsilon^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) \right] - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) N(-d_2^{\text{P}}(t, v^M, k_\epsilon)) \\
&\quad - \lambda_\epsilon \left[ N(-d_2^{\text{P}}(t, v^M, k_\epsilon)) 1_{\{k_\epsilon < K - F_T\}} + N(-d_2^{\text{P}}(t, v^M, K - F_T)) 1_{\{k_\epsilon \geq K - F_T\}} \right], \\
\pi^{\text{HV}}(t, v^M) &= \pi_t^M \cdot \frac{v^M (1 + (\partial/\partial v^M)P(t, v^M, K) - (\partial/\partial v^M)P(t, v^M, k_\epsilon) - (K - k_\epsilon)e^{-r(T-t)}(\partial/\partial v^M)\mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon))}{v^M + F_t + P(t, v^M, K) - P(t, v^M, k_\epsilon) - (K - k_\epsilon)e^{-r(T-t)}\mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon)}, \\
V_t^{\text{HV}} &= v^M + F_t + P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) - (K - k_\epsilon)e^{-r(T-t)}N(-d_2^{\text{Q}}(t, v^M, k_\epsilon)),
\end{aligned} \tag{14}$$

with  $\theta$ ,  $\pi_t^M$  as in (3),  $F_t = F_T e^{-r(T-t)}$ , and where  $P(t, v^M, K)$  is short for  $P(\cdot, v^M, k_\epsilon)$ , and the investor's wealth at  $t = T$  is specifically given by

$$V_T^{\text{HV}} = V_T^M + F_T + (K - V_T^M) 1_{\{k_\epsilon < V_T^M < K\}} \tag{15}$$

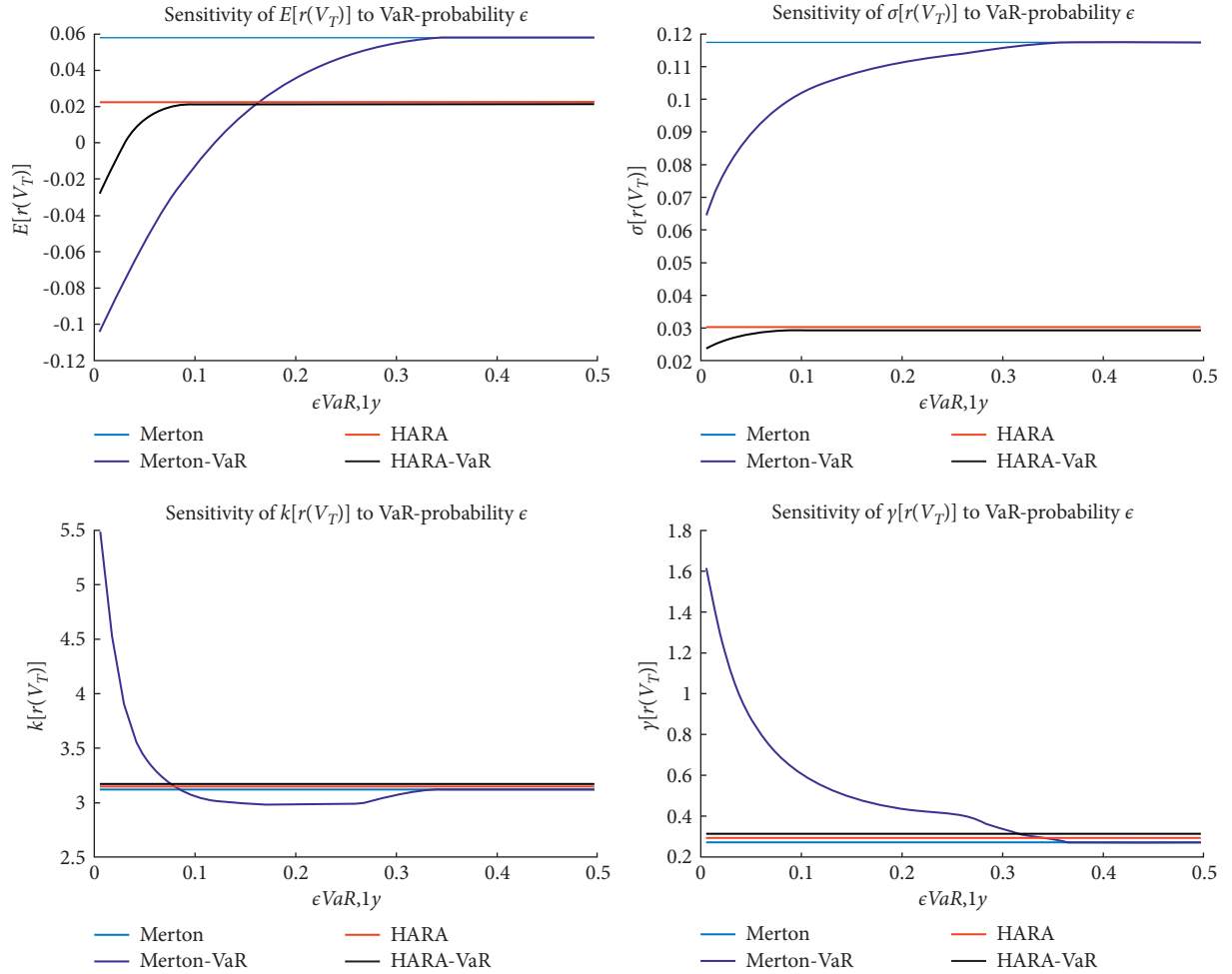


FIGURE 1: First four moments of sensitivity to  $\epsilon_{\text{VaR}}$  for Merton (Power), Merton-VaR (Power-VaR), HARA, and HARA-VaR.

and the required unconstrained wealth  $v_t^M$  as well as the lower strike  $k_\epsilon$  and the penalty term  $\lambda_\epsilon$  are given as the solution to the equation system:

$$\begin{aligned}
 & N\left(\frac{\ln(K - F_t) - (\ln v_t^M + (r + (\theta^2/(1 - \gamma)) - (1/2)(\theta^2/(1 - \gamma)^2))(T - t))}{(\theta/(1 - \gamma))\sqrt{T-t}}\right) 1_{\{k_\epsilon \geq K - F_t\}} \\
 & + N\left(\frac{\ln(k_\epsilon) - (\ln v_t^M + (r + (\theta^2/(1 - \gamma)) - (1/2)\theta^2/(1 - \gamma)^2)(T - t))}{\theta/(1 - \gamma)\sqrt{T-t}}\right) 1_{\{k_\epsilon < K - F_t\}} - \epsilon = 0,
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 & v_t^M + F_t + P\left(t, v_t^M, K, r, \frac{\theta}{1 - \gamma}\right) - P\left(t, v_t^M, k_\epsilon, r, \frac{\theta}{1 - \gamma}\right) \\
 & - (K - k_\epsilon)e^{-r(T-t)} N\left(\frac{\ln(k_\epsilon) - (\ln v_t^M + (r - (1/2)(\theta^2/(1 - \gamma)^2))(T - t))}{(\theta/(1 - \gamma))\sqrt{T-t}}\right) - v = 0,
 \end{aligned} \tag{17}$$

$$\frac{k_\epsilon^{\gamma-1}(K - k_\epsilon) - (1/\gamma)(K^\gamma - k_\epsilon^\gamma)}{((N'(d_2^Q(\cdot, v_0^M, K - F_t)))/(N'(d_2^Q(\cdot, v_0^M, k_\epsilon)))) 1_{\{k_\epsilon \geq K - F_t\}} + 1_{\{k_\epsilon < K - F_t\}}} - \lambda_\epsilon = 0. \tag{18}$$

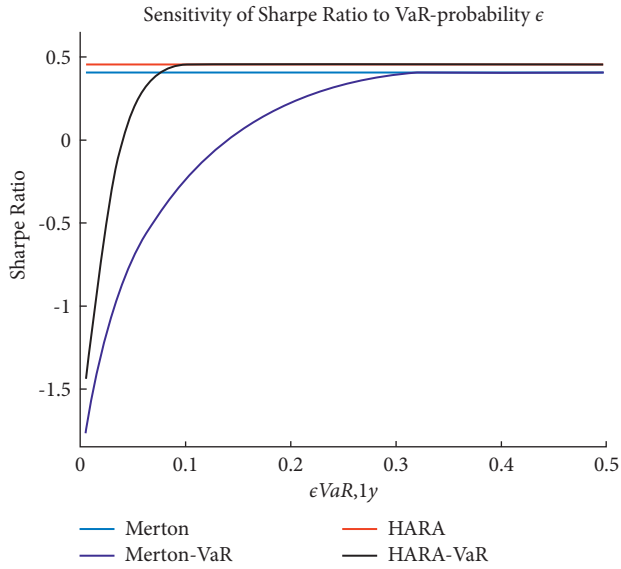


FIGURE 2: Sharpe ratio sensitivity to  $\epsilon_{VaR}$ .

For the proof, see the appendix.

### 3. Sensitivity Analysis of the Investment Strategies

So far, we have derived the closed-form solutions of the investment strategies Power unconstrained (P), Power-VaR (PV), HARA (H), and HARA-VaR (HV) by maximizing expected utility from terminal wealth. Analyzing the behavior and performance of the different strategies, we start by investigating the sensitivities of the solutions with respect to the main input parameters.

Table 1 provides an overview of the parameters, their value ranges, and their step size for which we compute sensitivities. We follow the notation for the sensitivity parameters in Table 1 but denote the financial market parameters  $(\mu_{1y}, \sigma_{1y}, r_{1y})$  explicitly as 1-year (annual) figures. Furthermore, let  $\epsilon_{VaR,Ty}$  denote the terminal (T-year) VaR

TABLE 3: Numerical results for variations in  $\epsilon_{VaR,1y}$ , HV case.

Risk measure	Merton wealth	Put strike	Protection ratio	Punishment
$\epsilon_{VaR,1y}$	$v_0^M$	$k_\epsilon$	$((v_0^M - k_\epsilon)/v_0^M)$	$\lambda_\epsilon^{HV}$
1%	41,510	6,162	85.16%	499
2%	41,867	8,349	80.05%	375
5%	42,009	10,898	74.06%	315
10%	43,340	15,519	64.42%	216
20%	51,981	23,820	54.18%	136
30%	61,672	37,612	39.01%	71.6
50%	70,447	70,598	0.00%	0.48

probability. As the investment strategies constitute solutions to expected terminal utility maximization, for every individual specification, we apply Monte Carlo (MC) simulations to simulate  $N_{MC} = 100,000$  paths and compute the triplets to allow for a sufficiently precise optimization. In particular, we simulate 42 parametric settings for different VaR probabilities to examine its impact on the optimal strategies. We interpolate between the data points (with one resulting data point for each setting and each strategy) by applying cubic spline fitting methods for illustrative purposes.

We measure the impact of three parameters of the investment strategies, i.e.,  $\epsilon_{VaR,1y}$  (VaR probability),  $K$  (soft-floor), and  $F$  (floor), on the first four (centralized) moments of the return of the terminal wealth (denoted  $r_j(V_T^*)$  for every setting  $j$ ). Specifically, we investigate the expected return  $\mathbb{E}[r_j(V_T^*)]$ , the standard deviation (volatility),  $\sigma[r_j(V_T^*)]$ , the skewness  $\gamma[r_j(V_T^*)]$ , and the kurtosis  $\kappa[r_j(V_T^*)]$ . We also compute the Sharpe ratio (SR) of the corresponding investment strategy, using the mean and standard deviation of the terminal return distribution, for every scenario  $j$ , denoted  $SR_j$ .

Further, let  $r_{i,j}(V_T^*)$  denote the total investment period return within  $[0, T]$  for simulation  $i$  and setting  $j$  and  $V_{T,i}^*$  the terminal wealth of strategy  $*$  and MC trajectory  $i$ . Then,  $SR_j$  is given by

$$SR_j = \frac{\mathbb{E}[r_j(V_T^*)] - e^{rT}}{\sigma[r_j(V_T^*)]} = \frac{(1/N_{MC}) \sum_{i=1}^{N_{MC}} [r_{i,j}(V_{T,i}^*)] - e^{rT}}{\sqrt{(1/(N-1)) \sum_{i=1}^{N_{MC}} (r_{i,j}(V_{T,i}^*) - (1/N) \sum_{i=1}^{N_{MC}} [r_{i,j}(V_{T,i}^*)])^2}} \quad (19)$$

The standard specifications for the analysis are provided in Table 2. We set the Black-Scholes market parameters  $(\mu_{1y} = 7\%, \sigma_{1y} = 20\%, r = 2\%)$  to represent long-term average values for a stock index such as the S&P 500. Note that the VaR barrier  $K$  and the capital guarantee  $F_T$  are expressed relative to the initial wealth  $v_0$ . We choose  $F_T = 75\%$  of initial wealth, as we consider such a hard level of capital guarantee as a reasonable proxy over a one-year period of time to protect against severe financial distress. Furthermore, choosing  $K = 85\%$  reflects the idea of the additional soft VaR barrier limiting the probability with which the wealth may hit  $K$  from above. The resulting combined

constraints ensure that the wealth cannot undergo 75% of initial wealth and undergoes 85% of initial wealth in less than  $\epsilon_{VaR,1y} = 5\%$  of times. We examine the investment strategies for  $T = 1$  to ensure easy interpretations of the results and avoid the need for annualizing the measures. Finally, we set the risk aversion  $\gamma = 0.45$  to assume moderately risk-averse investor preferences.

We start by investigating the effect of the VaR probability  $\epsilon_{VaR,1y}$ . For small VaR probabilities, the VaR and capital guaranteed investor HV naturally underperforms the standalone capital guaranteed HARA investor H, and PV underperforms P. For very low VaR probabilities, the



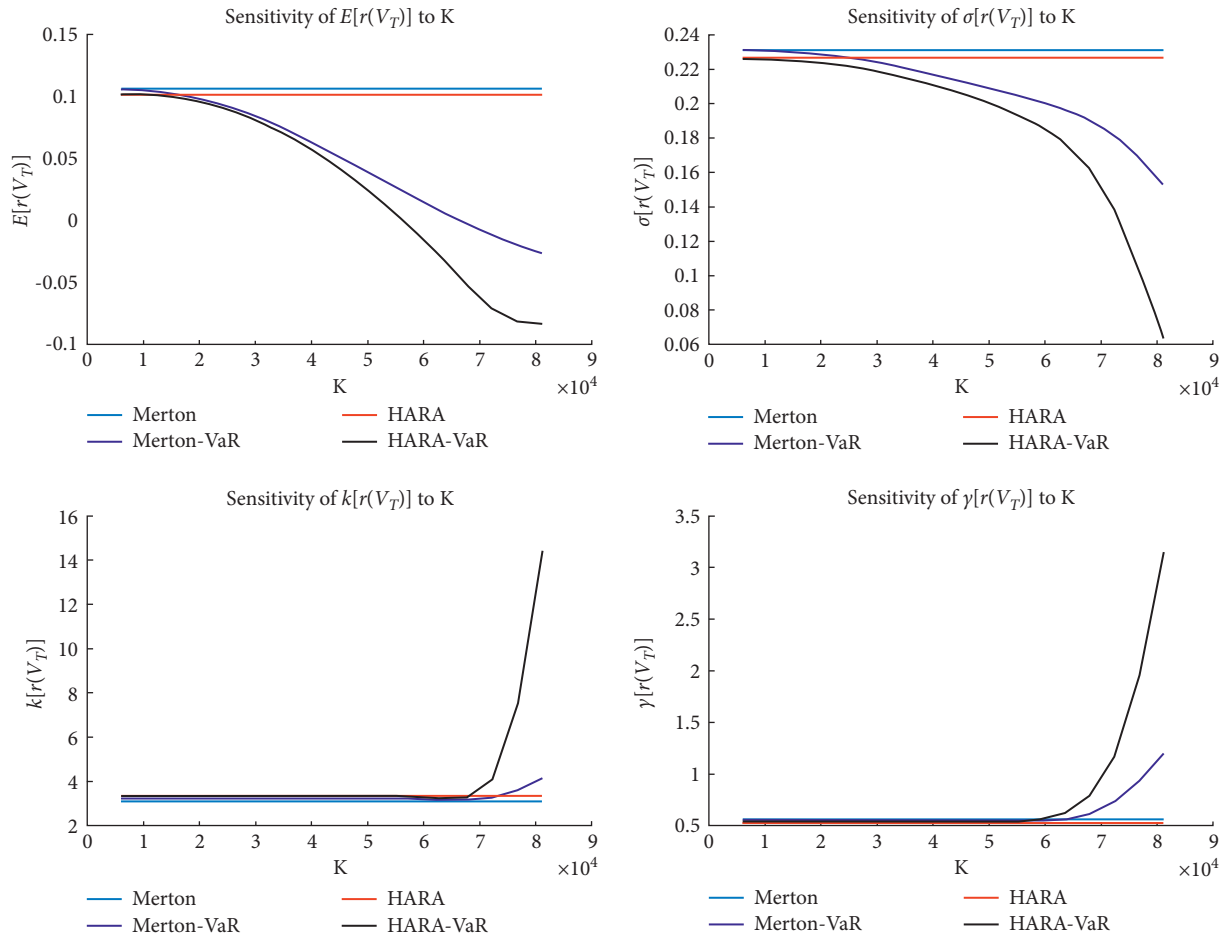


FIGURE 3: First four moments of sensitivity to  $K$  for Merton (Power), Merton-VaR (Power-VaR), HARA, and HARA-VaR.

expected returns are even negative: in the HV case, this result holds for  $\epsilon_{\text{VaR},1y} < 2.5\%$ ; in the PV case, this even holds for moderate VaR probabilities of  $\epsilon_{\text{VaR},1y} < 12.5\%$ . For larger VaR probabilities, the VaR constraint becomes less frequently binding and HV is converging to H and PV is converging to P. However, note that HV converges much faster to H than PV converges to P. This result holds for two reasons: one reason is that H is less risky and hence realizes a lower expected return than P, such that the convergence is naturally faster. Nevertheless, the sensitivity analysis indicates that the combination of VaR and minimum guarantee constraints improves the relative (and in this setting absolute) expected performance (as seen at  $t = 0$ ) compared to a standalone VaR constraint. One explanation could be that the problem of CPPI strategies (i.e., underperforming in markets transitioning from bearish to bullish) is less severe for a HV compared to the PV investor: Given that the risky asset decreases in value at the beginning of the investment horizon, with the risk appetite of a PV strategy initially being comparably larger than an additionally constrained HV strategy, the PV strategy must take more risk in order to satisfy the VaR constraint, hoping that the risky asset drift is realized within the remaining time  $(T - t)$ .

This is what [15] describes as risk transition periods. In contrast, the HV strategy initially has lower risk appetite and

a decline in the risky asset value is less painful. This potential mechanism is confirmed by a very high kurtosis of the PV strategy, compared to both HV and P. Although the expected return is negative in the PV case for a strictly constrained PV investor, the terminal wealth distribution shows less standard deviation in the final  $V_T^{\text{PV}}$  values. The VaR-induced gambling behavior as described by earlier research seems to be confirmed here for the PV case. Furthermore, the results indicate that there exist scenarios in which the optimal PV strategy may take more risk than the unconstrained P strategy in individual trajectories, depending on the path of the stochastic process  $V_t^M$  of the PV portfolio problem. Within the HV scenarios, we do not observe this very effect indicating that combining both constraints may lead to a correction of the partially adverse effects of the VaR constraint on the risk appetite. The HV portfolio strategy also does not show the high kurtosis observed for the PV strategy.

In summary, Figure 1 reads as follows: an investor who wants to preserve 75% of his initial capital guaranteed over a 1-year horizon and additionally does not want to fall short 15% of his initial capital in more than  $\epsilon = 5\%$  of the cases must “exchange” an increasing amount of his portfolio return for ensuring the VaR constraint and experiencing considerably lower volatility of terminal portfolio returns.



For low values of the VaR probability, this “price” in terms of less return naturally is larger than for large VaR probabilities. Also, given any level of VaR probability, the HV strategy is less risky than the PV strategy, measured by terminal return distribution, and in some cases ( $\epsilon_{\text{VaR},1y} < 15\%$ ), the returns are even higher in the HV case. This dominance of HV against PV certainly requires further analysis to be confirmed. Finally, it seems that the convergence of moments is faster for a HV compared to a H investor than for a PV compared to a P investor (except possibly for  $\sigma$ ), indicating that the minimum capital constraint already captures parts of the VaR constraint.

In terms of risk-adjusted performance, in our financial market setting, the SR convergence is also faster for HV to H than for PV to P, although the level of risk-adjusted performance is similar for both. Most importantly, we observe that for low levels of VaR probability, i.e., strict VaR constraints, the SR even becomes negative, indicating that, from a risk-adjusted performance view, a sufficiently restrictive regulatory environment of an already capital guaranteed portfolio further declines the risk-adjusted performance.

As the sensitivity of the strategies to the VaR probability  $\epsilon_{\text{VaR},1y}$  is central to this paper, we further investigate the downward protection of our portfolio in the optimal solution (see Figure 2): remember that (15) is the representation for the optimal terminal wealth  $V_T^{\text{HV}}$  of the HV portfolio problem. We hedge the portfolio against adverse movements from  $K$  until  $k_e$  (first-loss protection). Let  $k_e$  denote the lower Put strike for the unconstrained Merton wealth  $V_T^M$  for the HV case, with  $v_0^M$  (different than the initial wealth  $v_0$ ) computed at  $t = 0$  by the system of equations (16)-(18). Hence, the ratio  $((v_0^M - k_e)/v_0^M)$  indicates what portion of the Merton equivalent wealth is first-loss protected. This shows the amount of wealth that can be potentially insured against adverse portfolio movements by the VaR constraint.

Table 3 shows that both, the equivalent Merton wealth  $v_0^M$  and the lower strike  $k_e$ , decrease with a more restrictive VaR. As the lower strike  $k_e$  represents the amount from which the VaR does not seek downward protection, we observe that, from comparing  $v_0^M$  and  $k_e$ , we have a direct measure of how much of the  $v_0^M$  is downward protected. We observe that the more restrictive the VaR is, the higher the relative protection of the equivalent Merton wealth  $v_0^M$  is. Finally, for a very unrestrictive VaR probability (here  $\epsilon \geq 35\%$ ), we observe that the floor plus the equivalent Merton wealth converges to the initial capital, i.e.,  $\lim_{\epsilon \rightarrow 1} (F_0 + v_0^M) \rightarrow v_0$ .

The sensitivity analysis with respect to the VaR barrier (soft-floor)  $K$ , in Figure 3, confirms the expectation that the higher the VaR barrier  $K$  and hence, the more strict the VaR constraint, the lower the resulting terminal portfolio return expectations. Please note that the capital guarantee  $F_T$  is now modified to 2% of initial wealth (instead of 75% as per the table), as it must hold that  $F_T \leq K$ . In consequence, H and P converge as the minimum guarantee constraint diminishes. Interestingly, the combined capital and risk constraint leads to a worse terminal wealth return in the HV case compared to the standalone risk constraint (VaR), even if the floor is set

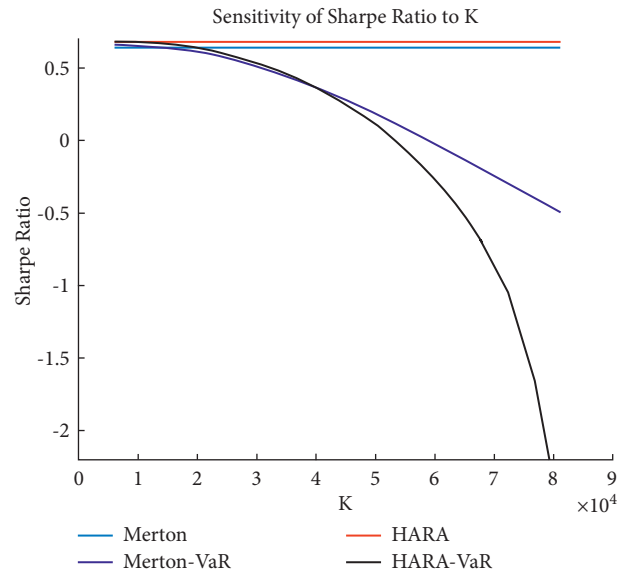


FIGURE 4: Sharpe ratio sensitivity to  $K$  for Merton (Power), Merton-VaR (Power-VaR), HARA, and HARA-VaR.

extremely low and hence HV and PV should coincide. This relation also holds for the distribution of the terminal returns. Higher moments indicate that skewness and kurtosis heavily increase for higher VaR barriers. We observe that the price that an investor must pay for reducing the spread of the terminal wealth is a decrease in performance as measured by the expected return.

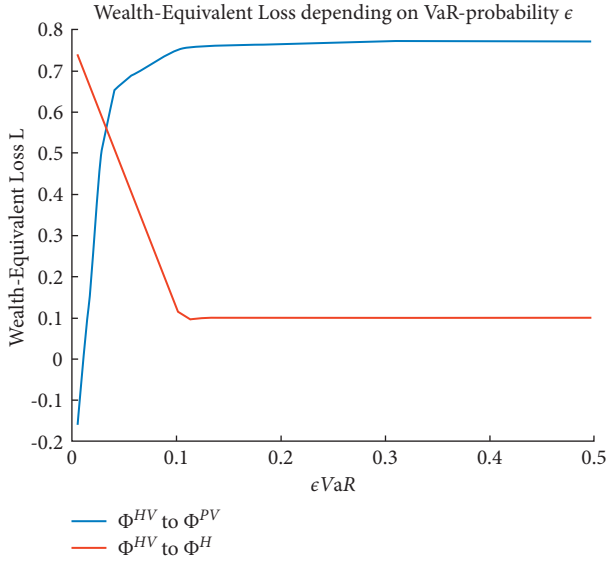
The effect of the decrease in performance in the HV case is also reflected in the SR of the terminal wealth as displayed in Figure 4; the risk-adjusted performance for the HV strategy decreases with an increasingly large VaR barrier  $K$ . The PV strategy also shows the tendency, but the magnitude of a decreasing SR is less severe.

#### 4. Wealth-Equivalent Loss Analysis

Suboptimality analysis investigates the effects of trading strategy restrictions (reflected in the admissible trading strategies  $\Lambda(v)$ ) on the resulting optimal wealth  $V_t$  of portfolio optimization problems. The resulting (c: constrained) admissible trading strategies now ensure that our risk and capital constraints are met, but the resulting optimal constrained strategy  $\pi^c$  leads to a lower (or same) level of expected utility  $\Phi^c(t, v_t)$  compared to the corresponding  $\Phi(t, v_t)$  of the unconstrained investment strategy. Put another way, constraints lead to a lower value  $\Phi^c(t, v_t)$  of the investment strategy:

$$\Phi^c(t, v) \leq \Phi(t, v). \tag{20}$$

Given we are interested in the degree of how much the constrained portfolio problem is worse than the unconstrained problem, we cannot simply compare  $(\Phi(t, v) - \Phi^c(t, v))$  since this distance is not stable to positive affine transformations of the utility functions. Instead, we follow an approach first investigated on intertemporal allocation problems by [17] and adapted to

FIGURE 5: Wealth-equivalent loss sensitivity to  $\epsilon_{\text{VaR},1y}$ .

constrained investment strategies by [18]. Let  $l_t$  be the wealth-equivalent loss (WEL) defined implicitly by

$$\Phi(t, v \cdot (1 - l_t)) = \Phi^c(t, v). \quad (21)$$

The parameter  $l_t, \forall t \in [0, T]$  can be interpreted as the amount of money the investor would have to sacrifice at time  $t$  if implementing the suboptimal strategy  $\pi^c$  instead of the optimal unconstrained strategy  $\pi^*$ . Hence, it follows that, to reach the same level of utility from the optimal investment strategy, the investor requires  $(1 - l_t)$  of the wealth  $v$  at time  $t$ . We can also interpret  $l_t$  as the wealth-equivalent utility loss from implementing risk measure or capital guarantee constraints.

In the following section, we will investigate the impact of a VaR constraint on a HARA utility investor from  $t = 0$ . The corresponding objective is to find the wealth-equivalent utility loss  $l_0 = L$  from the VaR constraint on the HARA utility function:

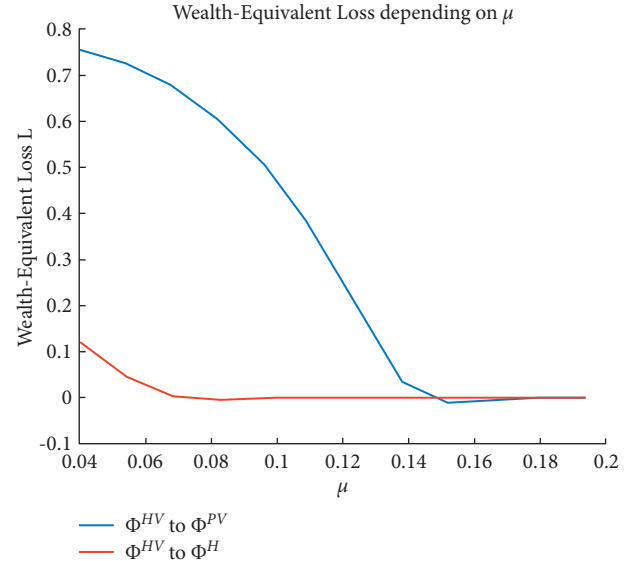
$$\Phi^H(0, v \cdot (1 - L)) = \Phi^{\text{HV}}(0, v). \quad (22)$$

The investigated effect will be a result of additionally imposing the VaR constraint on the already existing minimum capital guaranteed portfolio optimization problem (HARA) from Definition 2.2.

In addition, we can also investigate the effect of imposing the minimum capital guarantee  $F_T$  on a CRRa (power) investor following a VaR constraint:

$$\Phi^{\text{PV}}(0, v \cdot (1 - L)) = \Phi^{\text{HV}}(0, v). \quad (23)$$

We follow Section 3 and provide WEL analysis for all previously investigated sensitivities. The red line shows the percentage loss  $L$  from imposing the VaR constraint on a HARA investor (22), whereas the blue line shows the percentage loss  $L$  from imposing the capital guarantee (i.e., changing the utility function from Power to HARA) on a VaR portfolio problem (23).

FIGURE 6: Wealth-equivalent loss sensitivity to  $\mu_{1y}$ .

We start our WEL analysis by investigating the sub-optimality loss with regard to  $\epsilon_{\text{VaR},1y}$ , depicted in Figure 5. For very low levels of  $\epsilon_{\text{VaR},1y}$ , we observe a large WEL compared to the standalone minimum capital guarantee constraint strategy H. With increasing  $\epsilon_{\text{VaR},1y}$ , the WEL linearly declines and then remains on a constant level from  $\epsilon_{\text{VaR},1y} \geq 10\%$ , indicating that from this VaR probability there is some remaining WEL of 10% in our setting, which persists for less restrictive VaR settings. Hence, the WEL from a combined constraint is substantial for restrictive VaR levels and its magnitude tends to behave linearly across  $10\% \geq \epsilon_{\text{VaR},1y} \geq 0\%$ . On the other hand, the WEL from the minimum capital guarantee constraint HV to PV increases substantially within the region  $2\% \geq \epsilon_{\text{VaR},1y} \geq 0$ , in our case by 80%, while being even negative for extremely restrictive  $\epsilon_{\text{VaR},1y}$  levels. This means that given a very restrictive VaR constraint as, e.g., applied in reinsurance risk management, where a typical VaR probability below the default level of  $\epsilon_{\text{VaR},1y} = 0.5\%$  is applied, the combination of capital guarantee and VaR risk measure constraint may even lead to a wealth-equivalent gain. To handle such tail risk events, a commonly applied set of constraints may be preferred over a standalone VaR constraint. However, as the VaR constraint becomes less restrictive, the WEL increases dramatically even for low VaR levels of  $2\% \leq \epsilon_{\text{VaR},1y} \leq 5\%$  and converges in our case to roughly 75%.

The WEL relationship of the VaR and capital guarantee constraints with respect to the risky asset drift  $\mu_{1y}$  is depicted in Figure 6: the influence of a decrease in risky asset drift  $\mu_{1y}$  on the value of the constrained investment strategy is negative for both, the VaR constraint and the minimum capital constraint. However, a low risky asset drift leads to generally much larger WEL for the minimum capital constraint (blue line) than for the VaR constraint (red line). The WEL from HV to PV shows to be much larger than the WEL between HV and H, indicating that even for very high  $\mu_{1y}$  up to 15%, the additional capital constraint comes with positive

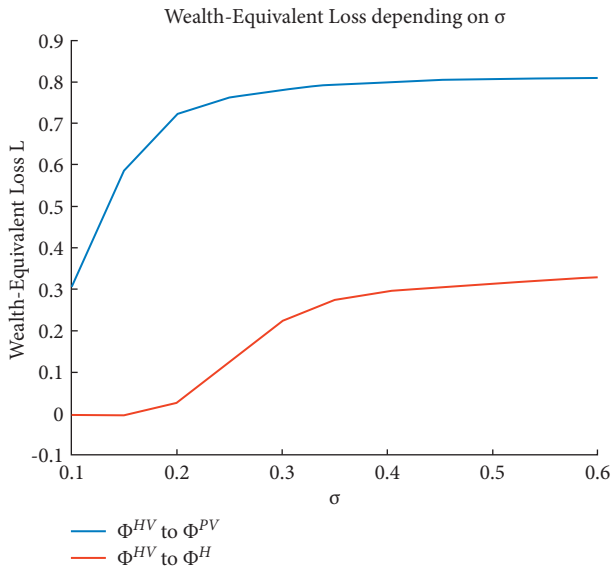


FIGURE 7: Wealth-equivalent loss sensitivity to  $\sigma_{1y}$ .

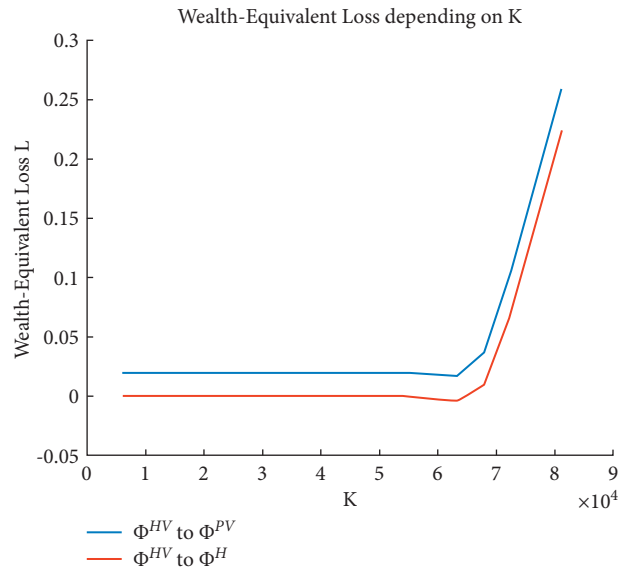


FIGURE 9: Wealth-equivalent loss sensitivity to  $K$ .

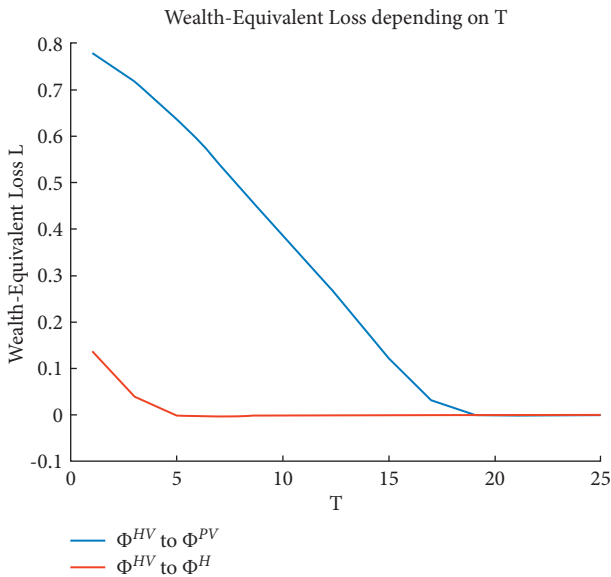


FIGURE 8: Wealth-equivalent loss sensitivity to  $T$ .

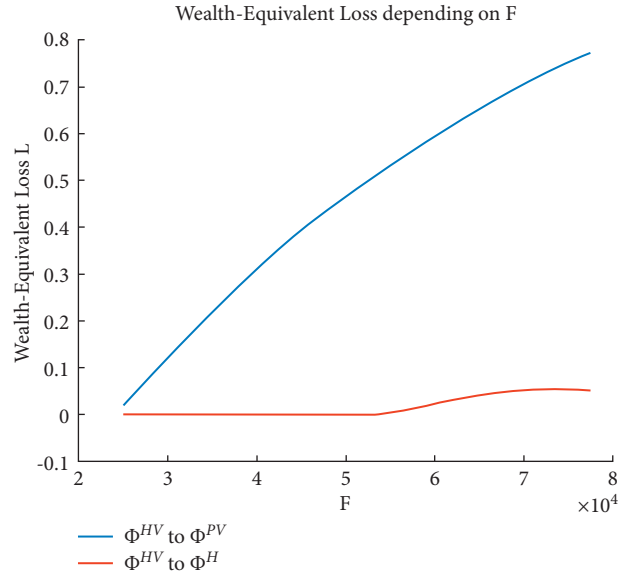


FIGURE 10: Wealth-equivalent loss sensitivity to  $F$ .

WEL costs. In contrast, an additional VaR constraint on an already capital guaranteed portfolio constraint comes relatively cheap as measured by WEL: for a risky asset drift of  $\mu_{1y} \geq 6.5\%$ , the WEL is zero in this scenario. However, we expect this attachment point from which WEL is zero to shift towards a higher  $\mu_{1y}$ ; the more restrictive the VaR constraint, the shorter the time horizon  $T$ , the higher the VaR barrier  $K$ , and the lower the minimum capital protection  $F_T$ . The latter holds because for already well capital-insured portfolios the additional costs of a VaR constraint should be lower than for a less well-protected portfolio. Please have in mind that the riskless asset drift  $r = 2\%$  and that we rather must interpret these results in terms of relative returns, i.e., excess return  $\mu_{1y} - r_{1y}$ . Hence, the larger the excess return is, the cheaper it is to impose an (additional) constraint.

Figure 7 illustrates the relationship between WEL for the examined portfolio constraints and the risky asset volatility  $\sigma_{1y}$ : the results indicate that a higher stock volatility increases the WEL for both strategies. This is natural as one should expect that risk or capital constraints become more costly as the need for insurance increases. The loss magnitude is generally much lower between HV and H than between HV and PV: imposing an additional capital guarantee to an already existing VaR constraint results in a WEL of more than 30% even for very small volatilities of  $\sigma_{1y} = 10\%$ . For stock index investors such as S&P investors, given a long-term index volatility of  $\sigma_{S\&P,1y} \approx 15\%$  p.a., the WEL would be already 60%. In contrast, when a VaR constraint is added to an already existing capital constraint, as in the HV vs. H case, we observe that the WEL for the

same index would still be almost zero, unless the stock volatility exceeds 20%. However, even then, the WEL does not exceed 30% in this setting, indicating that the capital guarantee element of the investment strategy deals well with increasing volatility and that the VaR does not influence WEL as strongly as the capital guarantee. This is equivalent to claiming that the VaR constraint is naturally cheaper on an already capital protected portfolio.

The effect of the time horizon  $T$  on capital and risk constraints must be generally interpreted with care: as the VaR as probabilistic risk constraint as well as the minimum capital guarantee as terminal wealth protection becomes by their definitions less restrictive with an increasing time horizon  $T$ , the results in Figure 8 should only be interpreted in a way that the WEL is declining with an increasing investment period  $T$ . Whereas imposing a capital constraint on a VaR-protected portfolio (blue line) translates into a relatively large WEL even for longer periods of time, imposing the VaR constraint on a capital protected portfolio only leads to a WEL for shorter time horizons of  $T \leq 5y$ . Interestingly, when adding a capital protection element which is expected to be more easily fulfilled for long investment horizons the WEL changes almost linearly in time but takes very long until the WEL effect vanishes.

Figure 9 illustrates the relationship between the portfolio constraint WEL and the VaR barrier: there is some WEL  $\approx 5\%$  between the HV and the PV strategy in this setting for  $K \leq 60\%$  (note the capital guarantee  $F_T$  is 2% of  $v_0$ , instead of 75%, as  $F_T \leq K$  must hold), reflecting the loss from the capital guarantee constraint. However, interestingly, the additional WEL, although observed on modest levels, increases with a more restrictive VaR constraint from an increased barrier  $K > 70\%$  of initial capital. This is surprising, as both the PV and HV must obey this constraint. Hence, the increased WEL for  $K \geq 70\%$  may also come from an interaction with the capital guarantee constraint, as otherwise, both portfolios obey the same VaR constraint. In contrast, comparing HV and H, the WEL only picks up an increasingly large  $K \geq 60\%$  and is zero for low VaR barriers.

Figure 10 depicts the resulting WEL from imposing a capital guarantee  $F_T$  on a VaR constrained portfolio (blue line). The results confirm the natural expectation that, given a larger degree of capital guarantee (a larger  $F_T$ ), the WEL increases to very high values (e.g., an  $F_T = 80\%$  results in a WEL of 80%). On the other hand, given that we impose a VaR on an already capital guaranteed solution, we would expect a generally smaller impact of WEL as the capital guarantee  $F_T$  grows; hence the portfolio risk is already reduced and the higher the guarantee level  $F_T$ , the lower the probability that the VaR constraint does not hold. However, we observe the opposite: for low guarantee levels  $F_T \leq 60\%$ , there is no additional WEL from imposing the VaR constraint (red line), whereas with a sufficiently large  $F_T \geq 70\%$ , we observe minor WEL. This means that the cost as measured by WEL of imposing an additional VaR constraint is larger for an already largely protected portfolio in this setting.

## 5. Conclusions

Reference [9] introduces an elegant way to incorporate risk constraints into the dynamic programming principle approach and to derive closed-form solutions to various optimization problems such as minimum capital guarantees and risk constraints. We apply their methodology to a setting of a HARA investor with VaR constraints, producing closed-form representations for the three objects keys in the analysis, namely, the optimal strategy, the optimal terminal wealth, and the value function.

We conduct numerical analyses to improve the understanding of this joint application of risk and capital guarantee constraints. In general, the analysis confirms that adding a VaR to an already existing capital guarantee (HARA-VaR) leads to a worse performance compared to the case of solely capital guarantee (HARA). On the other hand, VaR without capital guarantees (CRRR-VaR) could lead to even worse performances in terms of Sharpe ratios, kurtosis, variance, and expected returns. This gives evidence that capital guarantees mitigate the increase of risk of a stand-alone VaR constraint as described by [15]. Moreover, the more restrictive the soft-floor ( $K$ ), the better the performance of HARA-VaR versus CRRR-VaR.

We also identify circumstances in which the implementation of a VaR constraint to an already minimum capital guarantee portfolio comes with little additional cost and hence provides a relatively “cheap” protection against tail risks. For example, a HARA investor with a one-year terminal VaR constraint and  $\epsilon_{\text{VaR},1y} = 10\%$  would only experience a 10% WEL if the capital guarantee is 75% of his initial wealth (losses could be much higher for lower  $\epsilon_{\text{VaR},1y}$ ). Similarly, adopting a strategy with a minimum capital guarantee, VaR could come with huge additional cost to a VaR-protected CRRR investor, highlighting the cost of protection against tail risks. For example, a CRRR investor with a one-year terminal VaR constraint and  $\epsilon_{\text{VaR},1y} = 10\%$  would experience a 70% WEL if he modifies the strategy to accommodate a capital guarantee of 75% of his initial wealth, providing large additional downward protection against extreme events such as financial crises. Finally, an extension to multidimensions is derived theoretically in the Appendix.

## Appendix

### A. Complementary

*A.1. Reminder of Main Result from [9].* Let the price of a contingent claim  $f(\cdot)$  on  $V^M$  be denoted as

$$\Pi(t, v^M) = e^{-r(T-t)} \mathbb{E}^Q [f(V_T^M) | V_t^M = v^M], \quad \Pi(0, v^M) = v. \quad (\text{A.1})$$

The expected utility of the claim, based on utility  $u(\cdot)$ , is

$$U(t, v^M) = \mathbb{E} [u(f(V_T^M)) | V_t^M = v^M]. \quad (\text{A.2})$$

Reference [9] demonstrates that the wealth of a constrained problem can be represented in terms of the price  $\Pi$  of a contingent claim on the wealth of the unconstrained

problem  $V^M$ , and the value function  $\Phi^c$  can be represented in terms of the utility  $U$  on the contingent claim. The theorem next, Theorem 1 in [9], provides a condition such that the PDEs and terminal conditions associated with  $\Phi^c(t, \Pi(t, v))$  and  $U(t, v)$  coincide.

**Theorem 5.** *Let the following condition hold:*

$$\frac{vU_{vv}}{U_v} - \frac{v\Pi_{vv}}{\Pi_v} = -\frac{1-\gamma}{v}. \quad (\text{A.3})$$

Then,  $\Phi^c(t, \Pi(t, v)) = U(t, v)$ . Moreover the optimal stock allocation can be represented as

$$\pi^* = \frac{v\theta}{\sigma(1-\gamma)} \frac{\Pi_v}{\Pi}. \quad (\text{A.4})$$

## A.2. Proofs

*Proof.* of Proposition 4. We derive the optimal triplet for the HV case: We start by defining the auxiliary utility function using the simplified HARA utility function.

$$\tilde{u}^{\text{HV}}(v) = \frac{1}{\gamma}(v - F_T)^\gamma - \lambda_\epsilon 1_{\{v < K\}}, \quad (\text{A.5})$$

with the Lagrangian multiplier  $\lambda_\epsilon$  as a punishment to the resulting utility function implemented by the use of the indicator function  $1_{\{v < K\}}$ .  $F_T$  represents the floor of the CPPI strategy. In case the Lagrangian  $\lambda_\epsilon = 0$ , the auxiliary utility function coincides with a HARA (CPPI) investor utility function (and hence, the concavification argument is not required for an optimal solution).

We conjecture the claim function  $f^{\text{HV}}(v^M)$  to be characterized by

$$f^{\text{HV}}(v^M) = v^M + F_T + (K - v^M)1_{\{k_\epsilon < v^M < K\}}. \quad (\text{A.6})$$

The idea behind this conjecture is that the VaR can be interpreted as a combination of two Put options: Entering a long position in the higher strike Put with strike denoted as  $K$  and entering a short position in a lower strike Put with strike denoted by  $k_\epsilon$ . The smaller the VaR confidence level  $(1 - \epsilon)$ , the higher the strike  $k_\epsilon$  of the short Put, and the less the coverage for (extreme) losses. This interpretation of the VaR means that if the confidence level is below 100%, sharp (but very unlikely) losses below the confidence level  $(1 - \epsilon)$  are not considered to be relevant for the investor. The claim function now rewards the investor c.p. When the optimal unconstrained wealth  $v^M$  is below the upper strike  $K$  and above the lower strike  $k_\epsilon$  as the long Put option would be exercised. By nature of construction, the parameter  $K$  is above the floor  $F_T$ , as the CPPI portfolio value is bounded from below by the barrier  $F_T$ . Hence,  $F_T < K$ . The relationship between  $F_T$  and  $k_\epsilon$  is more difficult to illustrate as  $F_T$  is the floor relative to the total portfolio value  $V_t^{\text{HV}}$  whereas  $k_\epsilon$  corresponds to the Merton wealth  $V^M$  (claim).

To validate the conjecture of  $f^{\text{HV}}(v^M)$ , we will now first derive  $\Pi^{\text{HV}}(t, v^M)$  and  $\tilde{u}(f^{\text{HV}}(v^M))$  to derive  $U^{\text{HV}}(t, v^M)$ . Then, we will verify under which conditions Theorem I of [9] will hold to secure an optimal solution. Since the numerical solution of the conditions is nontrivial, we provide a detailed examination. Finally, we compute the optimal investment strategy  $\pi^{\text{HV}}$  and substitute  $\pi^{\text{HV}}$  into the portfolio dynamics to receive a representation for the optimal wealth dynamics  $dV_t^{\text{HV}}$ .

Deriving the optimal investors wealth  $\Pi^{\text{HV}}(t, v^M)$ , we start with

$$\begin{aligned} \Pi^{\text{HV}}(t, v^M) &= \mathbb{E}_{t, v^M}^Q \left[ e^{-r(T-t)} f(V_T^M) \right] = E_{t, v^M}^Q \left[ e^{-r(T-t)} \left( V_T^M + F_T + (K - V_T^M) 1_{\{k_\epsilon < V_T^M < K\}} \right) \right] \\ &= v^M + \mathbb{E}_{t, v^M}^Q \left[ e^{-r(T-t)} \left( F_T + (K - V_T^M) 1_{\{V_T^M < K\}} - (K - V_T^M) 1_{\{V_T^M < k_\epsilon\}} \right) \right] \\ &= v^M + F_t + P \left( t, v^M, K, r, \frac{\theta}{1-\gamma} \right) - P \left( t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma} \right) - (K - k_\epsilon) e^{-r(T-t)} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \\ &= v^M + F_t + P \left( t, v^M, K, r, \frac{\theta}{1-\gamma} \right) - P \left( t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma} \right) - (K - k_\epsilon) e^{-r(T-t)} N(-d_2^Q(t, v^M, k_\epsilon)), \end{aligned}$$

\leqno{\tf="OT310e2d14"{{(A.7)}}}

with  $F_t = F_T e^{-r(T-t)}$ ,  $d_2^Q(t, v^M, k_\epsilon)$  as defined in 9,  $P(t, v^M, r, \sigma, K)$  denoting the price of an European Put option with interest rate  $r$ , volatility  $\sigma$ , and strike  $K$ , given that the price of its underlying is  $v^M$  at time  $t$ .

The last step holds since, in the Black-Scholes regime,  $N(-d_2^Q(t, v^M, k_\epsilon))$  represents the risk-neutral probability of a Put option being exercised (see [19] for a detailed discussion). Hence,  $\mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) = N(-d_2^Q(t, v^M, k_\epsilon))$ .

Specifically, at time  $t = T$  it holds that

$$\begin{aligned}\Pi^{\text{HV}}(T, v^M) &= V_T^M + F_T + P\left(T, V_T^M, K, r, \frac{\theta}{1-\gamma}\right) - P\left(T, V_T^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) (K - k_\epsilon) e^{-r(T-T)} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \\ &= V_T^M + F_T + (K - V_T^M)^+ - (k_\epsilon - V_T^M)^+ - (K - k_\epsilon) 1_{\{V_T^M < k_\epsilon\}} \\ &\stackrel{\text{(A.7)}}{=} V_T^M + F_T + (K - V_T^M) 1_{\{K > V_T^M > k_\epsilon\}} = V_T^{\text{HV}}.\end{aligned}\tag{A.8}$$

The last simplifications hold since

$$\begin{aligned}V_T^M + F_T + \left\{ \begin{array}{ll} K - V_T^M - 0 - (K - k_\epsilon)0, & k_\epsilon \leq V_T^M \leq K \\ 0 - 0 - (K - k_\epsilon)0, & V_T^M > K \\ K - V_T^M - k_\epsilon + V_T^M - (K - k_\epsilon), & V_T^M < k_\epsilon \end{array} \right\} \\ = \left\{ \begin{array}{ll} V_T^M + F_T + (K - V_T^M), & k_\epsilon \leq V_T^M \leq K \\ V_T^M + F_T, & V_T^M > K \\ V_T^M + F_T, & V_T^M < k_\epsilon \end{array} \right\} \\ = V_T^M + F_T + (K - V_T^M) 1_{\{K > V_T^M > k_\epsilon\}}.\end{aligned}\tag{A.9}$$

Further, the derivative of  $\Pi^{\text{HV}}(t, v^M)$  with respect to  $v^M$  is given by

$$\Pi_{v^M}^{\text{HV}}(t, v^M) = 1 + \frac{\partial}{\partial v^M} P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) - (K - k_\epsilon) e^{-r(T-t)} \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon).\tag{A.10}$$

To compute the optimal investment strategy, we plug in the functions  $\Pi^{\text{HV}}(t, v^M)$  and  $\Pi_{v^M}^{\text{HV}}(t, v^M)$  into the optimal

implicit investment strategy representation of Theorem 1 in [9]:

$$\begin{aligned}\pi^{\text{HV}}(t, v^M) &= \frac{1}{1-\gamma} \frac{v^M \Pi_{v^M}(t, v^M)}{\Pi(t, v^M)} \frac{\theta}{\sigma} \\ &= \frac{\theta}{\sigma(1-\gamma)} \frac{v^M (1 + (\partial/\partial v^M) P(t, v^M, K) - (\partial/\partial v^M) P(t, v^M, k_\epsilon) - (K - k_\epsilon) e^{-r(T-t)} (\partial/\partial v^M) \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon))}{v^M + F_t + P(t, v^M, K) - P(t, v^M, k_\epsilon) - (K - k_\epsilon) e^{-r(T-t)} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon)} \\ &= \frac{\theta}{\sigma(1-\gamma)} \frac{v^M (1 + N(d_1^{\text{Q}}(t, v^M, K)) - N(d_1^{\text{Q}}(t, v^M, k_\epsilon)) + (K - k_\epsilon) e^{-r(T-t)} ((\gamma - 1)/\theta) ((N'(d_2^{\text{Q}}(t, v^M, k_\epsilon)))/(v^M \sqrt{T-t})))}{v^M + F_t + P(t, v^M, K) - P(t, v^M, k_\epsilon) - (K - k_\epsilon) e^{-r(T-t)} N(-d_2^{\text{Q}}(t, v^M, k_\epsilon))}.\end{aligned}\tag{A.11}$$

We start deriving the value function by plugging in the claim function  $f^{\text{HV}}(v^M)$  into the auxiliary utility function of the HV problem:

$$\begin{aligned}
 \tilde{u}(f^{\text{HV}}(v^M)) &= \frac{1}{\gamma}(f^{\text{HV}}(v^M) - F_T)^\gamma - \lambda_\epsilon \mathbf{1}_{\{f^{\text{HV}}(v^M) < K\}} \\
 &= \frac{1}{\gamma}(v^M + F_T + (K - v^M)\mathbf{1}_{\{k_\epsilon < v^M < K\}} - F_T)^\gamma - \lambda_\epsilon \mathbf{1}_{\{v^M + F_T + (K - v^M)\mathbf{1}_{\{k_\epsilon < v^M < K\}} < K\}} \\
 &= \frac{1}{\gamma}(v^M + (K - v^M)\mathbf{1}_{\{k_\epsilon < v^M < K\}})^\gamma - \lambda_\epsilon \mathbf{1}_{\{F_T + (K - v^M)\mathbf{1}_{\{k_\epsilon < v^M < K\}} < K - v^M\}} \\
 &= \frac{1}{\gamma}((v^M)^\gamma + (K^\gamma - (v^M)^\gamma)\mathbf{1}_{\{k_\epsilon^\gamma < (v^M)^\gamma < K^\gamma\}}) - \lambda_\epsilon \mathbf{1}_{\{v^M < \min\{k_\epsilon, K - F_T\}\}}.
 \end{aligned} \tag{A.12}$$

The last simplification holds due to

$$\begin{aligned}
 \mathbf{1}_{\{F_T + (K - v^M)\mathbf{1}_{\{k_\epsilon < v^M < K\}} < K - v^M\}} &= \mathbf{1} \left\{ \begin{array}{l} F_T < K - v^M, \quad v^M < k_\epsilon \\ F_T + K - v^M < K - v^M, \quad k_\epsilon \leq v^M \leq K \\ F_T < K - v^M, \quad v^M > K \end{array} \right\} \\
 &= \mathbf{1} \left\{ \begin{array}{l} F_T < K - v^M, \quad v^M < k_\epsilon \\ F_T < 0, \quad k_\epsilon \leq v^M \leq K \\ F_T < K - v^M, \quad v^M > K \end{array} \right\} = \left\{ \begin{array}{l} \mathbf{1}_{\{v^M < K - F_T\}}, \quad v^M < k_\epsilon \\ 0, \quad k_\epsilon \leq v^M \leq K \\ 0, \quad v^M > K \end{array} \right\} \\
 &= \mathbf{1}_{\{v^M < K - F_T\}} \mathbf{1}_{\{v^M < k_\epsilon\}} = \mathbf{1}_{\{v^M < \min\{k_\epsilon, K - F_T\}\}}.
 \end{aligned} \tag{A.13}$$

Thus, the expression for the value function  $\Phi_t^{\text{HV}} = U^{\text{HV}}(t, \Pi(t, v^M))$  becomes

$$\begin{aligned}
 U^{\text{HV}}(t, v^M) &= \mathbb{E}_{t, v^M}^{\mathbb{P}} [\tilde{u}(f^{\text{HV}}(V_T^M))] \\
 &= \mathbb{E}_{t, v^M}^{\mathbb{P}} \left[ \frac{1}{\gamma}((V_T^M)^\gamma + (K^\gamma - (V_T^M)^\gamma)\mathbf{1}_{\{k_\epsilon^\gamma < (V_T^M)^\gamma < K^\gamma\}}) - \lambda_\epsilon \mathbf{1}_{\{v^M < \min\{k_\epsilon, K - F_T\}\}} \right] \\
 &= \frac{e^{\tilde{r}(T-t)}}{\gamma} \mathbb{E}_{t, v^M}^{\mathbb{P}} \left[ e^{-\tilde{r}(T-t)} \left( (V_T^M)^\gamma + (K^\gamma - (V_T^M)^\gamma)\mathbf{1}_{\{k_\epsilon^\gamma < (V_T^M)^\gamma < K^\gamma\}} \right) \right] - \lambda_\epsilon \mathbb{P}_{t, v^M}(V_T^M < \min\{k_\epsilon, K - F_T\}) \\
 &= \frac{e^{\tilde{r}(T-t)}}{\gamma} \left( (v^M)^\gamma + P\left(t, (v^M)^\gamma, K^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) - P\left(t, (v^M)^\gamma, k_\epsilon^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) \right) - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) \mathbb{P}_{t, v^M}(V_T^M < k_\epsilon) - \lambda_\epsilon \mathbb{P}_{t, v^M}(V_T^M < \min\{k_\epsilon, K - F_T\}) \\
 &= \frac{e^{\tilde{r}(T-t)}}{\gamma} \left( (v^M)^\gamma + P\left(t, (v^M)^\gamma, K^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) - P\left(t, (v^M)^\gamma, k_\epsilon^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) \right) - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) \mathbb{P}_{t, v^M}(V_T^M < k_\epsilon) \\
 &\quad - \lambda_\epsilon \left[ \mathbb{P}_{t, v^M}(V_T^M < k_\epsilon)\mathbf{1}_{\{k_\epsilon < K - F_T\}} + \mathbb{P}_{t, v^M}(V_T^M < K - F_T)\mathbf{1}_{\{k_\epsilon \geq K - F_T\}} \right] \\
 &= \frac{e^{\tilde{r}(T-t)}}{\gamma} \left[ (v^M)^\gamma + P\left(t, (v^M)^\gamma, K^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) - P\left(t, (v^M)^\gamma, k_\epsilon^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) \right] - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) N(-d_2^\mathbb{P}(t, v^M, k_\epsilon)) \\
 &\quad - \lambda_\epsilon \left[ N(-d_2^\mathbb{P}(t, v^M, k_\epsilon))\mathbf{1}_{\{k_\epsilon < K - F_T\}} + N(-d_2^\mathbb{P}(t, v^M, K - F_T))\mathbf{1}_{\{k_\epsilon \geq K - F_T\}} \right],
 \end{aligned} \tag{A.14}$$



with  $d_2^{\mathbb{P}}(t, v^M, K - F_T)$  as defined in 9. Now, we check whether Theorem 1 in [9] holds by verifying that the following equation holds:

$$U_{v^M}^{\text{HV}}(t, v^M) = h(t)(v^M)^{\gamma-1} \Pi_{v^M}^{\text{HV}}(t, v^M). \tag{A.15}$$

The derivatives of  $U(t, v^M)$  and  $\Pi(t, v^M)$  are given by

$$\begin{aligned} U_{v^M}^{\text{HV}} &= \frac{e^{\tilde{r}(T-t)}}{\gamma} \left( \gamma(v^M)^{\gamma-1} + \frac{\partial}{\partial v^M} P\left(t, (v^M)^\gamma, K^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, (v^M)^\gamma, k_\epsilon^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) \right) \\ &\quad - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) \frac{\partial}{\partial v^M} \mathbb{P}_{t, v^M}(V_T^M < k_\epsilon) - \lambda_\epsilon \left[ \frac{\partial}{\partial v^M} \mathbb{P}_{t, v^M}(V_T^M < k_\epsilon) \mathbf{1}_{\{k_\epsilon < K - F_T\}} + \frac{\partial}{\partial v^M} \mathbb{P}_{t, v^M}(V_T^M < K - F_T) \mathbf{1}_{\{k_\epsilon \geq K - F_T\}} \right] \\ &= \frac{e^{\tilde{r}(T-t)}}{\gamma} \left( \gamma(v^M)^{\gamma-1} + \frac{\partial}{\partial v^M} P\left(t, (v^M)^\gamma, K^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, (v^M)^\gamma, k_\epsilon^\gamma, \tilde{r}, \frac{\gamma\theta}{1-\gamma}\right) \right) \\ &\quad - \frac{e^{\tilde{r}(T-t)}}{\gamma} (K^\gamma - k_\epsilon^\gamma) e^{-\tilde{r}(T-t)} \frac{\partial}{\partial v^M} \mathbb{P}_{t, v^M}(V_T^M < k_\epsilon) - e^{\tilde{r}(T-t)} \lambda_\epsilon \left[ e^{-\tilde{r}(T-t)} \frac{\partial}{\partial v^M} \mathbb{P}_{t, v^M}(V_T^M < k_\epsilon) \mathbf{1}_{\{k_\epsilon < K - F_T\}} + e^{-\tilde{r}(T-t)} \frac{\partial}{\partial v^M} \mathbb{P}_{t, v^M}(V_T^M < K - F_T) \mathbf{1}_{\{k_\epsilon \geq K - F_T\}} \right] \\ &= \frac{e^{\tilde{r}(T-t)}}{\gamma} \left( \gamma(v^M)^{\gamma-1} + \gamma(v^M)^{\gamma-1} \frac{\partial}{\partial v^M} P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) \right) - \gamma(v^M)^{\gamma-1} \frac{\partial}{\partial v^M} P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) \\ &\quad - \frac{e^{\tilde{r}(T-t)}}{\gamma} (K^\gamma - k_\epsilon^\gamma) \frac{1}{k_\epsilon^{\gamma-1}} (v^M)^{\gamma-1} e^{-r(T-t)} \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \\ &\quad - e^{\tilde{r}(T-t)} \frac{(v^M)^{\gamma-1}}{k_\epsilon^{\gamma-1}} e^{-r(T-t)} \lambda_\epsilon \left[ \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \mathbf{1}_{\{k_\epsilon < K - F_T\}} + \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < K - F_T) \mathbf{1}_{\{k_\epsilon \geq K - F_T\}} \right] \\ &= e^{\tilde{r}(T-t)} (v^M)^{\gamma-1} \left[ 1 + \frac{\partial}{\partial v^M} P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) \right] \\ &\quad - e^{\tilde{r}(T-t)} (v^M)^{\gamma-1} \frac{1}{k_\epsilon^{\gamma-1}} e^{-r(T-t)} \left( \begin{aligned} &\frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \\ &+ \lambda_\epsilon \left( \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \mathbf{1}_{\{k_\epsilon < K - F_T\}} + \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < K - F_T) \mathbf{1}_{\{k_\epsilon \geq K - F_T\}} \right) \end{aligned} \right) \\ &= e^{\tilde{r}(T-t)} (v^M)^{\gamma-1} \left[ \begin{aligned} &1 + \frac{\partial}{\partial v^M} P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) \\ &\frac{1}{k_\epsilon^{\gamma-1}} e^{-r(T-t)} \left( \begin{aligned} &\frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) + \lambda_\epsilon \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \mathbf{1}_{\{k_\epsilon < K - F_T\}} \\ &+ \lambda_\epsilon \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < K - F_T) \mathbf{1}_{\{k_\epsilon \geq K - F_T\}} \end{aligned} \right) \end{aligned} \right] \\ \Pi_{v^M}^{\text{HV}} &= 1 + \frac{\partial}{\partial v^M} P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) - (K - k_\epsilon) e^{-r(T-t)} \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon). \end{aligned} \tag{A.16}$$

We again check whether Theorem 1 holds by verifying that condition 9 of [9] holds with  $h(t) = e^{\tilde{r}(T-t)}$ :

$$\begin{aligned}
U_{v^M}^{\text{HV}}(t, v^M) &= h(t)(v^M)^{\gamma-1} \Pi_{v^M, v^M}^{\text{HV}}(t, v^M) \\
&\Leftrightarrow e^{\tilde{r}(T-t)} (v^M)^{\gamma-1} \left[ \begin{array}{c} 1 + \frac{\partial}{\partial v^M} P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) \\ \frac{1}{k_\epsilon^{\gamma-1}} e^{-r(T-t)} \left( \begin{array}{c} \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) + \lambda_\epsilon \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \mathbf{1}_{\{k_\epsilon < K-F_T\}} \\ + \lambda_\epsilon \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < K-F_T) \mathbf{1}_{\{k_\epsilon \geq K-F_T\}} \end{array} \right) \end{array} \right] \\
&= (v^M)^{\gamma-1} h(t) \left[ 1 + \frac{\partial}{\partial v^M} P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) - (K - k_\epsilon) e^{-r(T-t)} \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \right] \\
&\Leftrightarrow 1 + \frac{\partial}{\partial v^M} P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) - \frac{1}{k_\epsilon^{\gamma-1}} e^{-r(T-t)} \left( \begin{array}{c} \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \left[ \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) + \lambda_\epsilon \mathbf{1}_{\{k_\epsilon < K-F_T\}} \right] \\ + \lambda_\epsilon \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < K-F_T) \mathbf{1}_{\{k_\epsilon \geq K-F_T\}} \end{array} \right) \\
&= 1 + \frac{\partial}{\partial v^M} P\left(t, v^M, K, r, \frac{\theta}{1-\gamma}\right) - \frac{\partial}{\partial v^M} P\left(t, v^M, k_\epsilon, r, \frac{\theta}{1-\gamma}\right) - (K - k_\epsilon) e^{-r(T-t)} \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \\
&\Leftrightarrow \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \left[ \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) + \lambda_\epsilon \mathbf{1}_{\{k_\epsilon < K-F_T\}} \right] + \lambda_\epsilon \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < K-F_T) \mathbf{1}_{\{k_\epsilon \geq K-F_T\}} \\
&= k_\epsilon^{\gamma-1} (K - k_\epsilon) \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \\
&\Leftrightarrow \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < K-F_T) \lambda_\epsilon \mathbf{1}_{\{k_\epsilon \geq K-F_T\}} \\
&= \frac{\partial}{\partial v^M} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon) \left[ k_\epsilon^{\gamma-1} (K - k_\epsilon) - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) - \lambda_\epsilon \mathbf{1}_{\{k_\epsilon < K-F_T\}} \right] \\
&\quad - \frac{1-\gamma}{\theta} \frac{N'(d_2^Q(t, v^M, K-F_T))}{V_t^M \sqrt{T-t}} \lambda_\epsilon \mathbf{1}_{\{k_\epsilon \geq K-F_T\}} = -\frac{1-\gamma}{\theta} \frac{N'(d_2^Q(t, v^M, k_\epsilon))}{V_t^M \sqrt{T-t}} \left[ k_\epsilon^{\gamma-1} (K - k_\epsilon) - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) - \lambda_\epsilon \mathbf{1}_{\{k_\epsilon < K-F_T\}} \right] \\
&\Leftrightarrow \frac{N'(d_2^Q(\cdot, v^M, K-F_T))}{N'(d_2^Q(\cdot, v^M, k_\epsilon))} \lambda_\epsilon \mathbf{1}_{\{k_\epsilon \geq K-F_T\}} = k_\epsilon^{\gamma-1} (K - k_\epsilon) - \lambda_\epsilon \mathbf{1}_{\{k_\epsilon < K-F_T\}} - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) \\
&\Leftrightarrow \lambda_\epsilon \left[ \frac{N'(d_2^Q(\cdot, v^M, K-F_T))}{N'(d_2^Q(\cdot, v^M, k_\epsilon))} \mathbf{1}_{\{k_\epsilon \geq K-F_T\}} + \mathbf{1}_{\{k_\epsilon < K-F_T\}} \right] = k_\epsilon^{\gamma-1} (K - k_\epsilon) - \frac{1}{\gamma} (K^\gamma - k_\epsilon^\gamma) \\
&\Leftrightarrow \lambda_\epsilon = \frac{k_\epsilon^{\gamma-1} (K - k_\epsilon) - (1/\gamma) (K^\gamma - k_\epsilon^\gamma)}{(N'(d_2^Q(\cdot, v^M, K-F_T)))/(N'(d_2^Q(\cdot, v^M, k_\epsilon))) \mathbf{1}_{\{k_\epsilon \geq K-F_T\}} + \mathbf{1}_{\{k_\epsilon < K-F_T\}}}
\end{aligned} \tag{A.17}$$

It remains to solve the numerical equations in order to guarantee an optimal solution under Theorem 1 of [9]. The

numerical equations are given by the VaR constraint, investor wealth, and condition (A.17), respectively:

$$\begin{aligned}
\mathbb{P}_{0, v_0^M}(f^{\text{HV}}(V_T^M) < K) - \epsilon &= 0 \\
\Pi(0, v_0^M) - v_0 &= 0 \\
\frac{k_\epsilon^{\gamma-1} (K - k_\epsilon) - (1/\gamma) (K^\gamma - k_\epsilon^\gamma)}{(N'(d_2^Q(\cdot, v_0^M, K-F_T)))/(N'(d_2^Q(\cdot, v_0^M, k_\epsilon))) \mathbf{1}_{\{k_\epsilon \geq K-F_T\}} + \mathbf{1}_{\{k_\epsilon < K-F_T\}}} - \lambda_\epsilon &= 0.
\end{aligned} \tag{A.18}$$

The VaR constraint is satisfied given the conjecture  $f^{\text{HV}}(V_T^M)$ . The initial investor wealth  $v_0$  is linked to the value

of the claim function  $\Pi(0, v_0^M)$ . Note that  $\Pi(0, v_0^M)$  encapsulates the idea of a risk-neutral pricing depending on the

stochastic process  $V_t^M$  and its price at  $t = 0$  must be equal to the initial investor wealth  $v_0$ . Finally, equation (A.17) represents the link between  $k_\epsilon$  and the concavification parameter  $\lambda_\epsilon$ . The equation system shows that we must find  $k_\epsilon$  and  $\lambda_\epsilon$  such that the VaR constraint of equation (A.17) is met. Please note that it is sufficient to solve this equation

system at  $t = 0$ : Equation (A.17) is derived from Theorem 1 in [9]. As the concavification parameter  $\lambda_\epsilon$  is set at  $t = 0$ , we plug in  $v_0^M$  into equation (A.17). The solution to the equation system  $(v_0^M, k_\epsilon, \lambda_\epsilon)$  ensures the correct representations of Proposition 4 for all  $t \in [0, T]$ .

The VaR can be simplified to

$$\begin{aligned} \mathbb{P}_{0, v_0^M}(f^{\text{HV}}(V_T^M) < K) - \epsilon &= \mathbb{P}_{0, v_0^M}(V_T^M + F_T + (K - V_T^M)1_{\{k_\epsilon < V_T^M < K\}} < K) - \epsilon \\ &= \left\{ \begin{array}{l} \mathbb{P}_{0, v_0^M}(V_T^M + F_T < K), \quad V_T^M \leq k_\epsilon \\ \mathbb{P}_{0, v_0^M}(F_T < 0) = 0, \quad k_\epsilon \leq V_T^M \leq K \\ \mathbb{P}_{0, v_0^M}(V_T^M + F_T < K) = 0, \quad V_T^M \geq K \end{array} \right\} - \epsilon \\ &= \mathbb{P}_{0, v_0^M}(V_T^M < K - F_T)1_{\{k_\epsilon \geq K - F_T\}} + \mathbb{P}_{0, v_0^M}(V_T^M < k_\epsilon)1_{\{k_\epsilon < K - F_T\}} - \epsilon = 0, \end{aligned} \quad (\text{A.19})$$

for our application with  $F_T \leq k_\epsilon < K$  and  $F_T \geq 0$ .

We can further simplify the results using the distribution of the Merton wealth dynamics. Hence, the equation can be simplified to

$$\begin{aligned} \mathbb{P}_{0, v_0^M}(V_T^M < K - F_T)1_{\{k_\epsilon \geq K - F_T\}} + \mathbb{P}_{0, v_0^M}(V_T^M < k_\epsilon)1_{\{k_\epsilon < K - F_T\}} - \epsilon &= 0 \\ \Leftrightarrow \mathbb{P}_{0, v_0^M}(\ln(V_T^M) < \ln(K - F_T))1_{\{k_\epsilon \geq K - F_T\}} + \mathbb{P}_{0, v_0^M}(\ln(V_T^M) < \ln(k_\epsilon))1_{\{k_\epsilon < K - F_T\}} - \epsilon &= 0 \\ \Leftrightarrow N\left(\frac{\ln(K - F_T) - \mu_{\ln V_T^M}}{\sigma_{\ln(V_T^M)}}\right)1_{\{k_\epsilon \geq K - F_T\}} + N\left(\frac{\ln(k_\epsilon) - \mu_{\ln V_T^M}}{\sigma_{\ln(V_T^M)}}\right)1_{\{k_\epsilon < K - F_T\}} - \epsilon &= 0 \\ \Leftrightarrow N\left(\frac{\ln(K - F_T) - (\ln V_0^M + (r + (\theta^2/(1 - \gamma)) - (1/2)(\theta^2/((1 - \gamma)^2)))T)}{(\theta/(1 - \gamma))\sqrt{T}}\right)1_{\{k_\epsilon \geq K - F_T\}} - \epsilon & \\ + N\left(\frac{\ln(k_\epsilon) - (\ln V_0^M + (r + (\theta^2/(1 - \gamma)) - (1/2)(\theta^2/((1 - \gamma)^2)))T)}{(\theta/(1 - \gamma))\sqrt{T}}\right)1_{\{k_\epsilon < K - F_T\}} - \epsilon &= 0. \end{aligned} \quad (\text{A.20})$$

Furthermore, equation (A.7) leads to

$$V_0^M + e^{-rT}F_T + P\left(0, V_0^M, K, r, \frac{\theta}{1 - \gamma}\right) - P\left(0, V_0^M, k_\epsilon, r, \frac{\theta}{1 - \gamma}\right) - (K - k_\epsilon)e^{-rT}N\left(\frac{\ln(k_\epsilon) - (\ln V_0^M + (r - (1/2)(\theta^2/((1 - \gamma)^2)))T)}{(\theta/(1 - \gamma))\sqrt{T}}\right) - v_0 = 0. \quad (\text{A.21})$$

Finally, the three equations can be solved using a nonlinear equations solver in the corresponding environment (we use 'lsqnonlin' in MATLAB).  $\square$

**A.3. Multidimensional Extension.** Consider a vector of  $n$  risky assets with price process  $S_t$  evolving according to

$$dS_t = \text{diag}(S_t)(\mu dt + \Sigma dW_t), \quad S_0 = s > 0, \quad (\text{A.22})$$

with  $\mu = (\mu_1, \dots, \mu_n)'$ ,  $\mu_i > r$ ,  $i = 1, \dots, n$ ,  $W_t$  is a  $n$ -dimensional vector of independent Brownian motions, and  $\Sigma$  is a positive definite matrix. The investor allocates her funds  $v$  towards the risky assets and riskless asset. Let the process  $\pi = (\pi_B, \pi_S)' = (\pi_B(t), \pi_S(t))_{t \in [0, T]}$  with  $\pi(t) = (\pi_B(t), \pi_S(t))' \in \mathbb{R}^{n+1}$  be a self-financing relative portfolio process with  $\pi_B(t) = 1 - \pi_S(t)'$ . For  $t \in [0, T]$ , the wealth process  $V_t := V_t(\pi)$  with initial wealth  $V_0(\pi) = v$  evolves according to the following SDE:

$$dV_t = V_t [r + \pi'_S(\mu - r1)dt + \pi'_S \Sigma dW_t], V_0 = v. \quad (A.23)$$

The optimal allocation in risky assets ( $\pi_S$ ) for HARA is well known:

$$\pi^H(t, v) = \pi^M(t, v) \frac{v - F_t}{v}, \quad (A.24)$$

where  $F_t = F_T e^{-r(T-t)}$  and  $\pi^M(t, v)$  is the CRRA solution with

$$\pi^M(t, v) = (\Sigma \Sigma')^{-1} \frac{\mu - r1}{1 - \gamma}. \quad (A.25)$$

The solution in the presence of VaR constraints can be produced similarly to the 1-dimensional case, via the Merton solution, this is:

$$V_T^{HV} = V_T^M + F_T + (K - V_T^M) 1_{\{k_\epsilon < V_T^M < K\}}, \quad (A.26)$$

hence, similar to the proof of Proposition 4,

$$\Pi = V_t^{HV} = v^M + F_t + P\left(t, v^M, K, r, \frac{\theta_n}{1 - \gamma}\right) - P\left(t, v^M, k_\epsilon, r, \frac{\theta_n}{1 - \gamma}\right) - (K - k_\epsilon) e^{-r(T-t)} N(-d_2^Q(t, v^M, k_\epsilon)), \quad (A.27)$$

where  $\theta_n^2 = (\mu' - r1')(\Sigma' \Sigma)^{-1}(\mu - r1)$ . In terms of the optimal allocations, this leads to

$$\pi^* = \pi^{HV}(t, v^M) = \pi^M \cdot \frac{v \Pi_v}{\Pi}, \quad (A.28)$$

where

$$\frac{v \Pi_v}{\Pi} = \frac{v^M (1 + (\partial/\partial v^M)P(t, v^M, K) - (\partial/\partial v^M)P(t, v^M, k_\epsilon) - (K - k_\epsilon) e^{-r(T-t)} (\partial/\partial v^M) \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon))}{v^M + F_t + P(t, v^M, K) - P(t, v^M, k_\epsilon) - (K - k_\epsilon) e^{-r(T-t)} \mathbb{Q}_{t, v^M}(V_T^M < k_\epsilon)}, \quad (A.29)$$

and  $P(t, v^M, K)$  is short for  $P(t, v^M, K, r, (\theta_n/(1 - \gamma)))$ . As a reminder, the triple  $(v^M, k_\epsilon, \lambda_\epsilon)$  must solve the VaR constraint, investor wealth, and condition (A.17), respectively:

$$\begin{aligned} \mathbb{P}_{0, v^M}(f^{HV}(V_T^M) < K) - \epsilon &= 0, \\ \Pi(0, v^M) - v &= 0, \end{aligned} \quad (A.30)$$

$$\frac{k_\epsilon^{\gamma-1} (K - k_\epsilon) - (1/\gamma)(K^\gamma - k_\epsilon^\gamma)}{((N'(d_2^Q(\cdot, v^M, K - F_T)))/(N'(d_2^Q(\cdot, v^M, k_\epsilon)))) 1_{\{k_\epsilon \geq K - F_T\}} + 1_{\{k_\epsilon < K - F_T\}}} - \lambda_\epsilon = 0.$$

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## References

- [1] R. C. Merton, "Lifetime portfolio selection under uncertainty: the continuous-time case," *The Review of Economics and Statistics*, vol. 51, no. 3, p. 247, 1969.
- [2] J. A. Lopez, "Regulatory evaluation of value-at-risk models," *FRB of New York Staff Report*, no. 33, 1997.
- [3] M. Escobar, P. Kriebel, M. Wahl, and R. Zagst, "Portfolio optimization under solvency II," *Annals of Operations Research*, vol. 281, no. 1-2, pp. 193–227, 2018.

- [4] D. Laas and C. F. Siegel, "Basel III versus solvency II: an analysis of regulatory consistency under the new capital standards," *Journal of Risk & Insurance*, vol. 84, no. 4, pp. 1231–1267, 2017.
- [5] M. Kratz, Y. H. Lok, and A. J. McNeil, "Multinomial var backtests: a simple implicit approach to backtesting expected shortfall," *Journal of Banking & Finance*, vol. 88, pp. 393–407, 2018.
- [6] R. Zagst and J. Kraus, "Stochastic dominance of portfolio insurance strategies," *Annals of Operations Research*, vol. 185, no. 1, pp. 75–103, 2011.
- [7] F. Black and R. W. Jones, "Simplifying portfolio insurance," *Journal of Portfolio Management*, vol. 14, no. 1, pp. 48–51, 1987.
- [8] F. Black and F. André, "Theory of constant proportion portfolio insurance," *Journal of Economic Dynamics and Control*, vol. 16, no. 3-4, pp. 403–426, 1992.
- [9] H. Kraft and M. Steffensen, "A dynamic programming approach to constrained portfolios," *European Journal of Operational Research*, vol. 229, no. 2, pp. 453–461, 2013.

- [10] J. Annaert, S. Van Ossaer, and B. Verstraete, "Performance evaluation of portfolio insurance strategies using stochastic dominance criteria," *Journal of Banking & Finance*, vol. 33, no. 2, pp. 272–280, 2009.
- [11] P. Bertrand and J.-L. Prigent, "Omega performance measure and portfolio insurance," *Journal of Banking & Finance*, vol. 35, no. 7, pp. 1811–1823, 2011.
- [12] P. Bertrand and J.-L. Prigent, "Portfolio insurance strategies: obpi versus CPPI," *Finance*, vol. 26, no. 1, pp. 5–32, 2005.
- [13] Y. Zhu and R. C. Kavee, "Performance of portfolio insurance strategies," *Journal of Portfolio Management*, vol. 14, no. 3, pp. 48–54, 1988.
- [14] A. Chen, T. Nguyen, and M. Stadje, "Optimal investment under var-regulation and minimum insurance," *Insurance: Mathematics and Economics*, vol. 79, pp. 194–209, 2018.
- [15] S. Basak and A. Shapiro, "Value-at-risk-based risk management: optimal policies and asset prices," *Review of Financial Studies*, vol. 14, no. 2, pp. 371–405, 2001.
- [16] R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model," in *Stochastic Optimization Models in Finance*, pp. 621–661, Elsevier, Amsterdam, Netherlands, 1975.
- [17] J. H. Cochrane, "The sensitivity of tests of the intertemporal allocation of consumption to near-rational alternatives," Technical report, National Bureau of Economic Research, Cambridge, MA, USA, 1988.
- [18] B. Björn, H. Kraft, and C. Munk, "Solving constrained consumption–investment problems by simulation of artificial market strategies," *Management Science*, vol. 59, no. 2, pp. 485–503, 2013.
- [19] L. T. Nielsen, "Understanding  $N(d_1)$  and  $N(d_2)$ : risk-adjusted probabilities in the black-scholes model," INSEAD, Fontainebleau, France, INSEAD Working Paper, 1992.