Research Article

A Linear-Time Algorithm for 4-Coloring Some Classes of Planar Graphs

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Every graph $G = (V, E)$ considered in this paper consists of a finite set V of vertices and a finite set E of edges, together with an incidence function that associates each edge $e \in E$ of G with an unordered pair of vertices of G which are called the ends of the edge e. A graph is said to be a planar graph if it can be drawn in the plane so that its edges intersect only at their ends. A proper k-vertex-coloring of a graph $G = (V, E)$ is a mapping $c: V \longrightarrow S$ (S is a set of k colors) such that no two adjacent vertices are assigned the same colors. The famous Four Color Theorem states that a planar graph has a proper vertex-coloring with four colors. However, the current known proof for the Four Color Theorem is computer assisted. In addition, the correctness of the proof is still lengthy and complicated. In 2010, a simple $O(n^2)$ time algorithm was provided to 4-color a 3-colorable planar graph. In this paper, we give an improved linear-time algorithm to either output a proper 4-coloring of G or conclude that G is not 3-colorable when an arbitrary planar graph G is given. Using this algorithm, we can get the proper 4-colorings of 3-colorable planar graphs, planar graphs with maximum degree at most five, and claw-free planar graphs.

1. Introduction

Every *graph* $G = (V, E)$ considered here consists of a finite set V of vertices and a finite set E of edges, together with an incidence function that associates each edge of G with an unordered pair of vertices of G. If $\{u, v\}$ is the unordered pair of vertices corresponding to the edge e of G, then e is denoted by $e = uv$ and u is said to be *adjacent* to v. In addition, if $e = uv$, e is said to be *incident* to u and v and u and v are called the *ends* of e. For the standard terminology not given here, we refer the reader to [1]. The number of vertices of G is called the *order* of G. For a vertex $v \in V$, the *open neighborhood* $N(v)$ of v is defined as the set of vertices adjacent to v. The *closed neighborhood* $N[v]$ of v is defined as $N[v] = N(v) \cup \{v\}$. The *degree* of v is equal to $|N(v)|$, denoted by $d_G(v)$ or simply $d(v)$. By $\delta(G)$ and $\Delta(G)$, we denote the *minimum degree* and the *maximum degree* of graph G, respectively. A *k-regular graph* G is a graph such that every vertex of G has the degree k. For a subset $S \subseteq V$, the

induced subgraph, denoted by $G[S]$, is the subgraph of G whose vertex set is S and whose edge set consists of all edges of G which have both ends in S. A complete graph is a simple graph in which any two vertices are adjacent. Let K_n denote the complete graph on n vertices. Usually, K_3 is called a *triangle*. A *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and are nonadjacent otherwise. A graph G is bipartite if its vertex set can be partitioned into two sets V_1 and V_2 so that every edge has one end in V_1 and the other one in V_2. Specially, if every vertex in V_1 is adjacent to every vertex in V_2, then G is called a complete bipartite graph. As usual, $K_{m,n}$ denotes a complete bipartite graph with classes of cardinality m and n. The graph $K_{1,3}$ is also called a *claw*. Given a graph F, a graph G is *F-free* if it does not contain F as an induced subgraph. In particular, a $K_{1,3}$-free graph is *claw-free*. By starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of

H, one obtains the *join* of G and H, denoted by $G \vee H$. The join $C_n \vee K_1$ of a cycle C_n and a single vertex is referred to as a *n-wheel* denoted by W_n.

A graph is said to be a *planar graph* if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is called a *planar embedding* of the graph. Any such particular embedding is called a *plane graph*. A proper *k-vertex-coloring*, or simply a *proper k-coloring*, of a graph $G = (V, E)$ is a mapping $c: V \longrightarrow S$ (S is a set of k colors) such that no two adjacent vertices are assigned the same colors. In this paper, we refer to a proper coloring as a "coloring" and to a proper k-coloring as a "k-coloring." The famous Four Color Theorem states that a planar graph has a proper coloring with four colors. The original proof of the Four Color Theorem by Appel and Haken [2] and Appel et al. [3] relies heavily on the computer for checking details involved in finding an unavoidable set and verifying that all configurations in that set are reducible. It employs no fewer than 487 discharging rules, resulting in a set of over 1400 unavoidable configurations. The more recent proof by Robertson et al. [4], although also dependent on the computer, is simpler in many ways. In their proof, thirty-two discharging rules are needed, generating a list of 633 unavoidable configurations.

In 2010, a simple $O(n^2)$ time algorithm is provided to 4-color a 3-colorable planar graph by Kawarabayashi and Ozeki [5]. In this paper, given a planar graph G, we design an improved linear-time algorithm to either output a 4-coloring of G or conclude that G is not 3-colorable. Using this algorithm, we can get the 4-colorings of 3-colorable planar graphs, planar graphs with maximum degree at most five, and claw-free planar graphs.

2. Proof and Algorithm

First, we give some definitions and lemmas which are related to our algorithm. A plane graph G partitions the plane into a number of arcwise-connected open sets which are called the *faces* of G. We call f a *k-face* if f is incident to k edges of G. For a simple plane graph G, we call a vertex v of degree five *bad* if all faces incident with v, except for the at most one, are triangles and the exceptional face has size at most five. Moreover, v is Type I, Type II, and Type III if the exceptional face is a triangle, a 4-face and a 5-face, respectively, as shown in Figure 1. Obviously, if a plane G has a bad vertex of Type I, then G is not 3-colorable. Note that, if v is a bad vertex of Type I of the plane graph, the induced subgraph $G[N[v]]$ is not necessarily a 5-wheel since $G[N(v)]$ may have a triangle and thus $G[N[v]]$ has a K_4. To *identify* nonadjacent vertices x and y of a graph G is to replace the two vertices by a single vertex incident to all the edges which are incident to either x or y. About the bad vertex v of Type II or III, the following observation is obvious.

Observation 1. Let v be a bad vertex of Type II or III of G (see Figure 1). Then, u_1 and u_2 are contained in the same color class for any 3-coloring of G. Let G' be the graph obtained from G by identifying u_1 and u_2. Then, G is 3-colorable if and only if G' is 3-colorable.

The following lemmas are useful for our proofs.

Lemma 2 (see [5]). *Every simple planar graph G contains (i) a vertex of degree at most 4 or (ii) a bad vertex.*

Lemma 3 (see [5]). *Let G be a plane graph with a 4-coloring c', and let f be a face of size at least four. Take four vertices x_1, x_2, x_3, and x_4 (along clockwise order) in f. Then, G also has a 4-coloring such that at most three colors are used for x_1, x_2, x_3, and x_4. Moreover, given the graph G and the coloring c', we can find such a 4-coloring of G in $O(n)$ time, where $n = |G|$.*

Lemma 4 (see [6]). *If v is a vertex of planar graph G and $G[N[v]]$ is {claw, K_4}-free, then $d(v) \le 5$ and $G[N[v]]$ is a 5-wheel if $d(v) = 5$.*

The *icosahedron* is the 5-regular planar graph in Figure 2. In order to give our algorithm, we first give a well known fact about planar graphs.

Observation 5. The icosahedron is the unique 5-regular planar graph G such that, for every vertex $v \in V(G)$, the induced subgraph $G[N[v]]$ is a 5-wheel.

Further, we can prove the following theorem.

Theorem 6. *Let G be a claw-free planar graph. If G is not the icosahedron, then G has a vertex of degree at most four.*

Proof. Let $G = (V, E)$ be a claw-free planar graph. If G has no vertex of degree at most four, then $\delta(G) \ge 5$. In the following, we just prove that G is the icosahedron. If G has no K_4, by Lemma 4, every vertex v of G induces a 5-wheel of G. So, by Observation 5, G is the icosahedron. If not, let $[x_1 x_2 x_3 x_4]$ be a K_4 of G and assume that x_1 is inside the triangle $[x_2 x_3 x_4]$ in the embedding of G on the plane. Without losing of the generality, we may assume that there is no other K_4 inside the triangle $[x_2 x_3 x_4]$. Suppose not, we can continue to find another K_4 inside the triangle $[x_2 x_3 x_4]$ and consider the inner situation in this K_4 until finding such a triangle since G is a finite graph. Note that $d(x_1) \ge 5$. Assume that some neighbors of x_1 are inside the triangle $[x_1 x_2 x_3]$. Let G^* be the plane graph induced by the vertices on and inside $[x_1 x_2 x_3]$. We have the following claim. □

Claim 1. For every vertex x inside $[x_1 x_2 x_3]$, $G[N[x]] = G^*[N[x]]$ and $G[N[x]]$ is isomorphic to a 5-wheel by Lemma 4 and the assumption that there is no K_4 in G^*.

Take any vertex v in $V(G^*) - \{x_1, x_2, x_3\}$. $G[N[v]]$ is isomorphic to a 5-wheel by Claim 1. Let $C_v = v_1 v_2 v_3 v_4 v_5 v_1$ be the cycle of 5-wheel $G[N[v]]$. Note that at least three neighbors of v are inside the triangle $[x_1 x_2 x_3]$ since it is impossible that three neighbors of v are the vertices of the triangle $[x_1 x_2 x_3]$. Without losing of the generality, we assume that v_1, v_2, and v_3 are inside $[x_1 x_2 x_3]$. Then, $G[N[v_1]]$, $G[N[v_2]]$, and $G[N[v_3]]$ are isomorphic to the 5-wheel by Claim 1. Let $C_{v_1} = v_2 v v_5 v_6 v_7 v_2$ be the cycle of 5-wheel $G[N[v_1]]$ and $C_{v_2} = v_3 v v_1 v_7 v_8 v_3$ be the cycle of 5-wheel
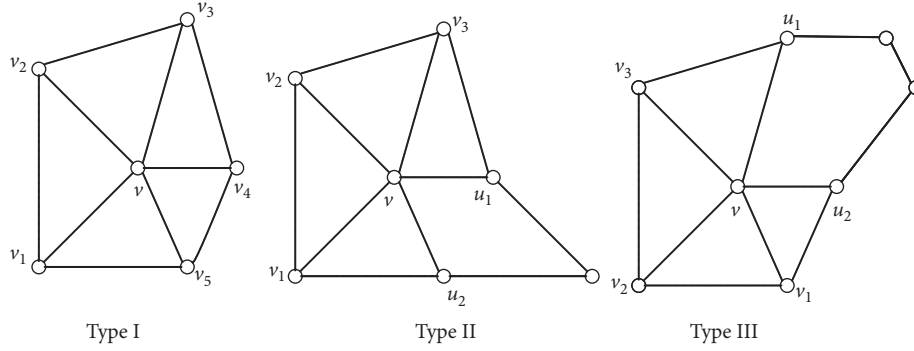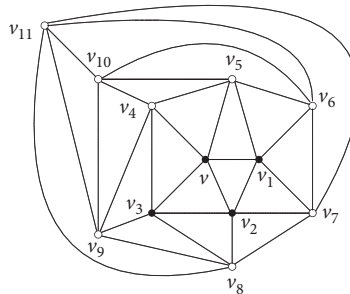
Figure 1: A bad vertex.

Figure 2: The icosahedron.

$G[N[v_2]]$ (see Figure 3). Further, let $C_{v_3} = v_4vv_2v_8v_9v_4$ be the cycle of 5-wheel $G[N[v_3]]$ (see Figure 4). Note that v_4 is not adjacent to v_8 in Figure 4, without losing of the generality, assume that v_4 is inside $[x_1x_2x_3]$. Then, $G[N[v_4]]$ is a 5-wheel, and we claim that neither v_6 nor v_7 is the fifth neighbor of v_4. If v_7 is the fifth neighbor of v_4, by the fact that $G[N[v_4]]$ is a 5-wheel, v_5 is also adjacent to v_7, a contradiction to the fact that $G[N[v_1]]$ is a 5-wheel. If v_6 is the fifth neighbor of v_4, by the fact that $G[N[v_4]]$ is a 5-wheel, v_9 is also adjacent to v_6. Then, no matter how we draw the two edges v_4v_6 and v_9v_6 in the plane, we can get a vertex of degree four by the claw-freeness, a contradiction. So, let $C_{v_4} = v_5vv_3v_9v_{10}v_5$ be the cycle of 5-wheel $G[N[v_4]]$. By the claw-freeness, v_{10} is adjacent to v_6. We claim that v_5 is also inside $[x_1x_2x_3]$, and thus $G[N[v_5]]$ is a 5-wheel (see Figure 5). Suppose not, if $v_5 = x_1$, $\{x_4, v_5(x_1), v_4, v_1\}$ would induce a claw, a contradiction. Consider the vertices in $\{v_6, v_7, v_8, v_9, v_{10}\}$ in Figure 5. Without losing of the generality, assume that v_{10} is inside $[x_1x_2x_3]$. Then, $d_{G^*}(v_{10}) = 5$, and $G[N[v_{10}]]$ is a 5-wheel. We can easily see that neither v_7 nor v_8 is the fifth neighbor of v_{10}. So, let $C_{v_{10}} = v_9v_4v_5v_6v_{11}v_9$ be the cycle of 5-wheel $G[N[v_{10}]]$. Hence, $\{v_{11}v_6, v_{11}v_9\} \subset E(G^*)$ by the claw-freeness and both v_6 and v_9 are inside $[x_1x_2x_3]$. Further, we have that $\{v_{11}v_7, v_{11}v_8\} \subset E(G^*)$ by the claw-freeness (see Figure 2). Further, by the claw-freeness, no one in $\{v_7, v_8, v_{11}\}$ is in the set $\{x_1, x_2, x_3\}$. Suppose not, if $v_{11} = x_1$, $\{x_4, v_{11}(x_1), v_6, v_9\}$ would induce a claw, a contradiction. Then, all the vertices inside the triangle $[x_1x_2x_3]$ induce the icosahedron (see Figure 2), still a contradiction to the assumption that some neighbor of x_1 is inside the triangle $[x_1x_2x_3]$.
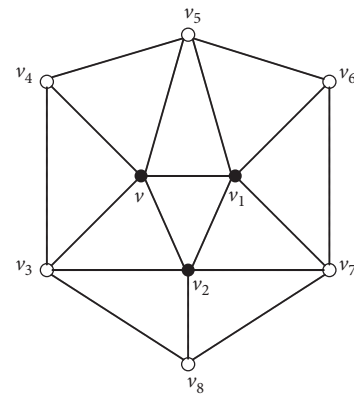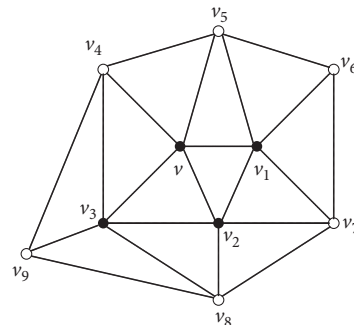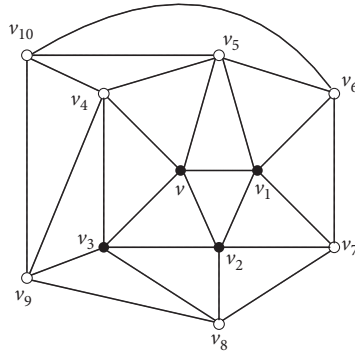

Figure 3: $G[N[\{v, v_1, v_2\}]]$.

Figure 4: $G[N[\{v, v_1, v_2, v_3\}]]$.

FIGURE 5: $G[N[\{v, v_1, v_2, v_3, v_4, v_5\}]]$.

Input. A planar graph $G = (V, E)$.
Output. A 4-coloring of G or the decision that $\chi(G) > 3$.
Step 1. Give a partition $V(G) = V_1 \cup V_2 \cup V_3$ such that the vertices in V_1 have degree at most four and the vertices in V_2 are bad vertices of Type II and III.
Step 2. If the current graph is the icosahedron or have at most four vertices, we give a 4-coloring of G directly. If not, first take a vertex v in V_1. Let $G' = G - v$ and carry out $A(G')$, turn to step 3. If there is no such a vertex, take a bad vertex v of Type II or III. Identifying u_1 and u_2 and denoting it by G', then we carry out $A(G')$, turn to step 5. If there is no such a vertex, stop the algorithm and output that $\chi(G) > 3$.
Step 3. If we get a 4-color of G', turn to step 4. If we get that $\chi(G') > 3$, then stop the algorithm $A(G)$ and output that $\chi(G) > 3$.
Step 4. Find a 4-color of $G - v$ such that at most three colors are used in $N(v)$ (see Lemma 3). Give a 4-coloring of G. Stop the algorithm.
Step 5. If we get a 4-color of G', extend the coloring of G' into a 4-coloring of G by assigning u_1 and u_2 the same color as in G', stop the algorithm. If we get that $\chi(G') > 3$, then stop the algorithm $A(G)$ and output that $\chi(G) > 3$.

ALGORITHM 1: An algorithm of coloring planar graphs.

Immediately, we have the following corollary.

Corollary 7. *The icosahedron is the unique 5-regular planar graph such that every vertex is a bad vertex of Type I.*

Proof. Let $G = (V, E)$ be a 5-regular planar graph such that every vertex is a bad vertex of Type I. Then, G is also a claw-free planar graph. By Theorem 6, G is the icosahedron.

Based on Observation 1, Lemma 2, Lemma 3, Corollary 7, and Theorem 6, we design an algorithm as follows. $\qquad\square$

Theorem 8. *Given a planar graph G, Algorithm 1 is a linear-time algorithm to output either a 4-coloring of G or the decision that G is not 3-colorable. In addition, if G is a 3-colorable planar graph or a planar graph with maximum degree at most five or a claw-free planar graph, Algorithm 1 gives a 4-coloring of G.*

Proof. Based on Lemmas 2 and 3 and the fact that identifying u_1 and u_2 in step 2 keeps the 3-colorability of G, we can say that Algorithm 1 outputs either a 4-coloring of G or the decision that G is not 3-colorable. Now, we prove that Algorithm 1 is a linear-time algorithm by induction on the order of G. Assume that it holds when the order of G is less than n. Then, let $|G| = n$. In step 1, we can find such a

partition in at most $O(n)$ time. In step 2, we can give a 4-coloring of G directly in constant time if G is the icosahedron or has at most four vertices. If not, we can construct a graph G' in constant time at most. Note that $|G'| = n - 1$. By induction, $A(G')$ needs $O(n - 1)$ time which is the total time of all repeated steps from 1 to 5 in $A(G')$. In step 3, it needs only the constant time by deciding the result of $A(G')$. In step 4, we can give a 4-coloring of G from the 4-coloring of G' in $O(n)$ time by Lemma 3. In step 5, it needs only the constant time. Thus, the total time is also $O(n)$.

If G is a 3-colorable planar graph, we are sure to get a 4-coloring of G by Observation 1 and Lemmas 2 and 3. If G is a planar graph with maximum degree at most five, we just consider the case that a vertex of degree greater than five occurs in G' by identifying u_1 and u_2 of G in step 2. When a vertex of degree greater than five occurs in G', then v becomes a vertex of degree four of G' (see Figure 1). According to our algorithm, we first take a vertex v of degree at most 4 in step 2 when we carry out $A(G')$. Hence, after several steps, the vertex of degree greater than five will disappear. So, the final G' will be the icosahedron or a graph with at most four vertices by Corollary 7, and we get a 4-coloring of G at last. If G is a claw-free planar graph, we can get a 4-coloring of G by Lemma 3 and Theorem 6. $\qquad\square$

Data Availability

Our result is supported by the rigorous proofs in the submitted paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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