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Research Article

Statistical Analysis for Competing Risks' Model with Two Dependent Failure Modes from Marshall–Olkin Bivariate Gompertz Distribution

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The bivariate or multivariate distribution can be used to account for the dependence structure between different failure modes. This paper considers two dependent competing failure modes from Gompertz distribution, and the dependence structure of these two failure modes is handled by the Marshall–Olkin bivariate distribution. We obtain the maximum likelihood estimates (MLEs) based on classical likelihood theory and the associated bootstrap confidence intervals (CIs). The posterior density function based on the conjugate prior and noninformative (Jeffreys and Reference) priors are studied; we obtain the Bayesian estimates in explicit forms and construct the associated highest posterior density (HPD) CIs. The performance of the proposed methods is assessed by numerical illustration.

1. Introduction

It is extremely common that the failure of a product or a system contains several competing failure modes in reliability engineering; any failure mode will lead to the failure result. Competing risks' data contain the failure time and the corresponding failure mode, which can be modeled by the competing risks' model and has been commonly performed in many research fields, such as engineering and medical statistics. Previous studies have mostly assumed the competing failure modes to be independent; Wang et al. [1], Ren and Gui [2], and Qin and Gui [3] focused on the independent competing risks' model under progressively hybrid censoring from Weibull and Burr-XII distributions. Objective Bayesian analysis for the competing risks' model with Wiener degradation phenomena and catastrophic failures was studied by Guan et al. [4]. In practice, the independency relationship between different failure modes is a very special case; a more common situation is dependency. That is, the failure mechanisms are interactive and interdependent; the occurrence of one failure mode will affect the occurrence of other failure modes. For example, a ship fixed carbon dioxide fire extinguishing system can fail due to pressure gauge, distribution value, cylinder group, and so on; these failure modes are dependent because they all are related to the storage environment. Therefore, it is more reasonable to assume dependency among different competing failure modes. The competing risks' model considers the product or system with multiple dependent competing failure modes, any one of which will cause the occurrence of failure. The dependent competing risks' model has been extensively studied. Zhang et al. [5] and Zhang et al. [6] studied the dependent competing risks' model under accelerated life testing (ALT) by copula function to measure the dependence between different competing failure modes; the results indicate the copula construction method has good accuracy and universality. Wang and Yan [7] and Wu et al. [8] also studied this model under ALT and progressively hybrid-censoring scheme using Clayton copula and Gumbel copula, respectively. For other related works, see the works of Lo and Wike [9] and Fang et al. [10].

In addition to using copula function to handle the relationship between different competing failure modes, the bivariate or multivariate distribution also can be used to account for the correlation between different failure modes. The Marshall–Olkin distribution [11], which has many good properties, is the best-known bivariate distribution and has been discussed extensively; it has a parameter to describe the dependence structure. Li et al. [12], Kundu and Gupta [13], and Bai et al. [14] provided reviews on Marshall–Olkin–Weibull distribution; Kundu and Gupta [13] obtained the explicit forms of the unknown parameters when the shape parameter is known; when the shape parameter is unknown, they used the importance sampling to compute the Bayesian estimates of the unknown parameters. Bai et al. [14] discussed the statistical analysis for the accelerated dependent competing risks' model under Type-II hybrid censoring schemes. Guan et al. [15] studied objective Bayesian analysis for the Marshall–Olkin exponential distribution based on reference priors; they also found that some of the reference priors are also matching priors and the posterior distributions based on these priors are proper.

The Gompertz distribution is a widely used growth model which has been studied extensively; Ismail [16] studied the Bayesian analysis of Gompertz distribution parameters and acceleration factor in the case of partially accelerated life testing under Type-I censoring scheme. Ghitany et al. [17] considered a progressively censored sample from Gompertz distribution; they discussed the existence and uniqueness of the MLEs of the unknown parameters. The Gompertz distribution plays an important role in fitting clinical trials' data in medical science and can be used to the theory of extreme-order statistics. In this paper, we will study the dependent competing risks' model from the Marshall–Olkin bivariate Gompertz (MOGP) distribution, which is a bivariate distribution with Gompertz marginal distributions. We focus our attention on the statistical analysis of the model parameters, including classical likelihood inference, Bayesian analysis, and objective Bayesian analysis. Because the Bayesian analysis based on conjugate prior is sensitive to the hyperparameters, inappropriate choice of it will cause bad priors. Based on this reason, we propose the objective Bayesian analysis based on noninformative priors for comparison. The objective Bayesian inference has been studied by Guan et al. [14], Bernardo [18], and Berger and Bernardo [19] based on Reference and Jeffreys priors.

In the rest of this paper, we will present the model description and some properties. Section 3 presents the MLEs and associated bootstrap CIs. In Section 4, Bayesian estimates and associated HPD CIs based on conjugate prior, Jeffreys prior [20], and reference priors [18] are obtained, and these priors lead to proper posteriors which are proved. Section 5 presents some results obtained from simulation study and illustrative analysis. Section 6 gives some final concluding remarks.

2. Model Description

Suppose that $f(t;\lambda,\theta)$ is a Gompertz distribution; the density function and reliability function of it are

$$\begin{aligned} f(t;\lambda,\theta) &= \theta e^{(\lambda t-\theta(e^{\lambda t}-1)/\lambda)},\ \lambda,\ \theta>0,\ t>0,\\ S(t;\lambda,\theta) &= e^{(-\theta(e^{\lambda t}-1)/\lambda)},\ \lambda,\ \theta>0,\ t>0, \end{aligned} \tag{1}$$

where λ is shape parameter and θ is scale parameter.

Suppose U_0, U_1, and U_2 are three independent Gompertz variables with different scale parameters, that is, $U_0 \sim GP(\lambda,\theta_0)$, $U_1 \sim GP(\lambda,\theta_1)$, and $U_2 \sim GP(\lambda,\theta_2)$. Let $T_1=\min(U_0,U_1)$ and $T_2=\min(U_0,U_2)$; we obtain $T_1 \sim GP(\lambda,\theta_0+\theta_1)$ and $T_2 \sim GP(\lambda,\theta_0+\theta_2)$. Then, the pair of variables (T_1,T_2) follows the MOGP distribution denoted by $(T_1,T_2) \sim \mathrm{MOGP}(\lambda,\theta_0,\theta_1,\theta_2)$. When $\theta_0=0$, the two variables T_1 and T_2 are independent and T_1 and T_2 will be dependent when $\theta_0>0$; hence, θ_0 can be regarded as a correlation coefficient between T_1 and T_2.

The joint PDF of $(T_1,\ T_2)$ can be written as

$$f_{T_1,T_2}(t_1,t_2;\lambda,\theta_0,\theta_1,\theta_2) = \begin{cases} f(t_1;\lambda,\theta_0+\theta_1)f(t_2;\lambda,\theta_2) & t_1>t_2 \\ f(t_1;\lambda,\theta_1)f(t_2;\lambda,\theta_0+\theta_2) & t_1<t_2. \\ \left(\dfrac{\theta_0}{(\theta_0+\theta_1+\theta_2)}\right)f(t;\lambda,\theta_0+\theta_1+\theta_2) & t_1=t_2=t \end{cases} \tag{2}$$

The surface plots of $f_{T_1,T_2}(t_1,t_2;\lambda,\theta_0,\theta_1,\theta_2)$ are presented in Figure 1. From Figure 1, we can see that $f_{T_1,T_2}(t_1,t_2;\lambda,\theta_0,\theta_1,\theta_2)$ is a unimodal function.

Put n identical products into test, and each product has two dependent failure modes with lifetimes T_1, T_2, $(T_1,T_2) \sim \mathrm{MOGP}(\lambda,\theta_0,\theta_1,\theta_2)$. Then, the system lifetime is $X=\min(T_1,T_2) \sim \mathrm{MOGP}(\lambda,\theta_0+\theta_1+\theta_2)$. Let $\delta_{0l}=I(T_{1l}=T_{2l})$, $\delta_{1l}=I(T_{1l}<T_{2l})$, and $\delta_{2l}=I(T_{1l}>T_{2l})$, for $l=1,\cdots,n$, where $I(\cdot)$ is an indicator function. Then, we can compute $n_0=\sum_l \delta_{0l}$, $n_1=\sum_l \delta_{1l}$, $n_2=\sum_l \delta_{2l}$, and $n=n_0+n_1+n_2$.

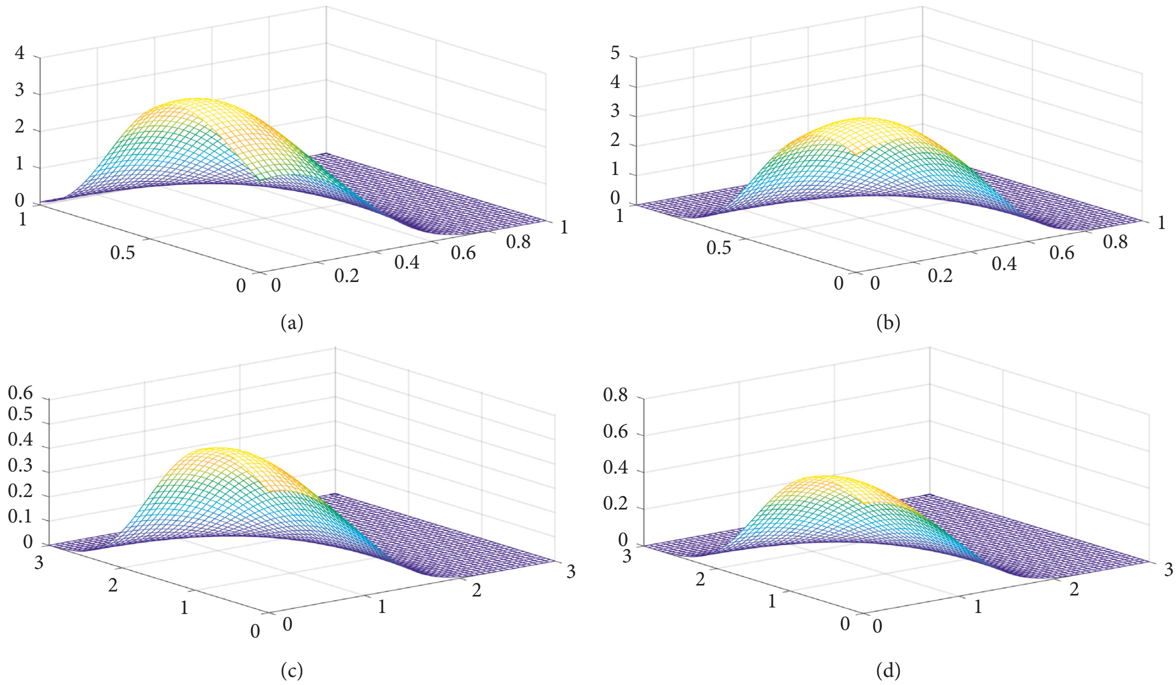

FIGURE 1: Surface plot of $f_{T_1,T_2}(t_1,t_2;\lambda,\theta_0,\theta_1,\theta_2)$ with different values of λ, θ_0, θ_1, θ_2. (a) $(\lambda, \theta_0, \theta_1, \theta_2) = (3, 0.5, 2, 1)$. (b) $(\lambda, \theta_0, \theta_1, \theta_2) = (3, 1.5, 0.5, 2)$. (c) $(\lambda, \theta_0, \theta_1, \theta_2) = (1, 0.5, 0.5, 0.5)$. (d) $(\lambda, \theta_0, \theta_1, \theta_2) = (1, 0.2, 0.8, 0.6)$.

Theorem 1. *For* $l = 1,\cdots,n$, $\delta_{0l} = I(T_{1l} = T_{2l})$, $\delta_{1l} = I(T_{1l} < T_{2l})$, *and* $\delta_{2l} = I(T_{1l} > T_{2l})$, *We have*

$$(\delta_{0l}, \delta_{1l}, \delta_{2l}) \sim \text{Multinomial}\left(1; \frac{\theta_0}{\theta_0+\theta_1+\theta_2}, \frac{\theta_1}{\theta_0+\theta_1+\theta_2}, \frac{\theta_2}{\theta_0+\theta_1+\theta_2}\right), l = 1,\cdots,n. \tag{3}$$

Proof. For $l = 1,\cdots,n$, we have $\delta_{0l} + \delta_{1l} + \delta_{2l} = 1$,

$$\begin{aligned} P(T_1 < T_2) &= \int_0^\infty \int_0^{t_2} f(t_1;\lambda,\theta_1) f(t_2;\lambda,\theta_0+\theta_2)\mathrm{d}t_1\mathrm{d}t_2 = \frac{\theta_1}{\theta_0+\theta_1+\theta_2}, \\ P(T_1 > T_2) &= \int_0^\infty \int_0^{t_1} f(t_1;\lambda,\theta_0+\theta_1) f(t_2;\lambda,\theta_2)\mathrm{d}t_2\mathrm{d}t_1 = \frac{\theta_2}{\theta_0+\theta_1+\theta_2}, \\ P(T_1 = T_2) &= 1 - \frac{\theta_1}{\theta_0+\theta_1+\theta_2} - \frac{\theta_2}{\theta_0+\theta_1+\theta_2} = \frac{\theta_0}{\theta_0+\theta_1+\theta_2}. \end{aligned} \tag{4}$$

Therefore, $(\delta_{0l}, \delta_{1l}, \delta_{2l}) \sim \text{Multinomial}(1; \theta_0/(\theta_0+\theta_1+\theta_2), \theta_1/(\theta_0+\theta_1+\theta_2), \theta_2/(\theta_0+\theta_1+\theta_2))$.

The likelihood function is

$$\begin{aligned} L(\lambda,\theta_0,\theta_1,\theta_2) = \prod_l \left[f_{T_1,T_2}(x_l,x_l;\lambda,\theta_0,\theta_1,\theta_2)\right]^{\delta_{0l}} \left[-\frac{\partial S_{T_1,T_2}(t_1,t_2;\lambda,\theta_0,\theta_1,\theta_2)}{\partial t_1}\Big|_{(x_l,x_l)}\right]^{\delta_{1l}}, \\ \times \left[-\frac{\partial S_{T_1,T_2}(t_1,t_2;\lambda,\theta_0,\theta_1,\theta_2)}{\partial t_2}\Big|_{(x_l,x_l)}\right]^{\delta_{2l}}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} f_{T_1,T_2}(x_l,x_l;\lambda,\theta_0,\theta_1,\theta_2) &= \frac{\theta_0}{\theta_0+\theta_1+\theta_2} f(t;\lambda,\theta_0+\theta_1+\theta_2), \\ &= \theta_0 \exp\left\{\lambda x_l - \frac{\theta_0+\theta_1+\theta_2}{\lambda}\left(e^{\lambda x_l}-1\right)\right\}, \\ &- \frac{\partial S_{T_1,T_2}(t_1,t_2;\lambda,\theta_0,\theta_1,\theta_2)}{\partial t_1}\Big|_{(x_l,x_l)}, \\ &= \theta_1 \exp\left\{\lambda x_l - \frac{\theta_0+\theta_1+\theta_2}{\lambda}\left(e^{\lambda x_l}-1\right)\right\}, \\ &- \frac{\partial S_{T_1,T_2}(t_1,t_2;\lambda,\theta_0,\theta_1,\theta_2)}{\partial t_2}\Big|_{(x_l,x_l)}, \\ &= \theta_2 \exp\left\{\lambda x_l - \frac{\theta_0+\theta_1+\theta_2}{\lambda}\left(e^{\lambda x_l}-1\right)\right\}. \end{aligned} \tag{6}$$

Then, we obtain

$$L(x;\lambda,\theta_0,\theta_1,\theta_2) = \theta_0^{n_0}\theta_1^{n_1}\theta_2^{n_2} \exp\left\{\lambda\sum_l x_l - \frac{\theta_0+\theta_1+\theta_2}{\lambda}\left[\sum_l\left(e^{\lambda x_l}-1\right)\right]\right\}. \tag{7}$$

□

3. Classical Inference

3.1. Maximum Likelihood Estimates (MLEs). The MLEs of θ_0, θ_1, θ_2, and λ can be obtained by maximizing the logarithm of $L(x;\lambda,\theta_0,\theta_1,\theta_2)$. Set the first partial derivation of $\log L(x;\lambda,\theta_0,\theta_1,\theta_2)$ about θ_0, θ_1, θ_2, λ to 0, i.e.,

$$\begin{aligned} \frac{\partial \log L(x;\lambda,\theta_0,\theta_1,\theta_2)}{\partial\lambda} &= \sum_l x_l + \frac{\theta_0+\theta_1+\theta_2}{\lambda^2}\left[\sum_l\left(e^{\lambda x_l}-1\right)\right] - \frac{\theta_0+\theta_1+\theta_2}{\lambda}\left(\sum_l x_l e^{\lambda x_l}\right) = 0, \\ \frac{\partial \log L(x;\lambda,\theta_0,\theta_1,\theta_2)}{\partial\theta_j} &= \frac{n_j}{\theta_j} - \frac{1}{\lambda}\left[\sum_l\left(e^{\lambda x_l}-1\right)\right] = 0, \quad j = 0,1,2. \end{aligned} \tag{8}$$

From (8), we get the MLEs of θ_0, θ_1, and θ_2 as

$$\widehat{\theta}_j(\lambda) = \frac{n_j\lambda}{\left[\sum_l\left(e^{\lambda x_l}-1\right)\right]}, \quad j = 0,1,2. \tag{9}$$

Substituting $\widehat{\theta}_j(\lambda)$ into $\log L(x;\lambda,\theta_0,\theta_1,\theta_2)$, we obtain

$$h(\lambda) = \lambda\sum_l x_l + \sum_{j=0}^{2}\ln\left(\frac{n_j\lambda}{\sum_l\left(e^{\lambda x_l}-1\right)}\right)^{n_j}, \tag{10}$$

which is the profile logarithm likelihood function of λ.

We can show that $\partial^2 h(\lambda)/\partial\lambda^2 < 0$, which implies that $h(\lambda)$ is concave. Some iterative schemes can be used to find the MLE for λ, such as Newton–Raphson algorithm.

3.2. Bootstrap Confidence Intervals (CIs). Since it is hard to construct the exact CIs for the unknown parameters, we consider the Bootstrap method to construct CIs for parameters θ_0, θ_1, θ_2, and λ. The Bootstrap method is a resampling method to estimate some statistical characteristics for the unknown parameters by taking samples from the original samples repeatedly; the obtained samples are called Bootstrap samples. This method has a great practical value since it does not need to assume the overall distribution or construct the pivot quantity. We generate the Bootstrap sample by the following three steps:

Step 1: for the fixed value of n and observed data $(x_1, x_2, \cdots, x_n)$, we get the estimates $\hat{\lambda}$, $\hat{\theta}_0$, $\hat{\theta}_1$, and $\hat{\theta}_2$ based on the maximum likelihood method.

Step 2: for the values of $n, \hat{\lambda}$, $\hat{\theta}_0$, $\hat{\theta}_1$, and $\hat{\theta}_2$, we generate the sample $(x_1^*, x_2^*, \cdots, x_n^*)$. Then, get the MLEs $\hat{\lambda}''$, $\hat{\theta}''_0$, $\hat{\theta}''_1$, and $\hat{\theta}''_2$.

Step 3: repeat Step 2 M times to obtain M sets of the values $\hat{\lambda}''$, $\hat{\theta}''_0$, $\hat{\theta}''_1$, and $\hat{\theta}''_2$. Arrange them as follows to get the Bootstrap sample:

$$\left\{\hat{\theta}''_{0[1]} < \cdots < \hat{\theta}''_{0[M]}, \hat{\theta}''_{1[1]} < \cdots < \hat{\theta}''_{1[M]}, \hat{\theta}''_{2[1]} < \cdots < \hat{\theta}''_{2[M]}, \hat{\lambda}''_{[1]} < \cdots < \hat{\lambda}''_{[M]}\right\}. \tag{11}$$

Based on the Bootstrap sample and by percentile Bootstrap (Boot-P) method, we construct the Boot-P CIs for θ_0, θ_1, θ_2, λ at $1-\gamma$ confidence level as

$$\left(\hat{\theta}''_{0[M\gamma/2]}, \hat{\theta}''_{0[M(1-\gamma/2)]}\right), \left(\hat{\theta}''_{1[M\gamma/2]}, \hat{\theta}''_{1[M(1-\gamma/2)]}\right), \left(\hat{\theta}''_{2[M\gamma/2]}, \hat{\theta}''_{2[M(1-\gamma/2)]}\right), \left(\lambda''_{[M\gamma/2]}, \lambda''_{[M(1-\gamma/2)]}\right). \tag{12}$$

4. Bayesian Inference and HPD CIs

4.1. Conjugate Prior. In this section, we suppose the shape parameter λ is known. Denote $\theta = \theta_0 + \theta_1 + \theta_2$, which has a Gamma prior with hyperparameters a and b as

$$\pi(\theta) = \left(\frac{b^a}{\Gamma(a)}\right)\theta^{a-1}e^{-b\theta}, \quad a > 0,\ b > 0,\ \theta > 0. \tag{13}$$

Due to $\theta_0/\theta + \theta_1/\theta + \theta_2/\theta = 1$, so given θ, $(\theta_1/\theta, \theta_2/\theta)$ follows a Dirichlet prior with hyper parameters c_0, c_1, and c_2, that is,

$$\pi_D\left(\frac{\theta_1}{\theta}, \frac{\theta_2}{\theta}|\theta\right) = \frac{\Gamma\left(\sum_{i=0}^{2} c_i\right)}{\prod_{i=0}^{2}\Gamma(c_i)}\prod_{i=0}^{2}\left(\frac{\theta_i}{\theta}\right)^{c_i-1}, \quad \theta_i > 0,\ c_i > 0,\ i = 0, 1, 2. \tag{14}$$

Therefore, the joint prior of θ_0, θ_1, and θ_2 becomes

$$\pi_1(\theta_0, \theta_1, \theta_2; a, b, c_0, c_1, c_2) = \frac{\Gamma(c)}{\Gamma(a)}(b\theta)^{a-c}\prod_{i=0}^{2}\frac{b^{c_i}}{\Gamma(c_i)}\theta_i^{c_i-1}\exp(-b\theta_i), \tag{15}$$

where $c = c_0 + c_1 + c_2$.

4.2. Jeffreys Prior. According to Jeffreys [20], Jeffreys prior is proportional to the square root of the determinant of the Fisher information matrix. From (7), we obtain the Fisher information matrix of $(\theta_0, \theta_1, \theta_2)$ as

$$I = I(\theta_0, \theta_1, \theta_2) = \begin{pmatrix} \frac{n_0}{\theta_0^2} & 0 & 0 \\ 0 & \frac{n_1}{\theta_1^2} & 0 \\ 0 & 0 & \frac{n_2}{\theta_2^2} \end{pmatrix}. \tag{16}$$

From Theorem 1, we have $n_i = n \cdot \theta_i/(\theta_0 + \theta_1 + \theta_2)$, $i = 0, 1, 2$, so $I(\theta_0, \theta_1, \theta_2)$ can be written as

$$I = n\begin{pmatrix} \frac{1}{(\theta_0\theta)} & 0 & 0 \\ 0 & \frac{1}{(\theta_1\theta)} & 0 \\ 0 & 0 & \frac{1}{(\theta_2\theta)} \end{pmatrix}. \quad (17)$$

Thus, the Jeffreys prior is given by

$$\pi_2(\theta_0, \theta_1, \theta_2) \propto \sqrt{\frac{1}{\theta_0\theta_1\theta_2\theta^3}}. \quad (18)$$

Theorem 2. *Based on the Jeffreys prior $\pi_2(\theta_0, \theta_1, \theta_2)$, the joint posterior distribution of $(\theta_0, \theta_1, \theta_2)$ is proper.*

Proof. From (6) and (7), we obtain the joint posterior distribution of $(\theta_0, \theta_1, \theta_2)$ based on $\pi_2(\theta_0, \theta_1, \theta_2)$ as

$$\begin{aligned}\pi_2(\theta_0, \theta_1, \theta_2 | x) &= \frac{L(x; \lambda, \theta_0, \theta_1, \theta_2)\pi_2(\theta_0, \theta_1, \theta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(x; \lambda, \theta_0, \theta_1, \theta_2)\pi_2(\theta_0, \theta_1, \theta_2)d\theta_0 d\theta_1 d\theta_2}, \\ &\propto \theta^{(-3/2)}\theta_0^{(n_0-1/2)}\theta_1^{(n_1-1/2)}\theta_2^{(n_2-1/2)}e^{(-A\theta/\lambda)}.\end{aligned} \quad (19)$$

Integrating $\pi_2(\theta_0, \theta_1, \theta_2 | x)$ with respect to θ_0, θ_1, and θ_2, we obtain

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty \theta^{-3/2}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}e^{(-A\theta/\lambda)}d\theta_0 d\theta_1 d\theta_2, \\ &= \int_0^\infty\int_0^\infty\int_0^\infty \left(\frac{\theta_0}{\theta}\right)^{n_0-1/2}\left(\frac{\theta_1}{\theta}\right)^{n_1-1/2}\left(\frac{\theta_2}{\theta}\right)^{n_2-1/2}\theta^{n-3}\exp\left\{-\left(\frac{A}{\lambda}\right)\theta\right\}d\theta_0 d\theta_1 d\theta_2, \\ &= \int_{0<\theta_0/\theta+\theta_1/\theta<1} \prod_{i=0}^{1}\left(\frac{\theta_i}{\theta}\right)^{n_i-1/2}\left(1-\sum_{i=0}^{1}\frac{\theta_i}{\theta}\right)^{n_2-1/2} d\frac{\theta_0}{\theta}d\frac{\theta_1}{\theta}\cdot\int_0^\infty \theta^{n-1}\exp\left\{-\frac{A}{\lambda}\theta\right\}d\theta, \\ &= B\left(n_0+\frac{1}{2}, n_1+n_2+1\right)B\left(n_1+\frac{1}{2}, n_2+\frac{1}{2}\right)\frac{\Gamma(n)}{(A/\lambda)^n} < \infty,\end{aligned} \quad (20)$$

where $A = \sum_{i=1}^{n}(e^{\lambda x_i} - 1)$ and $B(\cdot, \cdot)$ is a beta function.

Thus, the joint posterior distribution of $(\theta_0, \theta_1, \theta_2)$ based on $\pi_2(\theta_0, \theta_1, \theta_2)$ is proper. □

4.3. Reference Priors. Bernardo [18] and Berger and Bernardo [19] proposed the reference prior which plays a vital role in the objective Bayesian inference. We set $\mu_0 \equiv \theta = \theta_0 + \theta_1 + \theta_2$, $\mu_1 = \theta_0/\theta$, and $\mu_2 = \theta_1/\theta$; the transformation from $(\theta_0, \theta_1, \theta_2)$ to (μ_0, μ_1, μ_2) is one-to-one with the inverse transformation $\theta_0 = \mu_0\mu_1$, $\theta_1 = \mu_0\mu_2$, and $\theta_2 = \mu_0(1 - \mu_1 - \mu_2)$. The Jacobian matrix of the transformation has the form

$$J = \begin{pmatrix} \mu_1 & \mu_0 & 0 \\ \mu_2 & 0 & \mu_0 \\ 1-\mu_1-\mu_2 & -\mu_0 & -\mu_0 \end{pmatrix}, 0 < \mu_0 < \infty,\ 0 < \mu_1 + \mu_2 < 1. \quad (21)$$

The likelihood function (3) becomes

$$L(x; \lambda, \mu_0, \mu_1, \mu_2) = \mu_1^{n_0}\mu_2^{n_1}(1 - \mu_1 - \mu_2)^{n_2}\mu_0^{n}\exp\left\{\lambda\sum_l x_l - \left(\frac{\mu_0}{\lambda}\right)\left[\sum_l\left(e^{\lambda x_l} - 1\right)\right]\right\}. \quad (22)$$

The Fisher information matrix of (μ_0, μ_1, μ_2) can be written as

$$I_1 = J' I J = n\begin{pmatrix} 1/\mu_0^2 & 0 & 0 \\ 0 & \dfrac{1}{\mu_1} + \dfrac{1}{(1-\mu_1-\mu_2)} & \dfrac{1}{(1-\mu_1-\mu_2)} \\ 0 & \dfrac{1}{(1-\mu_1-\mu_2)} & \dfrac{1}{\mu_2} + \dfrac{1}{(1-\mu_1-\mu_2)} \end{pmatrix}. \quad (23)$$

Theorem 3.

(i) Under the ordering groups $\{\mu_0, (\mu_1, \mu_2)\}$ and $\{(\mu_1, \mu_2), \mu_0\}$, the reference priors are the same, which is given by $\omega_{R_1}(\mu_0, \mu_1, \mu_2) = \sqrt{1/\mu_0^2\mu_1\mu_2(1-\mu_1-\mu_2)}$; the corresponding reference prior for $(\theta_0, \theta_1, \theta_2)$ is $\pi_2(\theta_0, \theta_1, \theta_2) = \sqrt{1/\theta_0\theta_1\theta_2\theta^3}$

(ii) Under the ordering groups $\{\mu_0, \mu_1, \mu_2\}$, $\{\mu_0, \mu_2, \mu_1\}$, $\{\mu_1, \mu_0, \mu_2\}$, and $\{\mu_1, \mu_2, \mu_0\}$, the reference priors are the same, which is given by $\omega_{R_2}(\mu_0, \mu_1, \mu_2) = \sqrt{1/\mu_0^2\mu_1\mu_2(1-\mu_1-\mu_2)(1-\mu_1)}$; the corresponding reference prior for $(\theta_0, \theta_1, \theta_2)$ is $\pi_3(\theta_0, \theta_1, \theta_2) = \sqrt{1/\theta^2\theta_0\theta_1\theta_2(\theta_1+\theta_2)}$

(iii) Under the ordering groups $\{\mu_2, \mu_0, \mu_1\}$ and $\{\mu_2, \mu_1, \mu_0\}$, the reference priors are the same, which is given by $\omega_{R_3}(\mu_0, \mu_1, \mu_2) = \sqrt{1/\mu_0^2\mu_1\mu_2(1-\mu_1-\mu_2)(1-\mu_2)}$; the corresponding reference prior for $(\theta_0, \theta_1, \theta_2)$ is $\pi_4(\theta_0, \theta_1, \theta_2)) = \sqrt{1/\theta^2\theta_0\theta_1\theta_2(\theta_0+\theta_2)}$

Proof.

(i) The Fisher information matrix of (μ_0, μ_1, μ_2) is

$$I_1 = \begin{pmatrix} \sum_{11} & 0 \\ 0 & \sum_{22} \end{pmatrix}, \quad (24)$$

where $\sum_{11} = n/\mu_0^2$ and $\sum_{22} = n\begin{pmatrix} 1/\mu_1 + 1/(1-\mu_1-\mu_2) & 1/(1-\mu_1-\mu_2) \\ 1/(1-\mu_1-\mu_2) & 1/\mu_2 + 1/(1-\mu_1-\mu_2) \end{pmatrix}$. The reference prior for the ordering groups $\{\mu_0, (\mu_1, \mu_2)\}$ and $\{(\mu_1, \mu_2), \mu_0\}$ is the same as in [21], which is given by

$$\omega_{R_1}(\mu_0, \mu_1, \mu_2) \propto \left|\sum_{11}\right|^{1/2}\left|\sum_{22}\right|^{1/2} \propto \sqrt{\frac{1}{\mu_0^2\mu_1\mu_2(1-\mu_1-\mu_2)}}. \quad (25)$$

(ii) The inverse of I_1 is

$$H = \frac{1}{n}\begin{pmatrix} \mu_0^2 & 0 & 0 \\ 0 & \mu_1(1-\mu_1) & -\mu_1\mu_2 \\ 0 & -\mu_1\mu_2 & \mu_2(1-\mu_2) \end{pmatrix}. \quad (26)$$

(iii) According the notations in [18], we obtain $h_1 = 1/\mu_0^2$, $h_2 = 1/\mu_1(1-\mu_1)$, and $h_3 = (1-\mu_1)/(\mu_2(1-\mu_1-\mu_2))$.

Choose the compact sets $\Omega_k = \{(\mu_0, \mu_1, \mu_2) | a_{0k} < \mu_0 < b_{0k}, a_{1k} < \mu_1, a_{2k} < \mu_2, \mu_1 + \mu_2 < d_k\}$, such that $a_{0k}, a_{1k}, a_{2k} \longrightarrow 0$, $b_{0k} \longrightarrow \infty$, and $d_k \longrightarrow 1$, as $k \longrightarrow \infty$. Then, we have

$$\pi^k(\mu_0, \mu_1, \mu_2) = \frac{\sqrt{|h_1|}\sqrt{|h_2|}\sqrt{|h_3|}}{\int_{a_{0k}}^{b_{0k}}\sqrt{|h_1|}d\mu_0 \cdot \int_{a_{1k}}^{d_k-\mu_2^0}\sqrt{|h_2|}d\mu_1 \cdot \int_{a_{2k}}^{d_k-\mu_1}\sqrt{|h_3|}d\mu_2} I_{\Omega_k}(\mu_0, \mu_1, \mu_2), \quad (27)$$

where $\int_{a_{0k}}^{b_{0k}}\sqrt{|h_1|}d\mu_0 = \int_{a_{0k}}^{b_{0k}} 1/\mu_0 d\mu_0 = \log b_{0k} - \log a_{0k}$:

$$\begin{aligned} \int_{a_{1k}}^{d_k-\mu_2^0}\sqrt{|h_2|}d\mu_1 &= \int_{a_{1k}}^{d_k-\mu_2^0}\sqrt{\frac{1}{\mu_1(1-\mu_1)}}d\mu_1 = -\arcsin\left(1-2\left(d_k-\mu_2^0\right)\right) + \arcsin(1-2a_{1k}), \\ \int_{a_{2k}}^{d_k-\mu_1}\sqrt{|h_3|}d\mu_2 &= \int_{a_{2k}}^{d_k-\mu_1}\sqrt{\frac{1-\mu_1}{\mu_2(1-\mu_1-\mu_2)}}d\mu_2, \\ &= (1-\mu_1)^{1/2}\left(-\arcsin\left(\frac{1-\mu_1-2(d_k-\mu_1)}{1-\mu_1}\right) + \arcsin\left(\frac{1-\mu_1-2a_{2k}}{1-\mu_1}\right)\right). \end{aligned} \quad (28)$$

Then, we get the reference prior as

$$\omega_{R_2}(\mu_0, \mu_1, \mu_2) = \lim_{k \longrightarrow \infty} \frac{\pi^k(\mu_0, \mu_1, \mu_2)}{\pi^k(\mu_0^*, \mu_1^*, \mu_2^*)} \propto \sqrt{\frac{1}{\mu_0^2 \mu_1 \mu_2 (1 - \mu_1 - \mu_2)(1 - \mu_1)}}, \tag{29}$$

where $(\mu_0^*, \mu_1^*, \mu_2^*)$ is an inner point of Ω_k.

Similarly, under the ordering group $\{\mu_0, \mu_2, \mu_1\}$, the reference prior is $\omega_{R_2}(\mu_0, \mu_1, \mu_2)$.

The Fisher information matrix of $\{\mu_1, \mu_0, \mu_2\}$ is

$$I_2 = n \begin{pmatrix} \frac{1}{\mu_1} + \frac{1}{(1 - \mu_1 - \mu_2)} & 0 & \frac{1}{(1 - \mu_1 - \mu_2)} \\ 0 & \frac{1}{\mu_0^2} & 0 \\ \frac{1}{(1 - \mu_1 - \mu_2)} & 0 & \frac{1}{\mu_2} + \frac{1}{(1 - \mu_1 - \mu_2)} \end{pmatrix}. \tag{30}$$

The inverse of I_2 is

$$H_1 = \frac{1}{n} \begin{pmatrix} \mu_1(1 - \mu_1) & 0 & -\mu_1\mu_2 \\ 0 & \mu_0^2 & 0 \\ -\mu_1\mu_2 & 0 & \mu_2(1 - \mu_2) \end{pmatrix}. \tag{31}$$

Similarly, we obtain $h_1 = 1/\mu_1(1 - \mu_1)$, $h_2 = 1/\mu_0^2$, and $h_3 = (1 - \mu_1)/(\mu_2(1 - \mu_1 - \mu_2))$.

Choose the compact sets $\Omega_k = \{(\mu_1, \mu_0, \mu_2) | a_{0k} < \mu_1, \quad a_{1k} < \mu_0 < b_{1k}, a_{2k} < \mu_2, \mu_1 + \mu_2 < d_k\}$, such that $a_{0k}, a_{1k}, a_{2k} \longrightarrow 0$, $b_{1k} \longrightarrow \infty$, and $d_k \longrightarrow 1$, as $k \longrightarrow \infty$. Then, we have

$$\pi^k(\mu_1, \mu_0, \mu_2) = \frac{\sqrt{|h_1|}\sqrt{|h_2|}\sqrt{|h_3|}}{\int_{a_{0k}}^{d_k - u_2^0} \sqrt{|h_1|} d\mu_1 \cdot \int_{a_{1k}}^{b_{1k}} \sqrt{|h_2|} d\mu_0 \cdot \int_{a_{2k}}^{d_k - \mu_1} \sqrt{|h_3|} d\mu_2} I_{\Omega_k}(\mu_1, \mu_0, \mu_2), \tag{32}$$

where $\int_{a_{0k}}^{d_k - u_2^0} \sqrt{|h_1|} d\mu_1 = \int_{a_{0k}}^{d_k - u_2^0} \sqrt{1/\mu_1(1 - \mu_1)} d\mu_1 = -\arcsin(1 - 2(d_k - u_2^0)) + \arcsin(1 - 2a_{0k})$,

$$\begin{aligned} \int_{a_{1k}}^{b_{1k}} \sqrt{|h_2|} d\mu_0 &= \int_{a_{1k}}^{b_{1k}} \frac{1}{\mu_0} d\mu_0 = \log b_{1k} - \log a_{1k}, \\ \int_{a_{2k}}^{d_k - \mu_1} \sqrt{|h_3|} d\mu_2 &= \int_{a_{2k}}^{d_k - \mu_1} \sqrt{\frac{1 - \mu_1}{\mu_2(1 - \mu_1 - \mu_2)}} d\mu_2, \\ &= (1 - \mu_1)^{1/2} \left(-\arcsin\left(\frac{1 - \mu_1 - 2(d_k - u_1)}{1 - \mu_1} \right) + \arcsin\left(\frac{1 - \mu_1 - 2a_{2k}}{1 - \mu_1} \right) \right). \end{aligned} \tag{33}$$

Let $(\mu_1^*, \mu_0^*, \mu_2^*)$ be an inner point of Ω_k; we get the reference prior as

$$\omega_{R_2}(\mu_0, \mu_1, \mu_2) = \lim_{k \longrightarrow \infty} \frac{\pi^k(\mu_1, \mu_0, \mu_2)}{\pi^k(\mu_1^*, \mu_0^*, \mu_2^*)} \propto \sqrt{\frac{1}{\mu_0^2 \mu_1 \mu_2 (1 - \mu_1 - \mu_2)(1 - \mu_1)}}. \tag{34}$$

Similarly, under the ordering group $\{\mu_1, \mu_2, \mu_0\}$, the reference prior is $\omega_{R_2}(\mu_0, \mu_1, \mu_2)$.

(v) The Fisher information matrix of $\{\mu_2, \mu_1, \mu_0\}$ is

$$I_3 = n\begin{pmatrix} \frac{1}{\mu_2} + \frac{1}{(1-\mu_1-\mu_2)} & \frac{1}{(1-\mu_1-\mu_2)} & 0 \\ \frac{1}{(1-\mu_1-\mu_2)} & \frac{1}{\mu_1} + \frac{1}{(1-\mu_1-\mu_2)} & 0 \\ 0 & 0 & \frac{1}{\mu_0^2} \end{pmatrix}. \quad (35)$$

The inverse of I_3 is

$$H_2 = \frac{1}{n}\begin{pmatrix} \mu_2(1-\mu_2) & -\mu_1\mu_2 & 0 \\ -\mu_1\mu_2 & \mu_1(1-\mu_1) & 0 \\ 0 & 0 & \mu_0^2 \end{pmatrix}. \quad (36)$$

Then, we obtain $h_1 = 1/\mu_2(1-\mu_2)$, $h_2 = (1-\mu_2)/(\mu_1(1-\mu_1-\mu_2))$, and $h_3 = 1/\mu_0^2$.

Choose the compact sets $\Omega_k = \{(\mu_2, \mu_1, \mu_0) | a_{0k} < \mu_2, a_{1k} < \mu_1, \mu_2 + \mu_1 < d_k, a_{2k} < \mu_0 < b_{2k}\}$, such that $a_{0k}, a_{1k}, a_{2k} \longrightarrow 0$, $b_{2k} \longrightarrow \infty$, and $d_k \longrightarrow 1$, as $k \longrightarrow \infty$. Then, we have

$$\pi^k(\mu_2, \mu_1, \mu_0) = \frac{\sqrt{|h_1|}\sqrt{|h_2|}\sqrt{|h_3|}}{\int_{a_{0k}}^{d_k-u_1^0}\sqrt{|h_1|}d\mu_2 \cdot \int_{a_{1k}}^{d_k-\mu_2}\sqrt{|h_2|}d\mu_1 \cdot \int_{a_{2k}}^{b_{2k}}\sqrt{|h_3|}d\mu_0} I_{\Omega_k}(\mu_2, \mu_1, \mu_0), \quad (37)$$

where $\int_{a_{0k}}^{d_k-u_1^0}\sqrt{|h_1|}d\mu_2 = \int_{a_{0k}}^{d_k-u_1^0}\sqrt{1/\mu_2(1-\mu_2)}d\mu_2 = -\arcsin(1-2(d_k-u_1^0)) + \arcsin(1-2a_{0k})$,

$$\begin{aligned} \int_{a_{1k}}^{d_k-\mu_2}\sqrt{|h_2|}d\mu_1 &= \int_{a_{1k}}^{d_k-\mu_2}\sqrt{\frac{1-\mu_2}{\mu_1(1-\mu_1-\mu_2)}}d\mu_1, \\ &= (1-\mu_2)^{1/2}\left(-\arcsin\left(\frac{1-\mu_2-2(d_k-u_2)}{1-\mu_2}\right) + \arcsin\left(\frac{1-\mu_2-2a_{1k}}{1-\mu_2}\right)\right), \\ \int_{a_{2k}}^{b_{2k}}\sqrt{|h_3|}d\mu_0 &= \int_{a_{2k}}^{b_{2k}}\frac{1}{\mu_0}d\mu_0 = \log b_{2k} - \log a_{2k}. \end{aligned} \quad (38)$$

Let $(\mu_2^*, \mu_1^*, \mu_0^*)$ be an inner point of Ω_k, we obtain the reference prior as

$$\omega_{R_3}(\mu_0, \mu_1, \mu_2) = \lim_{k\longrightarrow\infty}\frac{\pi^k(\mu_2, \mu_1, \mu_0)}{\pi^k(\mu_2^*, \mu_1^*, \mu_0^*)} \propto \sqrt{\frac{1}{\mu_0^2\mu_1\mu_2(1-\mu_1-\mu_2)(1-\mu_2)}}. \quad (39)$$

Similarly, under the ordering group $\{\mu_2, \mu_0, \mu_1\}$, the reference prior is $\omega_{R_3}(\mu_0, \mu_1, \mu_2)$. According to the one-to-one transformation from (μ_0, μ_1, μ_2) to $(\theta_0, \theta_1, \theta_2)$, we can obtain the reference priors $\pi_2(\mu_0, \mu_1, \mu_2)$, $\pi_3(\mu_0, \mu_1, \mu_2)$, $\pi_4(\mu_0, \mu_1, \mu_2)$ from ω_{R_1}, ω_{R_2}, and ω_{R_3}, respectively. □

Theorem 4. *Based on the reference priors $\pi_3(\theta_0, \theta_1, \theta_2)$ and $\pi_4(\theta_0, \theta_1, \theta_2)$, the posterior distributions of $(\theta_0, \theta_1, \theta_2)$ are proper.*

Proof. The joint posterior distributions of $(\theta_0, \theta_1, \theta_2)$ based on reference prior $\pi_3(\theta_0, \theta_1, \theta_2)$ and $\pi_4(\theta_0, \theta_1, \theta_2)$ are, respectively, as

$$\begin{aligned}\pi_3(\theta_0,\theta_1,\theta_2|x) &= \frac{L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_3(\theta_0,\theta_1,\theta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_3(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2},\\ &\propto \theta^{-1}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}(\theta_1+\theta_2)^{-1/2}\exp\left\{-\frac{A\theta}{\lambda}\right\},\\ \pi_4(\theta_0,\theta_1,\theta_2|x) &= \frac{L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_4(\theta_0,\theta_1,\theta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_4(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2},\\ &\propto \theta^{-1}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}(\theta_0+\theta_2)^{-1/2}\exp\left\{-\frac{A\theta}{\lambda}\right\}.\end{aligned} \tag{40}$$

Integrating $\pi_3(\theta_0,\theta_1,\theta_2|x)$ and $\pi_4(\theta_0,\theta_1,\theta_2|x)$ with respect to θ_0,θ_1, and θ_2, respectively, we obtain

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty \theta^{-1}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}(\theta_1+\theta_2)^{-1/2}\exp\left\{-\frac{A\theta}{\lambda}\right\}d\theta_0 d\theta_1 d\theta_2,\\ &= B\left(n_0+\frac{1}{2},n_1+n_2+\frac{1}{2}\right)B\left(n_1+\frac{1}{2},n_2+\frac{1}{2}\right)\frac{\Gamma(n)}{(A/\lambda)^n}<\infty,\\ &\int_0^\infty\int_0^\infty\int_0^\infty \theta^{-1}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}(\theta_0+\theta_2)^{-1/2}\exp\left\{-\frac{A\theta}{\lambda}\right\}d\theta_0 d\theta_1 d\theta_2,\\ &= B\left(n_1+\frac{1}{2},n_0+n_2+\frac{1}{2}\right)B\left(n_0+\frac{1}{2},n_2+\frac{1}{2}\right)\frac{\Gamma(n)}{(A/\lambda)^n}<\infty.\end{aligned} \tag{41}$$

Thus, the posterior distributions of $(\theta_0,\theta_1,\theta_2)$ based on $\pi_3(\theta_0,\theta_1,\theta_2)$ and $\pi_4(\theta_0,\theta_1,\theta_2)$ are proper. □

4.4. Bayesian Estimates. The joint posterior distributions of $(\theta_0,\theta_1,\theta_2)$ based on π_1, π_2, π_3, and π_4 are, respectively, as

$$\pi_1(\theta_0,\theta_1,\theta_2|x) = \frac{L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_1(\theta_0,\theta_1,\theta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_1(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2}, \tag{42}$$

where

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_1(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2\\ &= w_1 w_2 \exp\left\{\lambda\sum_l x_l\right\}\int_{0<\theta_0/\theta+\theta_1/\theta<1}\prod_{i=0}^{1}\left(\frac{\theta_i}{\theta}\right)^{n_i+c_i-1}\left(1-\sum_{i=0}^{1}\frac{\theta_i}{\theta}\right)^{n_2+c_2-1} d\frac{\theta_0}{\theta}d\frac{\theta_1}{\theta},\int_0^\infty \theta^{n+a+1}e^{-(A/\lambda+b)\theta}d\theta\\ &= w_1 w_2 \exp\left\{\lambda\sum_l x_l\right\}B(n_0+c_0,n_1+c_1+n_2+c_2)B(n_1+c_1,n_2+c_2)\frac{\Gamma(n+a+2)}{(A/\lambda+b)^{n+a+2}},\end{aligned} \tag{43}$$

where $w_1 = \Gamma(\sum_{i=0}^{2} c_i) b^{a-c_0-c_1-c_2}/\Gamma(a)$ and $w_2 = \prod_{i=0}^{2} b^{c_i}/\Gamma(c_i)$. Thus, we obtain

$$\pi_1(\theta_0, \theta_1, \theta_2|x) = \frac{\theta^{a-c_0-c_1-c_2}\theta_0^{n_0+c_0-1}\theta_1^{n_1+c_1-1}\theta_2^{n_2+c_2-1}\exp\{-(A/\lambda+b)\theta\}}{B(n_0+c_0, n_1+c_1+n_2+c_2)B(n_1+c_1, n_2+c_2)\Gamma(n+a)/(A/\lambda+b)^{n+a}}. \tag{44}$$

Similarly,

$$\pi_2(\theta_0, \theta_1, \theta_2|x) = \frac{L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_2(\theta_0,\theta_1,\theta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_2(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2}, \tag{45}$$

where

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_2(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2,\\
&= \int_0^\infty\int_0^\infty\int_0^\infty \theta^{-3/2}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}\exp\left\{\lambda\sum_l x_l - \frac{A}{\lambda}\theta\right\}d\theta_0 d\theta_1 d\theta_2,\\
&= \exp\left\{\lambda\sum_l x_l\right\}B\left(n_0+\frac{1}{2}, n_1+n_2+1\right)B\left(n_1+\frac{1}{2}, n_2+\frac{1}{2}\right)\frac{\Gamma(n)}{(A/\lambda)^n}.
\end{aligned} \tag{46}$$

We obtain

$$\begin{aligned}
\pi_2(\theta_0, \theta_1, \theta_2|x) &= \frac{\theta^{-3/2}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}\exp\{-A\theta/\lambda\}}{B(n_0+1/2, n_1+n_2+1)B(n_1+1/2, n_2+1/2)\Gamma(n)/(A/\lambda)^n},\\
\pi_3(\theta_0, \theta_1, \theta_2|x) &= \frac{L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_3(\theta_0,\theta_1,\theta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_3(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2},
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_3(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2,\\
&= \int_0^\infty\int_0^\infty\int_0^\infty \theta^{-1}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}(\theta_1+\theta_2)^{-1/2}\exp\left\{\lambda\sum_l x_l - \frac{A\theta}{\lambda}\right\}d\theta_0 d\theta_1 d\theta_2,\\
&= \exp\left\{\lambda\sum_l x_l\right\}B\left(n_0+\frac{1}{2}, n_1+n_2+\frac{1}{2}\right)B\left(n_1+\frac{1}{2}, n_2+\frac{1}{2}\right)\frac{\Gamma(n)}{(A/\lambda)^n}.
\end{aligned} \tag{48}$$

We obtain

$$\pi_3(\theta_0,\theta_1,\theta_2|x) = \frac{\theta^{-1}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}(\theta_1+\theta_2)^{-1/2}\exp\{-A\theta/\lambda\}}{B(n_0+1/2,n_1+n_2+1/2)B(n_1+1/2,n_2+1/2)\Gamma(n)/(A/\lambda)^n},$$
$$\pi_4(\theta_0,\theta_1,\theta_2|x) = \frac{L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_4(\theta_0,\theta_1,\theta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_4(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2}, \tag{49}$$

where

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty L(x;\lambda,\theta_0,\theta_1,\theta_2)\pi_4(\theta_0,\theta_1,\theta_2)d\theta_0 d\theta_1 d\theta_2,\\ &= \int_0^\infty\int_0^\infty\int_0^\infty \theta^{-1}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}(\theta_0+\theta_2)^{-1/2}\exp\left\{\lambda\sum_l x_l - \frac{A\theta}{\lambda}\right\}d\theta_0 d\theta_1 d\theta_2,\\ &= \exp\left\{\lambda\sum_l x_l\right\}B\left(n_1+\frac{1}{2},n_0+n_2+\frac{1}{2}\right)B\left(n_0+\frac{1}{2},n_2+\frac{1}{2}\right)\frac{\Gamma(n)}{(A/\lambda)^n}.\end{aligned} \tag{50}$$

Then, we have

$$\pi_4(\theta_0,\theta_1,\theta_2|x) = \frac{\theta^{-1}\theta_0^{n_0-1/2}\theta_1^{n_1-1/2}\theta_2^{n_2-1/2}(\theta_0+\theta_2)^{-1/2}\exp\{-A\theta/\lambda\}}{B(n_1+1/2,n_0+n_2+1/2)B(n_0+1/2,n_2+1/2)\Gamma(n)/(A/\lambda)^n}. \tag{51}$$

From (9)–(12), we get the Bayesian estimates of parameters θ_0, θ_1, θ_2, and θ against squared error loss function based on π_1, π_2, π_3, and π_4, respectively, which are listed in Table 1.

4.5. HPD Credible Intervals. The HPD credible intervals of parameters θ_0, θ_1, θ_2, and θ can be constructed by the Monte Carlo method studied by Chen and Shao [22].

Step 1: given the value of n and the observed data $(x_1,x_2,\cdots,x_n)$, compute the Bayesian estimates of $\widehat{\theta}_0$, $\widehat{\theta}_1$, $\widehat{\theta}_2$, and $\widehat{\theta}$ based on π_1, π_2, π_3, and π_4, respectively.

Step 2: repeat Step 1 M times; we obtain M sets of the values $\widehat{\theta}_0$, $\widehat{\theta}_1$, $\widehat{\theta}_2$, and $\widehat{\theta}$ based on π_1, π_2, π_3, and π_4, respectively. Arrange them in the ascending order, we obtain

$$\widehat{\theta}_{0\pi_k[1]}<\cdots<\widehat{\theta}_{0\pi_k[M]},\widehat{\theta}_{1\pi_k[1]}<\cdots<\widehat{\theta}_{1\pi_k[M]},\widehat{\theta}_{2\pi_k[1]}<\cdots<\widehat{\theta}_{2\pi_k[M]},\widehat{\theta}_{\pi_k[1]}<\cdots<\widehat{\theta}_{\pi_k[M]},\ k=1,2,3,4. \tag{52}$$

Step 3: compute the CIs at $1-\gamma$ confidence level as

$$\left(\widehat{\theta}_{v\pi_k[w]},\widehat{\theta}_{v\pi_k[w+(1-\gamma)M]}\right),\left(\widehat{\theta}_{\pi_k[w]},\widehat{\theta}_{\pi_k[w+(1-\gamma)M]}\right),v=0,1,2;w=1,2,\cdots,M-(1-\gamma)M;k=1,2,3,4. \tag{53}$$

Step 4: the HPD CIs for θ_v, $v=0,1,2$, and θ are the shortest intervals among $(\widehat{\theta}_{v\pi_k[w]},\widehat{\theta}_{v\pi_k[w+(1-\gamma)M]})$, $(\widehat{\theta}_{\pi_k[w]},\widehat{\theta}_{\pi_k[w+(1-\gamma)M]})$, and $w=1,2,\cdots,M-(1-\gamma)M$, respectively.

Table 1: Bayesian estimates of parameters based on different priors.

Prior	θ_0	θ_1	θ_2	θ
π_1	$\lambda(n_0+c_0)(n+a)/(n+c_0+c_1+c_2)(A+b\lambda)$	$\lambda(n_1+c_1)(n+a)/(n+c_0+c_1+c_2)(A+b\lambda)$	$\lambda(n_2+c_2)(n+a)/(n+c_0+c_1+c_2)(b\lambda+A)$	$(n+a)\lambda/A+b\lambda$
π_2	$n\lambda(2n_0+1)/A(2n+3)$	$n\lambda(2n_1+1)/A(2n+3)$	$n\lambda(2n_2+1)/A(2n+3)$	$n\lambda/A$
π_3	$n\lambda(2n_0+1)/2A(n+1)$	$n\lambda(2n_1+2n_2+1)(2n_1+1)/4A(n+1)(n_1+n_2+1)$	$n\lambda(2n_1+2n_2+1)(2n_2+1)/4A(n+1)(n_1+n_2+1)$	$n\lambda/A$
π_4	$n\lambda(2n_0+2n_2+1)(2n_0+1)/4A(n+1)(n_0+n_2+1)$	$n\lambda(2n_1+1)/2A(n+1)$	$n\lambda(2n_0+2n_2+1)(2n_2+1)/2A(n+1)(n_0+n_2+1)$	$n\lambda/A$

Table 2: MSEs, ALs, and CPs of θ_0, θ_1, θ_2, and θ ($n = 10$).

Method		Para.	θ_0	θ_1	θ_2	θ
MLE		MSE	0.4858	0.8030	0.4865	0.9374
		Boot-AL	2.2414	2.7146	2.2266	1.7920
		Boot-CP	0.9339	0.9294	0.9405	0.9321
Bayes	π_1	MSE	0.4850	0.8012	0.4857	0.9340
		HPD-AL	2.0388	2.5119	2.0425	1.9018
		HPD-CP	0.9663	0.9440	0.9645	0.9369
	π_2	MSE	0.4055	0.5903	0.4061	0.9374
		HPD-AL	1.7980	2.2183	1.8016	1.9034
		HPD-CP	0.9552	0.9399	0.9539	0.9335
	π_3	MSE	0.4678	0.5732	0.3909	0.9374
		HPD-AL	1.8797	2.2193	1.7850	1.9034
		HPD-CP	0.9481	0.9405	0.9569	0.9460
	π_4	MSE	0.3748	0.7042	0.3754	0.9374
		HPD-AL	1.7724	2.3192	1.7760	1.9034
		HPD-CP	0.9527	0.9468	0.9515	0.9405

Table 3: MSEs, ALs, and CPs of θ_0, θ_1, θ_2, and θ ($n = 20$).

Method		Para.	θ_0	θ_1	θ_2	θ
MLE		MSE	0.2505	0.4519	0.2523	0.6907
		Boot-AL	1.5795	1.9048	1.5807	1.2957
		Boot-CP	0.9488	0.9483	0.9412	0.9407
Bayes	π_1	MSE	0.2503	0.4512	0.2520	0.6893
		HPD-AL	1.4434	1.7573	1.4382	1.3635
		HPD-CP	0.9832	0.9692	0.9831	0.9415
	π_2	MSE	0.2335	0.3766	0.2350	0.6907
		HPD-AL	1.3512	1.6476	1.3462	1.3640
		HPD-CP	0.9746	0.9447	0.9762	0.9409
	π_3	MSE	0.2551	0.3662	0.2293	0.6907
		HPD-AL	1.3834	1.6486	1.3398	1.3640
		HPD-CP	0.9668	0.9506	0.9777	0.9598
	π_4	MSE	0.2201	0.4260	0.2216	0.6907
		HPD-AL	1.3399	1.6868	1.3348	1.3640
		HPD-CP	0.9614	0.9525	0.9613	0.9498

Table 4: MSEs, ALs, and CPs of θ_0, θ_1, θ_2, and θ ($n = 30$).

Method		Para.	θ_0	θ_1	θ_2	θ
MLE		MSE	0.1752	0.3345	0.1771	0.6049
		Boot-AL	1.2849	1.5451	1.2896	1.0510
		Boot-CP	0.9651	0.9516	0.9654	0.9415
Bayes	π_1	MSE	0.1751	0.3341	0.1770	0.6040
		HPD-AL	1.1710	1.4354	1.1727	1.1164
		HPD-CP	0.9919	0.9629	0.9901	0.9427
	π_2	MSE	0.1694	0.2922	0.1712	0.6049
		HPD-AL	1.1197	1.3745	1.1213	1.1167
		HPD-CP	0.9839	0.9723	0.9835	0.9418
	π_3	MSE	0.1814	0.2849	0.1679	0.6049
		HPD-AL	1.1377	1.3750	1.1177	1.1167
		HPD-CP	0.9783	0.9770	0.9851	0.9615
	π_4	MSE	0.1612	0.3228	0.1629	0.6049
		HPD-AL	1.1132	1.3966	1.1148	1.1167
		HPD-CP	0.9896	0.9581	0.9885	0.9638

Table 5: MSEs, ALs, and CPs of θ_0, θ_1, θ_2, and θ ($n = 50$).

Method		Para.	θ_0	θ_1	θ_2	θ
MLE		MSE	0.1158	0.2460	0.1161	0.5380
		Boot-AL	0.9947	1.1981	1.0018	0.8227
		Boot-CP	0.9829	0.9578	0.9831	0.9554
Bayes	π_1	MSE	0.1157	0.2458	0.1161	0.5375
		HPD-AL	0.9118	1.1075	0.9071	0.8677
		HPD-CP	0.9955	0.9822	0.9954	0.9724
	π_2	MSE	0.1150	0.2243	0.1154	0.5380
		HPD-AL	0.8874	1.0786	0.8828	0.8679
		HPD-CP	0.9884	0.9900	0.9870	0.9702
	π_3	MSE	0.1209	0.2196	0.1137	0.5380
		HPD-AL	0.8961	1.0791	0.8811	0.8679
		HPD-CP	0.9813	0.9933	0.9901	0.9721
	π_4	MSE	0.1105	0.2417	0.1109	0.5380
		HPD-AL	0.8842	1.0892	0.8796	0.8679
		HPD-CP	0.9938	0.9789	0.9931	0.9717

Table 6: The simulated data when $n = 25$.

(0.0019	1)	(0.0025	1)	(0.0062	1)	(0.0361	0)	(0.0651	2)

(0.0675	1)	(0.1108	2)	(0.1447	1)	(0.1509	1)	(0.1694	2)

(0.1737	0)	(0.1859	1)	(0.1900	2)	(0.2218	0)	(0.2307	2)

(0.2537	1)	(0.2558	2)	(0.2734	2)	(0.2750	0)	(0.3349	1)

(0.3528	1)	(0.3790	0)	(0.3824	2)	(0.3875	1)	(0.5336	1)

Table 7: Point estimates and 95% CIs of θ_0, θ_1, θ_2, and θ.

Method		Para.	θ_0	θ_1	θ_2	θ
MLE		MLE	0.8029	1.9270	1.2847	4.0146
		Boot-CI	(0.0408, 1.8099)	(0.1891, 2.8803)	(0.0642, 1.8178)	(0.7671, 4.4112)
Bayes	π_1	Bayes	0.8029	1.9267	1.2845	4.0141
		HPD CI	(0.0898, 1.6991)	(0.3473, 2.9083)	(0.0396, 1.7374)	(0.8141, 4.1876)
	π_2	Bayes	0.8332	1.8937	1.2877	4.0146
		HPD CI	(0.0694, 1.7457)	(0.3070, 2.8296)	(0.0364, 1.7697)	(0.7893, 4.4378)
	π_3	Bayes	0.8492	1.8841	1.2812	4.0146
		HPD CI	(0.0798, 1.7561)	(0.1251, 2.5045)	(0.0315, 1.4392)	(0.6368, 4.1407)
	π_4	Bayes	0.8189	1.9301	1.2656	4.0146
		HPD-CP	(0.0638, 1.4011)	(0.2448, 2.8554)	(0.0456, 1.7418)	(0.8474, 4.3484)

Table 8: Point estimates and 95% CIs of θ_0, θ_1, θ_2, and θ.

Method		Para.	θ_0	θ_1	θ_2	θ
MLE		MLE	1.0882e-2	0.4664e-2	1.3214e-2	2.8760e-2
		Boot-CI	(0.7354e-3, 1.2770e-2)	(0.3244e-2, 2.4914e-2)	(0.4077e-3, 1.5990e-2)	(0.7324e-2, 3.5454e-2)
Bayes	π_1	Bayes	1.1882e-2	0.6879e-2	1.3758e-2	3.2520e-2
		HPD CI	(0.2305e-2, 1.1210e-2)	(0.3998e-2, 1.9825e-2)	(0.1861e-2, 1.4366e-2)	(1.0088e-2, 3.8182e-2)
	π_2	Bayes	1.0832e-2	0.4856e-2	1.3073e-2	2.8760e-2
		HPD CI	(0.8317e-3, 1.1676e-2)	(0.2315e-2, 1.8702e-2)	(0.8807e-3, 1.3700e-2)	(0.5246e-2, 3.2002e-2)
	π_3	Bayes	1.0974e-2	0.4817e-2	1.2969e-2	2.8760e-2
		HPD CI	(0.8499e-3, 1.1535e-2)	(0.3160e-2, 1.7152e-2)	(0.8814e-3, 1.4144e-2)	(0.6994e-2, 3.2140e-2)
	π_4	Bayes	1.0803e-2	0.4919e-2	1.3038e-2	2.8760e-2
		HPD-CP	(0.8949e-3, 1.1372e-2)	(0.3365e-2, 1.8866e-2)	(0.6968e-3, 1.4164e-2)	(0.7224e-2, 3.0467e-2)

5. Numerical Simulation and Illustrative Example

5.1. Simulation. Suppose the common shape parameter λ is known. The initial values for parameters $(\lambda, \theta_0, \theta_1, \theta_2)$ are (3, 1, 2, 1). The initial values for the hyperparameters a, b, c_0, c_1, and c_2 are all 0.001. Take the sample size $n = 10$, 20, 30, and 50. Generate the random samples $(x_1, x_2, \cdots, x_n)$ from MOGP$(\lambda, \theta_0, \theta_1, \theta_2)$ by the following steps:

Step 1: for a fixed value n, generate n samples $u_{01}, u_{02}, \cdots, u_{0n}$ from $GP(\lambda, \theta_0)$, $u_{11}, u_{12}, \cdots, u_{1n}$ from $GP(\lambda, \theta_1)$, and $u_{21}, u_{22}, \cdots, u_{2n}$ from $GP(\lambda, \theta_2)$. Then, we obtain $T_{1l} = \min(u_{0l}, u_{1l})$ and $T_{2l} = \min(u_{0l}, u_{2l})$, $l = 1, 2, \cdots, n$.

Step 2: compute $(x_l, \delta_{0l}, \delta_{1l}, \delta_{2l})$, $l = 1, 2, \cdots, n$, where $x_l = \min(T_{1l}, T_{2l})$, $\delta_{0l} = I(T_{1l} = T_{2l})$, $\delta_{1l} = I(T_{1l} < T_{2l})$, and $\delta_{2l} = I(T_{1l} > T_{2l})$.

Repeat the procedures 10,000 times; we get the values of the mean squared errors (MSEs) of the MLEs, the average lengths (ALs), and coverage probabilities (CPs) of the 95% Boot-P CIs, and the MSEs of the Bayesian estimates, the ALs, and CPs of the 95% HPD CIs, which are shown in Table 2–5. From the results in Table 2–5, we can make the following conclusions.

The MSEs of MLEs and Bayesian estimates decrease as the sample size increases. For given sample size n, the Bayesian estimates based on π_1, π_2, and π_4 are smaller than the MSEs of MLEs. The MSEs of Bayesian estimates of θ_0 and θ_2 based on π_4 are smaller than that based on π_1, π_2, and π_3. The MSEs of Bayesian estimates of θ_1 based on π_3 are smaller than that based on π_1, π_2, and π_4. The MSEs of Bayesian estimates of θ based on π_1 are smaller than that based on π_2, π_3, and π_4.

The CPs of Boot-P and HPD CIs are all close to 0.95. The ALs of Boot-P and HPD CIs decrease; the associated CPs increase when the sample size increases. The CPs of HPD CIs based on Bayesian estimates are larger than the CPs of Boot-P CIs based on MLEs.

5.2. Illustrative Analysis

5.2.1. Simulated Data. For illustrative purposes, with initial value for parameters $(\lambda, \theta_0, \theta_1, \theta_2)$ as $(3, 1, 2, 1)$, we use the procedures mentioned above to generate U_0, U_1, and U_2 from $GP(3, 1)$, $GP(3, 2)$, and $GP(3, 1)$, respectively. We then get $T_1 = \min(U_0, U_1)$ and $T_2 = \min(U_0, U_2)$; the latent lifetime of the system is $\min(T_1, T_2)$. The simulated data are listed in Table 6. The MLEs, Bayesian estimates, and associated 95% CIs for parameters θ_0, θ_1, θ_2, and θ are shown in Table 7. From Table 7, all the MLEs and Bayesian estimates of $(\theta_0, \theta_1, \theta_2, \theta)$ are close to the true value.

5.2.2. Real Data. Use the procedures mentioned above to a real dataset. Kundu and Gupta [13] analyzed the football data of UEFA Champions' League data which are presented in Table 1. From the data, T_1 and T_2 can be regarded as two dependent failure modes, and $n_0 = 7, n_1 = 17$, and $n_2 = 13$. This data have been fitted by Marshall–Olkin bivariate Gompertz distribution (see Wang et al. [23]).

The MLEs, Bayesian estimates, and associated 95% CIs for parameters θ_0, θ_1, θ_2, and θ are shown in Table 8. From Tables 7 and 8, Bayesian estimates under different priors are close to MLEs, and the lengths of 95% Boot-p CIs associated to MLEs are longer than the lengths of 95% HPD CIs associated to Bayesian estimates.

6. Conclusion

This paper discussed the point estimates and CIs for the parameters of the dependent competing risks' model from MOGP distribution. We studied the appropriateness of the posteriors based on conjugate prior and Jeffreys and Reference priors, obtained the Bayesian estimates in closed forms, and constructed the associated HPD CIs. From the simulations results, the use of the Bayesian method can be recommended if the priors are available. The results of the illustrative analysis show that the proposed methods work well; from the lengths of CIs, we can conclude the Bayesian estimates are better than MLEs in general.

Data Availability

The data used to support the findings of the study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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