Research Article

Soft Substructures in Quantales and Their Approximations Based on Soft Relations

Huan Zhou, Saqib Mazher Qurashi, Muti Ur Rehman, Muhammad Shabir, Rani Sumaira Kanwal, and Ahmed Mostafa Khalil

1 Aviation Engineering School, Air Force Engineering University, Xi’an 710038, China
2 Government College University Faisalabad, Faisalabad, Pakistan
3 Quaid-i-Azam University Islamabad, Islamabad, Pakistan
4 Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt

Correspondence should be addressed to Saqib Mazher Qurashi; saqibmazhar@gcuf.edu.pk

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The aim of this research article is to derive a new relation between rough sets and soft sets with an algebraic structure quantale by using soft binary relations. The aftersets and foresets are utilized to define lower approximation and upper approximation of soft subsets of quantales. As a consequence of this new relation, different characterization of rough soft substructures of quantales is obtained. To emphasize and make a clear understanding, soft compatible and soft complete relations are focused, and these are interpreted by aftersets and foresets. Particularly, in our work, soft compatible and soft complete relations play an important role. Moreover, this concept generalizes the concept of rough soft substructures of other structures. Furthermore, the algebraic relations between the upper (lower) approximation of soft substructures of quantales and the upper (lower) approximation of their homomorphic images with the help of soft quantales homomorphism are examined. In comparison with the different type of approximations in different type of algebraic structures, it is concluded that this new study is much better.

1. Introduction

Quantale theory was proposed by Mulvey[1]. It is based on defining an algebraic structure on a complete lattice. Since quantale was defined on a complete lattice, there must be a correlation between linear logic and quantale theory which was studied by Yetter, in his study. He presented a new class of models for linear intuitionistic logic [2]. In recent years, quantale is applied in vast research areas, such as algebraic theory [3], rough set theory [4–7], topological theory [8], theoretical computer science [9], and linear logic [10].

In 1982, Pawlak developed the famous rough set theory [11], which is a mathematization of inadequate knowledge. The rough set deals with the categorization and investigation of inadequate information and knowledge. After Pawlak’s work, Zhu [12] provided some new views on the rough set theory. In [13], Ali et al. studied some properties of generalized rough sets. Nowadays, rough sets are applied in many different areas, such as cognitive sciences, machine learning, pattern recognition, and process control.

There are many problems that arise in different fields such as engineering, economics, and social sciences in which data have some sort of uncertainty. Well-known mathematical tools have so many limitations because these tools are introduced for particular circumstances. There are many theories to overcome uncertainty such as fuzzy set theory, probability theory, rough sets, and vague sets, but these are limited due to its design.

In 1998, Molodtsov present the idea of soft set theory, which is a mathematical tool to overcome the adversities affecting the above theories [14]. Many authors like Maji et al. present different operations on soft sets and try to consolidate the algebraic aspects of soft sets [15]. A new and different idea of operations was presented by Ali et al. [16]. Many soft algebraic structures such as soft modules [17], soft groups [18], soft rings [19], and soft ordered semigroups [20]
were studied. The basic theme and purpose of soft sets are to create the idea of parametrization, and this idea has been utilized to find soft binary relation (SBR) which is a parameterized collection of binary relations on a universe under consideration. This puts forward the consideration for complicated objects that may be perceived from different points of view. In [21–23], Feng et al. presented the relationship between soft, rough, and fuzzy sets and produced rough soft sets, soft rough sets, and soft-rough fuzzy sets.

By using aftersets and foresets notions associated with SBR, a new approximation space is widely utilized these days. By using generalized approximation space based on SBR, different soft substructures in semigroups were approximated by Kanaval and Shabir [24]. Motivated by the idea in [24], soft substructures in quantales are defined, and the aftersets and foresets are employed to construct the lower approximation and upper approximation of soft substructures. Since we are dealing with the approximation of soft subsets of quantale, further soft substructures are employed for further characterization.

There are several authors who introduced rough sets theory in algebraic structures and soft algebraic structures. Iwinski analyzes algebraic properties of rough sets [25]. Qurashi and Shabir present the idea of roughness in Q-module [5]. Idea of the generalized rough quantales (subquantales) was presented by Xiao and Li [6]. Rough prime (semiprime and primary) ideals in quantales were investigated by Yang and Xu [7]. Fuzzy ideals (prime, semiprime, and primary) in quantales were introduced by Luo and Wang [4]. Generalized roughness of fuzzy substructures in quantale is studied by Qurashi et al. [26]. In [27], Yamak et al. proposed the idea of set-valued mappings as the basis of the generalized upper (lower) approximations of a ring with the help of ideals. Rough prime bi Γ-hyper ideals of Γ-semihypergroups were proposed by Yaqoob et al. [28, 29]. Rough substructures of semigroups were studied by Kuroki [30].

The following scheme is designed for the rest of the paper. Some essential explanations related to quantales, its substructures, soft substructures, and their corresponding sequels are connected in Section 2. Notion of approximations of soft sets over quantale generated by soft binary relations is discussed in Section 3. In Section 4, by using these ideas, generalized soft substructures are defined and investigated further fundamental algebraic characteristics of these phenomena. Additionally, we extend this study to define the relationship between homomorphic images and their approximation by soft binary relation in Section 5.

2. Preliminaries

Let Θ be a nonempty finite set called the universe set and Ψ be an E.R (equivalence relation) over Θ. Let [q]Ψ denotes the equivalence class of the relation containing q. Any definable set in Θ would be written as finite union of equivalence classes of Θ. Let R ⊆ Θ in general R is not a definable set in Θ. However, the set R can be approximated by two definable sets in Θ. The first one is called Ψ-lower approximation (Ψ – Lapp) of R, and the second is called Ψ-upper approximation (Ψ – Uapp). They are defined as follows:

\[
Ψ(R) = \{q ∈ Θ: [q]Ψ ∩ R ≠ Θ\}. \tag{1}
\]

\[
Ψ(R) = \{q ∈ Θ: [q]Ψ \cap R ≠ Θ\}. \tag{2}
\]

The Ψ – Lapp of R in Θ is the greatest definable in Θ contained in R. The Ψ – Uapp of R in Θ is the least definable set in Θ containing R. For any nonempty subset R in Θ, Ψ(R) = (Ψ(R), Ψ(R)) is called rough set with respect to Ψ or simply a Ψ-rough subset of P(Θ) × P(Θ) if Ψ(R) ≠ Ψ(R), where P(Θ) denotes the set of all subsets of Θ.

Definition 1 (see [31]). Let Θ be a complete lattice. Define an associative binary relation ◦ on Θ satisfying

\[
I ◦ (V_{i∈I} w_i) = V_{i∈I} (I ◦ w_i) \text{ and } (V_{i∈I} l_i) ◦ w = V_{i∈I} (l_i ◦ w),
\]

∀ I, w, l_i, w_i ∈ Θ. Then, (Θ, ◦) is called quantale.

Let T_1, T_2, T_i ∈ Θ, i ∈ I. We define some notions as follows:

\[
T_1 ◦ T_2 = \{t_1 ◦ t_2: t_1 ∈ T_1, t_2 ∈ T_2\};
\]

\[
T_1 ∨ T_2 = \{t_1 ∨ t_2: t_1 ∈ T_1, t_2 ∈ T_2\};
\]

\[
\land_{i∈I} T_i = \{\land_{i∈I} t_i: t_i ∈ T_i\}.
\]

Throughout the paper, quantales are denoted by Θ_1 and Θ_2.

Let Ω ≠ W ⊆ Θ. Then, W is called a subquantale of Θ if the following holds:

(1) w_1 ◦ w_2 ∈ W, ∀ w_1, w_2 ∈ W.

(2) V_{i∈I} w_i ∈ W, ∀ w_i ∈ W.

That is, Θ closed under ◦ and arbitrary supremum.

Definition 2 (see [32]). Let Θ be a quantale, Ω ≠ I ⊆ Θ is called left (right) ideal if the following satisfied:

(1) u, v ∈ I implies u ∨ v ∈ I.

(2) p ∈ Θ, u ∈ I such that p ≤ u implies p ∈ I.

(3) q ∈ Θ and u ∈ I implies q ◦ u ∈ I(u ◦ q ∈ I).

A nonempty subset I ⊆ Θ is called ideal of Θ if it is left as well right ideal.

Example 1. Let Θ = {0, p, q, r, 1} complete lattices are shown in Figure 1. We define ◦ be the associative binary operation on Θ as shown in Table 1.

Then, Θ is a quantale. Then, {0}, {0, p}, {0, q}, {0, p, q, r}, and Θ are all I of quantale Θ.

Definition 3 (see [32]). Let Ω ≠ I ⊆ Θ be an ideal. I is called prime ideal if, ∀ u, v ∈ Ω, u ∨ v ∈ I or u, v ∈ I. I is called semiprime I if, ∀ u ∈ Ω, u ◦ u ∈ I is called primary I if, ∀ u, v ∈ Ω, u ◦ v ∈ I and u ∉ I implies v^n ∈ I for some n ∈ N.
Let \( (\Psi, C) \) be a soft set over \( \Theta \times \Theta' \), where \( C \subseteq E \) (parametric set). Then, \( \Psi : C \rightarrow P(\Theta) \) and \( \Psi^F : C \rightarrow P(\Theta') \).

A SCPR \((\Psi, C)\) over \(\Theta \times \Theta'\) is satisfied only condition (1). A SCPR \((\Psi, C)\) over \(\Theta \times \Theta'\) is called soft prime ideal over \(\Theta\) iff \(\Psi^F(c)\) is a prime ideal of \(\Theta\), \(\forall c \in C\).

A SCPR \((\Lambda, C)\) over \(\Theta \times \Theta'\) is called soft complete relation (SCPR) with respect to the afterset if for all \(p, r, \in \Theta\), we have

\[
\Lambda(p, r) \in \Psi^F(c) \Rightarrow (p \circ r) \in \Psi^F(c)
\]

A SCPR \((\Lambda, C)\) is called \(\nu\)-complete w.r.t the afterset if it satisfies only condition (1).

A SCPR \((\Lambda, C)\) is called \(\nu\)-complete w.r.t the afterset if it satisfies only condition (1).
A SCPR $(\Lambda, C)$ is called $\omega$-complete w.r.t. the foresets if it satisfies only condition (2).

**Theorem 1.** Let $(\Psi, C)$ be a SCPR with respect to the afterset over $\Theta_1 \times \Theta_2$. Then, for any two soft sets $(F_1, C)$ and $(F_2, C)$ over $\Theta_2$, we have

\begin{align*}
(1) & \quad (\Psi^{F_1}, C) \circ (\Psi^{F_2}, C) \subseteq (\Psi^{F_1 
abla F_2}, C), \\
(2) & \quad (\Psi^{F_1}, C) \vee (\Psi^{F_2}, C) \subseteq (\Psi^{F_1 \vee F_2}, C)
\end{align*}

**Proof.** For arbitrary $c \in C$, let $x \in \Psi^{F_1}(c) \circ \Psi^{F_2}(c)$. Then, $x = y_1 \circ y_2$ for some $y_1 \in \Psi^{F_1}(c)$ and $y_2 \in \Psi^{F_2}(c)$. This implies that $y_1 \Psi(c) \cap F_1(c) \neq \emptyset$ and $y_2 \Psi(c) \cap F_1(c) \neq \emptyset$, so there exist elements $l, m \in \Theta_2$ such that $l \in y_1 \Psi(c) \cap F_1(c)$ and $m \in y_2 \Psi(c) \cap F_2(c)$. Thus, $l \in y_1 \Psi(c) \cap \Psi(c) \cap F_1(c)$ and $m \in F_2(c)$. So $(l, m) \in \Psi(c) \cap F_1(c)$ and $(y_1, y_2, l \circ m) \in \Psi(c)$; that is, $(l, m) \in y_1 \Psi(c) \cap \Psi(c) \cap F_1(c)$ and $m \in F_2(c)$. Therefore, $l \circ m \in \Psi(c) \cap \Psi(c) \cap F_1(c) \cap F_2(c)$. This shows that $x = y_1 \circ y_2 \in \Psi^{F_1 \vee F_2}(c)$.

For arbitrary $c \in C$, let $x \in \Psi^{F_1}(c) \cap \Psi^{F_2}(c)$. Then, $x = y_1 \cap y_2$ for some $y_1 \in \Psi^{F_1}(c)$ and $y_2 \in \Psi^{F_2}(c)$. This implies that $\emptyset \neq y_1 \Psi(c) \subseteq F_1(c)$ and $\emptyset \neq y_2 \Psi(c) \subseteq F_2(c)$. As $(y_1 \cap y_2) \Psi(c) = y_1 \Psi(c) \cap y_2 \Psi(c) \subseteq F_1(c) \cap F_2(c)$, this shows that $x = y_1 \cap y_2 \in \Psi^{F_1 \wedge F_2}(c)$. Hence, (2) is proved.

**Theorem 4.** Let $(\Psi, C)$ be a SCTR with respect to the afterset over $\Theta_1 \times \Theta_2$. Then, for any two soft sets $(L_1, C)$ and $(L_2, C)$ over $\Theta_1$, we have

\begin{align*}
(1) & \quad (\ast L_1, C) \circ (\ast L_2, C) \subseteq (\ast L_1 \ast L_2, C), \\
(2) & \quad (\ast L_1, C) \vee (\ast L_2, C) \subseteq (\ast L_1 \vee L_2, C)
\end{align*}

**Proof.** The proof is obvious.

**4. Approximation of Soft Substructures in Quantales**

In this section, we consider two quantales $\Theta_1$ and $\Theta_2$ and approximate different soft substructures of quantales by using different SBR over $\Theta_1 \times \Theta_2$. We will show that $U_{\text{app}}$ of a soft substructure of quantales by using SCPR is again a soft substructure of quantales and provide counter examples to support the argument that the converse is not true. Also, we will show that $L_{\text{app}}$ of a soft substructure of quantales by using SCTR is again a soft substructure of quantales and provide a counter example to support the argument that the converse is not true.

Throughout this section, we consider $(\Psi, C)$ to be the SBR over $\Theta_1 \times \Theta_2$ and $x \Psi(c) \neq \emptyset$ for all $x \in \Theta_1, c \in C$, and $\Psi(c) y \neq \emptyset$ for all $y \in \Theta_2, c \in C$ unless otherwise specified.

**Definition 11.** Let $(\Psi, C)$ be a SBR over $\Theta_1 \times \Theta_2$ and $(F, C)$ be a soft set over $\Theta_2$. If $U_{\text{app}}(\Psi, C)$ is a soft subquantale of $\Theta_1$, then $(F, C)$ is called generalized upper soft (GUS) subquantale of $\Theta_1$ w.r.t. the aftersets. If $U_{\text{app}}(\Psi, C)$ is a soft ideal (prime ideal, semiprime ideal, and primary ideal) of $\Theta_1$, then $(F, C)$ is called GUS ideal (prime ideal, semiprime ideal, and primary ideal) of $\Theta_1$ w.r.t. the aftersets.

**Definition 12.** Let $(\Psi, C)$ be a SBR over $\Theta_1 \times \Theta_2$ and $(L, C)$ be a soft set over $\Theta_2$. If $U_{\text{app}}(\Psi, C)$ is a soft subquantale of $\Theta_2$, then $(L, C)$ is called generalized upper soft (GUS) subquantale of $\Theta_2$ w.r.t. the foresets. If $U_{\text{app}}(\Psi, C)$ is a soft ideal (prime ideal, semiprime ideal, and primary ideal) of $\Theta_2$, then $(L, C)$ is called GUS ideal (prime ideal, semiprime ideal, and primary ideal) of $\Theta_2$ w.r.t. the foresets.

**Theorem 5.** Let $(\Psi, C)$ be a SCPR over $\Theta_1 \times \Theta_2$. If $(F, C)$ is a soft subquantale of $\Theta_2$, then $(F, C)$ is a GUS subquantale of $\Theta_1$ w.r.t. the aftersets.

**Proof.** Suppose that $(F, C)$ is a soft subquantale, then $\emptyset \neq \Psi(c) \neq \emptyset$ for any $c \in C$. Let $p_i \in \Psi(c), i \in I$. Then, $p_i \Psi(c) \cap F(c) \neq \emptyset$. So, there exist $q_i \in p_i \Psi(c) \cap F(c)$. Thus, $q_i \in p_i \Psi(c) \neq F(c)$ since $(\Psi, C)$ is a SCPR. Therefore, $(p_i, q_i) \in \Psi(c), i \in I$ implies $(\forall_{i \in I} p_i, \forall_{i \in I} q_i) \in \Psi(c)$. This implies that $\forall_{i \in I} q_i \in \forall_{i \in I} p_i \Psi(c)$. Also, $\forall_{i \in I} q_i \in F(c)$ (as $(F, C)$ is a soft subquantale). So, $\forall_{i \in I} q_i \in \forall_{i \in I} p_i \Psi(c) \cap F(c)$. Hence, $\forall_{i \in I} p_i \in \Psi(c).$
Let \( p_1, p_2 \in \mathcal{F}(c) \). Then, \( p_1 \Psi(c) \cap F(c) \neq \emptyset \) and \( p_2 \Psi(c) \cap F(c) \neq \emptyset \). So, there exists \( q_1 \in p_1 \Psi(c) \cap F(c) \) and \( q_2 \in p_2 \Psi(c) \cap F(c) \). Thus, \( q_1 \in p_1 \Psi(c) \), \( q_2 \in p_2 \Psi(c) \), and \( q_2 \in F(c) \) since \( (\Psi, C) \) is a SCPR. Therefore, \( (p_1 \circ q_1, p_2 \circ q_2) \in \Psi(c) \) implies \( (p_1 \circ q_1, p_2 \circ q_2) \in \Psi(c) \). This implies that \( q_1 \circ q_2 \in p_1 \cap p_2 \Psi(c) \). Also, \( q_1 \circ q_2 \in (c) \) (as \( (c) \) is a soft subquandle). So, \( q_1 \circ q_2 \in p_1 \cap p_2 \Psi(c) \cap F(c) \). Hence, \( (p_1 \circ q_1) \in \Psi(c) \). This completes the proof.

With the same arguments, the next Theorem 6 can be achieved.

**Theorem 6.** Let \( (\Psi, C) \) be a SCPR over \( \Theta_1 \times \Theta_2 \). If \( (L, C) \) is a soft subquandle of \( \Theta_1 \), then \( (L, C) \) is a \( GU_p \) subquandle of \( \Theta_2 \) w.r.t the forests.

**Theorem 7.** Let \( (\Psi, C) \) be a soft \( \mathcal{V} \)-complete relation over \( \Theta_1 \times \Theta_2 \) w.r.t the aftersets. If \( (F, C) \) is a soft left (right) ideal of \( \Theta_2 \), then \( (F, C) \) is a \( GU_p \) left (right) ideal of \( \Theta_1 \) w.r.t the aftersets.

**Proof.** Suppose that \( (F, C) \) is a soft left ideal of \( \Theta_2 \), then \( \emptyset \neq \Psi(c) \) for any \( c \in C \). Let \( u_1, u_2 \in \Psi(c) \). Then, \( u_1 \Psi(c) \cap F(c) \neq \emptyset \) and \( u_2 \Psi(c) \cap F(c) \neq \emptyset \). So, there exists \( v_1 \in u_1 \Psi(c) \cap F(c) \) and \( v_2 \in u_2 \Psi(c) \cap F(c) \). Thus, \( v_1 \in u_1 \Psi(c) \) and \( v_2 \in u_2 \Psi(c) \) since \( (\Psi, C) \) is a SCPR. Therefore, \( (u_1 \cup v_1, u_2 \cup v_2) \in \Psi(c) \); that is, \( v_1 \cap v_2 \in (u_1 \cup v_1, u_2 \cup v_2) \Psi(c) \). Hence, \( v_1 \cap v_2 \in (u_1 \cup v_1, u_2 \cup v_2) \Psi(c) \) and \( u_1 \cup v_1, u_2 \cup v_2 \in \Psi(c) \).

Now, let \( u_1, u_2 \in \Theta_1 \) such that \( u_1 \cap u_2 \neq \emptyset \) and \( u_2 \in \Psi(c) \). So, \( u_1 \cup u_2 = u_1 \cap u_2 \). Since \( u_2 \in \Psi(c) \), there exist \( v_1 \in u_1 \Psi(c) \cap F(c) \). So, \( v_2 \in u_2 \Psi(c) \cup F(c) \). Since \( (\Psi, C) \) is a soft \( \mathcal{V} \)-complete relation, therefore, \( v_2 \in u_2 \Psi(c) \cup F(c) \). This implies that \( v_1 \cap v_2 \in (u_1 \cup v_1, u_2 \cup v_2) \Psi(c) \); that is, \( v_1 \cap v_2 \in (u_1 \cup v_1, u_2 \cup v_2) \Psi(c) \). Hence, \( u_1 \cup v_1, u_2 \cup v_2 \in \Psi(c) \). Therefore, \( (u_1 \cup v_1, u_2 \cup v_2) \in \Psi(c) \) and \( u_1 \cup v_1, u_2 \cup v_2 \in \Psi(c) \). Thus, \( s \leq v_1 \) and \( s \leq v_2 \) imply \( s \in F(c) \) (as \( F(c) \) is ideal). So, \( s \in u_1 \Psi(c) \cap F(c) \). Hence, \( u_1 \in \Psi(c) \).

Let \( p, x \in \Theta_1 \) and \( x \in \Psi(c) \). Then, \( x \Psi(c) \cap F(c) \neq \emptyset \). So, there exist \( q \in x \Psi(c) \cap F(c) \) and \( q \in \Psi(c) \). Since \( (F, C) \) is a soft left ideal so, \( y \in \Psi(c) \) for any \( y \in p \cap (x \Psi(c) \cap F(c)) \). This implies that \( (p, y) \in \Psi(c) \). So, \( p \circ x \in \Psi(c) \). Similarly, \( p \circ x \in \Psi(c) \); therefore, \( x \Psi(c) \cap F(c) \neq \emptyset \).

**Theorem 8.** Let \( (\Psi, C) \) be a SCTR over \( \Theta_1 \times \Theta_2 \) w.r.t the aftersets. If \( (F, C) \) is a soft prime ideal of \( \Theta_2 \), then \( (F, C) \) is a \( GU_p \) prime ideal of \( \Theta_1 \) w.r.t the aftersets.

**Proof.** Assume that \( (F, C) \) is a soft prime ideal of \( \Theta_2 \), then \( \emptyset \neq \Psi(c) \) for any \( c \in C \). Then, by Theorem 5, \( (F, C) \) is generalized upper soft ideal of \( \Theta_1 \). Let \( p_1, p_2 \in \Theta_1 \) such that \( p_1 \circ p_2 \in \Psi(c) \). Then, \( p_1 \circ p_2 \Psi(c) \cap F(c) \neq \emptyset \). So, there exist \( q \in (p_1 \circ p_2) \Psi(c) \cap F(c) \). This implies that \( q \in (p_1 \circ p_2) \Psi(c) \cap F(c) \). Since \( (\Psi, C) \) is a SCTR, \( q \in (p_1 \circ p_2) \Psi(c) \cap F(c) \) for some \( c \in p_1 \Psi(c) \). Thus, \( c \circ d \in F(c) \) and \( d \in p_2 \Psi(c) \). Hence, \( (p_1 \circ q_1) \in \Psi(c) \). This completes the proof.

**Theorem 9.** Let \( (\Psi, C) \) be a SCTR over \( \Theta_1 \times \Theta_2 \) w.r.t the aftersets. If \( (F, C) \) is a soft prime ideal of \( \Theta_2 \), then \( (F, C) \) is a \( GU_p \) prime ideal of \( \Theta_1 \) w.r.t the aftersets.

**Proof.** Assume that \( (F, C) \) is a soft prime ideal of \( \Theta_2 \), then \( \emptyset \neq \Psi(c) \) for any \( c \in C \). Then, by Theorem 5, \( (F, C) \) is generalized upper soft ideal of \( \Theta_1 \). Let \( p_1 \in \Theta_1 \) such that \( p_1 \circ p_2 \in \Psi(c) \). Then, \( (p_1 \circ p_2) \Psi(c) \cap F(c) \neq \emptyset \). So, there exist \( q \in (p_1 \circ p_2) \Psi(c) \cap F(c) \). This implies that \( q \in (p_1 \circ p_2) \Psi(c) \cap F(c) \). Since \( (\Psi, C) \) is a SCTR, \( q \in (p_1 \circ p_2) \Psi(c) \cap F(c) \) for some \( c \in p_1 \Psi(c) \). Thus, \( c \circ d \in F(c) \) and \( d \in p_2 \Psi(c) \). Hence, \( (p_1 \circ q_1) \in \Psi(c) \). This completes the proof.
Then, \((\Psi, C)\) is SCPR. The aftersets with respect to \(\Psi(c_1)\) and \(\Psi(c_2)\) are given as follows:

\[
\begin{align*}
0\Psi(c_1) &= \{0', s', p', q', r', 1'\}, \\
p\Psi(c_1) &= \{0', s', p', r', 1', q'\}, \\
q\Psi(c_1) &= \{r', s', 1', q'\}, \\
p\Psi(c_2) &= \{r', s', 1', q'\}, \\
q\Psi(c_2) &= \varnothing,
\end{align*}
\]

(9)

Define soft set \((F, C)\) over \(\Theta_2\) by the rule

\[
\begin{align*}
F(c_1) &= \{r', s'\}, \\
F(c_2) &= \{q', r'\}.
\end{align*}
\]

(10)

Then, \((F, C)\) is not a soft subquantale of \(\Theta_2\). But \(\Psi'(c_1) = \{0, p, q, 1\}\) and \(\Psi'(c_2) = \{0, p\}\) are subquantale of \(\Theta_1\). So, \((F, C)\) is a GU\(p\)SS\(q\) of \(\Theta_1\) w.r.t the aftersets.

Foresets with respect to \(\Psi(c_1)\) and \(\Psi(c_2)\) are given as follows:

\[
\begin{align*}
\Psi(c_1)0' &= \{0\}, \\
\Psi(c_2)0' &= \{0\}, \\
\Psi(c_1)p' &= \{0, q\}, \\
\Psi(c_2)p' &= \{0\}, \\
\Psi(c_1)q' &= \{0, p\}, \\
\Psi(c_2)q' &= \{0, p\}, \\
\Psi(c_1)r' &= \{0, p, q, 1\}, \\
\Psi(c_2)r' &= \{0, p, q, 1\}, \\
\Psi(c_1)s' &= \{0, p, q, 1\}, \\
\Psi(c_2)s' &= \{0, p, q, 1\}, \\
\Psi(c_1)1' &= \{0, p, q, 1\}, \\
\Psi(c_2)1' &= \{0, p, q, 1\}.
\end{align*}
\]

(11)

Define soft set \((L, C)\) over \(\Theta_1\) by the rule

\[
\begin{align*}
L(c_1) &= \{p, q\}, \\
L(c_2) &= \{0, p, q\}.
\end{align*}
\]

(12)

Then, \((L, C)\) is not a soft subquantale of \(\Theta_1\). But \(\Psi'(c_1) = \{p', q', r', s', 1'\}\) and \(\Psi'(c_2) = \{0', p', q', r', s', 1'\}\) are subquantale of \(\Theta_2\). So, \((L, C)\) is a GU\(p\)SS subquantale of \(\Theta_2\) w.r.t the foresets.

(2) Now, let \(C = \{c_1, c_2\}\) and define SBR \((\Psi, C)\) over \(\Theta_1 \times \Theta_2\) by the rule

\[
\Psi(c_1) = \{(q, 0'), (p, 0'), (q, p'), (0, p'), (1, 0'), (1, p'), (p, p'), (0, 0')\},
\]

(13)

\[
\Psi(c_2) = \{(p, q'), (0, s'), (0, q'), (q, s'), (1, s'), (1, q'), (q, q'), (p, s')\}.
\]

(14)

Aftersets with respect to \(\Psi(c_1)\) and \(\Psi(c_2)\) are given as follows:

\[
\begin{align*}
0\Psi(c_1) &= \{0', p'\}, \\
0\Psi(c_2) &= \{s', q'\}, \\
p\Psi(c_1) &= \{0', p'\}, \\
p\Psi(c_2) &= \{s', q'\}, \\
q\Psi(c_1) &= \{0', p'\}, \\
q\Psi(c_2) &= \{s', q'\}, \\
p\Psi(c_1) &= \{0', p'\}, \\
p\Psi(c_2) &= \{s', q'\}.
\end{align*}
\]

(15)
Then, \((\Psi, C)\) is \(\vee\)-complete relation over \(\Theta_1 \times \Theta_2\) w.r.t the aftersets. Define soft set \((F, C)\) over \(\Theta_2\) by the rule
\[
F(c_1) = \{s', p'\},
\]
\[
F(c_2) = \{s', r'\}.
\]
(16)

Then, \((F, C)\) is not a soft ideal of \(\Theta_2\). But \(\mathcal{P}^p(c_1) = \{0, p, q, 1\}\) and \(\mathcal{P}^q(c_2) = \{0, p, q, 1\}\) are ideal of \(\Theta_1\). So, \((F, C)\) is a \(G\cup p\) ideal of \(\Theta_1\) w.r.t the aftersets.

Now, define SBR \((\Psi, C)\) over \(\Theta_1 \times \Theta_2\) by the rule
\[
\Psi(c_1) = \left\{(0', q'), (p, q'), (0', 0'), (p, 0')\right\}
\]
\[
\Psi(c_2) = \left\{(0, 0'), (q, r'), (0, q'), (p, 0'), (p, 0'), (0, q'), (p, 0'), (q, s'), (0, s'), (p, r'), (q, p')\right\}
\]
\[
\Psi(c_2) = \left\{(0, 0'), (q, r'), (0, q'), (p, 0'), (p, 0'), (0, q'), (p, 0'), (q, s'), (0, s'), (p, r'), (q, p')\right\}
\]
(17)
(18)

Foresets with respect to \(\Psi(c_1)\) and \(\Psi(c_2)\) are given as follows:
\[
\Psi(c_1)0' = \{p, 0\},
\]
\[
\Psi(c_2)0' = \{q, 0\},
\]
\[
\Psi(c_1)p' = \{p, 0\},
\]
\[
\Psi(c_2)p' = \{0, q\},
\]
\[
\Psi(c_1)q' = \{p, 0\},
\]
\[
\Psi(c_2)q' = \{q, 0\},
\]
\[
\Psi(c_1)r' = \{0, p\},
\]
\[
\Psi(c_2)r' = \{q, 0\},
\]
\[
\Psi(c_1)s' = \{0, p\},
\]
\[
\Psi(c_2)s' = \{q, 0\},
\]
\[
\Psi(c_1)1' = \{0, p\},
\]
\[
\Psi(c_2)1' = \{0, q\}.
\]
(19)

Then, \((\Psi, C)\) is soft \(\vee\)-complete relation over \(\Theta_1 \times \Theta_2\) w.r.t the aftersets. Define soft set \((L, C)\) over \(\Theta_1\) by the rule
\[
L(c_1) = \{p, q\},
\]
\[
L(c_2) = \{0, p, q\}.
\]
(20)

Then, \((L, C)\) is not a soft ideal of \(\Theta_1\). But \(\mathcal{P}^p(c_1) = \{0', p', q', r', s', 1'\}\) and \(\mathcal{P}^q(c_2) = \{0', p', q', r', s', 1'\}\) are ideal of \(\Theta_2\). So, \((L, C)\) is a \(G\cup L, C\) ideal of \(\Theta_2\) w.r.t the aftersets.

Similar examples can be presented to justify that converse of Theorems 11 to 13 is not true.

**Definition 13.** Let \((\Psi, C)\) be a SBR over \(\Theta_1 \times \Theta_2\). Consider the soft set \((M, C)\) over \(\Theta_2\), if \(L_{\text{app}}(\Psi^M, C)\) is a soft subquantale of \(\Theta_1\), then \((M, C)\) is called generalized lower soft \((\text{GL}_s\cup, C)\) subquantale of \(\Theta_1\) w.r.t the aftersets.

**Definition 14.** Let \((\Psi, C)\) be a SBR over \(\Theta_1 \times \Theta_2\). Consider the soft set \((L, C)\) over \(\Theta_1\), if \(L_{\text{app}}(\Psi^M, C)\) is a soft subquantale of \(\Theta_1\), then \((L, C)\) is called \(G\cup L, C\) ideal of \(\Theta_1\) w.r.t the aftersets.

**Theorem 11.** Let \((\Psi, C)\) be a SCTR over \(\Theta_1 \times \Theta_2\) w.r.t the aftersets. If \((M, C)\) is a soft subquantale of \(\Theta_1\), then \((M, C)\) is a \(G\cup L, C\) subquantale of \(\Theta_1\) w.r.t the aftersets.

**Proof.** Suppose that \((M, C)\) is a soft subquantale of \(\Theta_1\) and \(\Psi^M(c) \neq \emptyset\) for any \(c \in C\). Let \(u_1 \in \Psi^M(c)\), \(i \in I\). Then, \(u_1 \Psi^M(c) \subseteq M(c)\). Since \((\Psi, C)\) is a SCTR, \(\bigvee_{i \in I} \left(u_1 \Psi^M(c)\right) \subseteq M(c)\). Hence, \(\bigvee_{i \in I} u_1 \in \Psi^M(c)\).

Now, let \(u_1, u_2 \in \Psi^M(c)\). Then, \(u_1 \Psi^M(c) \subseteq M(c)\) and \(u_2 \Psi^M(c) \subseteq M(c)\). Since \((\Psi, C)\) is a SCTR and \((M, C)\) is a soft subquantale, \(u_1 \Psi^M(c) \circ u_2 \Psi^M(c) \subseteq M(c)\) implies \(u_1 \circ u_2 \Psi^M(c) \subseteq M(c)\). Hence, \(u_1 \circ u_2 \in \Psi^M(c)\).

With the same arguments, next Theorem 12 can be achieved. \(\square\)

**Theorem 12.** Let \((\Psi, C)\) be a SCTR over \(\Theta_1 \times \Theta_2\) w.r.t the aftersets. If \((L, C)\) is a soft subquantale of \(\Theta_1\), then \((L, C)\) is a \(G\cup L, C\) subquantale of \(\Theta_1\) w.r.t the aftersets.

**Theorem 13.** Let \((\Psi, C)\) be a SCTR over \(\Theta_1 \times \Theta_2\) w.r.t the aftersets. If \((M, C)\) is a soft ideal of \(\Theta_2\), then \((M, C)\) is a \(G\cup L, C\) ideal of \(\Theta_1\) w.r.t the aftersets.

**Proof.** Suppose that \((M, C)\) is a soft ideal of \(\Theta_2\) and \(\Psi^M(c) \neq \emptyset\) for any \(c \in C\). Let \(u_1, u_2 \in \Psi^M(c)\). Then, \(u_1 \Psi^M(c) \subseteq M(c)\) and \(u_2 \Psi^M(c) \subseteq M(c)\). Since \((\Psi, C)\) is a SCTR and \((M, C)\) is a soft ideal of \(\Theta_2\) so \(u_1 \Psi^M(c) \circ u_2 \Psi^M(c) \subseteq M(c)\); that is, \(u_1 \circ u_2 \Psi^M(c) \subseteq M(c)\). Hence, \(u_1 \circ u_2 \in \Psi^M(c)\).

Now, let \(u_1, u_2 \in \Theta_1\) such that \(u_2 \subseteq u_1\), and \(u_1 \in \Psi^M(c)\). So, \(u_1 \circ u_2 \in \Psi^M(c)\). Let, \(v_1 \in u_1 \Psi^M(c)\) and \(v_2 \in u_2 \Psi^M(c)\). Then, \(v_1 \circ v_2 \in (u_1 \circ u_2)\Psi^M(c)\); that is, \(v_1 \circ v_2 \in u_1 \Psi^M(c) \subseteq M(c)\). Since \(M(c)\) is ideal so \(v_1 \subseteq v_1 \circ v_2 \subseteq M(c)\) implies \(v_1 \in M(c)\). Hence, \(u_1 \Psi^M(c) \subseteq M(c)\). Therefore, \(u_1 \in \Psi^M(c)\).

Now, let \(u, y \in \Theta_1\) and \(y \in \Psi^M(c)\). Then, \(\emptyset \neq y \Psi^M(c) \subseteq M(c)\). Consider \(v_1 \in u \Psi^M(c) \circ y \Psi^M(c)\) since \((\Psi, C)\) is a SCTR so \(v_1 \in u \Psi^M(c) \circ y \Psi^M(c)\). Thus, \(v_1 = c \circ d\) for some \(c \in u \Psi^M(c)\) and \(d \in y \Psi^M(c)\). But \(y \Psi^M(c) \subseteq M(c)\) so \(d \in M(c)\) and \((M, C)\) is a soft ideal of \(\Theta_2\); therefore, \(c \circ d \in M(c)\), that is, \(v_1 \in M(c)\). Thus, \(u \circ y \Psi^M(c) \subseteq M(c)\). Hence, \(u \circ y \in \Psi^M(c)\). Similarly, we can show that \(y \circ u \in \Psi^M(c)\). \(\square\)
Theorem 14. Let \((\Psi, C)\) be a SCTR over \(\Theta_1 \times \Theta_2\) w.r.t. the aftersets. If \((M, C)\) is a soft prime ideal of \(\Theta_2\), then \((M, C)\) is a GLWS prime ideal of \(\Theta_1\) w.r.t. the aftersets.

Proof. Assume that \((M, C)\) is a soft prime ideal of \(\Theta_2\) and \(\Psi M (c) \neq \emptyset\) for any \(c \in C\). Then, by Theorem 4.19, \((M, C)\) is GLWS ideal of \(\Theta_1\). Let \(u_1, u_2 \in \Theta_1\) such that \(u_1 \circ u_2 \in \Psi M (c)\). Then, \((u_1 \circ u_2) \Psi (c) \subseteq M (c)\). Consider \(v \in (u_1 \circ u_2) \Psi (c) \subseteq M (c)\). Since \((\Psi, C)\) is a SCTR, \(v \in (u_1 \circ u_2) \Psi (c) \subseteq u_1 \Psi (c) \circ u_2 \Psi (c)\). Thus, \(v = c \circ \delta d\) for some \(c \in u_1 \Psi (c)\) and \(d \in u_2 \Psi (c)\). This implies that \(v = c \circ \delta d \in M (c)\). As \((M, C)\) is a soft prime ideal so, \(c \in M (c)\) or \(d \in M (c)\). Thus, \(c \in u_1 \Psi (c) \subseteq M (c)\) or \(d \in u_2 \Psi (c) \subseteq M (c)\). Hence, \(u_1 \in \Psi M (c)\) or \(u_2 \in \Psi M (c)\).

Theorem 15. Let \((\Psi, C)\) be a SCTR over \(\Theta_1 \times \Theta_2\) w.r.t. the aftersets. If \((M, C)\) is a soft semiprime ideal of \(\Theta_2\), then \((M, C)\) is a GLWS semiprime ideal of \(\Theta_1\) w.r.t. the aftersets.

Proof. Suppose that \((M, C)\) is a soft semiprime ideal of \(\Theta_2\) and \(\Psi M (c) \neq \emptyset\) for any \(c \in C\). Then, by Theorem 4.19., \((M, C)\) is a GLWS ideal of \(\Theta_1\). Let \(u_1, u_2 \in \Theta_1\) such that \(u_1 \circ u_2 \in \Psi M (c)\). Then, \((u_1 \circ u_2) \Psi (c) \subseteq M (c)\). Consider \(v \in (u_1 \circ u_2) \Psi (c) \subseteq M (c)\). Since \((\Psi, C)\) is a SCTR so \(v \circ \delta \in (u_1 \circ u_2) \Psi (c) \subseteq M (c)\). Thus, \(v = c \circ \delta \delta d\) for some \(c \in u_1 \Psi (c)\) and \(d \in u_2 \Psi (c)\). Thus, \(v \circ \delta \delta d \in M (c)\) for some \(d \in \Psi (c)\) and \(n \in \Psi (c)\).

Remark 3. One can find examples like Example 2 to show that converse of Theorems 11 to 16 is not true.

5. Relationship between Soft Quantale Homomorphism and Their Approximation

In this section, we define soft weak quantale homomorphism (SWQH), and then, we established the relationship between homomorphic images and their approximation by SBR.

Definition 15 (see [4]). A function \(\eta: \Theta_1 \longrightarrow \Theta_2\) is called weak quantale homomorphism (WQH) if

\[ \eta (p \circ q) = \eta (p) \circ \eta (q)\]

and

\[ \eta (p \circ q) = \eta (p) \circ \eta (q)\]

where \((\Theta_1, \circ_1)\) and \((\Theta_2, \circ_2)\) are quantales. If \(\eta\) is one-one, then \(\eta\) is monomorphism, and if \(\eta\) is bijective, then \(\eta\) is called isomorphism between \((\Theta_1, \circ_1)\) and \((\Theta_2, \circ_2)\).

Definition 16. Let \((H, C_1)\) be a soft quantale over \(\Theta_1\) and \((F, C_2)\) be a soft quantale over \(\Theta_2\). Then, \((H, C_1)\) is said to be soft weak homomorphic to \((F, C_2)\) if there exist ordered pair of functions \((\eta, \zeta)\) satisfies the following

1. \(\eta: \Theta_1 \longrightarrow \Theta_2\) is onto WQH, that is, \(\eta (p \circ q) = \eta (p) \circ \eta (q)\) and \(\eta (p \circ q) = \eta (p) \circ \eta (q)\).

2. \(\zeta: C_1 \longrightarrow C_2\) is surjective.

3. \(\eta (H (c_1)) = F (\zeta (c_1))\), \(\forall c_1 \in C_1\).

The ordered pair \((\eta, \zeta)\) of functions is SWQH. If in ordered pair \((\eta, \zeta)\) both \(\eta\) and \(\zeta\) are one-to-one functions, then \((H, C_1)\) is said to only weak isomorphic to \((F, C_2)\) and \((\eta, \zeta)\) is called SWQI.

Lemma 1. Let \((H, C_1)\) be soft weak homomorphic to \((F, C_2)\) with SWQH \((\eta, \zeta)\). Let \((\Psi_1, C_2)\) be a SBR over \(\Theta_2\) and \((H_1, C_1) \subseteq (H, C_1)\).

Define \(\Psi_1 (c_1) = (x, y) \in \Theta_1 \times \Theta_1: \eta (x) (y) \in \Psi_2 (c_2)\) be a SBR over \(\Theta_1\). Then, the following holds:

1. \((\Psi_1, C_2)\) is SCPR if \((\Psi_2, C_2)\) is SCPR

2. If \((\eta, \zeta)\) is SWQH and \((\Psi_2, C_2)\) is SCPR w.r.t. the aftersets \(w.r.t. the foresets\), then \((\Psi_1, C_2)\) is SCPR w.r.t. the aftersets \(w.r.t. the foresets\).

3. \(\eta (\Psi_1 (c_1)) = \Psi_2 (H_1 (c_1))\)

4. \(\eta (H_1 (\Psi_1 (c_1))) \subseteq \Psi_2 (H_1 (c_1))\) and if \((\eta, \zeta)\) is SWQH, then \(\eta (\Psi_1 (H_1 (c_1))) \subseteq \Psi_2 (H_1 (c_1))\).

5. Let \((\eta, \zeta)\) be a SWQI. Then, \(\eta (x) \in \eta (\Psi_1 (c_1)) \Rightarrow x \in H_1 (\Psi_1 (c_1))\) and \(\eta (x) \in \eta (H_1 (\Psi_1 (c_1))) \Rightarrow x \in H_1 (\Psi_1 (c_1))\).

Proof.

(1) and (2) are obvious.

(3) Suppose \((H_1, C_1) \subseteq (H, C_1)\) and for any \(c_1 \in C_1\), \(z \in \eta (\Psi_1 (c_1))\) for some \(z \in \Theta_2\). Then, there exist \(a \in \Theta_1\) such that \(a \in H_1 (\Psi_1 (c_1))\) and \(\eta (a) = z\). Thus, \(a \in \Psi_1 (c_1) \cap H_1 (c_1)\). So, \(a \in \Psi_1 (c_1) \cap H_1 (c_1)\) and \(x \in H_1 (c_1)\). Thus, \(\eta (a) = \eta (\Psi_1 (c_1))\) and \(\eta (x) = \eta (H_1 (c_1))\).

(4) Suppose \((H_1, C_1) \subseteq (H, C_1)\) and for any \(c_1 \in C_1\), \(z \in \eta (\Psi_1 (c_1))\) for some \(z \in \Theta_2\). Then, there exist \(a \in \Theta_1\) such that \(a \in H_1 (\Psi_1 (c_1))\) and \(\eta (a) = z\). Thus, \(a \in \Psi_1 (c_1) \cap H_1 (c_1)\). Let \(x \in z \Psi_2 (c_2)\). Then, there exist \(x \in \Psi_1 (c_1) \cap H_1 (c_1)\). Thus, \(x \in \Psi_1 (c_1) \cap H_1 (c_1)\).
Proof

(1) Let $H, \exists_1$ be an ideal of $\Theta$, for any $c_3 \in C_3$. We have $\eta(H, \exists_1) = \eta(H, \exists_1''(c_3))$.

Let $p, q \in \eta(H, \exists_1)$. Then, there exist $u, v \in H, \exists_1$ such that $\eta(u) = p$ and $\eta(v) = q$. Since $H, \exists_1$ is an ideal and $\eta, \zeta$ is SWQI so $p \vee q = \eta(u) \vee \eta(v) = \eta(u \vee v) \in \eta(H, \exists_1)$. This implies that $H, \exists_1$ is an ideal of $\Theta$.

(2) Let $H, \exists_4$ be a subquantale of $\Theta$, then $\eta(H, \exists_4)$ is a subquantale of $\Theta$.

Similarly, we can show that $\eta(H, \exists_4) = \eta(H, \exists_4''(c_3))$.

Remark 4. With a similar technique, Lemma 1 can be proved but for the foresets.

Theorem 17. Let $(H, C_3)$ be SWQI $(\eta, \zeta)$, which $(\Psi_2, C_3)$ be a SCRT over $\Theta_2$ and $(H, C_3) \subseteq (H, C_3)$. Define $\eta(H, \exists_1) = (x, y) \in \Theta_3 \times \Theta_3 : (x, y)(x) \in \Psi_2(c_3)$ for any $c_3 \in C_3$. Then, the following holds:

(1) $H, \exists_1(c_3)$ is an ideal of $\Theta_1$ iff $\eta(H, \exists_1(c_3))$ is an ideal of $\Theta_2$ for all $c_3 \in C_3$.

(2) $H, \exists_1(c_3)$ is a subquantale of $\Theta_1$ iff $\eta(H, \exists_1(c_3))$ is a subquantale of $\Theta_2$ for all $c_3 \in C_3$.

(3) $H, \exists_1(c_3)$ is a prime ideal of $\Theta_1$ iff $\eta(H, \exists_1(c_3))$ is a prime ideal of $\Theta_2$ for all $c_3 \in C_3$.

(4) $H, \exists_1(c_3)$ is a semiprime ideal of $\Theta_1$ iff $\eta(H, \exists_1(c_3))$ is a semiprime ideal of $\Theta_2$ for all $c_3 \in C_3$.

(5) $H, \exists_1(c_3)$ is a primary ideal of $\Theta_1$ iff $\eta(H, \exists_1(c_3))$ is a primary ideal of $\Theta_2$ for all $c_3 \in C_3$.

Proof

(1) Let $H, \exists_1(c_3)$ be an ideal of $\Theta_1$, for any $c_3 \in C_3$. We have $\eta(H, \exists_1(c_3)) = \eta(H, \exists_1''(c_3))$.

Let $p, q \in \eta(H, \exists_1(c_3))$. Then, there exist $u, v \in H, \exists_1$ such that $\eta(u) = p$ and $\eta(v) = q$. Since $H, \exists_1$ is an ideal and $\eta, \zeta$ is SWQI so $p \vee q = \eta(u) \vee \eta(v) = \eta(u \vee v) \in \eta(H, \exists_1(c_3))$.

Now, let $p, q \in \Theta_2$ such that $p \leq q$ and $q \in \eta(H, \exists_1(c_3))$. Then, there exist $u, v \in H, \exists_1(c_3)$ such that $\eta(u) = p$ and $\eta(v) = q$. So, $\eta(u) \leq \eta(v)$ implies $\eta(u \vee v) = \eta(u) \vee \eta(v) = \eta(u) \in \eta(H, \exists_1(c_3))$. This implies that $\eta(u \otimes v) \in \eta(H, \exists_1(c_3))$. This implies that $\eta(u \otimes v) \in \eta(H, \exists_1(c_3))$.

Finally, let $p \in \Theta_2$ and $q \in \eta(H, \exists_1(c_3))$. Then, there exist $u \in \Theta_1$ and $v \in H, \exists_1(c_3)$ such that $\eta(u) = p$ and $\eta(v) = q$. Since $H, \exists_1(c_3)$ is an ideal, $u \otimes v \in H, \exists_1(c_3)$. Thus, $\eta(u \otimes v) = \eta(u) \otimes \eta(v) = (p \otimes q) \in \eta(H, \exists_1(c_3))$. Similarly, $q \otimes p \in \eta(H, \exists_1(c_3))$. Hence, $\eta(H, \exists_1(c_3))$ is ideal of $\Theta_2$.

Conversely, suppose that $\eta(H, \exists_1(c_3))$ is ideal of $\Theta_2$ for any $c_3 \in C_3$. We will show that $H, \exists_1(c_3)$ is ideal of $\Theta_1$.

Let $u, v \in H, \exists_1(c_3)$. Then, $\eta(u) \vee \eta(v) \in \eta(H, \exists_1(c_3))$. Since $\eta(H, \exists_1(c_3))$ is ideal so $\eta(u \vee v) = \eta(u) \vee \eta(v) \in \eta(H, \exists_1(c_3))$. This implies that $\eta(u) \leq \eta(v)$. Since $\eta(H, \exists_1(c_3))$ is ideal $\eta(u) \in \eta(H, \exists_1(c_3))$. Then, by Lemma 5.2(5), $u \vee v \in H, \exists_1(c_3)$. Now, let $u, v \in \Theta_1$ such that $u \leq v$ and $v \in H, \exists_1(c_3)$.

Then, $u \vee v = v \in H, \exists_1(c_3)$. Thus, $\eta(u \vee v) = \eta(u) \vee \eta(v) = \eta(u) \in \eta(H, \exists_1(c_3))$. This implies that $\eta(u) \leq \eta(v)$. Since $\eta(H, \exists_1(c_3))$ is ideal $\eta(u) \in \eta(H, \exists_1(c_3))$. Then, by Lemma 5.2(5), $u \in H, \exists_1(c_3)$. Finally, let $u \in \Theta_1$ and $v \in H, \exists_1(c_3)$. Then, $\eta(u) \in \Theta_2$ and $\eta(v) \in \eta(H, \exists_1(c_3))$. Since $\eta(H, \exists_1(c_3))$ is ideal, $\eta(u) \in \eta(H, \exists_1(c_3))$ that is, $\eta(u \otimes \eta(v) \in \eta(H, \exists_1(c_3))$. Thus, $u \otimes v \in H, \exists_1(c_3)$. In a similar way, we can show that $u \otimes v \in H, \exists_1(c_3)$. This completes the proof.

The proof of (2)–(5) is similar to the proof of (1).

6. Comparison

Yang and Xu [7] introduced rough approximations in quantale which is a kind of partially ordered algebraic structure with an associatve binary operation. The main idea of work in [7] is based on equivalence relation equipped with congruence relation in quantale. In fact, the generalization of Pawlak’s space is discussed in [7]. Further approximation of fuzzy substructures of quantale in crisp atmospheric spaces was discussed in [4]. Sometimes, it is difficult to find out an equivalence relation and then congruence while finding rough substructures in quantale. To remove this hurdle, soft binary relations are utilized in this paper. Since suitable soft binary relations are easy to find out, it is an easy approach to apply soft rough properties to approach different characterizations of soft
rough structures in quantale with the help of aftersets and foresets.

7. Conclusion

The new combined effect of an algebraic structure quantale with rough and soft sets is presented by using soft binary relation, in this paper. The soft substructures of quantales like soft subquintale and soft ideal are discussed. The approximation w.r.t aftersets and foresets of these substructures by SBR which is an extended notion of Pawlak’s rough approximation with respect to aftersets and foresets of these substruc-
tures by SBR over $\Theta_1 \times \Theta_2$ is employed. This new relation over $\Theta_1 \times \Theta_2$ enables us to use the concept of aftersets and foresets to express the lower and upper approximation. Important results regarding to the approximation of soft substructures of quantales under SBR with some essential algebraic conditions such as compatible and complete relations are introduced. To emphasize and make a clear understanding, soft compatible and soft complete relations are focused, and these are interpreted by aftersets and foresets. Particularly, in our work, soft compatible and soft complete relations play an important role. Crucx of these results is that whenever we approximate a soft algebraic structure of quantale, corresponding upper and lower approximations, are again the same kind of soft algebraic structure. Furthermore, we presented the soft quantale homomorphism and established the relationship of soft homomorphic images with their approximation under SBR.

In future, one can use this work and generalize it to different soft algebraic structures such as soft quantale modules, soft hypergroups, soft hyperquantaletes, and soft hyperrings. One can take motivation from our generalized approximation space and define new approximation spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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