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Research Article

Soft Substructures in Quantales and Their Approximations Based on Soft Relations

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The aim of this research article is to derive a new relation between rough sets and soft sets with an algebraic structure quantale by using soft binary relations. The aftersets and foresets are utilized to define lower approximation and upper approximation of soft subsets of quantales. As a consequence of this new relation, different characterization of rough soft substructures of quantales is obtained. To emphasize and make a clear understanding, soft compatible and soft complete relations are focused, and these are interpreted by aftersets and foresets. Particularly, in our work, soft compatible and soft complete relations play an important role. Moreover, this concept generalizes the concept of rough soft substructures of other structures. Furthermore, the algebraic relations between the upper (lower) approximation of soft substructures of quantales and the upper (lower) approximation of their homomorphic images with the help of soft quantales homomorphism are examined. In comparison with the different type of approximations in different type of algebraic structures, it is concluded that this new study is much better.

1. Introduction

Quantale theory was proposed by Mulvey[1]. It is based on defining an algebraic structure on a complete lattice. Since quantale was defined on a complete lattice, there must be a correlation between linear logic and quantale theory which was studied by Yetter, in his study. He presented a new class of models for linear intuitionistic logic [2]. In recent years, quantale is applied in vast research areas, such as algebraic theory [3], rough set theory [4–7], topological theory [8], theoretical computer science [9], and linear logic [10].

In 1982, Pawlak developed the famous rough set theory [11], which is a mathematization of inadequate knowledge. The rough set deals with the categorization and investigation of inadequate information and knowledge. After Pawlak's work, Zhu [12] provided some new views on the rough set theory. In [13], Ali et al. studied some properties of generalized rough sets. Nowadays, rough sets are applied in many different areas, such as cognitive sciences, machine learning, pattern recognition, and process control.

There are many problems that arise in different fields such as engineering, economics, and social sciences in which data have some sort of uncertainty. Well-known mathematical tools have so many limitations because these tools are introduced for particular circumstances. There are many theories to overcome uncertainty such as fuzzy set theory, probability theory, rough sets, and vague sets, but these are limited due to its design.

In 1998, Molodtsov present the idea of soft set theory, which is a mathematical tool to overcome the adversities affecting the above theories [14]. Many authors like Maji et al. present different operations on soft sets and try to consolidate the algebraic aspects of soft sets [15]. A new and different idea of operations was presented by Ali et al. [16]. Many soft algebraic structures such as soft modules [17], soft groups [18], soft rings [19], and soft ordered semigroups [20] were studied. The basic theme and purpose of soft sets are to create the idea of parametrization, and this idea has been utilized to find soft binary relation (SBR) which is a parameterized collection of binary relations on a universe under consideration. This puts forward the consideration for complicated objects that may be perceived from different points of view. In [21–23], Feng et al. presented the relationship between soft, rough, and fuzzy sets and produced rough soft sets, soft rough sets, and soft-rough fuzzy sets.

By using aftersets and foresets notions associated with SBR, a new approximation space is widely utilized these days. By using generalized approximation space based on SBR, different soft substructures in semigroups were approximated by Kanawal and Shabir [24]. Motivated by the idea in [24], soft substructures in quantales are defined, and the aftersets and foresets are employed to construct the lower approximation and upper approximation of soft substructures. Since we are dealing with the approximation of soft subsets of quantale, further soft substructures are employed for further characterization.

There are several authors who introduced rough sets theory in algebraic structures and soft algebraic structures. Iwinski analyzes algebraic properties of rough sets [25]. Qurashi and Shabir present the idea of roughness in Q-module [5]. Idea of the generalized rough quantales (subquantales) was presented by Xiao and Li [6]. Rough prime (semiprime and primary) ideals in quantales were investigated by Yang and Xu [7]. Fuzzy ideals (prime, semiprime, and primary) in quantales were introduced by Luo and Wang [4]. Generalized roughness of fuzzy substructures in quantale is studied by Qurashi et al. [26]. In [27], Yamak et al. proposed the idea of set-valued mappings as the basis of the generalized upper (lower) approximations of a ring with the help of ideals. Rough prime bi Γ-hyper ideals of Γ-semihypergroups were proposed by Yaqoob et al. [28, 29]. Rough substructures of semigroups were studied by Kuroki [30].

The following scheme is designed for the rest of the paper. Some essential explanations related to quantales, its substructures, soft substructures, and their corresponding sequels are connected in Section 2. Notion of approximations of soft sets over quantale generated by soft binary relations is discussed in Section 3. In Section 4, by using these ideas, generalized soft substructures are defined and investigated further fundamental algebraic characteristics of these phenomena. Additionally, we extend this study to define the relationship between homomorphic images and their approximation by soft binary relation in Section 5.

2. Preliminaries

Let Θ be a nonempty finite set called the universe set and Ψ be an E.R (equivalence relation) over Θ. Let $[q]_\Psi$ denotes the equivalence class of the relation containing q. Any definable set in Θ would be written as finite union of equivalence classes of Θ. Let $R \subseteq \Theta$ in general R is not a definable set in Θ. However, the set R can be approximated by two definable sets in Θ. The first one is called Ψ-lower approximation ($\Psi - \mathrm{L}_{\mathrm{appr}}$) of R, and the second is called Ψ-upper approximation ($\Psi - U_{\mathrm{appr}}$). They are defined as follows:

$$\underline{\Psi}(R) = \{q \in \Theta\colon\ [q]_\Psi \subseteq R\}. \quad (1)$$

$$\overline{\Psi}(R) = \{q \in \Theta\colon\ [q]_\Psi \cap R \neq \varnothing\}. \quad (2)$$

The $\Psi - L_{\mathrm{appr}}$ of R in Θ is the greatest definable in Θ contained in R. The $\Psi - U_{\mathrm{appr}}$ of R in Θ is the least definable set in Θ containing R. For any nonempty subset R in Θ, $\Psi(R) = (\underline{\Psi}\,(R), \overline{\Psi}R)$ is called rough set with respect to Ψ or simply a Ψ-rough subset of $P(\Theta) \times P(\Theta)$ if $\underline{\Psi}(R) \neq \overline{\Psi}(R)$, where $P(\Theta)$ denotes the set of all subsets of Θ.

Definition 1 (see [31]). Let Θ be a complete lattice. Define an associative binary relation $\circ$ on Θ satisfying

$$l \circ (\vee_{i\in I} w_i) = \vee_{i\in I} (l \circ w_i) \text{ and } (\vee_{i\in I} l_i) \circ w = \vee_{i\in I} (l_i \circ w), \quad (3)$$

$\forall\, l, w, l_i, w_i \in \Theta$. Then, $(\Theta, \circ)$ is called quantale.

Let $T_1, T_2, T_I \subset \Theta$, $i \in I$. We define some notions as follows:

$$\begin{aligned} T_1 {}^\circ \mathrm{T}_2 &= \{t_1 {}^\circ \mathrm{t}_2\colon\ \mathrm{t}_1 \in \mathrm{T}_1, \mathrm{t}_2 \in \mathrm{T}_2\};\\ T_1 \vee T_2 &= \{t_1 \vee t_2\colon\ t_1 \in T_1, t_2 \in T_2\};\\ \vee_{i\in I} T_i &= \{\vee_{i\in I} t_i\colon\ t_i \in T_i\}. \end{aligned} \quad (4)$$

Throughout the paper, quantales are denoted by Θ_1 and Θ_2.

Let $\varnothing \neq W \subseteq \Theta$. Then, $_W$ is called a subquantale of Θ if the following holds:

(1) $w_1 {}^\circ \mathrm{w}_2 \in \mathrm{W}$, $\forall w_1, w_2 \in W$.

(2) $\vee_{i\in I} w_i, \in W$, $\forall w_i, \in W$.

That is, Θ closed under $\circ$ and arbitrary supremum.

Definition 2 (see [32]). Let Θ be a quantale, $\varnothing \neq I \subseteq \Theta$ is called left (right) ideal if the following satisfied:

(1) $u, v \in I$ implies $u \vee v \in 1$

(2) $p \in \Theta$, $u \in I$ such that $p \leq u$ implies $p \in I$

(3) $q \in \Theta$ and $u \in I$ implies $q^\circ u \in I\,(u^\circ q \in I)$

A nonempty subset $I \subseteq \Theta$ is called ideal of Θ if it is left as well as right ideal.

Example 1. Let $\Theta = \{0, p, q, r, 1\}$ complete lattices are shown in Figure 1. We define $\circ$ be the associative binary operation on Θ as shown in Table 1.

Then, Θ is a quantale. Then, $\{0\}$, $\{0, p\}$, $\{0, q\}$, $\{0,\ p, q,\ r\}$, and Θ are all I of quantale Θ.

Definition 3 (see [32]). Let $\varnothing \neq I \subseteq \Theta$ be an ideal. I is called prime ideal if, $\forall u,\ v \in \Theta$, $u^\circ v \in I \Rightarrow u \in I$ or $v \in I$. I is called semiprime I if, $\forall\, u \in \Theta$, $u^\circ u \in I \Rightarrow u \in II$ is called primary I if, $\forall u,\ v \in \Theta$, $u^\circ v \in I$ and $u \notin I$ implies $v^n \in I$ for some $n \in \mathrm{N}$.

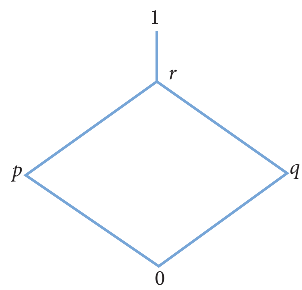

Figure 1: Illustration of Θ.

Table 1: Binary operation subject to Θ

$\circ_1$	0	p	q	r	1
0	0	p	q	r	1
p	0	p	q	r	1
q	0	p	q	r	1
r	0	p	q	r	1
1	0	p	q	r	1

Definition 4 (see [14]). A pair (Ψ, C) is called a soft set over Θ if $\Psi: C \longrightarrow P(\Theta)$ where C is a subset of E (the set of parameters).

Definition 5 (see [16]). Let (F, C_1) and (H, C_2) be two soft sets over Θ. Then, (F, C_1) soft subset (H, C_2) if the following conditions are fulfilled:

(1) $C_1 \subseteq C_2$
(2) $F(c) \subseteq H(c)$, $\forall c \in C_1$

Definition 6 (see [33]). Let (Ψ, C) be a soft set over $\Theta \times \Theta$, that is, $\Psi: C \longrightarrow P(\Theta \times \Theta)$. Then, (Ψ, C) is called a soft binary relation (SBR) over $\Theta \times \Theta$. A SBR over $\Theta_1 \times \Theta_2$ is a soft set (Ψ, C) over $\Theta_1 \times \Theta_2$. That is, $\Psi: C \longrightarrow P(\Theta_1 \times \Theta_2)$.

Definition 7. Let (Ψ, C) be a soft set over quantale Θ. Then,

(1) (Ψ, C) is called soft subquantale over Θ iff $\Psi(c)$ is a subquantale of Θ, $\forall c \in C$
(2) (Ψ, C) is called soft ideal over Θ iff $\Psi(c)$ is an ideal of Θ, $\forall c \in C$
(3) (Ψ, C) is called soft prime ideal over Θ iff $\Psi(c)$ is a prime ideal of Θ, $\forall c \in C$
(4) (Ψ, C) is called soft semiprime ideal over Θ iff $\Psi(c)$ is a semiprime ideal of Θ, $\forall c \in C$
(5) (Ψ, C) is called soft primary ideal over Θ iff $\Psi(c)$ is a primary ideal of Θ, $\forall c \in C$

3. Approximation of Soft Sets over Quantale by Soft Binary Relation

In this section, we present some important aspects regarding to the approximation of soft sets in quantale Θ by SBR. We utilized aftersets and foresets to approximate soft sets.

Definition 8 (see [34]). Let (Ψ, C) be a SBR over $\Theta_1 \times \Theta_2$, where $C \subseteq E$ (parametric set). Then, $\Psi: C \longrightarrow P(\Theta_1 \times \Theta_2)$. For a soft set (F, C) over Θ_2, the $L_{\text{appr}}(\underline{\Psi}^F, C)$ and $U_{\text{appr}}(\overline{\Psi}^F, C)$ of (F, C) w.r.t the afterset are essentially two soft sets over Θ_1, which is defined as

$$\underline{\Psi}^F(c) = \{q_1 \in \Theta_1: \varnothing \neq q_1\Psi(c) \subseteq F(c)\}, \tag{5}$$

$$\overline{\Psi}^F(c) = \{q_1 \in \Theta_1: q_1\Psi(c) \cap F(c) \neq \varnothing\}, \quad \forall, c \in C. \tag{6}$$

And for a soft set (H, C) over Θ_1, the $L_{\text{appr}}({}^H\underline{\Psi}, C)$ and $U_{\text{appr}}({}^H\overline{\Psi}, C)$ of (H, C) w.r.t the foreset are actually two soft sets over Θ_2, which is defined as

$$\begin{aligned} {}^H\underline{\Psi}(c) &= \{q_2 \in \Theta_2: \varnothing \neq \Psi(c)q_2 \subseteq H(c)\}, \\ {}^H\overline{\Psi}(c) &= \{q_2 \in \Theta_2: \Psi(c)q_2 \cap H(c) \neq \varnothing\}. \end{aligned} \tag{7}$$

For all $c \in C$, where $q_1\Psi(c) = \{q_2 \in \Theta_2: (q_1, q_2) \in \Psi(c)\}$ is called the afterset of q_1 and $\Psi(c)q_2 = \{q_1 \in \Theta_1: (q_1, q_2) \in \Psi(c)\}$ is called the foreset of q_2.

Remark 1

(1) For each soft set (F, C) over Θ_2, $\underline{\Psi}^F: C \longrightarrow P(\Theta_1)$ and $\overline{\Psi}^F: C \longrightarrow P(\Theta_1)$
(2) For each soft set (H, C) over Θ_1, ${}^H\underline{\Psi}: C \longrightarrow P(\Theta_2)$ and ${}^H\overline{\Psi}: C \longrightarrow P(\Theta_2)$

Definition 9. Let (Ψ, C) be a SBR over $\Theta_1 \times \Theta_2$, that is, $\Psi: C \longrightarrow P(\Theta_1 \times \Theta_2)$. Then, (Ψ, C) is called soft compatible relation (SCPR) if for all $p, r, j_i \in \Theta_1$ and $q, s, k_i \in \Theta_2$ $(i \in I)$, we have

(1) $(p, q), (r, s) \in \Psi(c) \Rightarrow (p \circ_1 r, q \circ_2 s) \in \Psi(c)$
(2) $(j_i, k_i) \in \Psi(c) \Rightarrow (\vee_{i \in I} j_i, \vee_{i \in I} k_i) \in \Psi(c)$

for every $c \in C$.

Definition 10. A SCPR (Ψ, C) over $\Theta_1 \times \Theta_2$ is called soft complete relation (SCTR) with respect to the afterset if, for all $p, r, \in \Theta_1$, we have

(1) $p\Psi(c) \vee r\Psi(c) = (p \vee r)\Psi(c)$
(2) $p\Psi(c) \circ_2 r\Psi(c) = (p \circ_1 r)\Psi(c)$

for all $c \in C$.

A SCPR (Ψ, C) is called ∨-complete w.r.t the aftersets if it satisfies only condition (1). A SCPR (Ψ, C) is called ∘-complete w.r.t the aftersets if it satisfies only condition (2).

A SCPR (Λ, C) over $\Theta_1 \times \Theta_2$ is called soft complete relation (SCTR) with respect to the foreset if for all $q, s \in \Theta_2$, and we have

(1) $\Lambda(c)q \vee \Lambda(c)s = \Lambda(c)(q \vee s)$
(2) $\Lambda(c)q \circ_1 \Lambda(c)s = \Lambda(c)(q \circ_2 s)$

for all $c \in C$.

A SCPR (Λ, C) is called ∨-complete w.r.t the foresets if it satisfies only condition (1).

A SCPR (Λ, C) is called $\circ$-complete w.r.t the foresets if it satisfies only condition (2).

Theorem 1. *Let (Ψ, C) be a SCPR with respect to the afterset over $\Theta_1 \times \Theta_2$. Then, for any two soft sets (F_1, C) and (F_2, C) over Θ_2, we have*

(1) $(\overline{\Psi}^{F_1}, C) \circ_1 (\overline{\Psi}^{F_2}, C) \subseteq (\overline{\Psi}^{F_1 \circ_2 F_2}, C)$

(2) $(\overline{\Psi}^{F_1}, C) \vee (\overline{\Psi}^{F_2}, C) \subseteq (\overline{\Psi}^{F_1 \vee F_2}, C)$

Proof. For arbitrary $c \in C$, let $x \in \overline{\Psi}^{F_1}(c) \circ_1 \overline{\Psi}^{F_2}(c)$. Then, $x = y_1 \circ_1 y_2$ for some $y_1 \in \overline{\Psi}^{F_1}(c)$ and $y_2 \in \overline{\Psi}^{F_2}(c)$. This implies that $y_1\Psi(c) \cap F_1(c) \neq \varnothing$ and $y_2\Psi(c) \cap F_2(c) \neq \varnothing$, so there exist elements $l, m \in \Theta_2$ such that $l \in y_1\Psi(c) \cap F_1(c)$ and $m \in y_2\Psi(c) \cap F_2(c)$. Thus, $l \in y_1\Psi(c)$, $m \in y_2\Psi(c)$, $l \in F_1(c)$ and $m \in F_2(c)$. So $(y_1, l) \in \Psi(c)$ and $(y_2, m) \in \Psi(c)$ imply $(y_1 \circ_1 y_2, l \circ_2 m) \in \Psi(c)$; that is, $(l \circ_2 m) \in (y_1 \circ_1 y_2)\Psi(c)$. Also, $l \circ_2 m \in F_1(c) \circ_2 F_2(c)$; therefore, $l \circ_2 m \in y_1 \circ_1 y_2 \Psi(c) \cap F_1(c) \circ_2 F_2(c)$. This shows that $x = y_1 \circ_1 y_2 \in \overline{\Psi}^{F_1 \circ_2 F_2}(c)$.

Now, for arbitrary $c \in C$, let $x \in \overline{\Psi}^{F_1}(c) \vee \overline{\Psi}^{F_2}(c)$. Then, $x = y_1 \vee y_2$ for some $y_1 \in \overline{\Psi}^{F_1}(c)$ and $y_2 \in \overline{\Psi}^{F_2}(c)$. This implies that $y_1\Psi(c) \cap F_1(c) \neq \varnothing$ and $y_2\Psi(c) \cap F_2(c) \neq \varnothing$, so there exist elements $l, m \in \Theta_2$ such that $l \in y_1\Psi(c) \cap F_1(c)$ and $m \in y_2\Psi(c) \cap F_2(c)$. Thus, $l \in y_1\Psi(c)$, $m \in y_2\Psi(c)$, $l \in F_1(c)$, and $m \in F_2(c)$. So $(y_1, l) \in \Psi(c)$ and $(y_2, m) \in \Psi(c)$ imply $(y_1 \vee y_2, l \vee m) \in \Psi(c)$; that is, $(l \vee m) \in (y_1 \vee y_2)\Psi(c)$. Also, $l \vee m \in F_1(c) \vee F_2(c)$; therefore, $l \vee m \in y_1 \vee y_2 \Psi(c) \cap F_1(c) \vee F_2(c)$. This shows that $x = y_1 \vee y_2 \in \overline{\Psi}^{F_1 \vee F_2}(c)$. □

Theorem 2. *Let (Ψ, C) be a SCPR with respect to the foreset over $\Theta_1 \times \Theta_2$. Then, for any two soft sets (L_1, C) and (L_2, C) over Θ_1, we have*

(1) $({}^{L_1}\overline{\Psi}, C) \circ_2 ({}^{L_2}\overline{\Psi}, C) \subseteq ({}^{L_1 \circ_1 L_2}\overline{\Psi}, C)$

(2) $({}^{L_1}\overline{\Psi}, C) \vee ({}^{L_2}\overline{\Psi}, C) \subseteq ({}^{L_1 \vee L_2}\overline{\Psi}, C)$

Proof. The proof is simple. □

Theorem 3. *Let (Ψ, C) be a SCTR w.r.t the afterset over $\Theta_1 \times \Theta_2$. Then, for any two soft sets (F_1, C) and (F_2, C) over Θ_2, we have*

(1) $(\underline{\Psi}^{F_1}, C) \circ_1 (\underline{\Psi}^{F_2}, C) \subseteq (\underline{\Psi}^{F_1 \circ_2 F_2}, C)$

(2) $(\underline{\Psi}^{F_1}, C) \vee (\underline{\Psi}^{F_2}, C) \subseteq (\underline{\Psi}^{F_1 \vee F_2}, C)$

Proof. For arbitrary $c \in C$, if at least one of $\Psi^{F_1}(c)$ and $\Psi^{F_2}(c)$ is empty, then (1) is obvious. Now, for arbitrary $c \in C$, consider that $\Psi^{F_1}(c) \neq \varnothing$ and $\Psi^{F_2}(c) \neq \varnothing$. Then, $\Psi^{F_1}(c) \circ_1 \Psi^{F_2}(c) \neq \varnothing$. So, let $x \in \Psi^{F_1}(c) \circ_1 \Psi^{F_2}(c)$. Then, $x = y_1 \circ_1 y_2$ for some $y_1 \in \Psi^{F_1}(c)$ and $y_2 \in \Psi^{F_2}(c)$. This implies that $\varnothing \neq y_1\Psi(c) \subseteq F_1(c)$ and $\varnothing \neq y_2\Psi(c) \subseteq F_2(c)$. As $(y_1 \circ_1 y_2)\Psi(c) = y_1\Psi(c) \circ_2 y_2\Psi(c) \subseteq F_1(c) \circ_2 F_2(c)$. This shows that $x = y_1 \circ_1 y_2 \in \Psi^{F_1 F_2}(c)$. Hence, (1) is proved.

For arbitrary $c \in C$, if at least one of $\Psi^{F_1}(c)$ and $\Psi^{F_2}(c)$ is empty, then (2) is obvious. Now, for arbitrary $c \in C$, consider that $\Psi^{F_1}(c) \neq \varnothing$ and $\Psi^{F_2}(c) \neq \varnothing$. Then, $\Psi^{F_1}(c) \vee \Psi^{F_2}(c) \neq \varnothing$. So, let $x \in \Psi^{F_1}(c) \vee \Psi^{F_2}(c)$. Then, $x = y_1 \vee y_2$ for some $y_1 \in \Psi^{F_1}(c)$ and $y_2 \in \Psi^{F_2}(c)$. This implies that $\varnothing \neq y_1\Psi(c) \subseteq F_1(c)$ and $\varnothing \neq y_2\Psi(c) \subseteq F_2(c)$. As $(y_1 \vee y_2)\Psi(c) = y_1\Psi(c) \vee y_2\Psi(c) \subseteq F_1(c) \vee F_2(c)$. This shows that $x = y_1 \vee y_2 \in \Psi^{F_1 \vee F_2}(c)$. Hence, (2) is proved. □

Theorem 4. *Let (Ψ, C) be a SCTR with respect to the foreset over $\Theta_1 \times \Theta_2$. Then, for any two soft sets (L_1, C) and (L_2, C) over Θ_1, we have*

(1) $({}^{L_1}\underline{\Psi}, C) \circ_2 ({}^{L_2}\underline{\Psi}, C) \subseteq ({}^{L_1 \circ_1 L_2}\underline{\Psi}, C)$

(2) $({}^{L_1}\underline{\Psi}, C) \vee ({}^{L_2}\underline{\Psi}, C) \subseteq ({}^{L_1 \vee L_2}\underline{\Psi}, C)$

Proof. The proof is obvious. □

4. Approximation of Soft Substructures in Quantales

In this section, we consider two quantales Θ_1 and Θ_2 and approximate different soft substructures of quantales by using different SBR over $\Theta_1 \times \Theta_2$. We will show that U_{appr} of a soft substructure of quantales by using SCPR is again a soft substructure of quantales and provide counter examples to support the argument that the converse is not true. Also, we will show that L_{appr} of a soft substructure of quantales by using SCTR is again a soft substructure of quantales and provide a counter example to support the argument that the converse is not true.

Throughout this section, we consider (Ψ, C) to be the SBR over $\Theta_1 \times \Theta_2$ and $x\Psi(c) \neq \varnothing$ for all $x \in \Theta_1$, $c \in C$, and $\Psi(c)y \neq \varnothing$ for all $y \in \Theta_2$, $c \in C$ unless otherwise specified.

Definition 11. Let (Ψ, C) be a SBR over $\Theta_1 \times \Theta_2$ and (F, C) be a soft set over Θ_2. If $U_{\text{appr}}.\ (\overline{\Psi}^F, C)$ is a soft subquantale of Θ_1, then (F, C) is called generalized upper soft (GU_pS) subquantale of Θ_1 w.r.t the aftersets. If $U_{\text{appr}}(\overline{\Psi}^F, C)$ is a soft ideal (prime ideal, semiprime ideal, and primary ideal) of Θ_1, then (F, C) is called GU_pS ideal (prime ideal, semiprime ideal, and primary ideal) of Θ_1 w.r.t the aftersets.

Definition 12. Let (Ψ, C) be a SBR over $\Theta_1 \times \Theta_2$ and (L, C) be a soft set over Θ_1. If $U_{\text{appr}}({}^{L}\overline{\Psi}, C)$ is a soft subquantale of Θ_2, then (L, C) is called generalized upper soft (GU_pS) subquantale of Θ_2 w.r.t the foresets. If $U_{\text{appr}}({}^{L}\overline{\Psi}, C)$ is a soft ideal (prime ideal, semiprime ideal, and primary ideal) of Θ_2, then (L, C) is called GU_pS ideal (prime ideal, semiprime ideal, and primary ideal) of Θ_2 w.r.t the foresets.

Theorem 5. *Let (Ψ, C) be a SCPR over $\Theta_1 \times \Theta_2$. If (F, C) is a soft subquantale of Θ_2, then (F, C) is a GU_pS subquantale of Θ_1 w.r.t the aftersets.*

Proof. Suppose that (F, C) is a soft subquantale, then $\varnothing \neq \overline{\Psi}^F(c)$ for any $c \in C$. Let $p_i \in \overline{\Psi}^F(c)$, $i \in I$. Then, $p_i\Psi(c) \cap F(c) \neq \varnothing$. So, there exists $q_i \in p_i\Psi(c) \cap F(c)$. Thus, $q_i \in p_i\Psi(c)$ and $q_i \in F(c)$ since (Ψ, C) is a SCPR. Therefore, $(p_i, q_i) \in \Psi(c)$, $i \in I$ implies $(\vee_{i\in I} p_i, \vee_{i\in I} q_i) \in \Psi(c)$. This implies that $\vee_{i\in I} q_i \in \vee_{i\in I} p_i \Psi(c)$. Also, $\vee_{i\in I} q_i \in F(c)$ (as (F, C) is a soft subquantale). So, $\vee_{i\in I} q_i \in \vee_{i\in I} p_i \in \Psi(c) \cap F(c)$. Hence, $\vee_{i\in I} p_i \in \overline{\Psi}^F(c)$.

Let $p_1, p_2 \in \overline{\Psi}^F(c)$. Then, $p_1\Psi(c) \cap F(c) \neq \varnothing$ and $p_2\overline{\Psi}(c) \cap F(c) \neq \varnothing$. So, there exists $q_1 \in p_1\Psi(c) \cap F(c)$ and $q_2 \in p_2\Psi(c) \cap F(c)$. Thus, $q_1 \in p_1\Psi(c)$, $q_1 \in F(c)$, $q_2 \in p_2\Psi(c)$, and $q_2 \in F(c)$ since (Ψ, C) is a SCPR. Therefore, $(p_1, q_1), (p_2, q_2) \in \Psi(c)$ implies $(p_1 \circ_1 q_1), (p_2 \circ_2 q_2) \in \Psi(c)$. This implies that $q_1 \circ_2 q_2 \in p_1 \circ p_2 \Psi(c)$. Also, $q_1 \circ_2 q_2 \in (c)$ (as (F, C) is a soft subquantale). So, $q_1 \circ_2 q_2 \in p_1 \circ p_2 \Psi(c) \cap F(c)$. Hence, $(p_1 \circ_1 q_1) \in \overline{\Psi}^F(c)$. This completes the proof.

With the same arguments, the next Theorem 6 can be achieved. □

Theorem 6. *Let (Ψ, C) be a SCPR over $\Theta_1 \times \Theta_2$. If (L, C) is a soft subquantale of Θ_1, then (L, C) is a GU_pS subquantale of Θ_2 w.r.t the foresets.*

Theorem 7. *Let (Ψ, C) be a soft $\vee$-complete relation over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. If (F, C) is a soft left (right) ideal of Θ_2, then (F, C) is a GU_pS left (right) ideal of Θ_1 w.r.t the aftersets.*

Proof. Suppose that (F, C) is a soft left ideal of Θ_2, then $\varnothing \neq \overline{\Psi}^F(c)$ for any $c \in C$. Let $u_1, u_2 \in \overline{\Psi}^F(c)$. Then, $u_1\Psi(c) \cap F(c) \neq \varnothing$ and $u_2\Psi(c) \cap F(c) \neq \varnothing$. So, there exists $v_1 \in u_1\Psi(c) \cap F(c)$ and $v_2 \in u_2\Psi(c) \cap F(c)$. Thus, $v_1 \in u_1\Psi(c)$, $v_1 \in F(c)$, $v_2 \in u_2\Psi(c)$, and $v_2 \in F(c)$ since (Ψ, C) is a SCPR. Therefore, $(u_1 \vee u_2, v_1 \vee v_2) \in \Psi(c)$; that is, $v_1 \vee v_2 \in (u_1 \vee u_2)\Psi(c)$. Also, $v_1 \vee v_2 \in F(c)$ (as (F, C) is a soft left ideal). So, $v_1 \vee v_2 \in (u_1 \vee u_2)\Psi(c) \cap F(c)$. Hence, $u_1 \vee u_2 \in \overline{\Psi}^F(c)$.

Now, let $u_1, u_2 \in \Theta_1$ such that $u_1 \leq u_2$ and $u_2 \in \overline{\Psi}^F(c)$. So, $u_1 \vee u_2 = u_2 \in \overline{\Psi}^F(c)$. Since $u_2 \in \overline{\Psi}^F(c)$, so there exist $v_2 \in u_2\Psi(c) \cap F(c)$. Thus, $v_2 \in u_2\Psi(c)$ and $v_2 \in F(c)$. Since (Ψ, C) is a soft $\vee$-complete relation, therefore, $v_2 \in u_2\Psi(c) = u_1 \vee u_2\Psi(c) = u_1\Psi(c) \vee u_2\Psi(c)$. This implies that $v_2 = s \vee t$, for some $s \in u_1\Psi(c)$ and $t \in u_2\Psi(c)$. Thus, $s \leq v_2$ and $v_2 \in F(c)$ imply $s \in F(c)$ (as $F(c)$ is ideal). So, $s \in u_1\Psi(c) \cap F(c)$. Hence, $u_1 \in \overline{\Psi}^F(c)$.

Let $p, x \in \Theta_1$ and $x \in \overline{\Psi}^F(c)$. Then, $x\Psi(c) \cap F(c) \neq \varnothing$. So, there exist $q \in x\Psi(c) \cap F(c)$. Thus, $q \in x\Psi(c)$ and $q \in F(c)$. Since (F, C) is a soft left ideal so, $y \circ_2 q \in F(c)$ for any $y \in p\Psi(c) \subseteq \Theta_2$. This implies that $(p, y) \in \Psi(c)$. So, $(p \circ_1 x, y \circ_2 q) \in \Psi(c)$; that is, $y \circ_2 q \in p \circ_1 x\Psi(c)$. So, $y \circ_2 q \in p \circ_1 x\Psi(c) \cap F(c)$. Hence, $p \circ_1 x \in \overline{\Psi}^F(c)$. Similarly, we can show that $x \circ_1 p \in \overline{\Psi}^F(c)$. □

Theorem 8. *Let (Ψ, C) be a SCTR over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. If (F, C) is a soft prime ideal of Θ_2, then (F, C) is a GU_pS prime ideal of Θ_1 w.r.t the aftersets.*

Proof. Assume that (F, C) is a soft prime ideal of Θ_2, then $\varnothing \neq \overline{\Psi}^F(c)$ for any $c \in C$. Then, by Theorem 5, (F, C) is generalized upper soft ideal of Θ_1. Let $p_1, p_2 \in \Theta_1$ such that $p_1 \circ_1 p_2 \in \overline{\Psi}^F(c)$. Then, $(p_1 \circ_1 p_2)\Psi(c) \cap F(c) \neq \varnothing$. So, there exist $q \in (p_1 \circ_1 p_2)\Psi(c) \cap F(c)$. This implies that $q \in (p_1 \circ_1 p_2)\Psi(c)$ and $q \in F(c)$. Since (Ψ, C) is a SCTR, $q \in (p_1 \circ_1 p_2)\Psi(c) = p_1\Psi(c) \circ_2 p_2\Psi(c)$. Thus, $q = c \circ_2 d$ for some $c \in p_1\Psi(c)$ and $d \in p_2\Psi(c)$. Thus, $c \circ_2 d \in F(c)$ and (F, C) is a soft prime ideal of Θ_2 so, $c \in F(c)$ or $d \in F(c)$. Thus, $c \in p_1\Psi(c) \cap F(c)$ or $d \in p_2\Psi(c) \cap F(c)$. Hence, $p_1 \in \overline{\Psi}^F(c)$ or $p_2 \in \overline{\Psi}^F(c)$. □

Theorem 9. *Let (Ψ, C) be a SCTR over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. If (F, C) is a soft semiprime ideal of Θ_2, then (F, C) is a GU_pS semiprime ideal of Θ_1 w.r.t the aftersets.*

Proof. Assume that (F, C) is a soft semiprime ideal of Θ_2, then $\varnothing \neq \overline{\Psi}^F(c)$ for any $c \in C$. Then, by Theorem 5, (F, C) is generalized upper soft ideal of Θ_1. Let $p_1 \in \Theta_1$ such that $p_1 \circ_1 p_1 \in \overline{\Psi}^F(c)$. Then, $(p_1 \circ_1 p_1)\Psi(c) \cap F(c) \neq \varnothing$. So, there exist $q \in (p_1 \circ_1 p_1)\Psi(c) \cap F(c)$. This implies that $q \in (p_1 \circ_1 p_1)\Psi(c)$ and $q \in F(c)$. Since (Ψ, C) is a SCTR, $q \in (p_1 \circ_1 p_1)\Psi(c) = p_1\Psi(c) \circ_2 p_1\Psi(c)$. Thus, $q = c \circ_2 c$ for some $c \in p_1\Psi(c)$. Thus, $c \circ_2 c \in F(c)$ and (F, C) is a soft semiprime ideal of Θ_2 so, $c \in F(c)$. Thus, $c \in p_1\Psi(c) \cap F(c)$. Hence, $p_1 \in \overline{\Psi}^F(c)$. □

Theorem 10. *Let (Ψ, C) be a SCTR over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. If (F, C) is a soft primary ideal of Θ_2, then (F, C) is a GU_pS primary ideal of Θ_1 w.r.t the aftersets.*

Proof. Assume that (F, C) is a soft primary ideal of Θ_2, then $\varnothing \neq \overline{\Psi}^F(c)$ for any $c \in C$. Then, by Theorem 5, (F, C) is generalized upper soft ideal of Θ_1. Let $p_1, p_2 \in \Theta_1$ such that $p_1 \circ_1 p_2 \in \overline{\Psi}^F(c)$ and $p_1 \notin \overline{\Psi}^F(c)$. Then, $(p_1 \circ_1 p_2)\Psi(c) \cap F(c) \neq \varnothing$. So, there exist $q \in (p_1 \circ_1 p_2)\Psi(c) \cap F(c)$. This implies that $q \in (p_1 \circ_1 p_2)\Psi(c)$ and $q \in F(c)$. Since (Ψ, C) is a SCTR, $q \in (p_1 \circ_1 p_2)\Psi(c) = p_1\Psi(c) \circ_2 p_2\Psi(c)$. Thus, $q = c \circ_2 d$ for some $c \in p_1\Psi(c)$ and $d \in p_2\Psi(c)$. Thus, $c \circ_2 d \in F(c)$ and (F, C) is a soft primary ideal of Θ_2 so $d^n \in F(c)$ for some $n \in \mathbb{N}$. Also, $d^n \in p_2^n\Psi(c)$ for $n \in \mathbb{N}$. Thus, $d^n \in p_2^n\Psi(c) \cap F(c)$. Hence, $p_2^n \in \overline{\Psi}^F(c)$. □

Remark 2. In general, the converse of the above theorem is not true. We will present examples to justify our claim as follows.

Example 2. Let $\Theta_1 = \{0, p, q, 1\}$ and $\Theta_2 = \{0', s', q', p', 1', r'\}$ be two complete lattices described in Figures 2 and 3, respectively.

We define $\circ_1$ and $\circ_2$ the associative binary operation on Θ_1 and Θ_2, respectively, as shown in Tables 2 and 3. Then, and are quantales.

(1) Let $C = \{c_1, c_2\}$ and define SBR (Ψ, C) over $\Theta_1 \times \Theta_2$ by the rule

$$\Psi(c_1) = \left\{ \begin{array}{c} (0, q'), (0, 0'), (p, s'), (0, s'), (q, p'), \\ (p, r'), (0, 1'), (0, p'), (p, 1'), (q, s'), \\ (q, 1'), (p, q'), (q, r'), (1, r'), (1, s'), \\ (1, 1'), (0, r') \end{array} \right\},$$

$$\Psi(c_2) = \left\{ \begin{array}{c} (0, r'), (0, 0'), (0, p'), (0, 1'), (0, s'), \\ (p, r'), (0, q'), (p, s'), (p, 1'), (p, q') \end{array} \right\}. \quad (8)$$

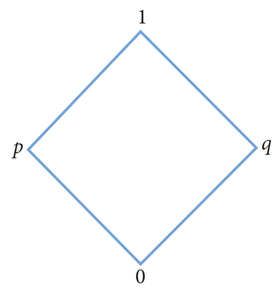

Figure 2: Illustration of Θ_1.

Then, (Ψ, C) is SCPR. The aftersets with respect to $\Psi(c_1)$ and $\Psi(c_2)$ are given as follows:

$$\begin{aligned} 0\Psi(c_1) &= \{0', s', p', q', r', 1'\},\\ 0\Psi(c_2) &= \{0', s', p', r', 1', q'\},\\ p\Psi(c_1) &= \{r', s', 1', q'\},\\ p\Psi(c_2) &= \{r', s', 1', q'\},\\ q\Psi(c_2) &= \varnothing,\\ 1\Psi(c_1) &= \{r', s', 1'\}\\ 1\Psi(c_2) &= \varnothing. \end{aligned} \tag{9}$$

Define soft set (F, C) over Θ_2 by the rule

$$\begin{aligned} F(c_1) &= \{r', s'\},\\ F(c_2) &= \{q', r'\}. \end{aligned} \tag{10}$$

Then, (F, C) is not a soft subquantale of Θ_2. But $\overline{\Psi}^F(c_1) = \{0, p, q, 1\}$ and $\overline{\Psi}^F(c_2) = \{0, p\}$ are subquantale of Θ_1. So (F, C) is a GU_pSS_Θ of Θ_1 w.r.t the aftersets.

Foresets with respect to $\Psi(c_1)$ and $\Psi(c_2)$ are given as follows:

$$\begin{aligned} \Psi(c_1)0' &= \{0\},\\ \Psi(c_2)0' &= \{0\},\\ \Psi(c_1)p' &= \{0, q\},\\ \Psi(c_2)p' &= \{0\},\\ \Psi(c_1)q' &= \{0, p\},\\ \Psi(c_2)q' &= \{0, p\},\\ \Psi(c_1)r' &= \{0, p, q, 1\},\\ \Psi(c_2)r' &= \{0, p\},\\ \Psi(c_1)s' &= \{0, p, q, 1\},\\ \Psi(c_2)s' &= \{0, p\},\\ \Psi(c_1)1' &= \{0, p, q, 1\},\\ \Psi(c_2)1' &= \{0, p\}. \end{aligned} \tag{11}$$

Define soft set (L, C) over Θ_1 by the rule

$$\begin{aligned} L(c_1) &= \{p, q\},\\ L(c_2) &= \{0, p, q\}. \end{aligned} \tag{12}$$

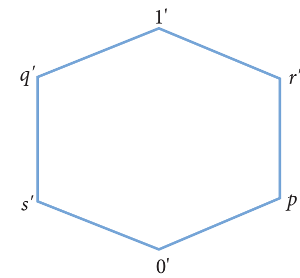

Figure 3: Illustration of Θ_2.

Table 2: Binary operation subject to Θ_1.

O_1	0	p	q	1
0	0	0	0	0
p	0	p	0	p
q	0	0	q	q
1	0	p	q	1

Table 3: Binary operation subject to Θ_2.

O_2	$0'$	s'	p'	q'	r'	$1'$
$0'$	$0'$	s'	p'	q'	r'	$1'$
s'	$0'$	s'	p'	q'	r'	$1'$
p'	$0'$	s'	p'	q'	r'	$1'$
q'	$0'$	s'	p'	q'	r'	$1'$
r'	$0'$	s'	p'	q'	r'	$1'$
$1'$	$0'$	s'	p'	q'	r'	$1'$

Then, (L, C) is not a soft subquantale of Θ_1. But ${}^L\overline{\Psi}(c_1) = \{p', q', r', s', 1'\}$ and ${}^L\overline{\Psi}(c_2) = \{0', p', q', r', s', 1'\}$ are subquantale of Θ_2. So, (L, C) is a GU_pS subquantale of Θ_2 w.r.t. the foresets.

(2) Now, let $C = \{c_1, c_2\}$ and define SBR (Ψ, C) over $\Theta_1 \times \Theta_2$ by the rule

$$\begin{aligned} \Psi(c_1) = \{&(q, 0'), (p, 0'), (q, p'), (0, p'),\\ &(1, 0'), (1, p'), (p, p'), (0, 0')\}. \end{aligned} \tag{13}$$

$$\begin{aligned} \Psi(c_2) = \{&(p, q'), (0, s'), (0, q'), (q, s'),\\ &(1, s'), (1, q'), (q, q'), (p, s')\}. \end{aligned} \tag{14}$$

Aftersets with respect to $\Psi(c_1)$ and $\Psi(c_2)$ are given as follows:

$$\begin{aligned} 0\Psi(c_1) &= \{0', p'\},\\ 0\Psi(c_2) &= \{s', q'\},\\ p\Psi(c_1) &= \{0', p'\},\\ p\Psi(c_2) &= \{s', q'\},\\ q\Psi(c_1) &= \{0', p'\},\\ q\Psi(c_2) &= \{s', q'\},\\ 1\Psi(c_1) &= \{0', p'\},\\ 1\Psi(c_2) &= \{s', q'\}. \end{aligned} \tag{15}$$

Then, (Ψ, C) is $\vee$-complete relation over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. Define soft set (F, C) over Θ_2 by the rule

$$F(c_1) = \{s', p'\},\quad F(c_2) = \{s', r'\}. \tag{16}$$

Then, (F, C) is not a soft ideal of Θ_2. But $\overline{\Psi}^F(c_1) = \{0, p, q, 1\}$ and $\overline{\Psi}^F(c_2) = \{0, p, q, 1\}$ are ideal of Θ_1. So, (F, C) is a GU_pS ideal of Θ_1 w.r.t the aftersets.

Now, define SBR (Ψ, C) over $\Theta_1 \times \Theta_2$ by the rule

$$\Psi(c_1) = \left\{ \begin{array}{c} (0, q'), (p, q'), (0, 0'), (p, 0'), \\ (p, s'), (p, r'), (0, 1'), (0, r'), \\ (p, p'), (0, s'), (p, 1'), (0, p') \end{array} \right\} \tag{17}$$

$$\Psi(c_2) = \left\{ \begin{array}{c} (0, 0'), (q, r'), (q, q'), (0, s'), \\ (0, 1'), (q, 0'), (0, r'), (q, p'), \\ (0, p'), (q, s'), (q, 1'), (0, q') \end{array} \right\} \tag{18}$$

Foresets with respect to $\Psi(c_1)$ and $\Psi(c_2)$ are given as follows:

$$\begin{aligned} \Psi(c_1)0' &= \{p, 0\}, \\ \Psi(c_2)0' &= \{q, 0\}, \\ \Psi(c_1)p' &= \{p, 0\}, \\ \Psi(c_2)p' &= \{0, q\}, \\ \Psi(c_1)q' &= \{p, 0\}, \\ \Psi(c_2)q' &= \{q, 0\}, \\ \Psi(c_1)r' &= \{0, p\}, \\ \Psi(c_2)r' &= \{q, 0\}, \\ \Psi(c_1)s' &= \{0, p\}, \\ \Psi(c_2)s' &= \{q, 0\}, \\ \Psi(c_1)1' &= \{0, p\}, \\ \Psi(c_2)1' &= \{0, q\}. \end{aligned} \tag{19}$$

Then, (Ψ, C) is soft $\vee$-complete relation over $\Theta_1 \times \Theta_2$ w.r.t the foresets. Define soft set (L, C) over Θ_1 by the rule

$$L(c_1) = \{p, q\},\quad L(c_2) = \{0, p, q\}. \tag{20}$$

Then, (L, C) is not a soft ideal of Θ_1. But ${}^L\overline{\Psi}(c_1) = \{0', p', q', r', s', 1'\}$ and ${}^L\overline{\Psi}(c_2) = \{0', p', q', r', s', 1'\}$ are ideal of Θ_2. So, (L, C) is a GU_pS ideal of Θ_2 w.r.t the foresets.

Similar examples can be presented to justify that converse of Theorems 11 to 13 is not true.

Definition 13. Let (Ψ, C) be a SBR over $\Theta_1 \times \Theta_2$. Consider the soft set (M, C) over Θ_2, if $L_{\text{appr}}(\underline{\Psi}^M, C)$ is a soft subquantale of Θ_1, then (M, C) is called generalized lower soft (GL_WS) subquantale of Θ_1 w.r.t the aftersets. If $L_{\text{appr}}(\underline{\Psi}^M, C)$ is a soft ideal (prime ideal, semiprime ideal, and primary ideal) of Θ_1, then (M, C) is called GU_pS ideal (prime ideal, semiprime ideal, and primary ideal) of Θ_1 w.r.t the aftersets.

Definition 14. Let (Ψ, C) be a SBR over $\Theta_1 \times \Theta_2$. Consider the soft set (L, C) over Θ_1, if $L_{\text{appr}}({}^L\underline{\Psi}, C)$ is a soft subquantale of Θ_2, then (L, C) is called GL_WS subquantale of Θ_2 w.r.t the foresets. If $L_{\text{appr}}({}^L\underline{\Psi}, C)$ is a soft ideal (prime ideal, semiprime ideal, and primary ideal) of Θ_2, then (L, C) is called GU_pS ideal (prime ideal, semiprime ideal, and primary ideal) of Θ_2 w.r.t the foresets.

Theorem 11. *Let (Ψ, C) be a SCTR over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. If (M, C) is a soft subquantale of Θ_2, then (M, C) is a GL_WS subquantale of Θ_1 w.r.t the aftersets.*

Proof. Suppose that (M, C) is a soft subquantale of Θ_2 and $\underline{\Psi}^M(c) \neq \varnothing$ for any $c \in C$. Let $u_i \in \underline{\Psi}^M(c)$, $i \in I$. Then, $u_i\Psi(c) \subseteq M(c)$. Since (Ψ, C) is a SCTR, therefore, $\vee_{i\in I}(u_i\Psi(c)) = (\vee_{i\in I}u_i)\Psi(c) \subseteq M(c)$. Hence, $\vee_{i\in I}u_i \in \underline{\Psi}^M(c)$.

Now, let $u_1, u_2 \in \underline{\Psi}^M(c)$. Then, $u_1\Psi(c) \subseteq M(c)$ and $u_2\Psi(c) \subseteq M(c)$. Since (Ψ, C) is a SCTR and (M, C) is a soft subquantale, therefore, $u_1\Psi(c) \circ_2 u_2\Psi(c) \subseteq M(c) \circ_2 M(c)$ implies $(u_1 \circ_1 u_2)\Psi(c) \subseteq M(c)$. Hence, $u_1 \circ_1 u_2 \in \underline{\Psi}^M(c)$.

With the same arguments, next Theorem 12 can be achieved. □

Theorem 12. *Let (Ψ, C) be a SCTR over $\Theta_1 \times \Theta_2$ w.r.t the foresets. If (L, C) is a soft subquantale of Θ_1, then (L, C) is a GL_WS subquantale of Θ_2 w.r.t the foresets.*

Theorem 13. *Let (Ψ, C) be a SCTR over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. If (M, C) is a soft ideal of Θ_2, then (M, C) is a GL_WS ideal of Θ_1 w.r.t the aftersets.*

Proof. Suppose that (M, C) is a soft ideal of Θ_2 and $\underline{\Psi}^M(c) \neq \varnothing$ for any $c \in C$. Let $u_1, u_2 \in \underline{\Psi}^M(c)$. Then, $u_1\Psi(c) \subseteq M(c)$ and $u_2\Psi(c) \subseteq M(c)$. Since (Ψ, C) is a SCTR and (M, C) is a soft ideal of Θ_2 so $u_1\Psi(c) \vee u_2\Psi(c) = (u_1 \vee u_2)\Psi(c) \subseteq M(c) \vee M(c)$; that is, $(u_1 \vee u_2)\Psi(c) \subseteq M(c)$ Hence, $u_1 \vee u_2 \in \underline{\Psi}^M(c)$.

Now, let $u_1, u_2 \in \Theta_1$ such that $u_1 \leq u_2$ and $u_2 \in \underline{\Psi}^M(c)$. So, $u_1 \vee u_2 = u_2 \in \underline{\Psi}^M(c)$. Let, $v_1 \in u_1\Psi(c)$ and $v_2 \in u_2\Psi(c) \subseteq M(c)$. So, $v_1 \vee v_2 \in (u_1 \vee u_2)\Psi(c)$, that is, $v_1 \vee v_2 \in u_2\Psi(c) \subseteq M(c)$. Since $M(c)$ is ideal so $v_1 \leq v_1 \vee v_2 \in M(c)$ implies $v_1 \in M(c)$. Thus, $u_1\Psi(c) \subseteq M(c)$. Hence, $u_1 \in \underline{\Psi}^M(c)$.

Now, let $u, y \in \Theta_1$ and $y \in \underline{\Psi}^M(c)$. Then, $\varnothing \neq y\Psi(c) \subseteq M(c)$. Consider $v_1 \in (u \circ_1 y)\Psi(c)$ since (Ψ, C) is a SCTR so $v_1 \in u\Psi(c) \circ_2 y\Psi(c)$. Thus, $v_1 = c \circ_2 d$ for some $c \in u\Psi(c)$ and $d \in y\Psi(c)$. But $y\Psi(c) \subseteq M(c)$ so $d \in M(c)$ and (M, C) is a soft ideal of Θ_2; therefore, $c \circ_2 d \in M(c)$, that is, $v_1 \in M(c)$. Thus, $(u \circ_1 y)\Psi(c) \subseteq M(c)$. Hence, $u \circ_1 y \in \underline{\Psi}^M(c)$. Similarly, we can show that $y \circ_1 u \in \underline{\Psi}^M(c)$. □

Theorem 14. *Let (Ψ, C) be a SCTR over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. If (M, C) is a soft prime ideal of Θ_2, then (M, C) is a $GL_W S$ prime ideal of Θ_1 w.r.t the aftersets.*

Proof. Assume that (M, C) is a soft prime ideal of Θ_2 and $\underline{\Psi}^M(c) \neq \varnothing$ for any $c \in C$. Then, by Theorem 4.19, (M, C) is $GL_W S$ ideal of Θ_1. Let $u_1, u_2 \in \Theta_1$ such that $u_1 \circ_1 u_2 \in \underline{\Psi}^M(c)$. Then, $(u_1 \circ_1 u_2)\Psi(c) \subseteq M(c)$. Consider $v \in (u_1 \circ_1 u_2)\Psi(c) \subseteq M(c)$. Since (Ψ, C) is a SCTR, $v \in (u_1 \circ_1 u_2)\Psi(c) = u_1\Psi(c) \circ_2 u_2\Psi(c)$. Thus, $v = c \circ_2 d$ for some $c \in u_1\Psi(c)$ and $d \in u_2\Psi(c)$. This implies that $v = c \circ_2 d \in M(c)$. As (M, C) is a soft prime ideal so, $c \in M(c)$ or $d \in M(c)$. Thus, $c \in u_1\Psi(c) \subseteq M(c)$ or $d \in u_2\Psi(c) \subseteq M(c)$. Hence, $u_1 \in \underline{\Psi}^M(c)$ or $u_2 \in \underline{\Psi}^M(c)$. □

Theorem 15. *Let (Ψ, C) be a SCTR over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. If (M, C) is a soft semiprime ideal of Θ_2, then (M, C) is a $GL_W S$ semiprime ideal of Θ_1 w.r.t the aftersets.*

Proof. Assume that (M, C) is a soft semiprime ideal of Θ_2 and $\underline{\Psi}^M(c) \neq \varnothing$ for any $c \in C$. Then, by Theorem 14, (M, C) is $GL_W S$ ideal of Θ_1. Let $u \in \Theta_1$ such that $u \circ_1 u \in \underline{\Psi}^M(c)$. Then, $(u \circ_1 u)\Psi(c) \subseteq M(c)$. Let $v \in u\Psi(c)$. As (Ψ, C) is a SCTR so $v \circ_2 v \in (u \circ_1 u)\Psi(c) \subseteq M(c)$. Since (M, C) is a soft semiprime ideal, $v \circ_2 v \in M(c)$ implies $v \in M(c)$. Thus, $u\Psi(c) \subseteq M(c)$. Hence, $u \in \underline{\Psi}^M(c)$. □

Theorem 16. *Let (Ψ, C) be a SCTR over $\Theta_1 \times \Theta_2$ w.r.t the aftersets. If (M, C) is a soft primary ideal of Θ_2, then (M, C) is a $GL_W S$ primary ideal of Θ_1 w.r.t the aftersets.*

Proof. Suppose that (M, C) is a soft primary ideal of Θ_2 and $\varnothing \neq \underline{\Psi}^M(c)$ for any $c \in C$. Then, by Theorem 4.19., (M, C) is a $GL_W S$ ideal of Θ_1. Let $u_1, u_2 \in \Theta_1$ such that $u_1 \circ_1 u_2 \in \underline{\Psi}^M(c)$ and $u_1 \notin \underline{\Psi}^M(c)$. Then, $(u_1 \circ_1 u_2)\Psi(c) \subseteq M(c)$. Let $v \in (u_1 \circ_1 u_2)\Psi(c)$. Since (Ψ, C) is a SCTR, $v \in u_1\Psi(c) \circ_2 u_2\Psi(c)$. Thus, $v = c \circ_2 d$ for some $c \in u_1\Psi(c)$ and $d \in u_2\Psi(c)$. Thus, $d^n \in u_2^n\Psi(c)$ for some $n \in \mathbb{N}$. Also, $c \circ_2 d \in M(c)$. As (M, C) is a soft primary ideal, $c \notin M(c)$ and $d^n \in M(c)$. Thus, $u_2^n\Psi(c) \subseteq M(c)$. Hence, $u_2^n \in \underline{\Psi}^M(c)$ for some $n \in \mathbb{N}$. □

Remark 3. One can find examples like Example 2 to show that converse of Theorems 11 to 16 is not true.

5. Relationship between Soft Quantale Homomorphism and Their Approximation

In this section, we define soft weak quantale homomorphism (SWQH), and then, we established the relationship between homomorphic images and their approximation by SBR.

Definition 15 (see [4]). A function $\eta: \Theta_1 \longrightarrow \Theta_2$ is called weak quantale homomorphism (WQH) if $\eta(p \circ_1 q) = \eta(p) \circ_2 \eta(q)$ and $\eta(p \vee q) = \eta(p) \vee \eta(q)$, where $(\Theta_1, \circ_1)$ and $(\Theta_2, \circ_2)$ are quantales. If η is one-one, then η is monomorphism. If η is onto, then η is called epimorphism, and if η is bijective, then η is called isomorphism between $(\Theta_1, \circ_1)$ and $(\Theta_2, \circ_2)$.

Definition 16. Let (H, C_1) be a soft quantale over Θ_1 and (F, C_2) be a soft quantale over Θ_2. Then, (H, C_1) is said to soft weak homomorphic to (F, C_2) if there exist ordered pair of functions (η, ζ) satisfies the following

(1) $\eta: \Theta_1 \longrightarrow \Theta_2$ is onto WQH, that is, $\eta(p \circ_1 q) = \eta(p) \circ_2 \eta(q)$ and $\eta(p \vee q) = \eta(p) \vee \eta(q)$
(2) $\zeta: C_1 \longrightarrow C_2$ is surjective
(3) $\eta(H(c_1)) = F(\zeta(c_1)), \forall c_1 \in C_1$

The ordered pair (η, ζ) of functions is SWQH. If in ordered pair (η, ζ) both η and ζ are one-to-one functions, then (H, C_1) is said to soft weak isomorphic to (F, C_2) and (η, ζ) is called SWQI.

Lemma 1. *Let (H, C_1) be soft weak homomorphic to (F, C_2) with SWQH (η, ζ). Let (Ψ_2, C_3) be a SBR over Θ_2 and $(H_1, C_1') \subseteq (H, C_1)$. Define $\Psi_1(c_3) = (x, y) \in \Theta_1 \times \Theta_1: (\eta(x), \eta(y)) \in \Psi_2(c_3)$ be a SBR over Θ_1. Then, the following holds:*

(1) (Ψ_1, C_3) is SCPR if (Ψ_2, C_3) is SCPR
(2) If (η, ζ) is SWQI and (Ψ_2, C_3) is SCPR w.r.t the aftersets (w.r.t the foresets), then (Ψ_1, C_3) is SCPR w.r.t the aftersets (w.r.t the foresets)
(3) $\eta({}^{H_1}\overline{\Psi}_1(c_3)) = \overline{\Psi}_2^{\eta(H_1)}(c_3)$
(4) $\eta({}^{H_1}\underline{\Psi}_1(c_3)) \subseteq \underline{\Psi}_2^{\eta(H_1)}(c_3)$ and if (η, ζ) is SWQI, then $\eta({}^{H_1}\underline{\Psi}_1(c_3)) = \underline{\Psi}_2^{\eta(H_1)}(c_3)$
(5) Let (η, ζ) be a SWQI. Then, $\eta(x) \in \eta({}^{H_1}\overline{\Psi}_1(c_3)) \Leftrightarrow x \in {}^{H_1}\overline{\Psi}_1(c_3)$ and $\eta(x) \in \eta({}^{H_1}\underline{\Psi}_1(c_3)) \Leftrightarrow x \in {}^{H_1}\underline{\Psi}_1(c_3)$

Proof

(1) and (2) are obvious
(3) Suppose $(H_1, C_1') \subseteq (H, C_1)$ and for any $c_3 \in C_3$, $z \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$ for some $z \in \Theta_2$. Then, there exist $a \in \Theta_1$ such that $a \in {}^{H_1}\overline{\Psi}_1(c_3)$ and $\eta(a) = z$. Thus, $x \in a\Psi_1(c_3) \cap H_1(c_1')$. So, $(a, x) \in \Psi_1(c_3)$ and $x \in H_1(c_1')$. Thus, $(\eta(a), \eta(x)) \in \Psi_2(c_3)$, that is, $\eta(x) \in \eta(a)\Psi_2(c_3)$. Also, $\eta(x) \in \eta(H_1(c_1'))$. So, $\eta(a)\Psi_2(c_3) \cap \eta(H_1(c_1')) \neq \varnothing$. This implies that $\eta(a) \in \overline{\Psi}_2^{\eta(H_1)}(c_3)$. Hence, $\eta({}^{H_1}\underline{\Psi}_1(c_3)) \subseteq \overline{\Psi}_2^{\eta(H_1)}(c_3)$. Now, let $w \in \overline{\Psi}_2^{\eta(H_1)}(c_3)$. Then, $w\Psi_2(c_3) \cap \eta(H_1(c_1')) \neq \varnothing$. This implies that $y \in w\Psi_2(c_3) \cap \eta(H_1(c_1'))$. Thus, $y \in w\Psi_2(c_3)$ and $y \in \eta(H_1(c_1'))$. This implies that there exists $x \in H_1(c_1') \subseteq \Theta_1$ and $x_1 \in \Theta_1$ such that $\eta(x) = y$ and $\eta(x_1) = w$. So, $(w, y) = (\eta(x_1), \eta(x)) \in \Psi_2(c_3)$. This implies that $(x_1, x) \in \Psi_1(c_3)$. So, $x \in x_1\Psi_1(c_3) \cap H_1(c_1')$. Thus, $x_1 \in {}^{H_1}\overline{\Psi}_1(c_3)$. So, $w = \eta(x_1) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Hence, $\overline{\Psi}_2^{\eta(H_1)}(c_3) \subseteq \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Consequently, $\eta({}^{H_1}\overline{\Psi}_1(c_3)) = \overline{\Psi}_2^{\eta(H_1)}(c_3)$.
(4) Suppose $(H_1, C_1') \subseteq (H, C_1)$ and for any $c_3 \in C_3$, $z \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$ for some $z \in \Theta_2$. Then, there exist $a \in \Theta_1$ such that $a \in {}^{H_1}\overline{\Psi}_1(c_3)$ and $\eta(a) = z$. Thus, $a\Psi_1(c_3) \subseteq H_1(c_1')$. Let $x \in z\Psi_2(c_3)$. Then, there exist

$y \in \Theta_1$ such that $\eta(y) = x$. So, $\eta(y) \in \eta(a)\Psi_2(c_3)$, that is, $(\eta(a), \eta(y)) \in \Psi_2(c_3)$. So, $(a, y) \in \Psi_1(c_3)$, that is, $y \in a\Psi_1(c_3) \subseteq H_1(c_1')$. Thus, $\eta(y) \in \eta(H_1(c_1'))$. So, $\eta(a)\Psi_2(c_3) \subseteq \eta(H_1(c_1'))$. Thus, $z = \eta(a) \in \underline{\Psi}_2^{\eta(H_1)}(c_3)$. Hence, $\eta({}^{H_1}\underline{\Psi}_1(c_3)) \subseteq \underline{\Psi}_2^{\eta(H_1)}(c_3)$.

Now, let $z \in \underline{\Psi}_2^{\eta(H_1)}(c_3)$. Then, there exist unique $a \in \Theta_1$ such that $\eta(a) = z$ and $\eta(a)\Psi_2(c_3) \subseteq \eta(H_1(c_1'))$. Let $x \in a\Psi_1(c_3)$, that is, $(a, x) \in \Psi_1(c_3)$. Then, $(\eta(a), \eta(x)) \in \Psi_2(c_3)$. Then, $\eta(x) \in \eta(a)\Psi_2(c_3) \subseteq \eta(H_1(c_1'))$. So, $\eta(x) \in \eta(H_1(c_1'))$. This implies that $x \in H_1(c_1')$. So, $a\Psi_1(c_3) \subseteq H_1(c_1')$. Then, $a \in {}^{H_1}\overline{\Psi}_1(c_3)$. So, $z = \eta(a) \in \eta({}^{H_1}\underline{\Psi}_1(c_3))$. Hence, $\underline{\Psi}_2^{\eta(H_1)}(c_3) \subseteq \eta({}^{H_1}\underline{\Psi}_1(c_3))$. Consequently, $\eta({}^{H_1}\underline{\Psi}_1(c_3)) = \underline{\Psi}_2^{\eta(H_1)}(c_3)$.

(6) Let $x \in {}^{H_1}\overline{\Psi}_1(c_3)$ for any $c_3 \in C_3$. Then, $\eta(x) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Conversely, suppose that $\eta(x) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. As η is bijection so $x \in {}^{H_1}\overline{\Psi}_1(c_3)$. Similarly, we can show that $\eta(x) \in \eta({}^{H_1}\underline{\Psi}_1(c_3)) \Leftrightarrow x \in {}^{H_1}\underline{\Psi}_1(c_3)$. □

Remark 4. With a similar technique, Lemma 1 can be proved but for the foresets.

Theorem 17. *Let (H, C_1) be soft weak isomorphic to (F, C_2) with SWQI (η, ζ). Let (Ψ_2, C_3) be a SCPR over Θ_2 and $(H_1, C_1') \subseteq (H, C_1)$. Define $\Psi_1(c_3) = (x, y) \in \Theta_1 \times \Theta_1$: $(\eta(x), \eta(y)) \in \Psi_2(c_3)$ for any $c_3 \in C_3$. Then, the following holds:*

(1) ${}^{H_1}\overline{\Psi}_1(c_3)$ is an ideal of Θ_1 iff $\overline{\Psi}_2^{\eta(H_1)}(c_3)$ is an ideal of Θ_2 for all $c_3 \in C_3$

(2) ${}^{H_1}\overline{\Psi}_1(c_3)$ is a subquantale of Θ_1 iff $\overline{\Psi}_2^{\eta(H_1)}(c_3)$ is a subquantale of Θ_2 for all $c_3 \in C_3$

(3) ${}^{H_1}\overline{\Psi}_1(c_3)$ is a prime ideal of Θ_1 iff $\overline{\Psi}_2^{\eta(H_1)}(c_3)$ is a prime ideal of Θ_2 for all $c_3 \in C_3$

(4) ${}^{H_1}\overline{\Psi}_1(c_3)$ is a semiprime ideal of Θ_1 iff $\overline{\Psi}_2^{\eta(H_1)}(c_3)$ is a semiprime ideal of Θ_2 for all $c_3 \in C_3$

(5) ${}^{H_1}\overline{\Psi}_1(c_3)$ is a primary ideal of Θ_1 iff $\overline{\Psi}_2^{\eta(H_1)}(c_3)$ is a primary ideal of Θ_2 for all $c_3 \in C_3$

Proof

(1) Let ${}^{H_1}\overline{\Psi}_1(c_3)$ be an ideal of Θ_1 for any $c_3 \in C_3$. We will show that $\overline{\Psi}_2^{\eta(H_1)}(c_3)$ is an ideal of Θ_2. By Lemma 1 (3), we have $\eta({}^{H_1}\overline{\Psi}_1(c_3)) = \overline{\Psi}_2^{\eta(H_1)}(c_3)$.

Let $p, q \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Then, there exist $u, v \in {}^{H_1}\overline{\Psi}_1(c_3)$ such that $\eta(u) = p$ and $\eta(v) = q$. Since ${}^{H_1}\overline{\Psi}_1(c_3)$ is ideal and (η, ζ) is SWQI so $p \vee q = \eta(u) \vee \eta(v) = \eta(u \vee v) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$.

Now, let $p, q \in \Theta_2$ such that $p \leq q$ and $q \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Then, there exist $u \in \Theta_1$ and $v \in {}^{H_1}\overline{\Psi}_1(c_3)$ such that $\eta(u) = p$ and $\eta(v) = q$. So, $\eta(u) \leq \eta(v)$ implies $\eta(u \vee v) = \eta(u) \vee \eta(v) = \eta(v) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. This implies that $u \vee v = v \in {}^{H_1}\overline{\Psi}_1(c_3)$. This implies that $u \leq v$ and ${}^{H_1}\overline{\Psi}_1(c_3)$ are ideal so $u \in {}^{H_1}\overline{\Psi}_1(c_3)$. Thus, $\eta(u) = p \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$.

Finally, let $p \in \Theta_2$ and $q \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Then, there exist $u \in \Theta_1$ and $v \in {}^{H_1}\overline{\Psi}_1(c_3)$ such that $\eta(u) = p$ and $\eta(v) = q$. Since ${}^{H_1}\overline{\Psi}_1(c_3)$ ideal, $u \circ_1 v \in {}^{H_1}\overline{\Psi}_1(c_3)$. Thus, $\eta(u \circ_1 v) = \eta(u) \circ_2 \eta(v) = (p \circ_2 q) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Similarly, $q \circ_2 p \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Hence, $\overline{\Psi}_2^{\eta(H_1)}(c_3)$ is ideal of Θ_2.

Conversely, suppose that $\overline{\Psi}_2^{\eta(H_1)}(c_3) = \eta({}^{H_1}\overline{\Psi}_1(c_3))$ be an ideal of Θ_2 for any $c_3 \in C_3$. We will show that ${}^{H_1}\overline{\Psi}_1(c_3)$ is ideal of Θ_1.

Let $u, v \in {}^{H_1}\overline{\Psi}_1(c_3)$. Then, $\eta(u), \eta(v) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Since $\eta({}^{H_1}\overline{\Psi}_1(c_3))$ is ideal so $\eta(u \vee v) = \eta(u) \vee \eta(v) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Then, by Lemma 5.2(5), $u \vee v \in {}^{H_1}\overline{\Psi}_1(c_3)$.

Now, let $u, v \in \Theta_1$ such that $u \leq v$ and $v \in {}^{H_1}\overline{\Psi}_1(c_3)$. Then, $u \vee v = v \in {}^{H_1}\overline{\Psi}_1(c_3)$. Thus, $\eta(u \vee v) = \eta(u) \vee \eta(v) = \eta(v) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. This implies that $\eta(u) \leq \eta(v)$. Since $\eta({}^{H_1}\overline{\Psi}_1(c_3))$ is ideal $\eta(u) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Then, by Lemma 5.2(5), $u \in {}^{H_1}\overline{\Psi}_1(c_3)$. Finally, let $u \in \Theta_1$ and $v \in {}^{H_1}\overline{\Psi}_1(c_3)$. Then, $\eta(u) \in \Theta_2$ and $\eta(v) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Since $\eta({}^{H_1}\overline{\Psi}_1(c_3))$ is ideal, $\eta(u) \circ_2 \eta(v) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$, that is, $\eta(u \circ_1 v) \in \eta({}^{H_1}\overline{\Psi}_1(c_3))$. Thus, $u \circ_1 v \in {}^{H_1}\overline{\Psi}_1(c_3)$. In a similar way, we can show that $v \circ_1 u \in {}^{H_1}\overline{\Psi}_1(c_3)$. This completes the proof.

The proof of (2)–(5) is similar to the proof of (1). □

With the same arguments, the next Theorem 18 can be achieved.

Theorem 18. *Let (H, C_1) be soft weak isomorphic to (F, C_2) with SWQI (η, ζ). Let (Ψ_2, C_3) be a SCTR over Θ_2 and $(H_1, C_1') \subseteq (H, C_1)$. Define $\Psi_1(c_3) = (x, y) \in \Theta_1 \times \Theta_1$: $\eta(x)$, $\eta y \in \Psi_2(c_3)$ for any $c_3 \in C_3$. Then, the following holds:*

(1) ${}^{H_1}\underline{\Psi}_1(c_3)$ is an ideal of Θ_1 iff $\underline{\Psi}_2^{\eta(H_1)}(c_3)$ is an ideal of Θ_2 for all $c_3 \in C_3$

(2) ${}^{H_1}\underline{\Psi}_1(c_3)$ is a subquantale of Θ_1 iff $\underline{\Psi}_2^{\eta(H_1)}(c_3)$ is a subquantale of Θ_2 for all $c_3 \in C_3$

(3) ${}^{H_1}\underline{\Psi}_1(c_3)$ is a prime ideal of Θ_1 iff $\underline{\Psi}_2^{\eta(H_1)}(c_3)$ is a prime ideal of Θ_2 for all $c_3 \in C_3$

(4) ${}^{H_1}\underline{\Psi}_1(c_3)$ is a semiprime ideal of Θ_1 iff $\underline{\Psi}_2^{\eta(H_1)}(c_3)$ is a semiprime ideal of Θ_2 for all $c_3 \in C_3$

(5) ${}^{H_1}\underline{\Psi}_1(c_3)$ is a primary ideal of Θ_1 iff $\underline{\Psi}_2^{\eta(H_1)}(c_3)$ is a primary ideal of Θ_2 for all $c_3 \in C_3$

6. Comparison

Yang and Xu [7] introduced rough approximations in quantale which is a kind of partially ordered algebraic structure with an associative binary operation. The main idea of work in [7] is based on equivalence relation equipped with congruence relation in quantale. In fact, the generalization of Pawlak's space is discussed in [7]. Further approximation of fuzzy substructures of quantale in crisp atmospheric space was discussed in [4]. Sometimes, it is difficult to find out an equivalence relation and then congruence while finding rough substructures in quantale. To remove this hurdle, soft binary relations are utilized in this paper. Since suitable soft binary relations are easy to find out, it is an easy approach to apply soft rough properties to approach different characterizations of soft rough structures in quantale with the help of aftersets and foresets.

7. Conclusion

The new combined effect of an algebraic structure quantale with rough and soft sets is presented by using soft binary relation, in this paper. The soft substructures of quantales like soft subquantale and soft ideal are discussed. The approximation w.r.t aftersets and foresets of these substructures by SBR which is an extended notion of Pawlak's rough approximation space are presented. The more generalized version of approximation space implied from SBR over $\Theta_1 \times \Theta_2$ is employed. This new relation over $\Theta_1 \times \Theta_2$ enables us to use the concept of aftersets and foresets to express the lower and upper approximation. Important results regarding to the approximation of soft substructures of quantales under SBR with some essential algebraic conditions such as compatible and complete relations are introduced. To emphasize and make a clear understanding, soft compatible and soft complete relations are focused, and these are interpreted by aftersets and foresets. Particularly, in our work, soft compatible and soft complete relations play an important role. Crux of these results is that whenever we approximate a soft algebraic structure of quantale, corresponding upper and lower approximations, are again the same kind of soft algebraic structure. Furthermore, we presented the soft quantale homomorphism and established the relationship of soft homomorphic images with their approximation under SBR.

In future, one can use this work and generalize it to different soft algebraic structures such as soft quantale modules, soft hypergroups, soft hyperquantales, and soft hyperrings. One can take motivation from our generalized approximation space and define new approximation spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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