



Research Article

Generalized β – Hyers–Ulam–Rassias Stability of Impulsive Difference Equations

Yahya Almalki ¹, Gul Rahmat,² Atta Ullah,² Fatima Shehryar,² Muhammad Numan,³ and Muhammad Usman Ali ³

¹Department of Mathematics, College of Sciences, King Khalid University, Abha 61413, Saudi Arabia

²Department of Mathematics, Islamia College Peshawar, Peshawar, Pakistan

³Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock, Pakistan

Correspondence should be addressed to Muhammad Usman Ali; muh_usman_ali@yahoo.com

Received 23 May 2022; Revised 6 July 2022; Accepted 30 July 2022; Published 15 September 2022

Academic Editor: Baiyuan Ding

Copyright © 2022 Yahya Almalki et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper describes the existence and uniqueness of the solution, β -Hyers–Ulam–Rassias stability and generalized β -Hyers–Ulam–Rassias stability of an impulsive difference system on bounded and unbounded discrete intervals. At the end, an example is given to illustrate the theoretical result.

1. Introduction

Many physical problems can be expressed in mathematical models using differential equations. Differential equations enable us to study the rapid changes in physical problems, for example, blood flows, river flows, biological systems, control theory, and mechanical systems with impact. A system of differential equations with impulses can be used to model several above-listed problems. A few existing results for a general class of impulsive systems were discussed by Ahmad [1]. The theory of impulsive difference equations was studied in [2–4]. In [5], the existence of solutions for semilinear abstract differential equations without instantaneous impulses was discussed.

At the University of Wisconsin, Ulam [6] proposed the stability problem, stated as follows. Let us denote by H_1 the group and by H_2 the metric group with a metric δ and a constant $\nu > 0$. The problem is to study if there exists $\lambda > 0$ satisfies for every $h: H_1 \rightarrow H_2$ such that

$$\delta(h(\sigma\nu), h(\sigma)h(\nu)) \leq \lambda, \quad \forall \sigma, \nu \in H_1, \quad (1)$$

there exists a homomorphism $f: H_1 \rightarrow H_2$ that satisfies

$$\delta(h(\sigma), f(\sigma)) \leq \nu, \quad \forall \sigma \in H_1. \quad (2)$$

The linear functional equations, of the form $f(x + y) = f(x) + f(y)$, and their solutions have been discussed in several spaces. A linear transformation is a solution of a linear functional equation. By considering the H_1 and H_2 as Banach spaces, Hyers [7] discussed the above problem in terms of linear functional equations. Then, Aoki [8] and Rassias [9] extended the concept of Hyers and Ulam. In the last decade, we have seen some worthwhile generalizations in the direction of Ulam stability.

In 2012, Wang et al. [10] studied the Ulam-type stability of first-order nonlinear impulsive differential equations by utilizing the bounded interval with finite impulses. In 2014, Wang et al. [11] studied the Hyers–Ulam–Rassias stability and generalized Hyers–Ulam–Rassias stability for impulsive evolution equations on a closed and bounded interval. In 2015, Zada et al. [12] studied the Hyers–Ulam stability of differential systems in terms of a dichotomy. The existence and Hyers–Ulam stability of the periodic fractional stochastic and Riemann–Liouville fractional neutral functional stochastic impulsive differential equations were given [13, 14]. Recently, Rahmat et al. [15] studied the Hyers–Ulam stability of delay differential equations. In 2019, Hu and Zhu [16] presented the stability criteria for an impulsive stochastic functional differential system with

distributed delay-dependent impulsive effects. Furthermore, Hu et al. [17] provided the improved Razumikhin stability criteria for an impulsive stochastic delay differential system, and for a detail study, we refer to the readers to [17] and the references therein.

In this paper, we will explain the β -Hyers–Ulam–Rassias stability and generalized β -Hyers–Ulam–Rassias stability of the impulsive difference system of the form

$$\begin{cases} \Theta_{n+1} = H\Theta_n + B\zeta_n + f(n, \Theta_n, \zeta_n), & n \geq 0, \\ \Theta_0, & n = 0, \\ \Theta_{n_k+1} = \Theta_{n_k-1} + I_k(n, \Theta_{n_k-1}, \zeta_{n_k-1}), & k = 1, 2, 3, \dots, m, \end{cases} \quad (3)$$

where the constant matrix $H, B \in \mathbb{R}^{n \times n}$, $f \in \mathbb{C}(\mathbb{Z}_{+ \times X, X})$ and $\Theta_n \in B(\mathbb{Z}_+, \mathbb{X})$ space of bounded and convergent sequences, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\mathbb{X} = \mathbb{R}^n$, $I = \{0, 1, 2, \dots, n\}$. In fact, we are presenting a discrete version of the work given in [18], in which β -Hyers–Ulam–Rassias stability was discussed for differential equations. With the help of [15, 18], we find out β -Hyers–Ulam–Rassias stability of the difference equation.

2. Preliminaries

Here, we discuss some notation and definitions, which will be needed for our main work. The n -dimensional Euclidean space will be denoted by \mathbb{R}^n along with the vector norm $\|\cdot\|$, and $n \times n$ matrices with real-valued entries will be denoted by $\mathbb{R}^{n \times n}$. The vector infinite norm is defined as $\|v\| = \max_{1 \leq i \leq n} |v_i|$, and the matrix infinite-norm is given as $\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$ where $v \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, also v_i and a_{ij} are the elements of the vector v and the matrix A , respectively. $\mathbb{C}(\mathbf{I}, \mathbb{X})$ will be the space of all convergent sequences from \mathbf{I} to \mathbb{X} with norm $\|v\| = \sup_{n \in \mathbf{I}} \|v_n\|$. We will use \mathbb{R}, \mathbb{Z} , and \mathbb{Z}_+ for the set of all real, integer, and non-negative integer numbers, respectively. The next lemma is a basic result about the solution of the difference system (1).

Lemma 1. *The impulsive difference system (1) has the solution*

$$\begin{aligned} X_n = & H^n \Theta_0 + H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i + H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \\ & + \sum_{n_k=0}^n T(n - n_k) I_k(n_k, \Theta_{n_k}, \zeta_{n_k}), \quad n \in I. \end{aligned} \quad (4)$$

The solution can easily be obtained by consecutively placing the values of $n \in \{0, 1, 2, \dots\}$.

Definition 1. A function $\|\cdot\|_\beta: \mathbb{V} \rightarrow [0, \infty)$ is called β -norm, with $0 < \beta \leq 1$, where \mathbb{V} is a vector space over the field \mathbf{K} , if the function satisfied the following properties:

- (1) $\|\mathcal{H}\|_\beta = 0$ if and only $\mathcal{H} = 0$
- (2) $\|\kappa \mathcal{H}\|_\beta = |\kappa|^\beta \|\mathcal{H}\|_\beta$, for each $\kappa \in \mathbf{K}$ and $\mathcal{H} \in \mathbb{V}$
- (3) $\|\mathcal{H} + \mathcal{H}_1\|_\beta \leq \|\mathcal{H}\|_\beta + \|\mathcal{H}_1\|_\beta$, for all $\mathcal{H}, \mathcal{H}_1 \in \mathbb{V}$

And $(\mathbb{V}, \|\cdot\|_\beta)$ is said to be β -norm space.

Definition 2. Let $\epsilon > 0, \psi > 0$ and $\varphi_n \in B(I, X)$. A sequence Θ_n will be an ϵ -approximate solution of (1), if

$$\begin{cases} \|\Theta_{n+1} - H\Theta_n - B\zeta_n - f(n, \Theta_n, \zeta_n)\| \leq \epsilon \varphi_n, & n \geq 0, \\ \|\Theta_{n_k+1} - \Theta_{n_k-1} - I_k(n, \Theta_{n_k-1}, \zeta_{n_k-1})\| \leq \epsilon \psi, & k = 1, 2, 3, \dots, m, \end{cases} \quad (5)$$

Definition 3. System (1) is said to be β -Hyers–Ulam–Rassias stable if for every ϵ -approximate solution Y_n of system (1), there exists an exact solution Θ_n of (1) and a nonnegative real number $\mathcal{C}_{f, N, M, \varphi, \Psi}$ such that

$$\|Y_n - \Theta_n\|^\beta \leq \mathcal{C}_{f, N, M, \varphi, \Psi} \epsilon^\beta (\varphi_n^\beta + \psi^\beta), \text{ for all } n \in I. \quad (6)$$

Definition 4. System (1) is a generalized Hyers–Ulam–Rassias stable if for every ϵ -approximate solution Y_n of system (1), there will be an exact solution Θ_n of (1) and a nonnegative real scalar $\mathcal{F}_{M, \eta, \varphi, \eta_\varphi, f}$ such that

$$\|Y_n - \Theta_n\|^\beta \leq \mathcal{F}_{M, \eta, \varphi, \eta_\varphi, f} \epsilon^\beta (\varphi_n^\beta + \Psi_{k+1}^\beta), \quad n \in I. \quad (7)$$

Remark 1. From (2), it is clear that $Y \in \mathbb{C}(\mathbf{I}, \mathbf{X})$ satisfies (2) if and only if there exists $h \in \mathbb{C}(\mathbf{I}, \mathbf{X})$ and a sequence $h_k, k \in M$ satisfying

$$\begin{cases} \|h_n\| \leq \epsilon \psi_n, & n \in M, \\ \Theta_{n+1} = H\Theta_n + B\zeta_n + \mathbf{f}(n, \Theta_n, \zeta_n) + h_n, & n \in \mathbb{Z}_+, \\ Y_0 = \Theta_0 + h_0, \\ \Theta_{n_k+1} - \Theta_{n_k-1} = I_k(\Theta_{n_k-1}, \zeta_{n_k-1}) + h_{n_k}, & k \in M. \end{cases} \quad (8)$$

The solution of Remark 1 is

$$\begin{aligned} \Theta_n = & H^n (\Theta_0 + h_0) + H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i \\ & + H^{n-1} \sum_{i=0}^{n-1} H^{-i} (f(i, \Theta_i, \zeta_i) + h_i) \\ & + \sum_{n_k=0}^n T(n - n_k) (I_k(n_k, \Theta_{n_k}, \zeta_{n_k}) + h_{n_k}), \quad n \in I. \end{aligned} \quad (9)$$

Lemma 2 (see [19]). *for any $n \geq 0$ with*

$$\|U_n\| \leq a_n + \sum_{i=0}^n P_i U_i + \sum_{0 \leq n_k \leq n} \gamma_k U_{n_k-1}, \quad n \geq 0, \quad (10)$$

then, we have

$$U_n \leq a_n (1 + \gamma_k)^k \exp\left(\sum_{i=0}^n P_i\right), \quad \text{where } k \in I. \quad (11)$$

Remark 2. If we replace γ_k by γ_{k_n} , then

$$Y_n \leq c \prod_{n_k=0}^n (1 + \gamma_{k_n}) \exp\left(\sum_{i=0}^n P_i\right) \text{ for } n \geq 0. \quad (12)$$

G_2^* :

$$\left[M \sum_{i=0}^{n-1} \|H^{-i}\| \mathcal{L}_f + Nn\mathcal{L}_{I_k} \right] < 1. \quad (15)$$

3. Uniqueness and Existence of Solution of an Impulsive Difference System

To describe the uniqueness and existence of the solution of system (1), we will use the following assumptions:

G_1 : for $f, I_k: \mathbb{Z}_{+ \times X \times X} \rightarrow \mathbb{X}$, $k \in I$, there exist constants $\mathcal{L}_f > 0$ and $\mathcal{L}_{I_k} > 0$, such that

$$\begin{aligned} \|f(n, \mathfrak{F}, p) - f(n, \mathfrak{F}', p)\| &\leq \mathcal{L}_f \|\mathfrak{F} - \mathfrak{F}'\|, \\ \|I_k(n, w, y) - I_k(n, w', y)\| &\leq \mathcal{L}_{I_k} \|w - w'\|. \end{aligned} \quad (13)$$

G_2 :

$$M = \sup_{i \in I} \|H^i\|, \quad N = \max_{0 \leq n_k \leq n} \|T(n - n_k)\|. \quad (14)$$

Theorem 1. *If assumptions G_1 , G_2 , and G_2^* are held, then system (1) has a unique solution $\Theta \in \mathbb{C}(I, \mathbb{X})$.*

Proof. Define $\mathcal{A}: \mathbb{C}(I, \mathbb{X}) \rightarrow \mathbb{C}(I, \mathbb{X})$ by

$$\begin{aligned} \mathcal{A}\Theta_n &= H^n \Theta_0 + H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i + H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \\ &\quad + \sum_{n_k=0}^n T(n - n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}). \end{aligned} \quad (16)$$

Now, for $\Theta, \Theta' \in \mathbb{C}(I, \mathbb{X})$, we have

$$\begin{aligned} \|\mathcal{A}\Theta_n - \mathcal{A}\Theta'_n\| &= \left\| \begin{aligned} &H^n \Theta_0 + H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i + H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \\ &\quad + \sum_{n_k=0}^n T(n - n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) \\ &- H^n \Theta'_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta'_i, \zeta_i) \\ &\quad - \sum_{n_k=0}^n T(n - n_k) I_k(n_k, \Theta'_{n_k-1}, \zeta_{n_k-1}) \end{aligned} \right\|. \end{aligned} \quad (17)$$

This implies that

$$\begin{aligned} \|\mathcal{A}\Theta_n - \mathcal{A}\Theta'_n\| &= \left\| \begin{aligned} &H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) + \sum_{n_k=0}^n T(n - n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) \\ &- H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta'_i, \zeta_i) - \sum_{n_k=0}^n T(n - n_k) I_k(n_k, \Theta'_{n_k-1}, \zeta_{n_k-1}) \end{aligned} \right\| \\ &\leq + \sum_{n_k=0}^n \|T(n - n_k)\| \left\| I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) - I_k(n_k, \Theta'_{n_k-1}, \zeta_{n_k-1}) \right\| \\ &\leq M \sum_{i=0}^{n-1} \|H^{-i}\| \mathcal{L}_f \|\Theta_i - \Theta'_i\| + N \sum_{n_k=0}^n \mathcal{L}_{I_k} \|\Theta_{n_k-1} - \Theta'_{n_k-1}\| \\ &\leq \left[M \sum_{i=0}^{n-1} \|H^{-i}\| \mathcal{L}_f + Nn\mathcal{L}_{I_k} \right] \|\Theta - \Theta'\|. \end{aligned} \quad (18)$$

This implies that \mathcal{A} is a contraction map using the Banach contraction principle, we say that system (1) has a unique solution. \square

4. β -Hyers–Ulam–Rassias Stability on Bounded Discrete Interval

To determine β -Hyers–Ulam–Rassias stability on the bounded discrete interval, we have one more assumption:

G_3 : there exist a constant $\eta_\phi > 0$ and φ_n and a nondecreasing function $\varphi \in B(I, X)$ such that

$$\sum_{i=0}^{n-1} \psi_i \leq \eta_\phi \phi_n. \quad (19)$$

Theorem 2. *System (1) is β -Hyers–Ulam–Rassias stable over discrete bounded interval, if G_1 , G_2 and G_3 are satisfied.*

Proof. The solution of system (1) is as follows:

$$\begin{aligned} \Theta_n &= H^n \Theta_0 + H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i + H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \\ &\quad + \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}). \end{aligned} \quad (20)$$

Let Y_n be the solution of inequality (2), we have

$$\begin{aligned} &\left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^n H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^n H^{-i} f(i, Y_i, \zeta_i) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right\| \\ &= \left\| H^n h_n + H^{n-1} \sum_{i=0}^{n-1} H^{-i} h_i + \sum_{n_k=0}^n T(n-n_k) h_{n_k} \right\| \leq M \varepsilon \Psi + M^2 \delta \sum_{i=0}^{n-1} \Psi_i + N \varepsilon \sum_{n_k=0}^m \Psi \\ &= (M+N) \sum_{k=0}^m \delta \Psi + \sum_{i=0}^{n-1} M^2 \delta \Psi_i \\ &\leq (mM\delta + M^2 \varepsilon \eta_\phi) (\varphi_n + \varphi) \\ &= M\delta (m + M\eta_\phi) (\varphi_n + \varphi), \end{aligned} \quad (21)$$

thus, for each $n \in \{n_k, n_{k+1}, \dots\}$, we have

$$\begin{aligned} \|Y_n - \theta_n\|^\beta &= \left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \right. \\ &\quad \left. - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) \right\|^\beta \\ &= \left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \right. \\ &\quad \left. - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) + H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) - H^{n-1} \right. \\ &\quad \left. \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) + \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right. \\ &\quad \left. - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right\|^\beta \end{aligned}$$

$$\begin{aligned}
& \leq \left(\left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right\| \right)^\beta \\
& \quad + \left(\left\| H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \right\| \right)^\beta \\
& \quad + \left(\left\| \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) \right\| \right)^\beta \\
& \leq (M\varepsilon(m + M\eta_\phi)(\varphi_n + \varphi))^\beta + \left(M^2 \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \|Y_i - \Theta_i\| \right)^\beta + \left(N \sum_{k=1}^m \mathcal{L}_{I_{n_k}} \|Y_{n_k} - \Theta_{n_k}\| \right)^\beta \\
& \quad \|Y_n - \Theta_n\| \leq 3^{(1/\beta)-1} \left[M\varepsilon(m + M\eta_\phi)(\varphi_n + \varphi) + M^2 \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \|Y_i - \Theta_i\| + N \sum_{k=1}^m \mathcal{L}_{I_{n_k}} \|Y_{n_k} - \Theta_{n_k}\| \right], \tag{22}
\end{aligned}$$

by using the relation

$$\begin{aligned}
(x + y + z)^\gamma & \leq 3^{\gamma-1} (x^\gamma + y^\gamma + z^\gamma), \\
\text{where } k \quad x, y, z & \geq 0 \quad \text{and } \gamma > 1. \tag{23}
\end{aligned}$$

Now, using Gronwall Lemma 2, we get

$$\|Y_N - \Theta_n\| \leq 3^{(1/\beta)-1} [M\varepsilon(m + M\eta_\phi)(\varphi_n + \varphi)] (1 + 3^{(1/\beta)-1} N \mathcal{L}_{I_k})^k \exp \left(M^2 3^{(1/\beta)-1} \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \right). \tag{24}$$

Now,

$$\begin{aligned}
\|Y_n - \Theta_n\|^\beta & \leq 3^{1-\beta} [M\varepsilon(m + M\eta_\phi)(\varphi_n + \varphi)]^\beta (1 + 3^{(1/\beta)-1} N \mathcal{L}_{I_k})^{k\beta} \exp \left(M^2 \beta 3^{(1/\beta)-1} \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \right) \\
& \leq 3^{1-\beta} (M\varepsilon(m + M\eta_\phi))^\beta (\varphi_n + \varphi)^\beta (1 + 3^{(1/\beta)-1} N \mathcal{L}_{I_k})^{k\beta} \exp \left(M^2 \beta 3^{(1/\beta)-1} \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \right) \\
& \leq C_{f,N,M,\varphi,\Psi} \varepsilon^\beta ((\varphi_n)^\beta + \varphi^\beta), \tag{25}
\end{aligned}$$

using $(x + y)^r \leq (x^r + y^r)$, $x, y \geq 0$, for any $r \in \{0, 1\}$, where

$$C_{f,N,M,\varphi,\Psi} = 3^{1-\beta} (M\varepsilon(m + M\eta_\phi))^\beta (1 + 3^{(1/\beta)-1} N \mathcal{L}_{I_k})^{k\beta} \exp \left(M^2 \beta 3^{(1/\beta)-1} \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \right). \tag{26}$$

Hence, system (1) is β -Hyers–Ulam–Rassias stable. \square

5. β -Hyers–Ulam–Rassias Stability on Unbounded Discrete Interval

To explain β -Hyers–Ulam–Rassias stability on an unbounded discrete interval, we must need the following assumptions:

G_4 : the operators family $\|H^{n-i}\| \leq Me^{\omega(n-i)}$ and $\|T(n-n_k)\| \leq Me^{\omega(n-n_k)}$.

$$\begin{aligned} G_5: \sum_{i=0}^{n-1} \mathcal{L}_{f_i} &\leq \kappa_f n + \kappa_f, \\ G_6: \mathcal{L}_{\mathcal{J}} &= \left(1 + 3^{(1/\beta)-1} M \mathcal{L}_{I_{n_k}}\right)^k < \infty, \\ G_7: \sum_{i=0}^{n-1} e^{\omega(n-i)+3^{1/\beta-1} M \kappa_f n} \Psi_i &\leq \eta_\varphi \varphi_n, \\ G_8: M_1 &= \sum_{n_k=0}^m e^{\omega(n-n_k)+3^{1/\beta-1} M \kappa_f n}. \end{aligned} \quad (27)$$

Theorem 2. Assume that G_1 and $G_3 - G_8$ are holds, then system (1) is β -Hyers–Ulam–Rassias stable over a discrete unbounded interval.

Proof. The solution of system (1) is

$$\begin{aligned} \Theta_n &= H^n \Theta_0 + H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i + H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \\ &\quad + \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}). \end{aligned} \quad (28)$$

Let Y_n be the solution of inequality (2), we have

$$\begin{aligned} &\left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right\| \\ &= \left\| \sum_{i=0}^{n-1} H^{n-1-i} h_i + \sum_{n_k=0}^n T(n-n_k) h_{n_k} \right\| \leq \sum_{i=0}^{n-1} \|H^{n-1-i}\| \|h\|_i + \sum_{n_k=0}^n \|T(n-n_k)\| \|h_{n_k}\| \\ &\leq M \left(\sum_{i=0}^{n-1} e^{\omega(n-i)} \varepsilon \Psi_i + \sum_{n_k=0}^m e^{\omega(n-n_k)} \varepsilon \Psi \right). \end{aligned} \quad (29)$$

Now, for each $n \in \{n_k, n_{k+1}\}$, we have

$$\begin{aligned} \|Y_n - \theta_n\|^\beta &= \left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \right. \\ &\quad \left. - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) \right\|^\beta \\ &= \left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \right. \\ &\quad \left. - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) + H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) \right. \\ &\quad \left. - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) + \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right. \\ &\quad \left. - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right\|^\beta \end{aligned}$$

$$\begin{aligned}
& \leq \left(\left\| \begin{aligned} & Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) \\ & - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \end{aligned} \right\| \right)^\beta + \left(\left\| H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) \right\| \right) \\
& - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \right)^\beta + \left(\left\| \begin{aligned} & \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \\ & - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) \end{aligned} \right\| \right)^\beta \\
& \leq \left(M \left(\sum_{i=0}^{n-1} e^{\omega(n-i)} \varepsilon \Psi_i + \sum_{n_k=0}^m e^{\omega(n-n_k)} \varepsilon \Psi \right) \right)^\beta + \left(M \sum_{i=0}^{n-1} e^{\omega(n-i)} \mathcal{L}_{f_i} \|Y_i - \Theta_i\| \right)^\beta \\
& + \left(M \sum_{n_k=0}^n e^{\omega(n-n_k)} \mathcal{L}_{I_{n_k}} \|Y_{n_k-1} - \Theta_{n_k-1}\| \right)^\beta, \tag{30}
\end{aligned}$$

if we set $\bar{Y}_n = e^{-\omega n} Y_n$ and $\bar{\Theta}_n = e^{-\omega n} \Theta_n$, then we have

$$\begin{aligned}
\|\bar{Y}_n - \bar{\Theta}_n\|^\beta & \leq \left(M \left(\sum_{i=0}^{n-1} e^{-\omega i} \varepsilon \Psi_i + \sum_{n_k=0}^m e^{-\omega n_k} \varepsilon \Psi \right) \right)^\beta + \left(M \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \|\bar{Y}_i - \bar{\Theta}_i\| \right)^\beta \\
& + \left(M \sum_{n_k=0}^n \mathcal{L}_{I_{n_k}} \|\bar{Y}_{n_k-1} - \bar{\Theta}_{n_k-1}\| \right)^\beta, \tag{31}
\end{aligned}$$

with the help of relation

$$(x + y + z)^\gamma \leq 3^{\gamma-1} (x^\gamma + y^\gamma + z^\gamma), \quad \text{where } x, y, z \geq 0 \tag{32}$$

$\gamma > 1.$

we get

$$\begin{aligned}
\|\bar{Y}_n - \bar{\Theta}_n\| & \leq 3^{(1/\beta)-1} M \left(\sum_{i=0}^{n-1} e^{-\omega i} \varepsilon \Psi_i + \sum_{n_k=0}^m e^{-\omega n_k} \varepsilon \Psi \right) + 3^{(1/\beta)-1} M \left(\sum_{i=0}^{n-1} \mathcal{L}_{f_i} \|\bar{Y}_i - \bar{\Theta}_i\| \right) \\
& + 3^{1/\beta-1} M \left(\sum_{n_k=0}^n \mathcal{L}_{I_{n_k}} \|\bar{Y}_{n_k-1} - \bar{\Theta}_{n_k-1}\| \right). \tag{33}
\end{aligned}$$

Using Lemma 1, we have

$$\|\bar{Y}_n - \bar{\Theta}_n\| \leq 3^{(1/\beta)-1} M \varepsilon \left(\sum_{i=0}^{n-1} e^{-\omega i} \Psi_i + \sum_{n_k=0}^m e^{-\omega n_k} \Psi \right) \left(1 + 3^{(1/\beta)-1} M \mathcal{L}_{I_{n_k}} \right)^k \exp \left(3^{(1/\beta)-1} M \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \right). \tag{34}$$

Resubmitting the values, we have

$$\begin{aligned}
\|Y_n - \Theta_n\| &\leq 3^{(1/\beta)-1} M \varepsilon \left(\sum_{i=0}^{n-1} e^{\omega(n-i)} \Psi_i + \sum_{n_k=0}^m e^{\omega(n-n_k)} \Psi \right) \left(1 + 3^{(1/\beta)-1} M \mathcal{L}_{I_{n_k}} \right)^k \exp \left(3^{(1/\beta)-1} M \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \right) \\
&\leq 3^{1/\beta-1} M \varepsilon \mathcal{L}_{\mathcal{F}} \left(\sum_{i=0}^{n-1} e^{\omega(n-i)} \Psi_i + \sum_{n_k=0}^m e^{\omega(n-n_k)} \Psi \right) \exp \left(3^{(1/\beta)-1} M (\kappa_f n + \varkappa_f) \right) \\
&= 3^{(1/\beta)-1} M \varepsilon \mathcal{L}_{\mathcal{F}} \left(\sum_{i=0}^{n-1} e^{\omega(n-i)+3^{(1/\beta)-1} M (\kappa_f n + \varkappa_f)} \Psi_i + \sum_{n_k=0}^m e^{\omega(n-n_k)+3^{(1/\beta)-1} M (\kappa_f n + \varkappa_f)} \Psi \right) \\
&= 3^{(1/\beta)-1} M \varepsilon \mathcal{L}_{\mathcal{F}} e^{3^{(1/\beta)-1} M \kappa_f} \left(\sum_{i=0}^{n-1} e^{\omega(n-i)+3^{(1/\beta)-1} M \kappa_f n} \Psi_i + \sum_{n_k=0}^m e^{\omega(n-n_k)+3^{(1/\beta)-1} M \kappa_f n} \Psi \right) \\
&\leq 3^{(1/\beta)-1} M \varepsilon \mathcal{L}_{\mathcal{F}} e^{3^{1/\beta-1} M \varkappa_f} (\eta_\varphi \varphi + M_1 \Psi) \\
&\leq 3^{(1/\beta)-1} M \varepsilon \mathcal{L}_{\mathcal{F}} e^{3^{(1/\beta)-1} M \varkappa_f} (M_1 + \eta_\varphi) (\varphi_n + \Psi) \\
\|Y_n - \Theta_n\|^\beta &\leq \mathcal{H}_{M, \mathcal{L}_{\mathcal{F}}, \eta_\varphi, M_1} \varepsilon^\beta (\varphi_n^\beta + \Psi^\beta),
\end{aligned} \tag{35}$$

where

$$\mathcal{H}_{M, \mathcal{L}_{\mathcal{F}}, \eta_\varphi, M_1} = 3^{1-\beta} \left(M \mathcal{L}_{\mathcal{F}} e^{3^{(1/\beta)-1} M \varkappa_f} (M_1 + \eta_\varphi) \right)^\beta > 0. \tag{36}$$

Thus, system (1) is β -Hyers–Ulam–Rassias stable. \square

Remark 3. Wang et al. [18] studied the β -Hyers–Ulam stability and β -Hyers–Ulam–Rassias stability for a system of impulsive differential equations as we know that difference equations relate to differential equations as discrete mathematics relate to continuous mathematics. The system of

impulsive difference equations used in this article is analogous to the system of impulsive differential equations used in [18]. Thus, the findings of this article are the discrete version of the work of Wang et al. [18].

6. Generalized β -Hyers–Ulam–Rassias Stability

In this section, we present the generalized β -Hyers–Ulam–Rassias stability, for which we need the following assumptions:

$$\begin{aligned}
G_9: 3^{(1/\beta)-1} M \varepsilon \prod_{n_k=0}^n \left(1 + 3^{(1/\beta)-1} M \mathcal{L}_{I_{n_k}} \right) \left(\sum_{i=0}^{n-1} e^{\omega(n-i)+3^{(1/\beta)-1} M \sum_{i=0}^{n-1} \mathcal{L}_{f_i}} \Psi_i \right) &\leq \eta_\varphi \varphi_n, \\
G_{10}: 3^{(1/\beta)-1} M \varepsilon \prod_{n_k=0}^n \left(1 + 3^{(1/\beta)-1} M \mathcal{L}_{I_{n_k}} \right) \left(\sum_{n_k=0}^m e^{\omega(n-n_k)+3^{(1/\beta)-1} M \sum_{i=0}^{n-1} \mathcal{L}_{f_i}} \Psi \right) &\leq \eta_\Psi \Psi_{k+1}.
\end{aligned} \tag{37}$$

Theorem 3. Assume that G_1 and $G_3 - G_{10}$ are satisfied, then system (1) is generalized β -Hyers–Ulam–Rassias stable.

Proof. The solution of system (1) is as follows:

$$\begin{aligned}
\Theta_n &= H^n \Theta_0 + H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i + H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \\
&\quad + \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}).
\end{aligned} \tag{38}$$

Let Y_n be the solution of inequality (2), we have

$$\begin{aligned}
& \left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^n H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^n H^{-i} f(i, Y_i, \zeta_i) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right\| \\
&= \left\| \sum_{i=0}^{n-1} H^{n-1-i} h_i + \sum_{n_k=0}^n T(n-n_k) h_{n_k} \right\| \leq \sum_{i=0}^{n-1} \|H^{n-1-i}\| \|h_i\| + \sum_{n_k=0}^n \|T(n-n_k)\| \|h_{n_k}\| \\
&\leq M \left(\sum_{i=0}^{n-1} e^{\omega(n-i)} \varepsilon \Psi_i + \sum_{n_k=0}^m e^{\omega(n-n_k)} \varepsilon \Psi \right).
\end{aligned} \tag{39}$$

Now, for each $n \in \{n_k, n_{k+1}\}$, we have

$$\begin{aligned}
\|Y_n - \theta_n\|^\beta &= \left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) \right\|^\beta \\
&= \left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) + H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) + \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right\|^\beta \\
&\leq \left(\left\| Y_n - H^n \Theta_0 - H^{n-1} \sum_{i=0}^{n-1} H^{-i} B \zeta_i - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) \right\| \right)^\beta \\
&\quad + \left(\left\| H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, Y_i, \zeta_i) - H^{n-1} \sum_{i=0}^{n-1} H^{-i} f(i, \Theta_i, \zeta_i) \right\| \right)^\beta \\
&\quad + \left(\left\| \sum_{n_k=0}^n T(n-n_k) I_k(n_k, Y_{n_k-1}, \zeta_{n_k-1}) - \sum_{n_k=0}^n T(n-n_k) I_k(n_k, \Theta_{n_k-1}, \zeta_{n_k-1}) \right\| \right)^\beta \\
&\leq \left(M \left(\sum_{i=0}^{n-1} e^{\omega(n-i)} \varepsilon \Psi_i + \sum_{n_k=0}^m e^{\omega(n-n_k)} \varepsilon \Psi \right) \right)^\beta + \left(M \sum_{i=0}^{n-1} e^{\omega(n-i)} \mathcal{L}_{f_i} \|Y_i - \Theta_i\| \right)^\beta \\
&\quad + \left(M \sum_{n_k=0}^n e^{\omega(n-n_k)} \mathcal{L}_{I_{n_k}} \|Y_{n_k-1} - \Theta_{n_k-1}\| \right)^\beta,
\end{aligned} \tag{40}$$

if we set $\bar{Y}_n = e^{-\omega n} Y_n$ and $\bar{\Theta}_n = e^{-\omega n} \Theta_n$, then we have

$$\begin{aligned} \|\bar{Y}_n - \bar{\Theta}_n\|^\beta &\leq \left(M \left(\sum_{i=0}^{n-1} e^{-\omega i} \varepsilon \Psi_i + \sum_{n_k=0}^m e^{-\omega n_k} \varepsilon \Psi \right) + \left(M \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \|\bar{Y}_i - \bar{\Theta}_i\| \right) \right)^\beta \\ &\quad + \left(M \sum_{n_k=0}^n \mathcal{L}_{I_{n_k}} \|\bar{Y}_{n_k-1} - \bar{\Theta}_{n_k-1}\| \right)^\beta, \end{aligned} \quad (41)$$

with the help of relation

$$(x + y + z)^\gamma \leq 3^{\gamma-1} (x^\gamma + y^\gamma + z^\gamma), \quad \text{where } x, y, z \geq 0 \quad (42)$$

$$\gamma > 1,$$

we get

$$\begin{aligned} \|\bar{Y}_n - \bar{\Theta}_n\| &\leq 3^{(1/\beta)-1} M \left(\sum_{i=0}^{n-1} e^{-\omega i} \varepsilon \Psi_i + \sum_{n_k=0}^m e^{-\omega n_k} \varepsilon \Psi \right) + 3^{(1/\beta)-1} M \left(\sum_{i=0}^{n-1} \mathcal{L}_{f_i} \|\bar{Y}_i - \bar{\Theta}_i\| \right) \\ &\quad + 3^{(1/\beta)-1} M \left(\sum_{n_k=0}^n \mathcal{L}_{I_{n_k}} \|\bar{Y}_{n_k-1} - \bar{\Theta}_{n_k-1}\| \right), \end{aligned} \quad (43)$$

using, we have

$$\|\bar{Y}_n - \bar{\Theta}_n\| \leq 3^{(1/\beta)-1} M \varepsilon \prod_{n_k=0}^n \left(\sum_{i=0}^{n-1} e^{-\omega i} \Psi_i + \sum_{n_k=0}^m e^{-\omega n_k} \Psi \right) \left(1 + 3^{(1/\beta)-1} M \mathcal{L}_{I_{n_k}} \right) \exp \left(3^{(1/\beta)-1} M \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \right), \quad (44)$$

resubmitting the values, we have

$$\begin{aligned} \|Y_n - \Theta_n\| &\leq 3^{(1/\beta)-1} M \varepsilon \prod_{n_k=0}^n \left(\sum_{i=0}^{n-1} e^{\omega(n-i)} \Psi_i + \sum_{n_k=0}^m e^{\omega(n-n_k)} \Psi \right) \left(1 + 3^{(1/\beta)-1} M \mathcal{L}_{I_{n_k}} \right) \exp \left(3^{(1/\beta)-1} M \sum_{i=0}^{n-1} \mathcal{L}_{f_i} \right) \\ &= 3^{(1/\beta)-1} M \varepsilon \prod_{n_k=0}^n \left(\sum_{i=0}^{n-1} e^{\omega(n-i)+3^{(1/\beta)-1} M \sum_{i=0}^{n-1} \mathcal{L}_{f_i}} \Psi_i \right. \\ &\quad \left. + \sum_{n_k=0}^m e^{\omega(n-n_k)+3^{(1/\beta)-1} M \sum_{i=0}^{n-1} \mathcal{L}_{f_i}} \Psi \right) \left(1 + 3^{(1/\beta)-1} M \mathcal{L}_{I_{n_k}} \right), \end{aligned} \quad (45)$$

At last, we obtain

$$\begin{aligned} \|Y_n - \Theta_n\| &\leq 3^{1/\beta-1} M \varepsilon (\eta_\varphi \varphi_n + \eta_\Psi \Psi_{k+1}) \\ &\leq 3^{1/\beta-1} M \varepsilon (\eta_\varphi + \eta_\Psi) (\varphi_n + \Psi_{k+1}) \\ \|Y_n - \Theta_n\|^\beta &= 3^{1-\beta} M^\beta \varepsilon^\beta (\eta_\varphi + \eta_\Psi)^\beta (\varphi_n + \Psi_{k+1})^\beta \\ &= \mathcal{F}_{M, \eta_\varphi, \eta_\Psi, f} \varepsilon^\beta (\varphi_n^\beta + \Psi_{k+1}^\beta), \end{aligned} \quad (46)$$

where

$$\mathcal{F}_{M, \eta_\varphi, \eta_\Psi, f} = 3^{1-\beta} M^\beta (\eta_\varphi + \eta_\Psi)^\beta > 0. \quad (47)$$

Hence, system (1) is generalized β -Hyers-Ulam-Rassias stable. \square

7. Example

The impulsive difference system is as follows:

$$\begin{cases} \Theta_{n+1} = H\Theta_n + B\zeta_n + f(n, \Theta_n, \zeta_n), & n \in \{0, 1, 2, 3\}, \\ \Theta_0, & n = 0, \\ \Delta(\Theta_{n_k}) = I_k(n, \Theta_{n_{k-1}}, \zeta_{n_{k-1}}) = \frac{1}{3r^2}(\Theta_{n_{k-1}}), & k = 1, 2, 3, \dots, m. \end{cases} \quad (48)$$

Assumption G_1 and G_2 are holds if $N = 1$ and $M = 1$. Clearly, assumptions G_5 and G_6 are hold if $\kappa_f = 0$ and

$$\begin{aligned} \mathcal{L}_J &= \sup \prod_{r=1}^{\infty} \left(1 + 3^{(1/\beta)} M \mathcal{L}_{I_{n_k}} \right) \\ &= \sup \prod_{r=1}^{\infty} \left(1 + \frac{1}{r^2} \right) \leq e \sum_{r=1}^{\infty} (1/r^2) \leq e^{(\pi^2/6)}. \end{aligned} \quad (49)$$

Also, $\dot{u}_f = 0$. Now, set $\varphi_n = e^n$ and $\psi = 1$, then G_7 holds if $\eta_\phi = 1/2$. G_8 holds if $e^2/e^2 - 1$. Thus, system (3) is $1/2$ -Hyers-Ulam-Rassias stable with respect to $(\sqrt{e^n}, 1)$ on Z_+ and $\mathcal{H}_{M, \mathcal{L}_f, \eta_\phi, M_1} = \sqrt{3}e^{\pi^2/r^2} (1/2 + e^2/e^2 - 1)^{1/2}$.

8. Conclusion

Nowadays, studies on the qualitative behavior of impulsive difference equations have a significant contribution to the literature. In particular, the discussion regarding the β -Hyers-Ulam-Rassias stability of difference equations has been considered as one of the important topics of the literature, in which different types of conditions have been used in the form of inequalities, and most results have been obtained through discrete Gronwall inequality. In this paper, we have investigated the existence and uniqueness of the solution through the Banach contraction principle and β -Hyers-Ulam-Rassias stability of the impulsive difference system with the help of Gronwall inequality.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this article and approved the final manuscript.

Acknowledgments

The Deanship of Scientific Research at King Khalid University, Abha, Saudi Arabia, has funded this work through Large Groups Project under grant number RGP.2/211/43.

References

- [1] N. U. Ahmed, "Existence of optimal controls for a general class of impulsive systems on Banach spaces," *SIAM Journal on Control and Optimization*, vol. 42, no. 2, pp. 669–685, 2003.
- [2] D. D. Bainov, V. Lakshmikantham, and P. S. Simeonov, "Theory of impulsive differential equations," *Series in Modern Applied Mathematics; World scientific*, vol. 6, 1989.
- [3] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive differential equations and inclusions*, Hindawi Publishing Corporation, London UK, 2006.
- [4] E. Hernández, "On a new class of abstract impulsive differential equations," *Proceedings of the American Mathematical Society*, vol. 141, no. 5, pp. 1641–1649, 2012.
- [5] M. Pierri, "Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses," *Applied Mathematics and Computation*, vol. 219, no. 12, pp. 6743–6749, 2013.
- [6] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, Hoboken New Jersey, USA, 1960.
- [7] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences*, vol. 27, no. 4, pp. 222–224, 1941.
- [8] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, no. 1-2, pp. 64–66, 1950.
- [9] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
- [10] J. Wang, M. Feckan, and Y. Zhou, "Ulam's type stability of impulsive ordinary differential equations," *Journal of Mathematical Analysis and Applications*, vol. 395, no. 1, pp. 258–264, 2012.
- [11] J. Wang, Y. Zhou, and Z. Lin, "On a new class of impulsive fractional differential equations," *Applied Mathematics and Computation*, vol. 242, pp. 649–657, 2014.
- [12] A. Zada, O. Shah, and R. Shah, "Hyers-Ulam stability of non-autonomous systems in terms of boundedness of Cauchy problems," *Applied Mathematics and Computation*, vol. 271, pp. 512–518, 2015.
- [13] Y. Guo, M. Chen, X. B. Shu, and F. Xu, "The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm," *Stochastic Analysis and Applications*, vol. 39, no. 4, pp. 643–666, 2021.
- [14] Y. Guo, X. B. Shu, Y. Li, and F. Xu, "The existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$," *Boundary Value Problems*, vol. 1, p. 59, 2019.
- [15] G. Rahmat, A. Ullah, A. U. Rahman, M. Sarwar, T. Abdeljawad, and A. Mukheimer, "Hyers-Ulam stability of non-autonomous and nonsingular delay difference equations," *Advances in Difference Equations*, vol. 2021, p. 474, 2021.

- [16] W. Hu and Q. Zhu, "Stability criteria for impulsive stochastic functional differential systems with distributed-delay dependent impulsive effects," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 3, pp. 1–6, 2019.
- [17] W. Hu, Q. Zhu, and H. R. Karimi, "Some improved Razumikhin stability criteria for impulsive stochastic delay differential systems," *IEEE Transactions on Automatic Control*, vol. 64, no. 12, pp. 5207–5213, 2019.
- [18] X. Wang, M. Arif, and A. Zada, " β -Hyers-Ulam-Rassias stability of semilinear nonautonomous impulsive system," *Symmetry*, vol. 11, no. 2, p. 231, 2019.