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Research Article

Internal Synchronization Using Adaptive Control

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This paper mainly deals on the issue of a chaotic synchronization of a master and slave systems. It is generally the requirement of the synchronization that someone needs at least one to one master and slave systems. In the current study, the authors introduce the concept of a synchronization in which there is no need of slave/response system externally. Furthermore, the synchronization has been demonstrated here within a system among the subsystems of different orders. In addition, adaptive control is chosen for the synchronization among various combinations in multiswitching manner. For demonstration purpose, Lorenz Six Dimensional Hyper Chaotic System (L6DHCS) is chosen. There are three different kinds of possible switches presented by the authors formed within the considered system. The numerical simulations are carried out to validate the effectiveness of the analytical technique using Mathematica.

1. Introduction

Chaos synchronization is one of the widely studied phenomena of nonlinear science that has greatly expanded people's perception from all across the world. Because of the widespread presence of chaos in many systems, researchers have been seriously interested in studying chaotic systems for over three decades [1–3]. Chaos causes unpredictable and inappropriate system behavior and leads to irreparable losses. This is the reason why chaos synchronization has drawn the attention of scholars from all across. Various as well as numerous methods have been presented for synchronization of chaotic systems, such as active control [4–7], backstepping control [8, 9], fuzzy control [10, 11], impulsive control [12–14], event-triggered-based neural network [15, 16], output feedback control [17, 18], projective synchronization [19], and sliding mode control [20–22]. It is worth mentioning the recent contribution of various synchronization techniques that have been proposed. Its all aforementioned or therein were either based on a master and a slave system were synchronized or many generalized forms of synchronization formed from combining one or more techniques operating on a number of chaotic systems were formed. Among numerous, here we would like to share some of the worthwhile contributions read by the authors based on different adaptive approaches.

Shahzad [23] studied the issue of multiswitching synchronization (MSS) of chaotic systems using the adaptive sliding mode approach, and it has been shown that MSS is a general case of reduced/increased order of synchronization. For demonstration purposes, circular restricted three-body problem and the Lorenz system were used as a master and slave system, respectively. Shahzad [24] has investigated the improved results with Mathematica and further studied the effects of external uncertainty and disturbances on stability using adaptive sliding mode control. It has been seen that the synchronization of chaotic systems with unknown parameters is an issue. Adaptive control can be used to achieve synchronization to deal with uncertainty [20, 21]. Khan and Shikha [19] have studied the hybrid function projective synchronization using the adaptive control technique for unknown system parameters. In their study, both the master and slave systems are chosen in such a way that none of them can be derived from the members of the unified chaotic system. Chen et al. [25] studied the synchronization of multiple chaotic systems with unknown parameters using an adaptive control method, and two kinds of different synchronization modes were considered. In the first one, more response systems were synchronized with one drive system, and the second one is based on ring transmission synchronization that guarantees that all the chaotic systems can synchronize with each other. Zare et al. [26] proposed a robust adaptive control strategy to synchronize a class of uncertain chaotic systems with unknown time delays. Using Lyapunov theory and Lipschitz conditions in chaotic systems, the necessary adaptation rules for estimating uncertain parameters and unknown time delays are determined. Mobayen et al. [27] suggested a novel barrier function-based adaptive nonsingular terminal sliding mode control methodology for robust stability of disturbed nonlinear systems. It was proved that the barrier function-based control method can force the state trajectories to converge to a region near origin in the finite time. Moreover, a sufficient criterion was derived to satisfy the asymptotic stability of state trajectories. Alattas et al. [28] proposed an integral-type dynamic sliding mode control scheme to synchronize the hyperchaotic systems in the existence of uncertainty as quickly as possible that can be used for an extensive range of identical/nonidentical master-slave structures. Furthermore, for a new six-dimensional hyperchaotic system, it was exposed that the synchronization errors are completely compensated for by the new control scheme which has a better response compared to a similar controller. Mobayen et al. [29] constructed a family of nine new chameleon chaotic systems by introducing two parameters to the 3D chaotic systems with quadratic nonlinearities and exhibiting line equilibrium points. During analysis, three categories of hidden attractors (no equilibria, line of equilibria, and one stable equilibrium) and a self-excited attractor have been seen. The study motivates on the adaptive finite time sliding mode control of one category of these chameleon chaotic systems subjected to uncertainties and disturbances.

All the aforementioned studies are somehow based on two or more systems called drive-response or master-slave combination required for synchronization. Here in the current study, the authors aim to study the internal synchronization in which the subsystems of a chaotic dynamical system have been synchronized among each other in a multiswitching style. In addition to show the existence of internal synchronization, the authors have combined the idea of internal synchronization in multiswitching style with the adaptive control scheme and L6DHCS has been chosen for demonstration purposes. The adaptive control technique is one of the oldest and frequently used methods [30–33]. There is no other particular reason behind the selection of adaptive control for internal synchronization. Also, the authors have not found any issues in the implementation of the selected technique during internal synchronization. As it has been mentioned above that in internal synchronization, the part dynamics of a certain chaotic system represented by a single or more state variables are synchronized with other part dynamics represented by other different sets of the same number of state variables within the same system. This itself is the novelty of our proposed study. This is a kind of a unique way of synchronizing a selected subsystem that gives someone the freedom to synchronize without an external slave system. As far as the best author's information, the kind of synchronization has never been studied before in the past. The rest of the contents of the article are arranged as follows. In Section 2, a problem statement is presented. In Section 3, the IMSS for L6DHCS has been discussed that has three kinds of switches. Finally, some concluding remarks on the presented study can be seen in Section 4.

2. Problem Statement

As in the IMSS, the part dynamics of a certain system represented by a single or more state variables with other part dynamics represented by another set of the same number of state variables within the same system are synchronized, and moreover, there can be different kinds of switches depending on the error.

Let

$$\dot{x}(t) = f(x) + F(x)\boldsymbol{\alpha}, \tag{1}$$

be any chaotic system in which $x \in [x_1, x_2, x_3, \cdots, x_n]^T \in R^n$ are the state vector, $\boldsymbol{\alpha} \in R^m$ is the unknown constant parameter vector of the system, $f(x)$ is an $n \times 1$ matrix, $F(x)$ is an $n \times m$ matrix, and the elements $F_{ij}(x) \in L_\infty$ for $x \in R^n$ in matrix $F(x)$.

Now break the system into two subsystems as follows:

$$\text{Master subsystem}: \ \dot{x}_1(t) = f_1(x) + F_1(x)\boldsymbol{\alpha}_1, \tag{2}$$

$$\text{Slave subsystem}: \ \dot{x}_2(t) = f_2(x) + F_2(x)\boldsymbol{\alpha}_2 + u(t), \tag{3}$$

where $x_1, x_2 \subset x$ such that $x_1 \cap x_2 = \phi$ and $u(t) \in R^{n_1}$ is controlling vector, $x_1, x_2 \in R^{n_1}$ is the state vector, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \subset \boldsymbol{\alpha} \in R^{n_1}$ is the unknown constant parameter vector of the system, $f_1(x)$ and $f_2(x)$ are the $n_1 \times 1$ matrices, $F_1(x)$ and $F_2(x)$ are the $n_1 \times m_1$ ($n_1 \leq n/2$ and $m_1 \leq m/2$) matrices, and the elements $F_{ij}(x) \in L_\infty$ for $x_1, x_2 \in R^{n_1}$ in the matrices $F(x)$.

The IMSS problem can be transformed to the equivalent problem of stabilizing the error system from drive-response subsystems with the unknown parameters, and a suitable feedback control law $u(t)$ is designed such that the stability of the error subsystems can be achieved in the sense that $\lim_{t\longrightarrow\infty} \|x_j(t) - x_i(t)\| \longrightarrow 0$ for $x_i \in x_1$ and $x_j \in x_2$.

Theorem 1. *Let the control function be*

$$u(t) = -f_1(x) - f_2(x) - F_1(x)\widehat{\boldsymbol{\alpha}}_1 - F_2(x)\widehat{\boldsymbol{\alpha}}_2 - ke(t), \tag{4}$$

where the adaptive laws of parameters are defined as

$$\begin{aligned} \dot{\widehat{\boldsymbol{\alpha}}}_1 &= [F_1(x)]^T e, \\ \dot{\widehat{\boldsymbol{\alpha}}}_2 &= [F_2(x)]^T e, \end{aligned} \tag{5}$$

in which, $\widehat{\boldsymbol{\alpha}}_1$ and $\widehat{\boldsymbol{\alpha}}_2$ are the estimations of the unknown parameters of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, respectively. Then, the response subsystem (3) can synchronize the drive subsystem (2) for $\mathbf{k} > 0$, where $\mathbf{k}$ is the gain constant vector and it has been chosen positive.

Proof. The error dynamics from (2) and (3) can be written as follows:

$$\dot{e}(t) = F_1(x)(\boldsymbol{\alpha}_1 - \widehat{\boldsymbol{\alpha}}_1) + F_2(x)(\boldsymbol{\alpha}_2 - \widehat{\boldsymbol{\alpha}}_2) - \mathbf{k}\mathbf{e}. \quad (6)$$

Let us have a positive definite Lyapunov function (PDLF)

$$V(t) = \frac{1}{2}\left[e^T e + e_{\alpha_1}^T e_{\alpha_1} + e_{\alpha_2}^T e_{\alpha_2}\right], \quad (7)$$

where $\mathbf{e}_{\boldsymbol{\alpha}_1} = \boldsymbol{\alpha}_1 - \widehat{\boldsymbol{\alpha}}_1$ and $\mathbf{e}_{\boldsymbol{\alpha}_2} = \boldsymbol{\alpha}_2 - \widehat{\boldsymbol{\alpha}}_2$.

$$\dot{V}(t) = \mathbf{e}^T\dot{\mathbf{e}} + \mathbf{e}_{\boldsymbol{\alpha}_1}^T\dot{\mathbf{e}}_{\boldsymbol{\alpha}_1} + \mathbf{e}_{\boldsymbol{\alpha}_2}^T\dot{\mathbf{e}}_{\boldsymbol{\alpha}_2},$$

$$\begin{aligned}\dot{V}(t) = {} & \mathbf{e}^T[F_1(x)(\boldsymbol{\alpha}_1 - \widehat{\boldsymbol{\alpha}}_1) + F_2(x)(\boldsymbol{\alpha}_2 - \widehat{\boldsymbol{\alpha}}_2) - k\mathbf{e}] \\ & - \mathbf{e}_{\boldsymbol{\alpha}_1}^T[F_1(x)]^T\mathbf{e} - \mathbf{e}_{\boldsymbol{\alpha}_2}^T[F_2(x)]^T\mathbf{e},\end{aligned}$$

$$\dot{V}(t) = -k\mathbf{e}^T\mathbf{e} < 0. \quad (8)$$

□

3. IMSS in L6DHCS

As discussed above, in L6DHCS [34], there will be three types of switches, i.e., the switches based on single, double, and triple errors. In order to design the different kinds of switches which is based on the error, the dimensionless L6DHCS has been chosen.

$$\text{L6DHCS} = \begin{cases} \dot{x}_1 = a(x_2 - x_1) + x_4, \\ \dot{x}_2 = cx_1 - x_1x_3 - x_2 + x_5, \\ \dot{x}_3 = x_1x_2 - bx_3, \\ \dot{x}_4 = dx_4 - x_1x_3, \\ \dot{x}_5 = -kx_2, \\ \dot{x}_6 = lx_2 + hx_6, \end{cases} \quad (9)$$

where x_i (for $i = 1, 2, 3, 4, 5, 6$) is the state variable; a, b, c, h, k, and l are the parameters involved in the system used for demonstration and have been fixed at $a = 10$; $b = 8/3$; $c = 28$; $d = 2$; $k = 8.4$; $l = 1$; $h = 1$ for all the types of switches.

In the proposed study, the IMSS has been demonstrated through L6DHCS that has three kinds of internal synchronization characteristics, i.e., three kinds of switches as follows:

(1) Single error-based switches

$$\underbrace{\begin{bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_{30} \end{bmatrix} \longrightarrow \begin{bmatrix} e_{11} \\ e_{21} \\ \cdot \\ \cdot \\ \cdot \\ e_{301} \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_3 - x_4 \end{bmatrix}}_{\text{Switches based on single error}}. \quad (10)$$

(2) Double error-based switches

$$\underbrace{\begin{bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_{90} \end{bmatrix} \longrightarrow \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ e_{901} & e_{902} \end{bmatrix} = \begin{bmatrix} x_1 - x_3 & x_2 - x_4 \\ x_1 - x_4 & x_2 - x_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ x_3 - x_1 & x_4 - x_2 \end{bmatrix}}_{\text{Switches based on 2 errors}}. \quad (11)$$

(3)Triple error-based switches

$$\underbrace{\begin{bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_{20} \end{bmatrix} \longrightarrow \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ e_{201} & e_{202} & e_{203} \end{bmatrix} = \begin{bmatrix} x_4 - x_1 & x_5 - x_2 & x_6 - x_3 \\ x_3 - x_1 & x_5 - x_2 & x_6 - x_4 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x_3 - x_1 & x_4 - x_2 & x_5 - x_6 \end{bmatrix}}_{\text{Switches based on 3 errors}}. \quad (12)$$

3.1. IMSS Based on Single Error

3.1. IMSS Based on Single Error. In this subsection, it has been discussed that only first three switches are based on the single error and the rest of the switches can be carried out in the same way.

Switch-1: In order to synchronize x_1 and x_2, let us design the two subsystems from (9) representing the dynamics of x_1 and x_2:

$$\text{Master subsystem}: \quad \dot{x}_1 = -ax_1 + ax_2 + x_4, \quad (13)$$

$$\text{Slave subsystem}: \quad \dot{x}_2 = cx_1 - x_2 - x_1x_3 + x_5 + u_{11}. \quad (14)$$

From (10), (13) and (14), the error dynamics for switch-1 can be written as

$$\dot{e}_{11} = -(a + c)e_{11} - x_1x_3 + x_5 - x_4 + cx_2 - x_2 + u_{11}, \quad (15)$$

where

$$u_{11} = (\hat{a} + \hat{c})e_{11} + x_1x_3 - x_5 + x_4 - cx_2 + x_2 - k_{11}e_{11}, \quad (16)$$

Now, (15) can be written as

$$\dot{e}_{11} = -(a - \hat{a})e_{11} - (c - \hat{c})e_{11} - k_{11}e_{11}. \quad (17)$$

In order to describe the stability, let us have a PDLF

$$V(t) = \frac{1}{2}\left(e_{11}^2 + e_a^2 + e_c^2\right), \quad (18)$$

$$\begin{aligned} \dot{V}(t) &= e_{11}\dot{e}_{11} + e_a\dot{e}_a + e_c\dot{e}_c, \\ \dot{V}(t) &= -k_{11}e_{11}^2 - k_{12}e_a^2 - k_{13}e_c^2, \text{ for } k_{1j} > 0 \text{ for } j = 1, 2, 3, \end{aligned} \quad (19)$$

where $\dot{\hat{a}} = -e_{11}^2 + k_{12}e_a$ and $\dot{\hat{c}} = -e_{11}^2 + k_{13}e_c$.

Switch-2: In order to synchronize x_1 and x_4, let us design the two subsystems from (9) representing the dynamics of x_1 and x_4:

$$\text{Master subsystem}: \quad \dot{x}_1 = -ax_1 + ax_2 + x_4, \quad (20)$$

$$\text{Slave subsystem}: \quad \dot{x}_4 = dx_4 - x_1x_3 + u_{21}. \quad (21)$$

From (10), (20) and (21), the error dynamics for switch-2 can be written as

$$\dot{e}_{21} = (d - a)e_{21} - x_1x_3 - ax_2 - x_4 + dx_1 + ax_4 + u_{21}, \quad (22)$$

where

$$u_{21} = -\left(\hat{d} - \hat{a}\right)e_{21} + x_1x_3 + ax_2 + x_4 - dx_1 - ax_4 - k_{21}e_{21}, \quad (23)$$

Now, (22) can be written as

$$\dot{e}_{21} = -(a - \hat{a})e_{21} + \left(d - \hat{d}\right)e_{21} - k_{21}e_{21}. \quad (24)$$

In order to describe the stability, let us have a PDLF

$$V(t) = \frac{1}{2}\left(e_{21}^2 + e_a^2 + e_d^2\right), \quad (25)$$

$$\begin{aligned} \dot{V}(t) &= e_{21}\dot{e}_{21} + e_a\dot{e}_a + e_d\dot{e}_d, \\ \dot{V}(t) &= -k_{21}e_{21}^2 - k_{22}e_a^2 - k_{23}e_d^2, \text{ for } k_{2j} > 0 \text{ for } j = 1, 2, 3, \end{aligned} \quad (26)$$

where $\dot{\hat{a}} = -e_{21}^2 + k_{22}e_a$ and $\dot{\hat{d}} = e_{21}^2 + k_{23}e_d$.

Switch-3: In order to synchronize x_1 and x_5, let us design the two subsystems from (9) representing the dynamics of x_1 and x_5:

$$\text{Master subsystem}: \quad \dot{x}_1 = -ax_1 + ax_2 + x_4, \quad (27)$$

$$\text{Slave subsystem}: \quad \dot{x}_5 = -kx_2 + u_{31}. \quad (28)$$

From (10), (27) and (28), the error dynamics for switch-3 can be written as

$$\dot{e}_{31} = -ae_{31} - kx_2 - ax_2 - x_4 + ax_5 + u_{31}, \quad (29)$$

where

$$u_{31} = \hat{a}e_{31} + kx_2 + ax_2 + x_4 - ax_5 - k_{31}e_{31}. \quad (30)$$

Now, (29) can be written as

$$\dot{e}_{31} = -(a - \hat{a})e_{31} - k_{31}e_{31}. \quad (31)$$

In order to describe the stability, let us have a PDLF

$$V(t) = \frac{1}{2}\left(e_{31}^2 + e_a^2\right), \quad (32)$$

$$\begin{aligned} \dot{V}(t) &= e_{31}\dot{e}_{31} + e_a\dot{e}_a, \\ \dot{V}(t) &= -k_{31}e_{31}^2 - k_{32}e_a^2, \text{ for } k_{3j} > 0 \text{ for } j = 1, 2, \end{aligned} \quad (33)$$

where $\dot{\hat{a}} = -e_{31}^2 + k_{32}e_a$.

In this subsection, IMSS has been demonstrated for L6DHCS having the switches based on a single error. However, there will be such 30 switches (see equation (10)) but it has been discussed only first three switches. In the proposed study, for the simulation of three switches, the initial conditions $x_1(0) = -1$, $x_2(0) = 2$, $x_3(0) = 1$, $x_4(0) = -1$, $x_5(0) = -6.45$, and $x_6(0) = 1$ and parameters in the chosen controlling technique $k_{11} = 1$, $k_{12} = 1$, $k_{13} = 1$, $k_{21} = 0.21$, $k_{22} = 0.63$, $k_{23} = 1.001$, $k_{31} = 1$, $k_{32} = 1$, $k_{33} = 1$, $k_{41} = 1$, and $k_{42} = 1$ are chosen for simulation on Mathematica. The dynamics of the errors as well as state variables have been plotted (see Figures 1–4). All figures confirm that all the switches under study achieved internal synchronization. Furthermore, the time series of $\dot{V}(t)$ is smaller than or equal to zero for $t \geq 0$ (see Figure 5), a clear indication that stability is achieved for all the switches during synchronization internally.

3.2. IMSS Based on Two Errors. In this part, the authors discuss the IMSS between the two subsystems having two state variables of L6DHCS. It has been discussed that only the first two switches are based on two errors, and the rest of the switches can be carried out in the same way.

Switch-1: In the switch-1, let us design the two subsystems having two state variables from (9) in which $e_{11} = x_3 - x_1$ and $e_{12} = x_4 - x_2$:

$$\text{Master subsystem}: \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -ax_1 + ax_2 + x_4 \\ cx_1 - x_2 - x_1x_3 + x_5 \end{bmatrix}, \quad (34)$$

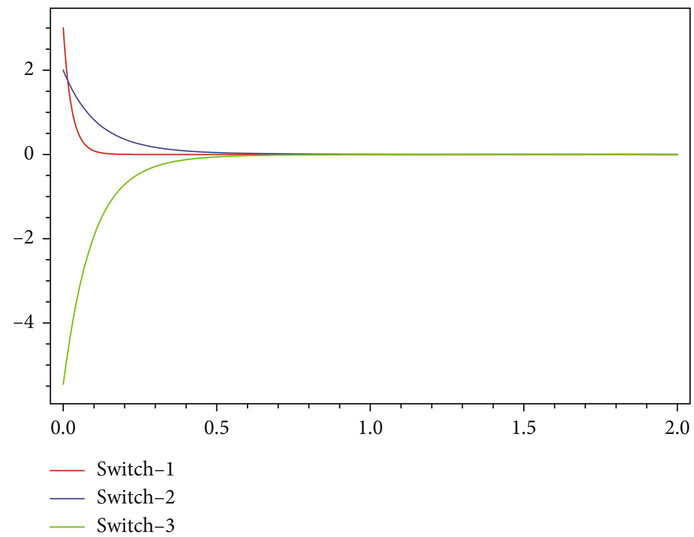

Figure 1: Time series of errors.

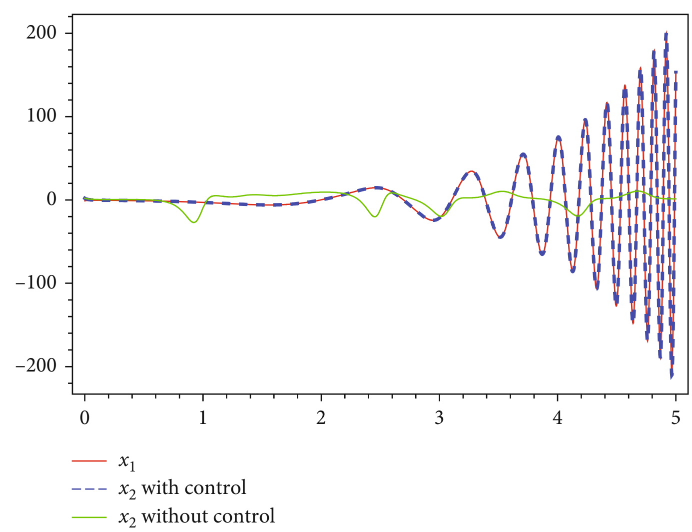

Figure 2: Time series of x_1 and x_2 for switch-1.

$$\text{Slave subsystem}: \quad \begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -bx_3 + x_1 x_2 \\ dx_4 - x_1 x_3 \end{bmatrix} + \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}. \tag{35}$$

From (11), (34) and (35), the error dynamics for switch-1 can be written as

$$\dot{e}_{11} = -(a+b)e_{11} + x_1 x_2 - ax_2 - x_4 - bx_1 + ax_3 + u_{11}, \tag{36}$$

$$\dot{e}_{12} = de_{12} - cx_1 + (1+d)x_2 - x_5 + u_{12}, \tag{37}$$

where

$$\begin{aligned} u_{11} &= \left(\widehat{a} + \widehat{b}\right)e_{11} - x_1 x_2 + ax_2 + x_4 + bx_1 - ax_3 - k_{11}e_{11}, \\ u_{12} &= -\widehat{d}e_{12} + cx_1 - (1+d)x_2 + x_5 - k_{12}e_{12}. \end{aligned} \tag{38}$$

Now, (36) and (37) can be written as

$$\begin{aligned} \dot{e}_{11} &= -\left(a + b - \widehat{a} - \widehat{b}\right)e_{11} - k_{11}e_{11}, \\ \dot{e}_{12} &= \left(d - \widehat{d}\right)e_{12} - k_{12}e_{12}. \end{aligned} \tag{39}$$

In order to describe the stability, let us have a PDLF

$$V(t) = \frac{1}{2}\left(e_{11}^2 + e_{12}^2 + e_a^2 + e_b^2 + e_d^2\right), \tag{40}$$

$$\begin{aligned} \dot{V}(t) &= e_{11}\dot{e}_{11} + e_{12}\dot{e}_{12} + e_a\dot{e}_a + e_b\dot{e}_b + e_d\dot{e}_d, \\ \dot{V}(t) &= -k_{11}e_{11}^2 - k_{12}e_{12}^2 - k_{13}e_a^2 - k_{14}e_b^2 - k_{15}e_d^2, \text{ for } k_{1j} > 0 \text{ for } j \\ &= 1, 2, \cdots, 5, \end{aligned} \tag{41}$$

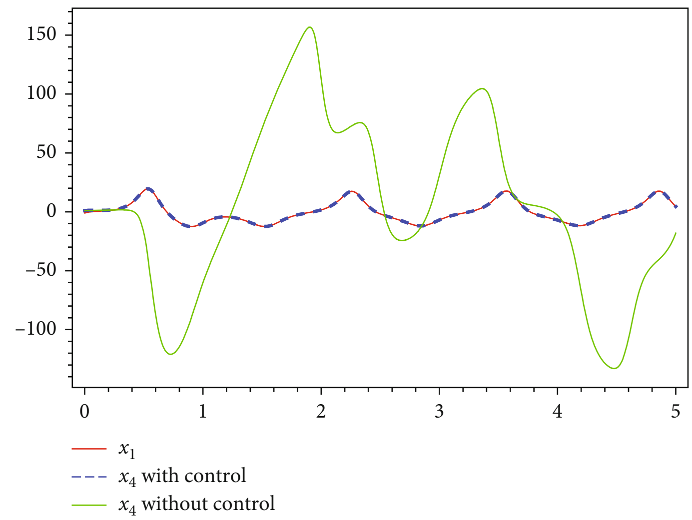

Figure 3: Time series of x_1 and x_4 for switch-3.

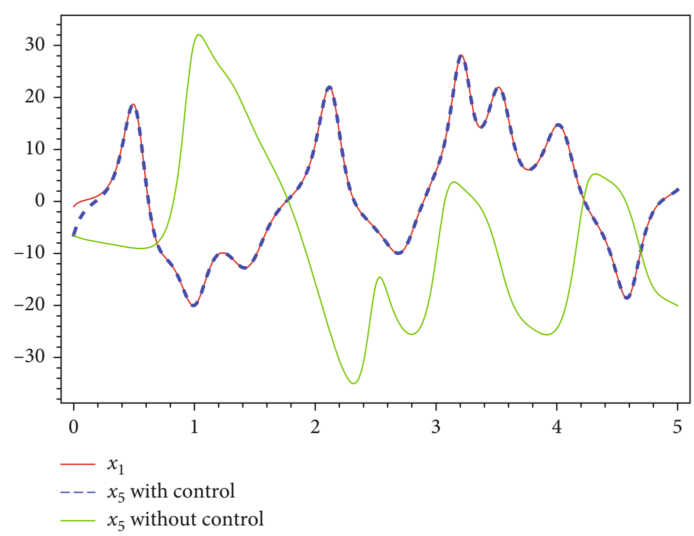

Figure 4: Time series of x_1 and x_5 for switch-4.

where $\dot{\hat{a}} = -e_{11}^2 + k_{13}e_a$, $\dot{\hat{b}} = -e_{11}^2 + k_{14}e_b$, and $\dot{\hat{d}} = e_{12}^2 + k_{15}e_d$.

Switch-2: In the switch-2, let us design the two subsystems having two state variables from (9) in which $e_{21} = x_4 - x_1$ and $e_{22} = x_5 - x_2$:

$$\text{Master subsystem}: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -ax_1 + ax_2 + x_4 \\ cx_1 - x_2 - x_1x_3 + x_5 \end{bmatrix}, \tag{42}$$

$$\text{Slave subsystem}: \begin{bmatrix} \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} dx_4 - x_1x_3 \\ -kx_2 \end{bmatrix} + \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}. \tag{43}$$

From (11), (42) and (43), the error dynamics for switch-2 can be written as

$$\dot{e}_{21} = (d - a)e_{21} - x_1x_3 - ax_2 + dx_1 + (a-1)x_4 + u_{21}, \tag{44}$$

$$\dot{e}_{22} = (k-1)e_{22} - cx_1 + x_1x_3 - kx_5 + u_{22}, \tag{45}$$

where

$$\begin{aligned} u_{21} &= -\left(\hat{d} - \hat{a}\right)e_{21} + x_1x_3 + ax_2 - dx_1 - (a-1)x_4 - k_{21}e_{21}, \\ u_{22} &= -\left(\hat{k} - 1\right)e_{22} + cx_1 - x_1x_3 + kx_5 - k_{22}e_{22}. \end{aligned} \tag{46}$$

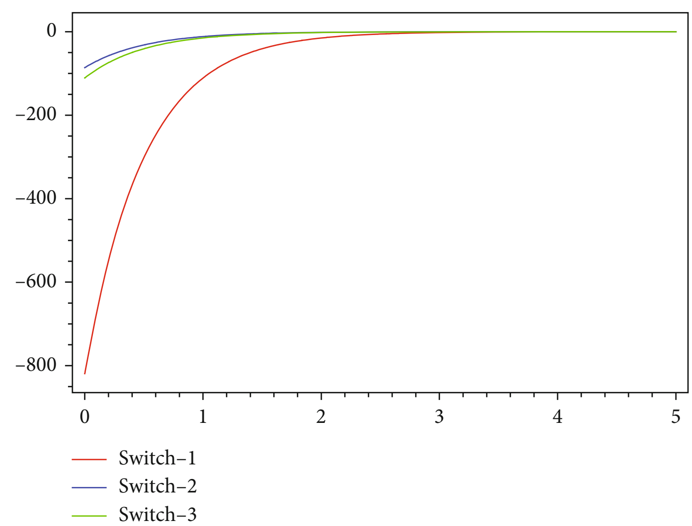

FIGURE 5: Time series of $V(t)$.

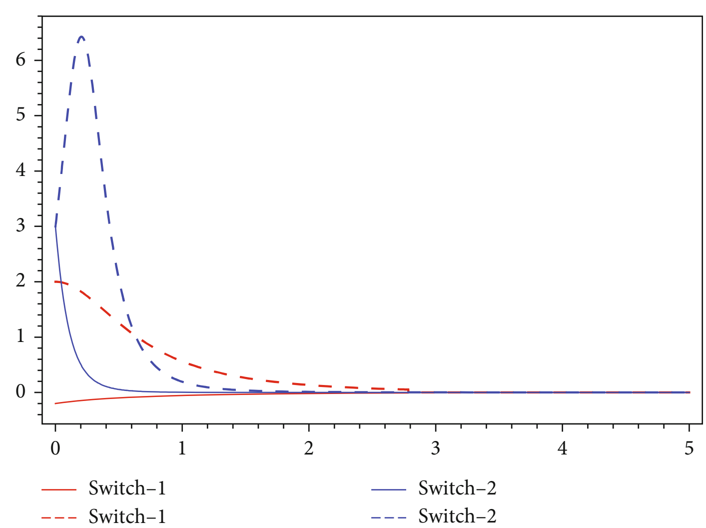

FIGURE 6: Time series of errors.

Now, (44) and (45) can be written as

$$\begin{aligned} \dot{e}_{21} &= \left(d - a - \hat{d} + \hat{a}\right)e_{21} - k_{21}e_{21}, \\ \dot{e}_{22} &= \left(k - \hat{k}\right)e_{22} - k_{22}e_{22}. \end{aligned} \tag{47}$$

In order to describe the stability, let us have a PDLF

$$V(t) = \frac{1}{2}\left(e_{21}^2 + e_{22}^2 + e_a^2 + e_d^2 + e_k^2\right), \tag{48}$$

$$\begin{aligned} \dot{V}(t) &= e_{21}\dot{e}_{21} + e_{22}\dot{e}_{22} + e_a\dot{e}_a + e_d\dot{e}_d + e_k\dot{e}_k, \\ \dot{V}(t) &= -k_{21}e_{21}^2 - k_{22}e_{22}^2 - k_{23}e_a^2 - k_{24}e_d^2 - k_{25}e_k^2, \text{ for } k_{2j} > 0 \text{ for } j = 1, 2, \cdots, 5, \end{aligned} \tag{49}$$

where $\dot{\hat{a}} = -e_{21}^2 + k_{23}e_a$, $\dot{\hat{d}} = e_{21}^2 + k_{24}e_d$, and $\dot{\hat{k}} = e_{22}^2 + k_{25}e_k$.

In this subsection, IMSS is presented for L6DHCS based on two errors for all switches, i.e., the master-slave subsystems both will have two state variables. However, there will be a total 90 of such kinds of switches (see equation (11)) but it has been discussed only two switches and the rest can be studied in the same way. In the proposed study, for the simulation of three switches, the initial conditions $x_1(0) = -2$, $x_2(0) = 2$, $x_3(0) = -2.2$, $x_4(0) = 4$, $x_5(0) = 5$, and $x_6(0) = 1$ and parameters in the chosen controlling technique $k_{11} = 1$, $k_{12} = 1$, $k_{13} = 0.01$, $k_{14} = 2$, $k_{15} = 1$, $k_{21} = 1$, $k_{22} = 1$, $k_{23} = 1$, $k_{24} = 1$, and $k_{25} = 1$ are chosen for simulation on Mathematica. The dynamics of the error systems and state vectors for the switches under study have been plotted in order to show the achievement of IMSS (see Figures 6–10). From all figures (Figures 6–10), it is very well clear that the IMSS is robust. Furthermore, from the time series of $\dot{V}(t)$ (see Figure 11), it is very well clear that there is robust stability for the switches under investigation.

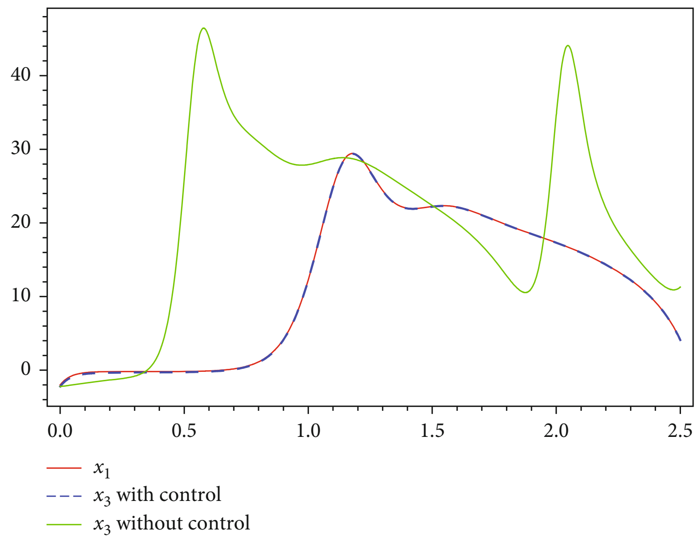

FIGURE 7: Time series of x_1 and x_3.

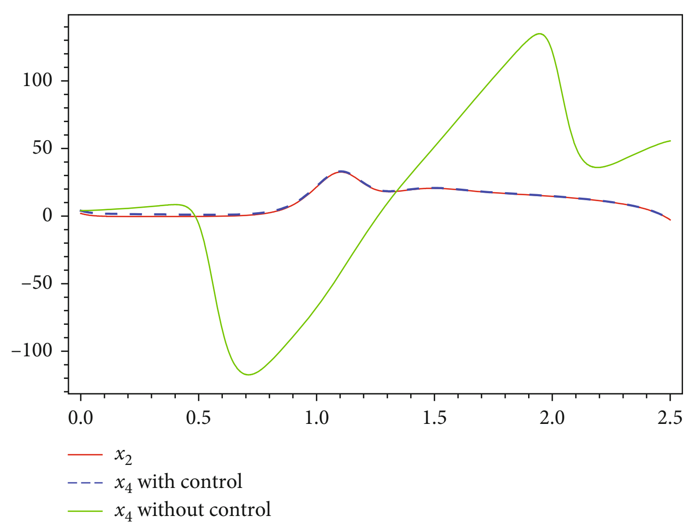

FIGURE 8: Time series of x_2 and x_4.

3.3. IMSS Based on Three Errors. In this part, the authors discuss the IMSS between the two subsystems having three state variables of L6DHCS. It has been discussed that only one switch is based on three errors, and the rest of the switches can be carried out in a same way.

Switch-1: In the switch-1, let us design the two subsystems having three state variables from (9) in which $e_{11} = x_4 - x_1$, $e_{12} = x_5 - x_2$, and $e_{13} = x_6 - x_3$:

$$\text{Master subsystem}: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -ax_1 + ax_2 + x_4 \\ cx_1 - x_2 - x_1x_3 + x_5 \\ -bx_1 + x_1x_2 \end{bmatrix}, \tag{50}$$

$$\text{Slave subsystem}: \begin{bmatrix} \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} dx_4 - x_1x_3 \\ -kx_2 \\ lx_2 + hx_6 \end{bmatrix} + \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix}. \tag{51}$$

From (12), (50) and (51), the error dynamics for switch-1 can be written as

$$\dot{e}_{11} = (d - a)e_{11} - x_1x_3 - ax_2 - x_4 + dx_1 + ax_4 + u_{11}, \tag{52}$$

$$\dot{e}_{12} = ke_{12} - cx_1 + x_2 + x_1x_3 - (1 + k)x_5 + u_{12}, \tag{53}$$

$$\dot{e}_{13} = (h - b)e_{13} + lx_2 - x_1x_2 + hx_3 + bx_6 + u_{13}, \tag{54}$$

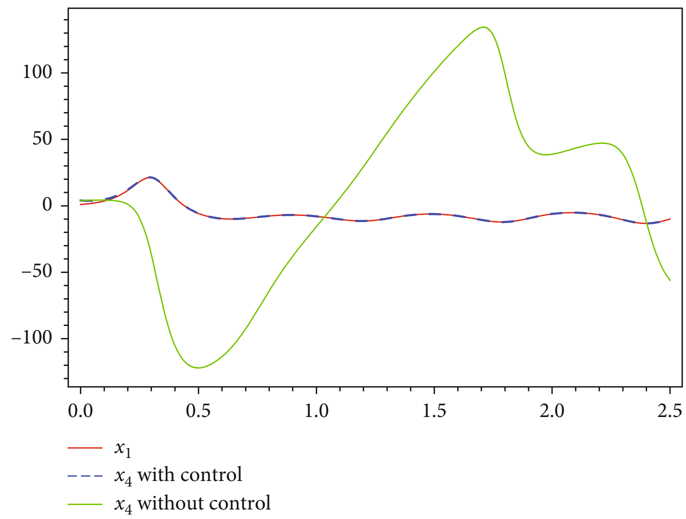

Figure 9: Time series of x_1 and x_4.

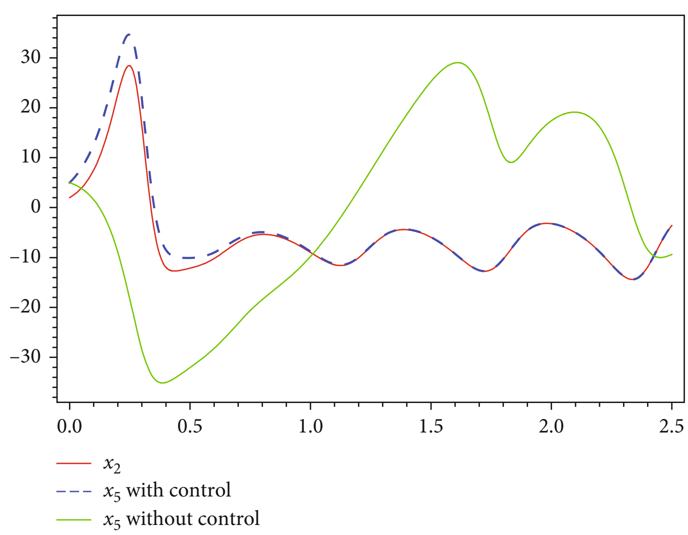

Figure 10: Time series of x_2 and x_5.

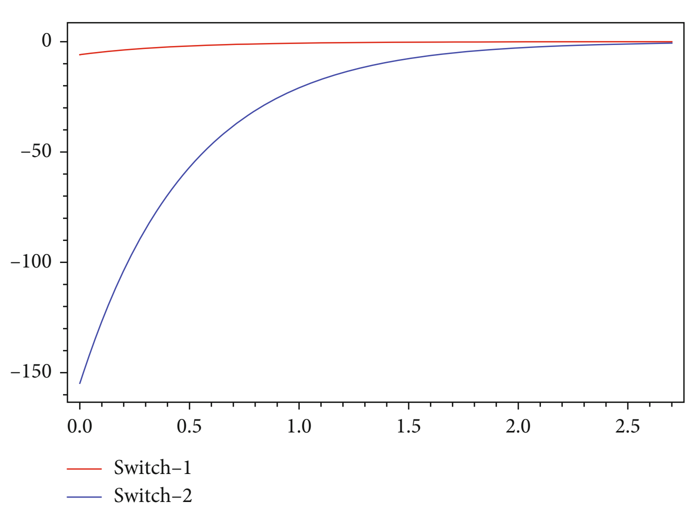

Figure 11: Time series of $V(t)$.

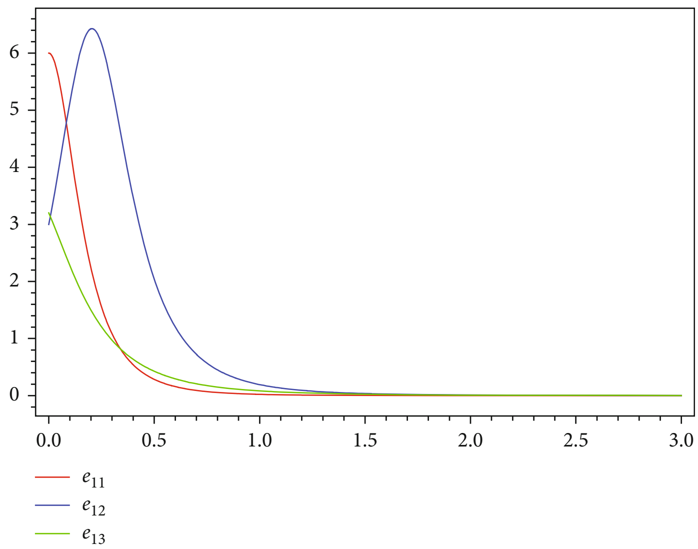

Figure 12: Time series of errors.

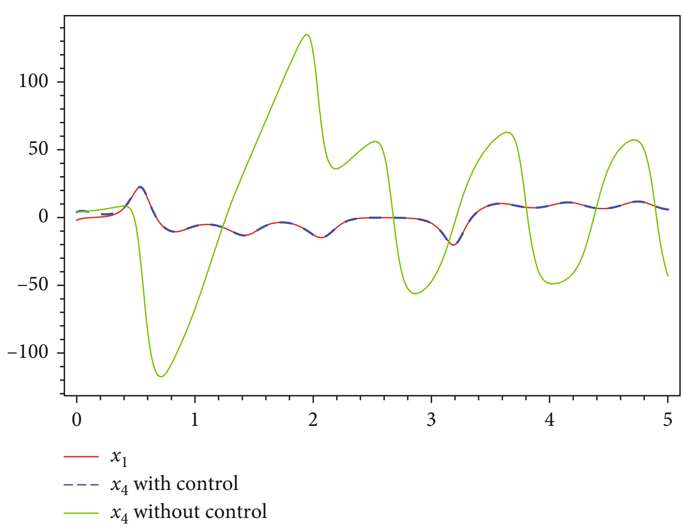

Figure 13: Time series of x_1 and x_4 for S_1.

where

$$\begin{aligned} u_{11} &= -\left(\widehat{d} - \widehat{a}\right) e_{11} + x_1 x_3 + a x_2 + x_4 - d x_1 - a x_4 - k_{11} e_{11}, \\ u_{12} &= -\widehat{k} e_{12} + c x_1 - x_2 - x_1 x_3 + (1+k) x_5 - k_{12} e_{12}, \\ u_{13} &= -\left(\widehat{h} - \widehat{b}\right) e_{13} - l x_2 + x_1 x_2 - h x_3 - b x_6 - k_{13} e_{13}. \end{aligned} \tag{55}$$

Now, (52), (53) and (54) can be written as

$$\begin{aligned} \dot{e}_{11} &= (e_d - e_a) e_{11} - k_{11} e_{11}, \\ \dot{e}_{12} &= e_k e_{12} - k_{12} e_{12}, \\ \dot{e}_{13} &= (e_h - e_b) e_{13} + k_{13} e_{13}. \end{aligned} \tag{56}$$

In order to describe the stability, let us have a PDLF

$$V(t) = \frac{1}{2}\left(e_{11}^2 + e_{12}^2 + e_{13}^2 + e_a^2 + e_b^2 + e_d^2 + e_h^2 + e_k^2\right), \tag{57}$$

$$\dot{V}(t) = e_{11}\dot{e}_{11} + e_{12}\dot{e}_{12} + e_{13}\dot{e}_{13} + e_a\dot{e}_a + e_b\dot{e}_b + e_d\dot{e}_d + e_h\dot{e}_h + e_k\dot{e}_k,$$

$$\begin{aligned} \dot{V}(t) = {} & -k_{11}e_{11}^2 - k_{12}e_{12}^2 - k_{13}e_{13}^2 - k_{14}e_a^2 - k_{15}e_b^2 - k_{16}e_d^2 \\ & - k_{17}e_h^2 - k_{18}e_k^2, \text{ for } k_{1j} > 0 \text{ for } j = 1, 2, \cdots, 8, \end{aligned} \tag{58}$$

where $\dot{\widehat{a}} = -e_{11}^2 + k_{14}e_a$, $\dot{\widehat{b}} = -e_{13}^2 + k_{15}e_b$, $\dot{\widehat{d}} = e_{11}^2 + k_{16}e_d$, $\dot{\widehat{h}} = e_{13}^2 + k_{17}e_h$, and $\dot{\widehat{k}} = e_{12}^2 + k_{18}e_k$.

In this subsection, IMSS is presented for L6DHCS based on three errors for all switches, i.e., the drive-response subsystems both will have three state variables. However, there will be total 20 of such kind of switches (see equation (12))

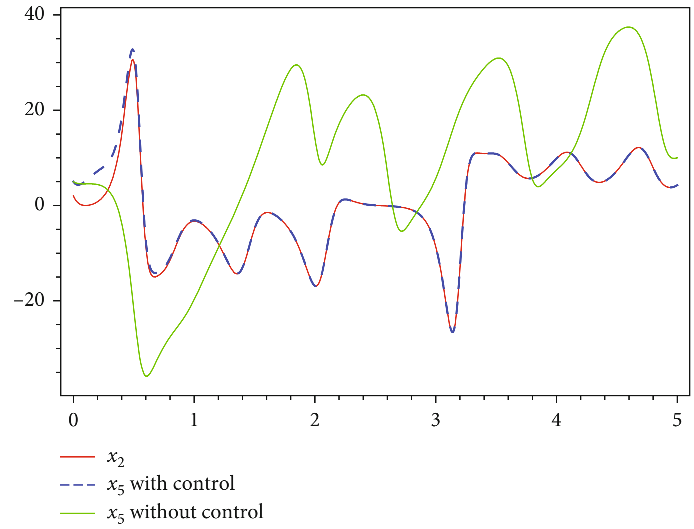

Figure 14: Time series of x_2 and x_5 for S_1.

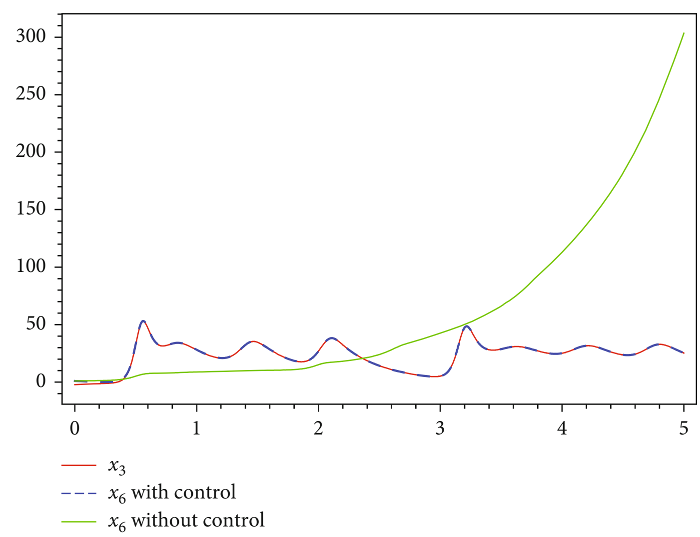

Figure 15: Time series of x_3 and x_6 for S_1.

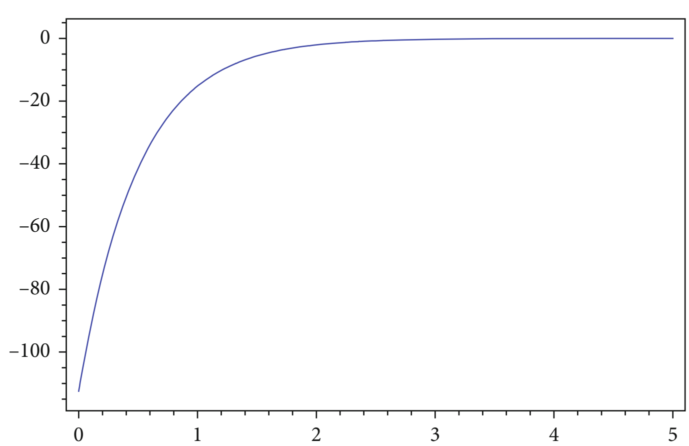

Figure 16: Time series of $V(t)$ for switch-1.

but it has been discussed that only one switch and the rest can be studied in the same way. In the proposed study, for the simulation of three switches, the initial conditions $x_1(0)=-2$, $x_2(0)=2$, $x_3(0)=-2.2$, $x_4(0)=4$, $x_5(0)=5$, and $x_6(0)=1$ and parameters in the chosen controlling technique $k_{11}=1$, $k_{12}=1$, $k_{13}=1$, $k_{14}=1$, $k_{15}=1$, $k_{16}=1$, $k_{17}=1$, $k_{18}=1$, $k_{21}=0.01$, $k_{22}=0.03$, $k_{23}=0.004$, $k_{24}=0.01$, $k_{25}=0.01$, $k_{26}=0.001$, $k_{27}=0.001$, and $k_{28}=0.004$ are chosen for simulation on Mathematica. The dynamics of the error systems and state vectors for the switch under study have been plotted in order to show the achievement of IMSS (see Figures 12–15). From all figures (Figures 12–15), it is very well clear that IMSS is achieved. Furthermore, from the time series of $\dot{V}(t)$ (see Figure 16), it is very well clear that there is a stability for the switch under investigation.

Remark 2. The gain constant vector (k) is however chosen positive but it is not affecting the time response for stability if we chose any positive number in the current study.

4. Conclusions

In this paper, the authors have presented the various possible subdynamics of a chaotic dynamical system that are synchronized with each other providing some multiswitches of different orders within the system. There are many different types of switches that can be designed on the basis of the error(s) involved during IMSS. The successful implementation of IMSS has been demonstrated through L6DHCS, which has three kinds of switches based on the error. For all the three types of designed switches, all computational work is carried out using Mathematica to validate the existence and achievement of IMSS. Below are the remarkable features of our proposed study:

(i) IMSS removes the external dependency from slave system, i.e., someone can break a system internally into master and slave systems as per the requirement

(ii) It can give less cost as someone can choose the appropriate type of switch as per the requirement

(iii) It can enhance the security level during secure communication

The current study opens the doors for the researchers as it gives the directions to synchronize the chaotic system without the external slave system.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

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