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Research Article

Numerical Solution of the Absolute Value Equation Using Modified Iteration Methods

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This article suggests two new modified iteration methods called the modified Gauss-Seidel (MGS) method and the modified fixed point (MFP) method to solve the absolute value equation. Using appropriate assumptions, we examine the convergence of the given methods. Lastly, numerical examples illustrate the usefulness of the new strategies.

1. Introduction

Let the absolute value equation (AVE) be

$$Ax - |x| = b. \tag{1}$$

In this case, the order of the A matrix is $n \times n$, $b \in R^n$, and $|x|$ describes the component-wise absolute value of x vector. The AVE is an important nondifferentiable and nonlinear problem in optimization, such as convex quadratic programming, linear complementarity problems (LCPs), and linear programming (see [1–12]).

The development of numerical procedures for AVE has been extensively studied recently, and several techniques have been presented. For example, Fakharzadeh and Shams [13] introduced the mixed-type splitting approach to determine AVE (1) and showed the new convergence properties under appropriate conditions. Edalatpour et al. [14] introduced the generalized form of the Gauss-Seidel approach for determining (1). The following summarizes this approach:

$$(D_A - E_A)x^{k+1} - \left|x^{k+1}\right| = F_A x^k + b.k = 0, 1, 2, \cdots, \tag{2}$$

where D_A, E_A, and F_A are expressed in Equation (4), respectively. Ke and Ma [15] offered an SOR-like strategy to obtain the AVE (1). This method is defined as follows:

$$\begin{cases} x^{j+1} = (1-\ddot{\omega})x^j + \ddot{\omega}A^{-1}\left(y^j + b\right), \\ y^{j+1} = (1-\ddot{\omega})y^j + \ddot{\omega}\left|x^{j+1}\right|, \\ \quad j = 0, 1, 2, \cdots, \end{cases} \tag{3}$$

where $x^0 \in R^n$, $y^0 \in R^n$, $A \in R^{n\times n}$, and $\ddot{\omega} > 0$. Chen et al. [16] showed the SOR-like strategy with optimal parameters and examined some novel convergence situations different from [15]. Zamani and Hladík [17] offered a new concave minimization approach for AVE (1), which addresses the deficiency of the system proposed in [18] and others (see [19–23]).

The remainder of this study is organized in the following manner. Section 2 presents various notations and definitions. Section 3 discusses the proposed methods as well as their convergence for AVE (1). We demonstrate the efficiency of the new techniques in Section 4 by providing numerical examples. In the last section, we make concluding remarks.

2. Preliminaries

The purpose of this section is to briefly review some of the symbols and concepts used in this article.

Suppose $A = (a_{ij}) \in R^{n\times n}$; we describe the spectral radius and absolute value of A as $\rho(A)$ and $|A| = (|a_{ij}|)$, respectively.

Lemma 1 (see [24]). *Let A be an invertible matrix. If* $\|A^{-1}\| < 1$, *then for any vector b, Equation (1) has a unique solution.*

3. Modified Iteration Methods

Here, we present a detailed discussion of the proposed modified methods for solving AVEs.

3.1. MGS Method for AVE. Let us divide matrix A as follows:

$$A = D_A - E_A - F_A, \tag{4}$$

where F_A, E_A, and D_A are strictly upper-triangular, strictly lower-triangular, and diagonal matrices of A, respectively. Based on (4), the MGS approach to solving (1) can be expressed as follows:

In the MGS method, let $A \in R^{n\times n}$ with $|A| \neq 0$ and $b \in R^n$. Based on an starting vector $x^0 \in R^n$ and for $k = 0, 1, 2, \cdots$, until the iterative sequence $\{x^k\}_{k=0}^{\infty}$ is convergent, calculate

$$\begin{cases} x^{k+(1/2)} = (D_A - E_A)^{-1}\left|x^k\right| + (D_A - E_A)^{-1}\left(F_A x^k + b\right), \\ x^{k+1} = (D_A - F_A)^{-1}\left|x^{k+(1/2)}\right| + (D_A - F_A)^{-1}\left(E_A x^{k+(1/2)} + b\right). \end{cases} \tag{5}$$

The next step is to examine the convergence of the MGS approach by utilizing the subsequent theorem.

Theorem 2. *Assume that AVE (1) is solvable, and matrix A satisfies Lemma 1. If*

$$\left|x^{k+1} - x^\star\right| \leq (GR + GRV + GJR + GJRV)\left|x^k - x^\star\right|, \tag{6}$$

where

$$G = \left|(D_A - E_A)^{-1}\right|, R = \left|(D_A - F_A)^{-1}\right|, J = |F_A| \text{ and } V = |E_A|. \tag{7}$$

Then, for any starting vector $x^{(0)}$, the sequence $\{x^k\}_{k=0}^{\infty}$ created by the MGS approach converges to the unique solution $x^\star$ of the Equation (1).

Proof. Suppose that $x^\star$ is the solution of Equation (1); then we obtain

$$\begin{cases} x^{k+(1/2)} - x^\star = (D_A - E_A)^{-1}\left(\left|x^k\right| - |x^\star|\right) + (D_A - E_A)^{-1} F_A\left(x^k - x^\star\right), \\ x^{k+1} - x^\star = (D_A - F_A)^{-1}\left(\left|x^{k+(1/2)}\right| - |x^\star|\right) + (D_A - F_A)^{-1} E_A\left(x^{k+(1/2)} - x^\star\right). \end{cases} \tag{8}$$

Using absolute values for each side of the first equation of (8), we get

$$\begin{aligned} &\left|x^{k+(1/2)} - x^\star\right| \\ &\quad\leq \left|(D_A - E_A)^{-1}\right|\left|\left|x^k\right| - |x^\star|\right| + \left|(D_A - E_A)^{-1}\right||F_A|\left|x^k - x^\star\right|, \\ &\quad\leq \left|(D_A - E_A)^{-1}\right|\left|x^k - x^\star\right| + \left|(D_A - E_A)^{-1}\right||F_A|\left|x^k - x^\star\right|, \end{aligned} \tag{9}$$

or equivalently

$$\left|x^{k+(1/2)} - x^\star\right| \leq G\left|x^k - x^\star\right| + GJ\left|x^k - x^\star\right|, \leq (G + GJ)\left|x^k - x^\star\right|. \tag{10}$$

Similarly, the second part of (8) indicates

$$\left|x^{k+1} - x^\star\right| \leq (R + RV)\left|x^{k+(1/2)} - x^\star\right|. \tag{11}$$

From (10) and (11), we obtain

$$\left|x^{k+1} - x^\star\right| \leq (G + GJ) \times (R + RV)\left|x^k - x^\star\right|. \tag{12}$$

So,

$$\left|x^{k+1} - x^\star\right| \leq (GR + GRV + GJR + GJRV)\left|x^k - x^\star\right|. \tag{13}$$

$(GR + GRV + GJR + GJRV)$ is nonnegative. Note that if $\rho(GR + GRV + GJR + GJRV) < 1$, then the sequence $\{x^k\}_{k=0}^{\infty}$ of MGS approach converges to the unique solution $x^\star$ of AVE. □

3.2. MFP Method for AVE. First, we will briefly discuss the fixed point method of determining the AVE. The AVE (1) is equivalent to the fixed point problem of solving

$$x = F(x), \tag{14}$$

such that

$$F(x) = x - \phi E[Ax - |x| - b], \tag{15}$$

where $\phi > 0$ and E is a positive diagonal matrix. If we take $\phi E = D_A^{-1}$, then, the fixed point method for solving AVE (1) is defined as follows:

$$x^{k+1} = x^k - D_A^{-1}\left[Ax^k - \left|x^k\right| - b\right], k = 0, 1, 2, \cdots. \tag{16}$$

The purpose of this article is to discuss the MFP method. Then, the offered method is expressed as follows:

In the MFP method, let $A \in R^{n\times n}$ with det $(A) \neq 0$ and $b \in R^n$. Based on an starting vector $x^0 \in R^n$ and for $k = 0, 1, 2, \cdots$, until the iterative sequence $\{x^k\}_{k=0}^{\infty}$ is convergent, calculate

$$\begin{cases} x^{k+(1/2)} = x^k - D_A^{-1}\left(Ax^k - \left|x^k\right| - b\right), \\ x^{k+1} = x^k - D_A^{-1}\left(Ax^{k+(1/2)} - \left|x^{k+(1/2)}\right| - b\right). \end{cases} \tag{17}$$

Table 1: Abbreviations of testing methods.

Methods	Description
SORLaopt	SOR-like approach with approximate optimal parameter [16]
SORLo	SOR-like method [15, 16]
SORLopt	SOR-like approach with optimal parameter [16]
GGS	Generalized Guass-Seidel method [14]
MGS	Modified Guass-Seidel method (our algorithm 3.1)
MFP	Modified fixed point method (our algorithm 3.2)

The next step is to examine the convergence of the MFP method by utilizing the subsequent theorem.

Theorem 3. *Let matrix A satisfy Lemma 1. If*

$$\left|x^{k+1} - x^*\right| \le \left(I - \left|D_A^{-1}A\right| + \left|D_A^{-1}\right|\right)^2\left|x^k - x^*\right|, \tag{18}$$

then the $\{x^k\}_{k=0}^{\infty}$ *sequence derived from (17) converges to the unique solution* x^*.

Proof. Suppose x^* represents the solution to AVE (1); then we obtain

$$\begin{cases} x^{k+(1/2)} - x^* = \left(x^k - x^*\right) - D_A^{-1}\left(A\left(x^k - x^*\right) - \left(\left|x^k\right| - |x^*|\right)\right), \\ x^{k+1} - x^* = \left(x^{k+(1/2)} - x^*\right) - D_A^{-1}\left(A\left(x^{k+(1/2)} - x^*\right) - \left(\left|x^{k+(1/2)}\right| - |x^*|\right)\right). \end{cases} \tag{19}$$

$$\begin{cases} x^{k+(1/2)} - x^* = \left(x^k - x^*\right) - D_A^{-1}A\left(x^k - x^*\right) + D_A^{-1}\left(\left|x^k\right| - |x^*|\right)\right), \\ x^{k+1} - x^* = \left(x^{k+(1/2)} - x^*\right) - D_A^{-1}A\left(x^{k+(1/2)} - x^*\right) + D_A^{-1}\left(\left|x^{k+(1/2)}\right| - |x^*|\right)\right). \end{cases} \tag{20}$$

Using absolute values for each side of the first equation of (20), we get

$$\begin{aligned} \left|x^{k+(1/2)} - x^*\right| &\le \left|x^k - x^*\right| - \left|D_A^{-1}A\right|\left|x^k - x^*\right| + \left|D_A^{-1}\right|\left|\left|x^k\right| - |x^*|\right|, \\ &\le \left|x^k - x^*\right| - \left|D_A^{-1}A\right|\left|x^k - x^*\right| + \left|D_A^{-1}\right|\left|x^k - x^*\right|, \end{aligned} \tag{21}$$

or equivalently

$$\left|x^{k+(1/2)} - x^*\right| \le \left|\left(I - \left|D_A^{-1}A\right| + \left|D_A^{-1}\right|\right)\left|x^k - x^*\right|\right.. \tag{22}$$

Similarly, the second part of (20) indicates

$$\left|x^{k+1} - x^*\right| \le \left|\left(I - \left|D_A^{-1}A\right| + \left|D_A^{-1}\right|\right)\left|x^{k+(1/2)} - x^*\right|\right.. \tag{23}$$

From (22) and (23), we obtain

$$\begin{aligned} \left|x^{k+1} - x^*\right| \le &\left|\left(I - \left|D_A^{-1}A\right| + \left|D_A^{-1}\right|\right)\right. \\ &\times \left(I - \left|D_A^{-1}A\right| + \left|D_A^{-1}\right|\right)\left|x^k - x^*\right|. \end{aligned} \tag{24}$$

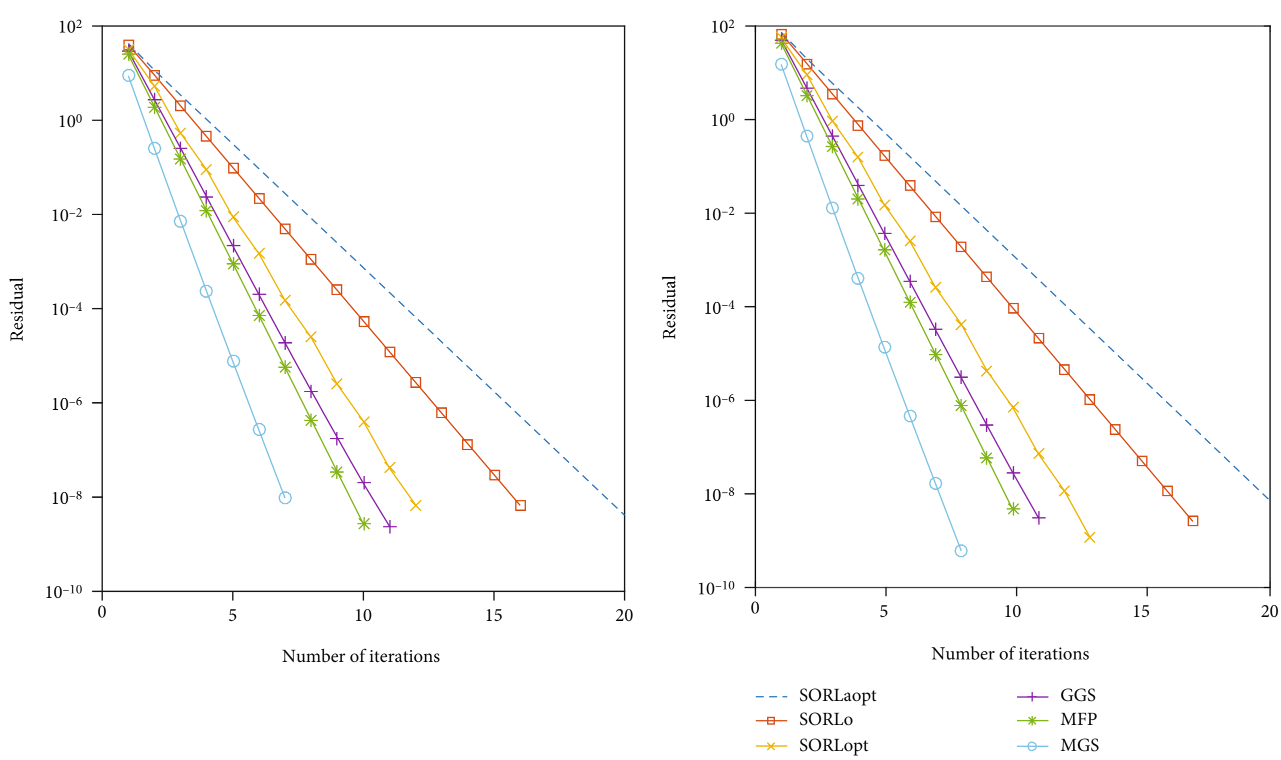
So,

$$\left|x^{k+1} - x^*\right| \le \left|\left(I - \left|D_A^{-1}A\right| + \left|D_A^{-1}\right|\right)^2\left|x^k - x^*\right|\right.. \tag{25}$$

Evidently, if $\rho((I - |D_A^{-1}A| + |D_A^{-1}|)^2) < 1$, the iteration sequence $\{x^k\}_{k=0}^{\infty}$ created by the MFP method is convergent (Table 1). □

Table 2: Calculations of Example 1.

Approaches	n	1000	2000	3000	4000	5000
SORLaopt	Iter	20	20	20	20	20
	Time	0.0014	0.0028	0.0048	0.0058	0.0073
	RES	$4.13e-09$	$5.84e-09$	$7.15e-09$	$8.26e-09$	$9.23e-09$
SORLo	Iter	16	16	17	17	17
	Time	0.0014	0.0023	0.0039	0.0057	0.0064
	RES	$6.27e-09$	$9.51e-09$	$2.59e-09$	$3.00e-09$	$3.35e-09$
SORLopt	Iter	12	12	13	13	13
	Time	0.0007	0.0019	0.0027	0.0036	0.0039
	RES	$5.75e-09$	$5.75e-09$	$5.75e-09$	$5.75e-09$	$5.75e-09$
	RES	$6.80e-09$	$9.63e-09$	$1.19e-09$	$1.37e-09$	$1.53e-09$
GGS	Iter	11	11	11	11	11
	Time	0.0037	0.0063	0.0086	0.0169	0.1921
	RES	$2.40e-09$	$2.76e-07$	$3.07e-09$	$3.36e-09$	$3.63e-09$
MFP	Iter	10	10	10	10	10
	Time	0.0006	0.0011	0.0017	0.0029	0.0033
	RES	$2.68e-09$	$3.791e-09$	$4.65e-09$	$5.37e-09$	$6.01e-09$
MGS	Iter	7	8	8	8	8
	Time	0.0003	0.0007	0.0012	0.0018	0.0031
	RES	$9.80e-09$	$5.09e-10$	$6.18e-10$	$7.11e-10$	$7.93e-10$

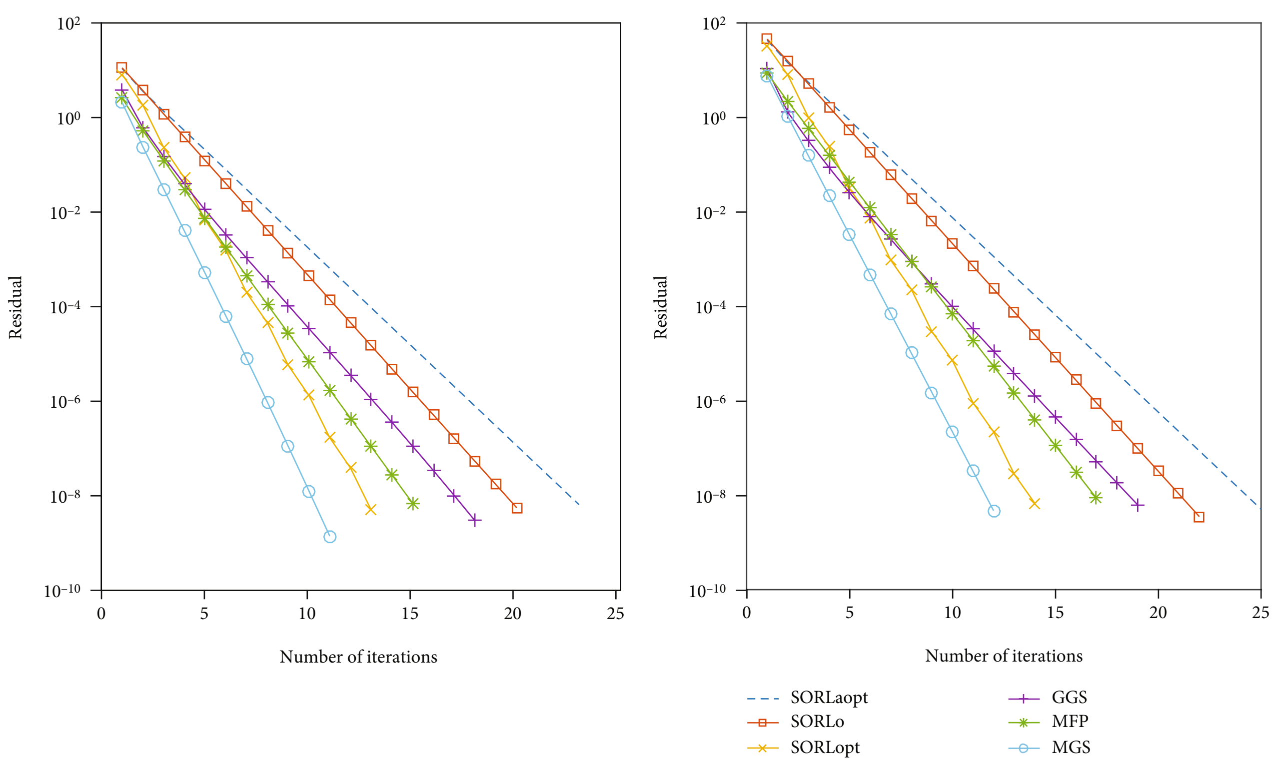


(a) $n = 1000$ (b) $n = 3000$

Figure 1: Comparison of different approaches in a graph form.

Table 3: Calculations of Example 2.

Method	n	64	256	1024	4096
SORLaopt	Iter	23	24	25	26
	Time	0.0036	0.0086	0.0283	0.1466
	RES	$4.24e-09$	$5.32e-09$	$4.92e-09$	$4.08e-09$
SORLo	Iter	20	21	22	22
	Time	0.0032	0.0081	0.0258	0.1162
	RES	$3.15e-09$	$4.03e-09$	$3.38e-09$	$7.41e-09$
SORLopt	Iter	13	14	14	15
	Time	0.0018	0.0047	0.0136	0.0855
	RES	$4.93e-07$	$3.01e-07$	$6.77e-07$	$1.82e-07$
GGS	Iter	18	19	19	19
	Time	0.0027	0.0068	0.0121	0.1107
	RES	$2.94e-09$	$3.62e-09$	$6.26e-09$	$9.55e-09$
MFP	Iter	15	17	17	18
	Time	0.0017	0.0039	0.0106	0.0671
	RES	$6.85e-09$	$2.99e-09$	$8.66e-09$	$5.57e-09$
MGS	Iter	11	12	12	13
	Time	0.0009	0.0013	0.0098	0.0192
	RES	$3.87e-09$	$2.05e-09$	$5.16e-09$	$1.69e-09$



(a) $n = 64$ (b) $n = 1024$

Figure 2: Comparison of different approaches in a graph form.

4. Numerical Tests

In this section, we investigate experimentally the performance of newly developed approaches for determining the AVEs. The computations have been performed on an Intel Core (TM) i7-10875H, with 32 GB memory, and a 5.1 GHz CPU, as well as using MATLAB (2017a). The initial guess is a zero vector, and the current iteration ends when it meets the requirement

$$\mathrm{RES} \coloneqq \left\| Ax^k - \left|x^k\right| - b \right\|_2 \leq 10^{-8}. \tag{26}$$

Example 1 (see [16]). Let

$$A \begin{pmatrix} 8 & -1 & & & \\ -1 & 8 & -1 & & \\ & -1 & 8 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 8 \end{pmatrix} \in R^{n\times n}, x = \begin{pmatrix} -1 \\ 1 \\ \vdots \\ -1 \\ 1 \end{pmatrix} \in R^n, \tag{27}$$

and vector $b = Ax - |x| \in R^n$. Table 2 summarizes the outcomes.

Listed in Table 2 are the iteration steps (Iter), CPU time in seconds (Time), and residuals (RES) for each method. Table 2 provides numerical results that show the suggested methods to be more efficient in terms of time and iteration steps than all other compared methods. Additionally, we compare all approaches graphically for $n = 1000$ and $n = 3000$. The graphical outcomes are shown in Figure 1.

Example 2 (see [16]). Let $M + 4I = A \in R^{n\times n}$ and $Ax - |x| = b \in R^n$ using

$$M \begin{pmatrix} S & -I & & & \\ -I & S & -I & & \\ & -I & S & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & S \end{pmatrix} \in R^{n\times n}, x = \begin{pmatrix} -1 \\ 1 \\ \vdots \\ -1 \\ 1 \end{pmatrix} \in R^n, \tag{28}$$

where $I \in R^{v\times v}$ is the identity matrix, $v^2 = n$, and

$$S = \mathrm{tridiag}(-1, 4, -1) = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 4 \end{pmatrix} \in R^{v\times v} \tag{29}$$

is a tridiagonal matrix. Table 3 outlines the results.

From Table 3, we can notice that the MGS method has higher accuracy for different values of n. The MGS method needs less time and a number of iterations as compared with other methods. Furthermore, we observe that the MFP method converges to the solution rapidly as compared with the SORLaopt method, SORLo method, and GGS method. However, the time of the MFP approach is better than the SORLopt strategy. Furthermore, we take $n = 64$ and $n = 1024$ for Example 2 and compare all approaches in a graph form. The graphical outcomes are shown in Figure 2.

The graphical forms in Figures 1 and 2 show the implementation of the offered methods. Graphically, representation displays that the convergence of the presented strategies is faster than other methods.

5. Conclusion

We examined two new iteration approaches for determining AVEs called the MGS method as well as the MFP method, and we discussed the necessary circumstances for convergence between both methods. To ensure the efficacy of the defined approaches, numerical examples have also been conducted. According to the theoretical analyses and numerical investigations, the presented strategies seem to be vowing to determine the AVEs.

Data Availability

The data utilized to support the findings of this analysis are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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