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Research Article

The Classification of the Exact Single Travelling Wave Solutions to the Constant Coefficient KP-mKP Equation Employing Complete Discrimination System for Polynomial Method

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The purpose of this article is to explore different types of solutions for the Kadomtsev-Petviashvili-modified Kadomtsev-Petviashvili (KP-mKP) equation which is termed as KP-Gardner equation, extensively used to model strong nonlinear internal waves in $(1+2)$-dimensions on the stratified ocean shelf. This evolution equation is also used to describe weakly nonlinear shallow-water wave and dispersive interracial waves traveling in a mildly rotating channel with slowly varying topography. Introducing Liu's approach regarding the complete discrimination system for polynomial and the trial equation technique, a set of new solutions to the KP-mKP equation containing Jacobi elliptic function have been derived. It is found that these analytical solutions numerically exhibit different nonlinear structures such as solitary waves, shock waves, and periodic wave profiles. The reliability and effectiveness are confirmed from the numerical graphs of the solutions. Finally, the existence and validity of the various topological structures of the solutions are confirmed from the phase portrait of the dynamical system. Based on this investigation, it is confirmed that the method is not only suited for obtaining the classification of the solutions but also for qualitative analysis, which means that it can also be extended to other fields of application.

1. Introduction

Nonlinear evolution equations (NLEEs) are the real treasure of the modern scientific world because various complex physical phenomena that appeared in the natural system are well described by NLEEs and so, these evolutions are applied to almost all branches of science such as physics, chemistry, biology, astronomy, plasma dynamics, water-wave phenomena, and ocean engineering [1–6]. Among the various NLEEs the Korteweg de Vries (KdV) is the basic and most popular equation discovered by Diederik Johannes Korteweg and his pupil Gustav De Vries to describe shallow-water waves. It is also found that weakly nonlinear KdV-like theories play a crucial role in describing many important features of unsteady internal waves in shallow water as well as in ocean water. Some ocean wave investigations especially, the Coastal Ocean Probe Experiment during 1995 in the Oregon Bay [7] shows that although the KdV framework is well approved for a wide range of parameters there is some parametric domain where the KdV model miserably fails. Actually, if symmetrical stratification appears then the coefficients of the nonlinear term in the KdV equation incorporated in long internal solitary wave modeling vanishes. Thus, the extension of small quadratic approximation of nonlinearity in the KdV model to higher-order nonlinearity by incorporating cubic nonlinear terms becomes important in many applications. For the first time, Miura addressed the Gardner equation about a century ago by expanding the KdV equation [8, 9]. The Gardner equation adopts the same type of behaviors as the standard KdV equation; however, the former claim the validity to the wider parametric domain for internal wave motion in a particular environment. The extension of the parametric domain for modeling of internal wave motion is found in [10–15]. But the KdV as well as the Gardner model can be used to study the theory of soliton in one dimension only. To overcome the restriction for studying wave dynamics in absolutely one-dimensional ZK and KP model arise. So, to study soliton theory in a two-dimension system KP equation is a widely used model [16–18]. However, as with the KdV model, there are situations in which the nonlinear coefficient of the KP equation disappears, at which point it will result in a singularity of infinite amplitude, which is unrealistic. Soliton in finite-amplitude requires strong nonlinearity, which is achieved by incorporating dual nonlinearities into the KP model. The KP-mKP equation is developed to provide soliton in finite-amplitude. In this article, we intend to study the KP-mKP equation in the following form:

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial \tau} + Pv\frac{\partial v}{\partial x} + Qv^2\frac{\partial v}{\partial x} + R\frac{\partial^3 v}{\partial x^3}\right] + S\frac{\partial^2 v}{\partial y^2} = 0. \quad (1)$$

This equation contains quadratic and cubic nonlinear terms along with a third-order dispersive term. Different types of complex physical phenomena in a diverse field, such as strong nonlinear internal waves on the ocean shelf in two dimension [19] and propagation of dust acoustic waves in plasma environment [20], are well described by the KP-mKP model.

Aslanova et al. [21] studied the propagating characteristic of dispersive shock waves through the cylindrical Gardner equation, which is derived from the $(2+1)$-dimensional KP-mKP equation by using a similarity reduction transformation. Boateng et al. [22] derived some trigonometric and hyperbolic trigonometric analytical solution of $(2+1)$-dimensional KP-mKP equation employing the modified extended direct algebraic method. Shakeel and Mohyud-Din [23] had constructed hyperbolic, trigonometric, and rational functions from the KP-mKP equation using the $(G'/G, 1/G)$ method and found that some of the results in their investigation become identical with the results published earlier when some parameters take certain values. Jawad et al. [19] obtained several forms of the solution, such as soliton solutions and hyperbolic solutions, etc. to the KP-mKP equation by employing improved (G'/G) expansion method considering the tanh-coth hypothesis. They also reported the constraint conditions for the existence of the solutions. Liu et al. [24] analyzed the phase portrait of the KP-mKP model utilizing the bifurcation theory of dynamical systems, and a class of exact traveling wave solutions such as solitary solution, periodic solution, kink (antikink) solution, and breaking wave solution, are derived for the said equation. Wazwaz obtained multiple singular solutions and multisolitary solutions for the KP-mKP equation utilizing Hirota's bilinear approach and exhibited the variety of the solutions from a numerical standpoint [25].

The investigations of exact and approximate solutions of nonlinear evolution equation is an important research topic in nonlinear science because solutions not only determines the behavior of an equation but also helps to understand the underlying nonlinear phenomena properly, where the equation use as a model. To achieve this goal, some excellent numerical investigations on the study of NLEEs are found in [26–30]. Several physicists and applied mathematician discovered many analytical techniques such as inverse scattering method [31], Jacobian elliptic function method [32, 33], bilinear transformation method [34, 35], tanh method [36], extended tanh method [37, 38], Adomian decomposition method [39–41], Reduction perturbation method [42, 43], homotopy perturbation method [44, 45], sine-cosine method [46, 47], variational iteration method [48, 49], homogeneous balance method [50, 51], multiple exp-function method [52, 53], and Fan's algebraic method [54, 55]. In order to explore all the exact travelling wave solutions for a nonlinear system, Liu introduced a new approach which is termed as complete discrimination system for polynomial method (CDSPM) [56, 57]. It is found that if a NLEE can be turned into an integral form then all possible exact solutions can be derived by this CDSPM [58]. Recently, this method has been used successively for solving many NLEEs [59–62]. Fan et al. applied this method to find all possible exact travelling wave solutions to the generalized Pochhammer-Chree equation [63]. Cao et al. employed this CDSPM to find all exact travelling wave solutions to the variable coefficient Gardner equation [64]. The tanh-coth method is used to acquire solitary and shock solutions to several nonlinear evolution equations by Wazwaz et al. in [5]. By using inverse scattering and the Hopscotch method, Hirota was able to arrive at analytical and numerical solutions to the KdV equation [4]. Based on a Weiss-Tabor-Carnevale approach, Kudryashov constructs solitary, shock, and Jacobi elliptic function solutions to the Kuramoto-Sivashinsky equations [3]. A detailed study of solitary and shock wave solutions of various nonlinear evolution equations has been conducted using Backlund transformation and inverse scaling inin [1, 2]. Compared to other existing techniques, the highest advantage of Liu's approach is that the original equation can be transformed into an integral form, from which all single traveling wave solutions may be derived, including the shock solution, solitary solution, and periodic solution containing the Jacobian elliptic function solution, that is very hard to be acquired by other technique.

In this article, the CDSPM is applied to the KP-mKP equation and the exact solutions are obtained. Initially, the KP-mKP equation was reduced to an ODE by adopting a traveling wave transformation. Further, we have employed the change of the variable and introduced CDSPM to obtain the corresponding integrals. Thus, we get the classification of all single traveling wave solutions to the KP-mKP equation. As a general rule, this technique can be applied only to determining exact solutions; however, it can also be applied to qualitative analysis of solutions. Additionally, the paper presents dynamic results, including bifurcation points and critical conditions. The remaining part of this article is constructed as follows. A brief introduction on the CDSPM is presented in Section 2. In Section 3, the original KP-mKP equation is converted into the ordinary differential equation and then solve it using the idea of the CDSPM. In Section 4, the KP-mKP equation is given as a concrete example to further display the powerfulness of this method in qualitative and quantitative analyses, especially classifying the equilibrium points and showing the bifurcation phenomena. Finally, some conclusions are drawn in Section 5.

2. Discrimination System

We consider a general nonlinear partial differential equation with the unknown $v = v(x, y, \tau)$ as

$$\mathcal{N}\left(v, v_\tau, v_x, v_{xx}, v_{\tau x}, v_{xxx}, v_{yy} \cdots\right) = 0. \tag{2}$$

Now, combining the real variables x, y, and τ, we introduce a new variable ζ such that

$$v(x, y, \tau) = \phi(\zeta), \zeta = k_1 x + k_2 y - c\tau, \tag{3}$$

where k_1, k_2 are constant and c stands for expressing the speed of the traveling wave and Equation (2) is transformed into an ordinary differential equation (ODE) as

$$\mathcal{M}\left(\phi, \phi', \phi'', \phi''', \cdots\right) = 0, \tag{4}$$

where $\mathcal{M}$ is a polynomial in ϕ and its derivatives and the symbol $(')$ denotes derivative with respect to ζ. After integrating (4), we can express

$$\left(\phi'\right)^2 = G(\phi), \tag{5}$$

where $G(\phi)$ may be polynomial or other kind of rational or irrational function. Then, we can write (5) into the integral form as

$$\pm(\zeta - \zeta_0) = \int \frac{d\phi}{\sqrt{G(\phi)}}, \tag{6}$$

where ζ_0 is an integral constant. Several significant results are achieved by the above described procedure.

3. All Travelling Wave Solutions to KP-mKP Equation

In this section, we investigate all travelling wave solutions of constant coefficient KP-mKP Equation (1). Substituting the transform (3) in Equation (1), we have

$$\begin{aligned} -ck_1\phi'' + Pk_1^2\left(\left(\phi'\right)^2 + \phi\phi''\right) + Qk_1^2\left(\phi^2\phi'' + 2\phi\left(\phi'\right)^2\right) \\ + Rk_1^4\phi'''' + Sk_2^2\phi'' = 0, \end{aligned} \tag{7}$$

integrating and taking integrating constant to be zero, we have

$$-ck_1\phi' + Pk_1^2\phi\phi' + Qk_1^2\phi^2\phi' + Rk_1^4\phi''' + Sk_2^2\phi' = 0, \tag{8}$$

again integrating, we have

$$-ck_1\phi + \frac{1}{2}Pk_1^2\phi^2 + \frac{1}{3}Qk_1^2\phi^3 + Rk_1^4\phi'' + Sk_2^2\phi = \frac{1}{2}c_1, \tag{9}$$

where c_1 is an integrating constant. Multiplying both sides by $2\phi'$ and then integrating, we have

$$-ck_1\phi^2 + \frac{1}{3}Pk_1^2\phi^3 + \frac{1}{6}Qk_1^2\phi^4 + Rk_1^4\phi'^2 + Sk_2^2\phi^2 = c_1\phi + c_2, \tag{10}$$

where c_2 is an integrating constant. The above equation can be written as

$$\left(\phi'\right)^2 = \frac{1}{Rk_1^4}\left[-\frac{1}{6}Qk_1^2\phi^4 - \frac{1}{3}Pk_1^2\phi^3 + \left(ck_1 - Sk_2^2\right)\phi^2 + c_1\phi + c_2\right], \tag{11}$$

$$\left(\phi'\right)^2 = \alpha_4\phi^4 + \alpha_3\phi^3 + \alpha_2\phi^2 + \alpha_1\phi + \alpha_0, \tag{12}$$

$$\Rightarrow$$

$$\pm(\zeta - \zeta_0) = \int \frac{d\phi}{\sqrt{\alpha_4\phi^4 + \alpha_3\phi^3 + \alpha_2\phi^2 + \alpha_1\phi + \alpha_0}}, \tag{13}$$

where $\alpha_4 = -Q/6Rk_1^2$, $\alpha_3 = -P/3Rk_1^2$, $\alpha_2 = (ck_1 - Sk_2^2)/Rk_1^4$, $\alpha_1 = c_1/Rk_1^4$, and $\alpha_0 = c_2/Rk_1^4$.

For $\alpha_4 > 0$, let $\Psi = (\alpha_4)^{1/4}(\phi + (\alpha_3/4\alpha_4))$ and $\zeta_1 = (\alpha_4)^{1/4}\zeta$, then (12) changes to

$$\Psi_{\zeta_1}^2 = \Psi^4 + p\Psi^2 + q\Psi + r, \tag{14}$$

and (13) becomes

$$\pm(\zeta_1 - \zeta_0) = \int \frac{d\Psi}{\sqrt{\Psi^4 + p\Psi^2 + q\Psi + r}}, \tag{15}$$

where $p = -(3\alpha_3^2/8\alpha_4\sqrt{\alpha_4}) + (\alpha_2/\sqrt{\alpha_4})$, $q = (\alpha_3^4/8\alpha_4^2\sqrt[4]{\alpha_4}) - (\alpha_3\alpha_2/2\alpha_4\sqrt[4]{\alpha_4}) + (\alpha_1/\sqrt[4]{\alpha_4})$, and $r = \alpha_0 - (3\alpha_3^4/256\alpha_4^3) + (\alpha_3^2\alpha_2/16\alpha_4^2) - (3\alpha_3\alpha_1/4\alpha_4)$.

For $\alpha_4 < 0$, let $\Psi = (-\alpha_4)^{1/4}(\phi + (\alpha_3/4\alpha_4))$ and $\zeta_1 = (-\alpha_4)^{1/4}\zeta$, then (12) changes to

$$\Psi_{\zeta_1}^2 = -\left(\Psi^4 + p\Psi^2 + q\Psi + r\right), \tag{16}$$

and (13) becomes

$$\pm(\zeta_1 - \zeta_0) = \int \frac{d\Psi}{\sqrt{-(\Psi^4 + p\Psi^2 + q\Psi + r)}}, \tag{17}$$

where $p = (3\alpha_3^2/8\alpha_4\sqrt{-\alpha_4}) - (\alpha_2/\sqrt{-\alpha_4})$, $q = -(\alpha_3^4/8\alpha_4^2\sqrt[4]{-\alpha_4}) + (\alpha_3\alpha_2/2\alpha_4\sqrt[4]{-\alpha_4}) - (\alpha_1/\sqrt[4]{-\alpha_4})$, and $r = -\alpha_0 + (3\alpha_3^4/256\alpha_4^3) - (\alpha_3^2\alpha_2/16\alpha_4^2) + (3\alpha_3\alpha_1/4\alpha_4)$.

Let $H(\Psi)=\Psi^4+p\Psi^2+q\Psi+r$, then its complete discrimination system can be expressed as [65]

$$\begin{gathered}D_1=4, D_2=-p, D_3=-2p^3+8pr-9q^2, D_4=-p^3q^2+4p^4r+36pq^2r-32p^2r^2-\frac{27}{4}q^4+64r^3,\\ E_2=9p^2-32pr.\end{gathered} \tag{18}$$

Again, to make the study effective and reliable, it is very necessary to find the stable and unstable regions of the solution for different values of discriminant quantities of the polynomial $H(\Psi)$. The stable and unstable parametric zones of the system are given in Table 1 [66].

According to the complete discrimination system for the polynomial of order four has total nine cases and to obtain solution of (17) and (13), we discussed all the cases separately as follows:

Case1. When $D_4=0$, $D_3=0$, and $D_2=0$, $H(\Psi)$ has only one root zero of multiplicity four. Then, $H(\Psi)$ becomes

$$H(\Psi)=\Psi^4, \tag{19}$$

for $\alpha_4>0$, from (15), we have

$$\zeta_1-\zeta_0=\int\frac{d\Psi}{\Psi^2}=-\Psi^{-1}, \tag{20}$$

where ζ_0 is an integral constant. So, the solutions of Equation (12) are of the form

$$\phi(\zeta)=\mp\alpha_4^{-1/4}\left(\alpha_4^{1/4}\zeta-\zeta_0\right)^{-1}-\frac{\alpha_3}{4\alpha_4}, \tag{21}$$

which is a rational function solution. For example, when $P=3, Q=6, R=-1, S=11/8, c_1=0, c_2=0, k_1=1, k_2=1, c=1$, and $\zeta_0=0$, then, we get rational function solution of (1) as (see Figure 1(a))

$$v(x,y,\tau)=-(x+y-\tau)^{-1}-\frac{1}{4}. \tag{22}$$

Case2. When $D_4=0$, $D_3=0$, $D_2>0$, and $E_2=0$, $H(\Psi)$ has two real roots of multiplicities three and one. Then, $H(\Psi)$ can be written in the following form as

$$H(\Psi)=(\Psi-r_1)^3(\Psi-r_2), \tag{23}$$

therefore, when $\alpha_4>0$ from (15), we have

$$\pm\zeta_1-\zeta_0=\int\frac{d\Psi}{(\Psi-r_1)\sqrt{(\Psi-r_1)(\Psi-r_2)}}=\frac{2}{r_2-r_1}\sqrt{\frac{\Psi-r_2}{\Psi-r_1}}, \tag{24}$$

when $\Psi>r_1, \Psi>r_2$ or $\Psi<r_1, \Psi<r_2$, the solution of (15) is of the form

$$\begin{gathered}\Psi=\frac{4(r_1-r_2)}{(r_2-r_1)^2(\zeta_1-\zeta_0)^2-4}+r_1,\\ \phi(\zeta)=\pm\alpha_4^{-1/4}\left[\frac{4(r_1-r_2)}{(r_2-r_1)^2\left(\alpha_4^{1/4}\zeta-\zeta_0\right)^2-4}+r_1\right]-\frac{\alpha_3}{4\alpha_4}.\end{gathered} \tag{25}$$

When $P=12, Q=6, R=-1, S=1, c_1=24, c_2=72, k_1=1, k_2=1, c=1$, and $\zeta_0=0$, then $r_1=1$ and $r_2=-3$, we get rational function solution of (1) as (see Figure 1(b))

$$v(x,y,\tau)=\frac{4}{4(x+y-\tau)^2-1}. \tag{26}$$

When $\alpha_4<0$ from (17), we have

$$\pm\zeta_1-\zeta_0=\int\frac{d\Psi}{(\Psi-r_1)\sqrt{(\Psi-r_1)(\Psi-r_2)}}=\frac{2}{r_1-r_2}\sqrt{\frac{r_2-\Psi}{\Psi-r_1}}. \tag{27}$$

When $\Psi>r_1, \Psi<r_2$ or $\Psi<r_1, \Psi>r_2$, the solution of (17) is of the form

$$\begin{gathered}\Psi=\frac{4(r_1-r_2)}{-(r_2-r_1)^2(\zeta_1-\zeta_0)^2-4}+r_1,\\ \phi(\zeta)=\pm(-\alpha_4)^{-\frac{1}{4}}\left[\frac{4(r_1-r_2)}{-(r_2-r_1)^2\left((-\alpha_4)^{1/4}\zeta-\zeta_0\right)^2-4}+r_1\right]-\frac{\alpha_3}{4\alpha_4}.\end{gathered} \tag{28}$$

Case3. When $D_4=0$, $D_3=0$, and $D_2<0$, $H(\Psi)$ has a pair of complex conjugate roots of multiplicities two. Then, $H(\Psi)$ can be expressed as in the following form as

$$H(\Psi)=\left((\Psi-\gamma)^2+\delta^2\right)^2, \tag{29}$$

where $\delta>0$, if $\alpha_4>0$ then from (15), we obtain

$$\zeta_1-\zeta_0=\int\frac{d\Psi}{(\Psi-\gamma)^2+\delta^2}=\frac{1}{\delta}\arctan\frac{\Psi-\gamma}{\delta}, \tag{30}$$

then, we get

$$\Psi=\delta\tan\left(\delta(\zeta_1-\zeta_0)\right)+\gamma. \tag{31}$$

Table 1: Stable and unstable region.

Stable region	Relative unstable region	Absolute unstable region
$D_i < 0 \cup D_i > 0 \cup E_2 > 0$	$D_i = 0 \cup E_2 = 0 \setminus (0,0,0)$	(0,0,0)
$i = 1,2,3,4$	$i = 1,2,3,4$	$i = 1,2,3,4$

We obtain solution as

$$\phi(\zeta) = \pm\alpha_4^{-1/4}\delta \tan\left(\delta\left(\alpha_4^{1/4}\zeta - \zeta_0\right)\right) + \gamma - \frac{\alpha_3}{4\alpha_4}. \tag{32}$$

When $P = 12, Q = 6, R = -1, S = 5, c_1 = 4, c_2 = 7, k_1 = 1, k_2 = 1$, $c = 1$, and $\zeta_0 = 0$, then $\gamma = 0$ and $\delta = 2$, we get solution of original Equation (1) as (see Figure 1(c))

$$v(x, y, \tau) = 2\tan\left(2(x + y - \tau)\right) - 1. \tag{33}$$

Case4. When $D_4 > 0$, $D_3 > 0$, and $D_2 > 0$, then $H(\Psi)$ has four distinct real roots. In this case, we write

$$H(\Psi) = (\Psi - r_1)(\Psi - r_2)(\Psi - r_3)(\Psi - r_4), \tag{34}$$

where r_1, r_2, r_3, and r_4 are all real numbers and let $r_1 > r_2 > r_3 > r_4$. When $a_4 > 0$, if $\Psi > r_1$ or $\Psi < r_4$, then we take the following transformation:

$$\Psi = \frac{r_2(r_1 - r_4)\sin^2\theta - r_1(r_2 - r_4)}{(r_1 - r_4)\sin^2\theta - (r_2 - r_4)}, \tag{35}$$

if $r_3 < \Psi < r_2$, then we take the following transformation:

$$\Psi = \frac{r_4(r_2 - r_3)\sin^2\theta - r_3(r_2 - r_4)}{(r_2 - r_3)\sin^2\theta - (r_2 - r_4)}. \tag{36}$$

Combining (35) or (36) with (15), we get

$$\zeta_1 - \zeta_0 = \int \frac{d\Psi}{\sqrt{(\Psi - r_1)(\Psi - r_2)(\Psi - r_3)(\Psi - r_4)}} = \frac{2}{(r_1 - r_3)(r_2 - r_4)}\int \frac{d\theta}{\sqrt{1 - m^2\sin^2\theta}}, \tag{37}$$

where $m^2 = (r_1 - r_4)(r_2 - r_3)/(r_1 - r_3)(r_2 - r_4)$, also from the definition of jacobi elliptic sine function, we get

$$\sin\theta = sn\left(\frac{\sqrt{(r_1 - r_3)(r_2 - r_4)}}{2}(\zeta_1 - \zeta_0), m\right). \tag{38}$$

Combining (38) with (35), we obtain solution of (15) as

$$\Psi = \frac{r_2(r_1 - r_4)sn^2\left(\left(\sqrt{(r_1 - r_3)(r_2 - r_4)}/2\right)(\zeta_1 - \zeta_0), m\right) - r_1(r_2 - r_4)}{(r_1 - r_4)sn^2\left(\left(\sqrt{(r_1 - r_3)(r_2 - r_4)}/2\right)(\zeta_1 - \zeta_0), m\right) - (r_2 - r_4)}, \tag{39}$$

and we can get elliptic function double solutions of Equation (12) as

$$\phi(\zeta) = \frac{\alpha_4^{-1/4}\left[r_2(r_1 - r_4)sn^2\left(\left(\sqrt{(r_1 - r_3)(r_2 - r_4)}/2\right)\left(\alpha_4^{1/4}\zeta - \zeta_0\right), m\right) - r_1(r_2 - r_4)\right]}{(r_1 - r_4)sn^2\left(\left(\sqrt{(r_1 - r_3)(r_2 - r_4)}/2\right)\left(\alpha_4^{1/4}\zeta - \zeta_0\right), m\right) - (r_2 - r_4)} - \frac{\alpha_3}{4\alpha_4}. \tag{40}$$

Combining (38) with (37), we obtain solution of (15) as

$$\Psi = \frac{r_4(r_2 - r_3)sn^2\left(\left(\sqrt{(r_1 - r_3)(r_2 - r_4)}/2\right)(\zeta_1 - \zeta_0), m\right) - r_3(r_2 - r_4)}{(r_2 - r_3)sn^2\left(\left(\sqrt{(r_1 - r_3)(r_2 - r_4)}/2\right)(\zeta_1 - \zeta_0), m\right) - (r_2 - r_4)}, \tag{41}$$

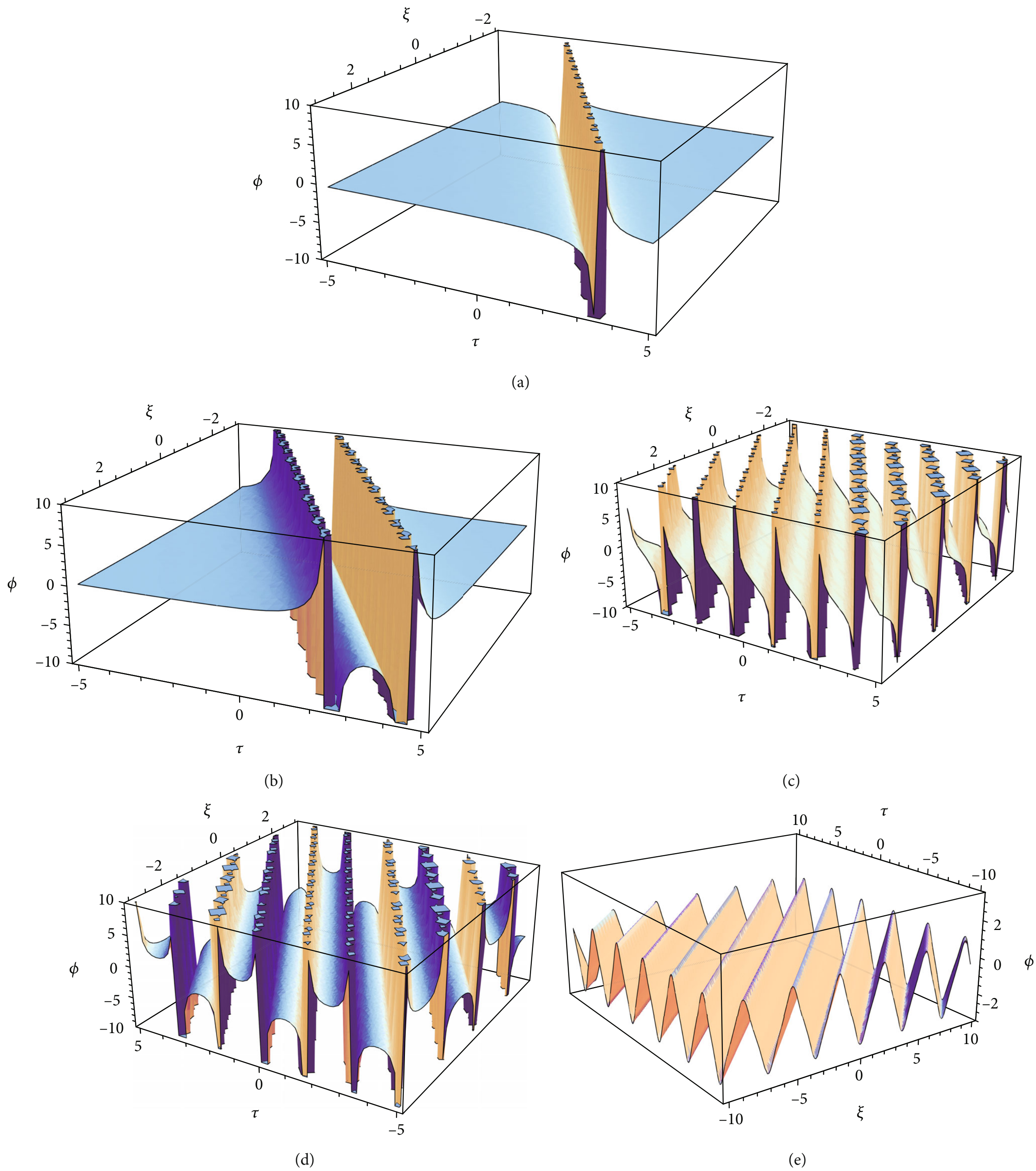

Figure 1: Continued.

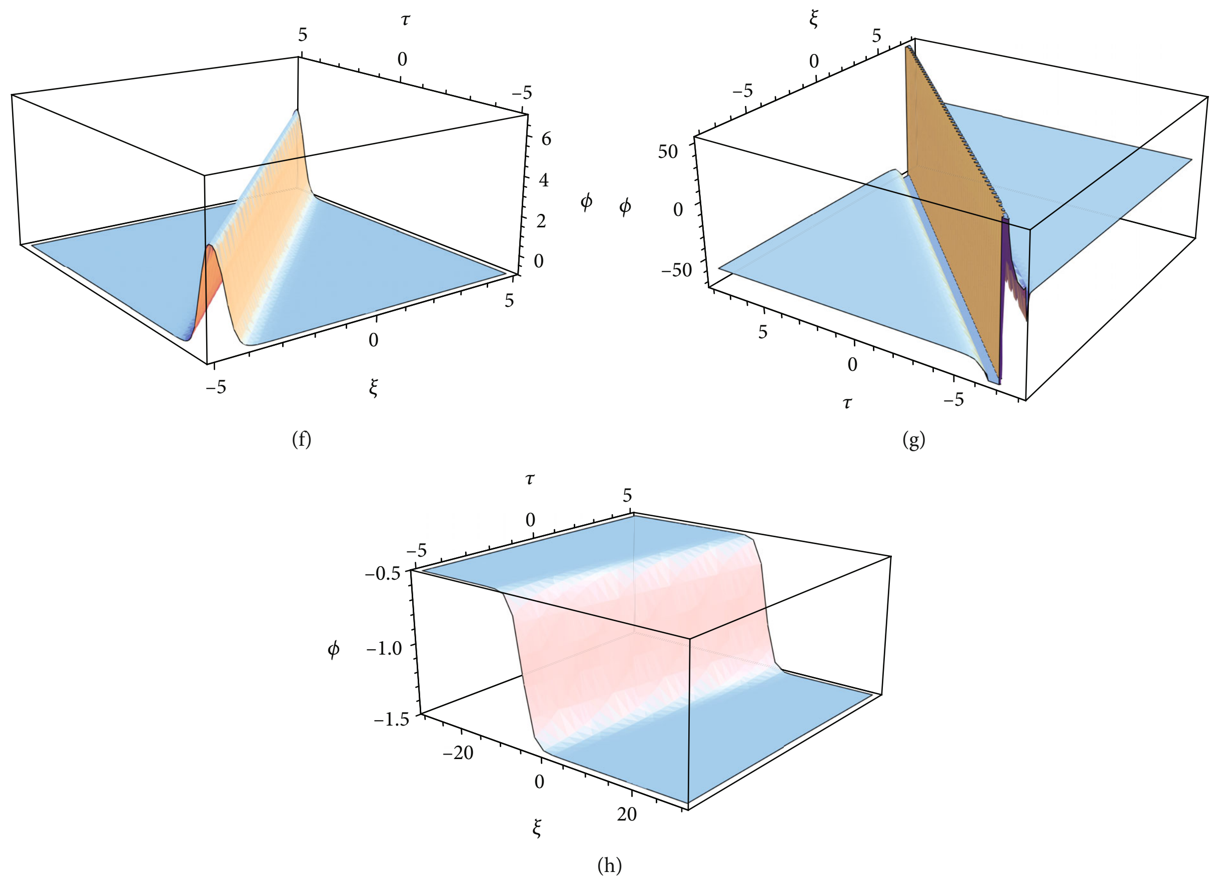

Figure 1: (a) 3D plot of (22), when $P = 3$, $Q = 6$, $R = -1$, $S = 11/8$, $c_1 = 0$, $c_2 = 0$, $k_1 = 1$, $k_2 = 1$, $c = 1$, and $\zeta_0 = 0$. (b) 3D plot of (26), when $P = 12$, $Q = 6$, $R = -1$, $S = 1$, $c_1 = 4$, $c_2 = 7$, $k_1 = 1$, $k_2 = 1$, $c = 1$, and $\zeta_0 = 0$. (c) 3D plot of (33) when $P = 12$, $Q = 6$, $R = -1$, $S = 5$, $c_1 = 24$, $c_2 = 72$, $k_1 = 1$, $k_2 = 1$, $c = 1$, and $\zeta_0 = 0$. (d) 3D plot of (43), when $P = 0$, $Q = 6$, $R = -1$, $S = 4$, $c_1 = -10$, $c_2 = -34$, $k_1 = 1$, $k_2 = 1$, $c = 1$, and $\zeta_0 = 0$. (e) 3D plot of (63), when $P = -12$, $Q = -6$, $R = 1$, $S = -10$, $c_1 = 64$, $c_2 = 104$, $k_1 = 1$, $k_2 = 1$, $c = 1$, $\zeta_0 = 0$. (f) 3D plot of (69), when $P = 12$, $Q = 6$, $R = -1$, $S = -12$, $c_1 = 64$, $c_2 = 104$, $k_1 = 1$, $k_2 = 1$, $c = 1$, and $\zeta_0 = 0$. (g) 3D plot of (75) when $P = 12$, $Q = 6$, $R = -1$, $S = 9$, $c_1 = 24$, $c_2 = 72$, $k_1 = 1$, $k_2 = 1$, $c = 1$, and $\zeta_0 = 0$. (h) 3D plot of (82) when $P = 12$, $Q = 6$, $R = -1$, $S = 13/2$, $c_1 = 21$, $c_2 = 1047/16$, $k_1 = 1$, $k_2 = 1$, $c = 1$, and $\zeta_0 = 0$.

and solutions of (12) as

$$\phi(\zeta) = \frac{\alpha_4^{-1/4}\left[r_4(r_2-r_3)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)\left(\alpha_4^{1/4}\zeta-\zeta_0\right), m\right) - r_3(r_2-r_4)\right]}{(r_2-r_3)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)\left(\alpha_4^{1/4}\zeta-\zeta_0\right), m\right) - (r_2-r_4)} - \frac{\alpha_3}{4\alpha_4}. \tag{42}$$

The expressions (40) and (42) are elliptic functions double periodic solutions. For instance, when $P = 0$, $Q = 6$, $R = -1$, $S = 4$, $c_1 = 0$, $c_2 = -4$, $k_1 = 1$, $k_2 = 1$, $c = 1$, and $\zeta_0 = 0$, then $r_1 = 2$, $r_2 = 1$, $r_3 = -1$, and $r_4 = -2$. So if $\Psi > r_1$ or $\Psi < r_4$, we get the elliptic function solution of (1) as (see Figure 1(d))

$$v(x, y, \tau) = \frac{4sn^2((3/2)(x+y-\tau), 8/9) - 6}{4sn^2(3/2(x+y-\tau), 8/9) - 3}. \tag{43}$$

For $\alpha_4 < 0$, if $r_1 > \Psi > r_2$, then we consider the following transformation:

$$\Psi = \frac{r_3(r_1-r_2)\sin^2\theta - r_2(r_1-r_3)}{(r_1-r_2)\sin^2\theta - (r_1-r_3)}, \tag{44}$$

and if $r_4 < \Psi < r_3$, then we consider the following transformation:

$$\Psi = \frac{r_1(r_3-r_4)\sin^2\theta - r_4(r_3-r_1)}{(r_3-r_4)\sin^2\theta - (r_3-r_1)}. \tag{45}$$

Similarly from (17), we have

$$\begin{aligned}
\Psi &= \frac{r_3(r_1-r_2)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)(\zeta_1-\zeta_0),\mathrm{m}\right)-r_2(r_1-r_3)}{(r_1-r_2)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)(\zeta_1-\zeta_0),\mathrm{m}\right)-(r_1-r_3)},\\
\phi(\zeta) &= \frac{(-\alpha_4)^{-1/4}\left[r_3(r_1-r_2)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)\left((-\alpha_4)^{1/4}\zeta-\zeta_0\right),m\right)-r_2(r_1-r_3)\right]}{(r_1-r_2)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)\left((-\alpha_4)^{1/4}\zeta-\zeta_0\right),m\right)-(r_1-r_3)}-\frac{\alpha_3}{4\alpha_4},\\
\Psi &= \frac{r_1(r_3-r_4)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)(\zeta_1-\zeta_0),m\right)-r_4(r_3-r_1)}{(r_3-r_4)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)(\zeta_1-\zeta_0),m\right)-(r_3-r_1)},\\
\phi(\zeta) &= \frac{(-\alpha_4)^{-1/4}\left[r_1(r_3-r_4)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)\left((-\alpha_4)^{1/4}\zeta-\zeta_0\right),m\right)-r_4(r_3-r_1)\right]}{(r_3-r_4)sn^2\left(\left(\sqrt{(r_1-r_3)(r_2-r_4)}/2\right)\left((-\alpha_4)^{1/4}\zeta-\zeta_0\right),m\right)-(r_3-r_1)}-\frac{\alpha_3}{4\alpha_4},
\end{aligned} \tag{46}$$

where $m^2 = (r_1-r_2)(r_3-r_4)/(r_1-r_3)(r_2-r_4)$.

Case5. When $D_4 < 0$ and $D_2D_3 \geq 0$, $H(\Psi)$ has ca pair of complex conjugate roots and two distinct real roots. Then, $H(\Psi)$ can be presented as

$$H(\Psi) = (\Psi-r_1)(\Psi-r_2)\left[(\Psi-\gamma)^2+\delta^2\right], \tag{47}$$

where r_1, r_2, γ and δ are numbers also $r_1 > r_2$ and $\delta > 0$. We consider the following transformation:

$$\Psi = \frac{e_1\cos\theta+e_2}{e_3\cos\theta+e_4}, \tag{48}$$

where

$$\begin{aligned}
e_1 &= \frac{1}{2}(r_1+r_2)e_3-\frac{1}{2}(r_1-r_2)e_4,\\
e_2 &= \frac{1}{2}(r_1+r_2)e_4-\frac{1}{2}(r_1-r_2)e_3,\\
e_3 &= r_1-\gamma-\frac{\delta}{f},\\
e_4 &= r_1-\gamma-\delta f,\\
f &= g\pm\sqrt{g^2+1},\\
g &= \frac{\delta^2+(r_1-\gamma)(r_2-\gamma)}{\delta(r_1-r_2)}.
\end{aligned} \tag{49}$$

Using (15) and the transformation (48), we get

$$\begin{aligned}
\zeta_1-\zeta_0 &= \int\frac{d\Psi}{\sqrt{\pm(\Psi-r_1)(\Psi-r_2)\left[(\Psi-\gamma)^2+\delta^2\right]}}\\
&= \frac{2fm}{\sqrt{\mp 2f\delta(r_1-r_2)}}\int\frac{d\theta}{\sqrt{1-m^2\sin^2\theta}},
\end{aligned} \tag{50}$$

where $m^2 = 1/(1+f^2)$. Using (50) and the Jacobi elliptic cosine function, we obtain

$$\cos\theta = cn\left(\frac{\sqrt{\mp 2f\delta(r_1-r_2)}}{2fm}(\zeta_1-\zeta_0),m\right). \tag{51}$$

Now, combining (50) and (50), we gain solution of (15) as

$$\Psi = \frac{e_1cn\left(\left(\sqrt{\mp 2f\delta(r_1-r_2)}/2fm\right)(\zeta_1-\zeta_0),m\right)+e_2}{e_3cn\left(\left(\sqrt{\mp 2f\delta(r_1-r_2)}/2fm\right)(\zeta_1-\zeta_0),m\right)+e_4}. \tag{52}$$

Therefore, the solution of (12) is

$$\phi(\zeta) = \frac{\alpha_4^{-1/4}\left[e_1cn\left(\left(\sqrt{\mp 2f\delta(r_1-r_2)}/2fm\right)\left(\alpha_4^{1/4}\zeta-\zeta_0\right),m\right)+e_2\right]}{e_3cn\left(\left(\sqrt{\mp 2f\delta(r_1-r_2)}/2fm\right)\left(a_4^{1/4}\zeta-\zeta_0\right),m\right)+e_4}-\frac{\alpha_3}{4\alpha_4}, \tag{53}$$

which is a Jacobi elliptic function double periodic solution. Particularly, When $P=-24, Q=-12, R=2, S=-5, c_1=52, c_2=148, k_1=1, k_2=1, c=1,$ and $\zeta_0=0$, then $r_1=2, r_2=-2, \gamma=0, \delta=1, f=1/2, e_1=-3, e_2=0, e_3=0, e_4=3/2$, and $m=4/5$, then we obtain jacobi elliptic function solution of (1) as

$$v(x,y,\tau) = -2cn\left(\frac{5}{2}(x+y-\tau),\frac{4}{5}\right)-1. \tag{54}$$

Case6. When $D_4 > 0$ and $D_2D_3 \leq 0$, then $H(\Psi)$ has two pairs of complex conjugate roots and this case we write $H(\Psi)$ as

$$H(\Psi) = \left[(\Psi-\gamma_1)^2+\delta_1^2\right]\left[(\Psi-\gamma_2)^2+\delta_2^2\right], \tag{55}$$

where $\gamma_1, \gamma_2, \delta_1$, and δ_2 are real numbers and $\delta_1 \geq \delta_2 > 0$.

For $a_4 > 0$, we take the following transformation:

$$\Psi = \frac{e_1 \tan\theta + e_2}{e_3 \tan\theta + e_4}, \tag{56}$$

where

$$\begin{aligned} e_1 &= \gamma_1 e_3 + \delta_1 e_4, \\ e_2 &= \gamma_1 e_4 - \delta_1 e_3, \\ e_3 &= -\delta_1 - \frac{\delta_2}{f}, \\ e_4 &= \gamma_1 - \gamma_2, \\ f &= g + \sqrt{g^2 - 1}, \\ g &= \frac{(\gamma_1 - \gamma_2)^2 + \delta_1^2 + \delta_2^2}{2\delta_1\delta_2}. \end{aligned} \tag{57}$$

Then, from (15), we get

$$\begin{aligned} \zeta_1 - \zeta_0 &= \int \frac{d\Psi}{\sqrt{\left((\Psi - \gamma_1)^2 + \delta_1^2\right)\left((\Psi - \gamma_2)^2 + \delta_2^2\right)}} \\ &= \frac{e_3^2 + e_4^2}{\delta_2\sqrt{\left(e_3^2 + e_4^2\right)\left(f^2 e_3^2 + e_4^2\right)}} \int \frac{d\theta}{\sqrt{1 - m^2 \sin^2\theta}}, \end{aligned} \tag{58}$$

where $m = (f^2 - 1)/f^2$. By using (58) and the definition of jacobi elliptic functions [67], we get

$$\sin\theta = sn\left(\frac{\delta_2\sqrt{\left(e_3^2 + e_4^2\right)\left(f^2 e_3^2 + e_4^2\right)}}{e_3^2 + e_4^2}(\zeta_1 - \zeta_0), m\right), \tag{59}$$

$$\cos\theta = cn\left(\frac{\delta_2\sqrt{\left(e_3^2 + e_4^2\right)\left(f^2 e_3^2 + e_4^2\right)}}{e_3^2 + e_4^2}(\zeta_1 - \zeta_0), m\right). \tag{60}$$

Combining (59) and (60) with (56), we have elliptic function double periodic solution as

$$\begin{aligned} \Psi &= \frac{e_1 sn(\xi(\zeta_1 - \zeta_0), m) + e_2 cn(\xi(\zeta_1 - \zeta_0), m)}{e_3 sn(\xi(\zeta_1 - \zeta_0), m) + e_4 cn(\xi(\zeta_1 - \zeta_0), m)}, \\ \phi(\zeta) &= \frac{\alpha_4^{-1/4}\left[e_1 sn\left(\xi\left(a_4^{1/4}\zeta - \zeta_0\right), m\right) + e_2 cn\left(\xi\left(\alpha_4^{1/4}\zeta - \zeta_0\right), m\right)\right]}{e_3 sn\left(\xi\left(a_4^{1/4}\zeta - \zeta_0\right), m\right) + e_4 cn\left(\xi\alpha_4^{1/4}\zeta - \zeta_0\right), m\right)} - \frac{\alpha_3}{4\alpha_4}, \end{aligned} \tag{61}$$

where

$$\xi = \frac{\delta_2\sqrt{\left(e_3^2 + e_4^2\right)\left(f^2 e_3^2 + e_4^2\right)}}{e_3^2 + e_4^2}. \tag{62}$$

For example, when $P = -12, Q = -6, R = 1, S = -10, c_1 = -10, c_2 = -34, k_1 = 1, k_2 = 1, c = 1, \text{and}\, \zeta_0 = 0$, then $\gamma_1 = 0, \delta_1 = 2, \gamma_2 = 0, \delta_2 = 1, f = 2, e_1 = 0, e_2 = 5, e_3 = -5/4, e_4 = 0$, and $\zeta = 2$, then we obtain the Jacobi elliptic function solution of (1) as (see Figure 1(e))

$$v(x, y, \tau) = -\frac{2cn(2(x + y - \tau), 3/4])}{sn(2(x + y - \tau), 3/4)} - \frac{1}{4}. \tag{63}$$

Case7. When $D_4 = 0$, $D_3 > 0$, and $D_2 > 0$, $H(\Psi)$ has a real root of multiplicities two and two single real roots. Then, $H(\Psi)$ is of the following form:

$$H(\Psi) = (\Psi - r_1)^2(\Psi - r_2)(\Psi - r_3), \tag{64}$$

where r_1, r_2, and r_3 are real numbers and $r_2 > r_3$, $r_1 = -(r_2 + r_3)/2$. When $\Psi > r_2, r_2 > r_1 > r_3$, we obtain the solution of (15) and (12), respectively, as

$$\Psi = \frac{2(r_1 - r_2)(r_1 - r_3)}{\pm(r_2 - r_3)\sin\left[\sqrt{-(r_1 - r_2)(r_1 - r_3)}(\zeta_1 - \zeta_0)\right] - (2r_1 - r_2 - r_3)}, \tag{65}$$

$$\phi(\zeta) = \frac{2\alpha_4^{-1/4}(r_1 - r_2)(r_1 - r_3)}{\pm(r_2 - r_3)\sin\left[\sqrt{-(r_1 - r_2)(r_1 - r_3)}\left(\alpha_4^{1/4}\zeta - \zeta_0\right)\right] - (2r_1 - r_2 - r_3)} - \frac{\alpha_3}{4\alpha_4}. \tag{66}$$

When $r_1 > r_2$ or $r_1 < r_3$, we obtain the solution of (15) and (12), respectively, as

$$\Psi = \frac{2(r_1 - r_2)(r_1 - r_3)}{(r_2 - r_3)\cosh\left[\sqrt{(r_1 - r_2)(r_1 - r_3)}(\zeta_1 - \zeta_0)\right] - (2r_1 - r_2 - r_3)}, \tag{67}$$

$$\phi(\zeta) = \frac{2\alpha_4^{-1/4}(r_1 - r_2)(r_1 - r_3)}{(r_2 - r_3)\cosh\left[\sqrt{(r_1 - r_2)(r_1 - r_3)}\left(\alpha_4^{1/4}\zeta - \zeta_0\right)\right] - (2r_1 - r_2 - r_3)} - \frac{\alpha_3}{4\alpha_4}. \tag{68}$$

The expressions (66) and (68) are solitary wave solutions. For instance, when $P = 12, Q = 6, R = -1, S = -12, c_1 = 64, c_2 = 104, k_1 = 1, k_2 = 1, c = 1, \text{and}\, \zeta_0 = 0$, then $r_1 = -3$, $r_2 = 4, \text{and}\, r_3 = 2$, we can get solitary solution of (1) as (see Figure 1(f))

$$u(x, y, \tau) = \frac{35}{\cosh\left[\sqrt{35}(x + y - \tau)\right] + 6} - 1. \tag{69}$$

Case8. When $D_4 = 0$, $D_3 D_2 < 0$, $H(\Psi)$ has a pair of complex conjugate roots and a real root of multiplicity two. Then, we can write $H(\Psi)$ as

$$H(\Psi) = (\Psi - r_1)^2\left[(\Psi - \gamma)^2 + \delta^2\right], \tag{70}$$

where r_1, γ and δ all are real numbers and $\delta \neq 0$. Then, from (15), we have

$$\pm\zeta_1-\zeta_0=\int\frac{d\Psi}{(\Psi-r_1)\sqrt{(\Psi-\gamma)^2+\delta^2}}=\frac{1}{\sqrt{(r_1-\gamma)^2+\delta^2}}\ln\left|\frac{\xi_1\Psi+\xi_2-\sqrt{(\Psi-\gamma)^2+\delta^2}}{\Psi-r_1}\right|, \tag{71}$$

where

$$\xi_1=\frac{r_1-2\gamma}{\sqrt{(r_1-\gamma)^2+\delta^2}},\quad \xi_2=\sqrt{(r_1-\gamma)^2+\delta^2}-\frac{r_1(r_1-2\gamma)}{\sqrt{(r_1-\gamma)^2+\delta^2}}. \tag{72}$$

Then, the solution of (15)

$$\Psi=\frac{\left(e^{\pm\sqrt{(r_1-\gamma)^2+\delta^2}(\zeta_1-\zeta_0)}-\xi_1\right)+\sqrt{(r_1-\gamma)^2+\delta^2}(2-\xi_1)}{\left(e^{\pm\sqrt{(r_1-\gamma)^2+\delta^2}(\zeta_1-\zeta_0)}-\xi_1\right)^2-1}, \tag{73}$$

hence,

$$\phi(\zeta)=\frac{\alpha_4^{-1/4}\left[\left(e^{\pm\sqrt{(r_1-\gamma)^2+\delta^2}\left(\alpha_4^{1/4}\zeta-\zeta_0\right)}-\xi_1\right)+\sqrt{(r_1-\gamma)^2+\delta^2}(2-\xi_1)\right]}{\left(e^{\pm\sqrt{(r_1-\gamma)^2+\delta^2}\left(\alpha_4^{1/4}\zeta-\zeta_0\right)}-\xi_1\right)^2-1}-\frac{\alpha_3}{4\alpha_4}. \tag{74}$$

For instance, when $P=12, Q=6, R=-1, S=9, c_1=24, c_2=72, k_1=1, k_2=1, c=1,$ and $\zeta_0=0$, then $r_1=1, \gamma=-1$, and $\delta=2$, we can obtain solution of (1) as (see Figure 1(g))

$$v(x,y,\tau)=\frac{\left(e^{\pm2\sqrt{2}(x+y-\tau)}-3/2\sqrt{2}\right)+2\sqrt{2}\left(2-3/2\sqrt{2}\right)}{\left(e^{\pm2\sqrt{2}(x+y-\tau)}-3/2\sqrt{2}\right)^2-1}-1. \tag{75}$$

Case9. When $D_4=0,\ D_3=0,\ D_2>0,$ and $E_2>0$, $H(\Psi)$ has two real roots of multiplicity two. Then, we can express $H(\Psi)$ as follows:

$$H(\Psi)=(\Psi-r_1)^2(\Psi-r_2)^2, \tag{76}$$

where r_1 and r_2 are real numbers and $r_1>r_2$. Therefore, from (15), we have

$$\pm\zeta_1-\zeta_0=\int\frac{d\Psi}{(\Psi-r_1)(\Psi-r_2)}=\frac{1}{r_1-r_2}\ln\left|\frac{\Psi-r_1}{\Psi-r_2}\right|. \tag{77}$$

When $\Psi>r_1$ or $\Psi<r_2$, the solution of (15)

$$\Psi=\frac{r_2-r_1}{e^{(r_1-r_2)(\zeta_1-\zeta_0)-1}}+r_2=\frac{r_2-r_1}{2}\left[\coth\frac{(r_1-r_2)(\zeta_1-\zeta_0)}{2}-1\right]+r_2, \tag{78}$$

and solution of (12) is given by

$$\phi(\zeta)=\alpha_4^{-1/4}\left[\frac{r_2-r_1}{2}\left[\coth\frac{(r_1-r_2)\left(\alpha_4^{1/4}\zeta-\zeta_0\right)}{2}-1\right]+r_2\right]-\frac{\alpha_3}{4\alpha_4}. \tag{79}$$

When $r_1>\Psi>r_2$, the solution of (15)

$$\Psi=\frac{r_2-r_1}{-e^{(r_1-r_2)(\zeta_1-\zeta_0)-1}}+r_2=\frac{r_2-r_1}{2}\left[\tanh\frac{(r_1-r_2)(\zeta_1-\zeta_0)}{2}-1\right]+r_2, \tag{80}$$

and of (12)

$$\phi(\zeta)=\alpha_4^{-1/4}\left[\frac{r_2-r_1}{2}\left[\tanh\frac{(r_1-r_2)\left(\alpha_4^{1/4}\zeta-\zeta_0\right)}{2}-1\right]+r_2\right]-\frac{\alpha_3}{4\alpha_4}. \tag{81}$$

So for this case, we have obtain hyperbolic solution (81) and (79) of (12). For instance, when $P=12, Q=6, R=-1, S=13/2, c_1=21, c_2=1047/16, k_1=1, k_2=1, c=1,$ and $\zeta_0=0$, then $r_1=1/2, r_2=-1/2$ and $-1<\Psi<1$, we get shock wave solution of Equation (1) as (see Figure 1(h))

$$v(x,y,\tau)=-\frac{1}{2}\tanh\left[\frac{1}{2}(x+y-\tau)\right]-1. \tag{82}$$

4. Dynamic Properties

Now, we observe the dynamical properties of KP-mKP Equation (1) through the CDSPM. Analyzing the phase portraits of the dynamic system [68, 69], it is observed that the topological structures of the solution profiles are changed due to the variations of the parameters involved in the system. Thus, the CDSPM is not only effective for acquiring various types of solutions but also could be utilized to conduct the qualitative analysis of the solutions. Applying the theory of dynamical system [68, 69], Equation (9) is stated equivalently to the following system:

$$\frac{d\phi}{d\zeta}=z,\quad \frac{dz}{d\zeta}=\frac{1}{Rk_1^4}\left[\frac{1}{2}c_1+\left(ck_1-Sk_2^2\right)\phi-\frac{1}{2}Pk_1^2\phi^2-\frac{1}{3}Qk_1^2\phi^3\right], \tag{83}$$

and it is equivalent to

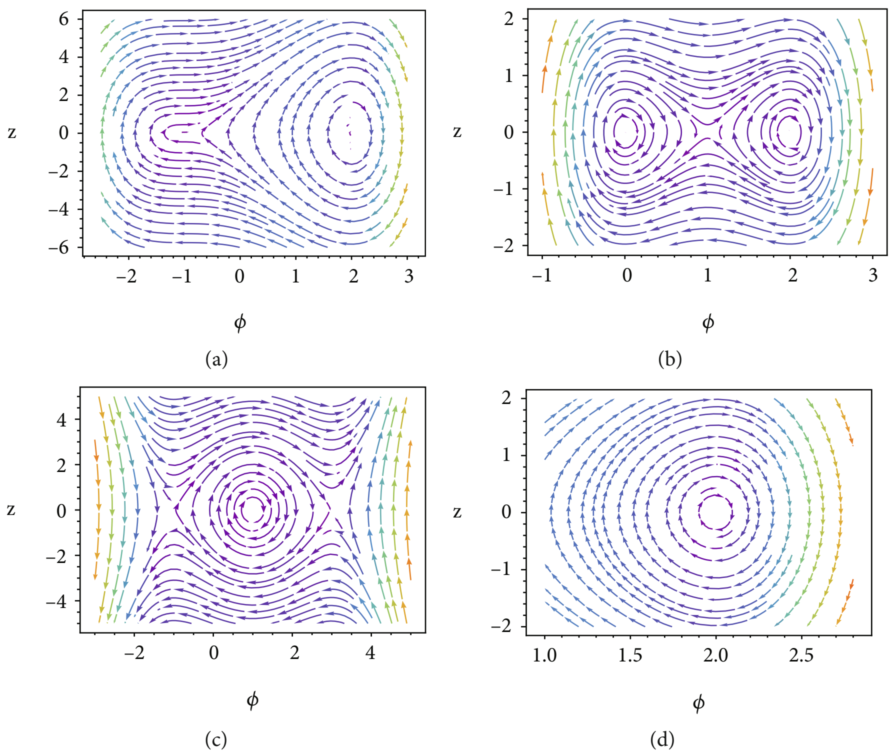

Figure 2: Phase portrait of dynamical dynamical system (83) in $(\phi, d\phi/d\zeta = z)$-plane: (a) when $\beta_3 = -2, \beta_2 = 0, \beta_1 = 6$, and $\beta_0 = 4$; (b) for $\beta_3 = -2, \beta_2 = 6, \beta_1 = -4$, and $\beta_0 = 0$; (c) when $\beta_3 = 1, \beta_2 = -3, \beta_1 = -1$, and $\beta_0 = 3$; and (d) when $\beta_3 = -2, \beta_2 = 4, \beta_1 = -2$, and $\beta_0 = 4$.

$$\frac{d\phi}{d\zeta} = z,$$
$$\frac{dz}{d\zeta} = \beta_3\phi^3 + \beta_2\phi^2 + \beta_1\phi + \beta_0, \tag{84}$$

where $\beta_3 = -Q/3Rk_1^2$, $\beta_2 = -P/2Rk_1^2$, $\beta_1 = (ck_1 - Sk_2^2)/Rk_1^4$, and $\beta_0 = c_1/2Rk_1^4$.

The existence of a homoclinic orbit in phase portrait in general corresponds to a solitary wave profile whereas the kink (antikink) wave solution is recognized by a heteroclinic orbit. Again, a periodic orbit confirms the presence of a periodic traveling wave solution. Varying the values of different parameters $\beta_0, \beta_1, \beta_2$, and β_3 involved in the system, we determine all homoclinic orbits, heteroclinic orbits, and periodic orbits of (84). Thus, the existence of solitary waves, kink (or antikink) waves, and periodic waves of Equation (1) are confirmed. The Hamiltonian function corresponding to the dynamical system (84) is defined as

$$H_2(\phi, z) = \frac{z^2}{2} - \left(\frac{\beta_3}{4}\phi^4 + \frac{\beta_2}{3}\phi^3 + \frac{\beta_1}{2}\phi^2 + \beta_0\phi + c_2\right), \tag{85}$$

which satisfies

$$\frac{\partial H_2}{\partial z} = \frac{d\phi}{d\zeta},$$
$$\frac{\partial H_2}{\partial \phi} = -\frac{dz}{d\zeta}. \tag{86}$$

Now, we call $M(a, 0)$ as the coefficient matrix of the system (84) and denote J as the determinant of $M(a, 0)$ at the equilibrium point $(a, 0)$. We take $T = \mathrm{trace}(M(a, 0))$ and $N = T^2 - 4J$.

Let $L(\phi) = \beta_3\phi^3 + \beta_2\phi^2 + \beta_1\phi + \beta_0$, whose complete discrimination system is as follows:

$$\Delta = \beta_2^2\beta_1^2 - 27\beta_3^2\beta_0^2 - 4\beta_1^3\beta_3 - 4\beta_2^3\beta_0 - 18\beta_3\beta_2\beta_1\beta_0. \tag{87}$$

Case 1. When $\Delta = 0$, $L(\phi)$ has a single real root together with another real root of multiplicity two, then

$$L(\phi) = \beta_3(\phi - a_1)^2(\phi - a_2), \tag{88}$$

then, the $(a, 0)$ and $(b, 0)$ are the equilibrium points of the system (84). For example when $\beta_3 = -2, \beta_2 = 0, \beta_1 = 6$, and $\beta_0 = 4$, we have $a_1 = -1$ and $a_2 = 2$. For the equilibrium point $(2, 0)$, we have $J = 18 > 0$, $T = 0$ and $N = -72 < 0$, so $(2, 0)$ is a center. Thus, it is confirmed that there exist a family of periodic orbits about $(2, 0)$. And at the equilibrium point $(-1, 0)$, we have $J = 0$, $T = 0$, so $(-1, 0)$ is a cusp (see Figure 2(a)).

Case 2. When $\Delta > 0$, $L(\phi)$ has three distinct single real root, then

$$L(\phi) = \beta_3(\phi - a_1)(\phi - a_2)(\phi - a_3), \tag{89}$$

then $(a_1 - 0)$, $(a_2 - 0)$ and $(a_3 - 0)$ become the three equilibrium points of the system. In particular when $\beta_3 = -2, \beta_2 = 6, \beta_1 = -4$, and $\beta_0 = 0$, we have $a_1 = 0$, $a_2 = 1$, and $a_3 = 2$.

At the equilibrium point $(0,0)$, we find $J=4>0$, $T=0$, and $N=-16<0$; thus, naturally $(0,0)$ becomes a center. On the other hand, at $(1,0)$, we have $J=-2<0$, $T=0$, and $N=8>0$, so $(1,0)$ becomes a saddle point. On the another equilibrium point $(2,0)$, we compute $J=4>0$, $T=0$ and $N=-16<0$, so $(2,0)$ is a center. Moreover, a couple of homoclinic orbits to $(1,0)$ encircling the centers $(0,0)$ and $(2,0)$ on both sides of the saddle point $(1,0)$ are found (see Figure 2(b)).

But, if $\beta_3=1$, $\beta_2=-3$, $\beta_1=-1$ and $\beta_0=3$, we acquire $a_1=1$, $a_2=3$, and $a_3=-1$. The equilibrium point $(1,0)$ becomes a center, as we see $J>0$, $T=0$, $N<0$. Again, at the equilibrium point $(3,0)$, we get $J<0$, $N>0$, and at $(-1,0)$, we find $J<0$, $N>0$, for the same values of the parameter. Thus, $(3,0)$ and $(-1,0)$ become saddle. And a pair of nonlinear heteroclinic orbits are observed joining two saddle points $(3,0)$ and $(-1,0)$ enfolding the center $(1,0)$ (see Figure 2(c)).

Case 3. When $\Delta<0$, $L(\phi)$ contains a real root as well as a pair of conjugate complex root, then

$$L(\phi)=\beta_3(\phi-a_1)\left[(\phi-\gamma)^2+\delta^2\right]. \tag{90}$$

Thus, $(a,0)$ is the only equilibrium point of the system and when $\beta_3=-2$, $\beta_2=4$, $\beta_1=-2$ and $\beta_0=4$, we find $a_1=2$. In that case, we have $J=10>0$, $T=0$, and $N=-40<0$, and thus, the point $(2,0)$ becomes a center. Then, a family of periodic orbits exist near about $(2,0)$ (see Figure 2(d)).

From the above analysis, we observe that the CDSPM might be possibly utilized to analyze the characteristic of the equilibrium points, and hence, the topological properties of the solution of the original equation could also be studied. Thus, we claim that if an equation is possibly presented in an integral form similar to (9); then, different characteristics of the said equation may be determined by the corresponding complete discrimination.

5. Conclusion

In this present investigation, employing the idea of the CDSPM, special kinds of exact analytical solutions for the KP-mKP equation are derived. Various wave features, such as solitary wave solution (Figure 1(f)), kink wave solution (Figure 1(h)), shock wave solution (Figures 1(a) and 1(g)), rational function solution (Figure 1(b)), exponential solution (Figure 1(g)), singular wave solution (Figure 1(c)), hyperbolic wave solution (Figure 1(f)), and periodic wave solution (Figures 1(d) and 1(e)) are explored from the KP-mKP equation. All these types of solutions in a combined manner could be scarcely acquired by any other technique as they include Jacobian elliptic functions. In particular, the existing popular method fail miserably in many cases to find any finite amplitude periodic solution for the evolution equation. In addition, we can also find stable ranges of the parameters involved in the equation. The qualitative properties of these solutions are analyzed through the numerical graphs which also show some new identities on Jacobian elliptic functions. Moreover, this article also demonstrates the strength of CDSPM in qualitative and quantitative analyses, by finding the critical domain for bifurcation and changing the type of solution, classifying the equilibrium points, and examining the phase portrait of topological characteristic. Based on this above analysis, the method can be applied not only to classify the solutions but also can be used for qualitative analysis, which opens the door to further promotion of the method. This result confirms the effectiveness and consequence of the CDSPM in solving evolution equations and the solutions obtained in this article could be realized through the significant applications in different scientific and engineering fields such as fluid dynamics, atmospheric phenomena, plasma science matter, and elastic media.

Nomenclature

P: Coefficient of quadratic nonlinear term of KP-mKP equation
Q: Coefficient of cubic nonlinear term of KP-mKP equation
R: Coefficient of dispersion term of KP-mKP equation
ζ_0: Integrating constant
c_i: Integrating constant, $i=1,2$
α_i: Coefficient of ϕ^i in biquadratic polynomial of ϕ for $i=0,..,4$
β_i: Coefficient of ϕ^i in in cubic polynomial of ϕ for $i=0,1,2,3$
p: Coefficient of Ψ^2 in biquadratic polynomial of Ψ
q: Coefficient of Ψ in polynomial of $H(\Psi)$
r: Coefficient of Ψ^0 in biquadratic polynomial of Ψ
r_i: Real roots of the polynomial $H(\Psi)$ for $i=1,2,3,4$
δ, γ: Real numbers
a_i: Real roots of $L(\phi)$ for $i=1,2,3$.

Data Availability

There is no data to support the findings of this study

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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