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Research Article

An Optimized Discrete Data Classification Method in *N*-Dimensional

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We propose a discrete data classification method of scattered data in N-dimensional by solving the minimax problem for a set of points. The current research is extended from 2-dimensional and 3-dimensional to N-dimensional. The problem can be applied to artificial intelligence classification problems (machine learning, deep learning), point data analysis problems (data science problem), the optimized design of nanoscale circuits, and the location of facility problems, circle detection on 2D image, or sphere detection on depth image. We generalized the discrete data classification methodology in N-dimensional. Finally, we resolved to find an exact solution of the location of a manifold for our suggested problem in N-dimensional.

1. Introduction

Data classification has been emphasized to developing artificial intelligence (machine learning, deep learning) performance from the big data. And from the image big data, image classification is an important part of the data classification [1, 2]. Some classification methods are support vector machine (SVM) [3], the decision tree [4], artificial neural network [5], and the naive Bayes classifier [6]. The measurement of data classification is key role of the performance of the methodology. In the measurement of data classification, we describe the minimax problem. Our *constrained optimization* problem is then formulated as an *unconstrained minimax problem* of finding $\mathbf{X}^*$ such that

$$J_\infty(\mathbf{X}^*) = \min_{\mathbf{X}\in\mathbb{R}^N} J_\infty(\mathbf{X}). \tag{1}$$

The organization of the paper is as follows. Section 1 is the introduction. 1.1. Literature review, Section 2 states the problem statement 2.1 and Innovative Research Contribution 2.2. The algorithm and the data classification methodology to solve our problem are described in Section 3. We prove that the algorithm searches an exact minimax solution. The data classification methodology is verified. Then, in Section 4, numerical results are presented. Our methodology is validated. Conclusions are given in Section 5.

1.1. Literature Review. A class of similar problems to (1) has a long history of development in operation research. It goes back to Pierre de Fermat who considered the case of $d_j(\mathbf{X}) = |\mathbf{P}_j - \mathbf{X}|$, $n = 2$ and $p = 1$ with equal weights whose solution is called *Fermat point*. Consider the case of $d_j(\mathbf{X}) = |\mathbf{P}_j - \mathbf{X}|$, for a general n and $p = 1$ with equal weights. Then, the problem is to find the geometric median of the set of points $\mathbf{P}_j$, $j = 1, \cdots, n$, which is a standard problem in facility location to minimize the cost of transportation. The minimizer $\mathbf{X}^*$ of $J_1(\mathbf{X})$ is known as the Fermat-Weber [7] point or 1-median. The Fermat-Weber problem has drawn much attention from mathematicians and facility location scientists and engineers; see, for instance [8–20] and the references therein. For Euclidean metric $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$ only, see the references [12–16, 18–21], and for rectilinear (Manhattan) metric $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ and Euclidean metric, see the references [10, 17]. For a survey paper on the Fermat-Weber problem, see Wesolowsky [22–25].

The case of $d_j(\mathbf{X}) = |\mathbf{P}_j - \mathbf{X}| - |\mathbf{Q}_1 - \mathbf{X}|$, with equal weights without the constraint on the circle passing through any point was dealt by Drezner et al. [21] for $p = 1, 2$, and ∞

and that with equal and unequal weights by Brimberg et al. [26] for $p = \infty$.

For nonlinear minimax problems with successive approximation methods for finding a stationary point is also studied by Demjanov [27]. This literature is helpful for us to suggest a new minimax model with a measurement for the general N-dimensional.

2. Problem Statement and Research Contribution

2.1. Problem Statement. We generalize the discrete data classification method from 2-dimensional and 3-dimensional to N-dimensional. In the N-dimensional case, let $\mathbf{P}_j(x_{1j}, x_{2j}, \cdots, x_{Nj})$, $j = 1, \cdots, n$, be a given set of discrete points on the space. Additionally, suppose that two additional points $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are given which are distinct from $\mathbf{P}_j$, $j = 1, \cdots, n$. We are interested in the *constrained optimization problem* of finding a N-dimensional manifold that is closest to all discrete points $\mathbf{P}_j$, $j = 1, \cdots, n$, among the manifolds that are constrained to pass through $\mathbf{Q}_1$ and $\mathbf{Q}_2$, see Figure 1 for the previous suggested problem setting in the 2-dimensional case. In 2 dimensions, the closeness of a circle to a set of discrete points is given by the weighted maximum distance from the circle to the points. In 3 dimensions, the closeness of a sphere to a set of discrete points is given by the weighted maximum distance from the sphere to the points, see Figure 2. Generally, the closeness of a manifold to a set of discrete points is given by the weighted maximum distance from the manifold to the points in the N-dimensional. Denote by $\mathbf{X}$ the center of a manifold which passes through the two points $\mathbf{Q}_1$ and $\mathbf{Q}_2$ and by $D_w(\mathbf{X})$ the n-dimensional vector

$$D_w(\mathbf{X}) = (w_1 d_1(\mathbf{X}), \cdots, w_n d_n(\mathbf{X})), \tag{2}$$

where $d_j(\mathbf{X}) = |\mathbf{P}_j - \mathbf{X}| - |\mathbf{Q}_1 - \mathbf{X}|$ and the weight $w_j \in (0, 1)$, $j = 1, \cdots, n$.

Set the objective function $J_p(\mathbf{X}) = \|D_w(\mathbf{X})\|_p$, where $\|\cdot\|_p$ denotes the ℓ^p-norm for $1 \le p \le \infty$.

Then, we resolve the minimax problem of (1).

Figure 2 is in the 3-dimensional case, and we introduce the 2-dimensional case in Figure 1. Since the spheres are constrained to pass $\mathbf{Q}_1$ and $\mathbf{Q}_2$, the centers should lie on the straight line which bisects the line segment $\overline{\mathbf{Q}_1\mathbf{Q}_2}$ perpendicularly. Those three points are on the same plane. We extended these problems to N-dimensional space. To simplify the problem, let us translate and rotate $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{P}_j$, $j = 1, \cdots, n$ so that the locations of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are $(a_1, a_2, \cdots, a_{N-1}, 0)$ and $(-a_1, -a_2, \cdots, -a_{N-1}, 0)$, where $2\sqrt{a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}^2}$ is the distance between $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Also denote the coordinates of $\mathbf{P}_j$ by $(x_{1j}, x_{2j}, \cdots, x_{Nj})$, $j = 1, \cdots, n$. Since the center of the N-dimensional manifold lies on the N-axis, problem (1) is then reduced to a one-dimensional problem. Denoting by $(0, 0, \cdots, t)$ the coordinate of N-dimensional, the radius of the manifold which passes through $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is $\sqrt{t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2}$.

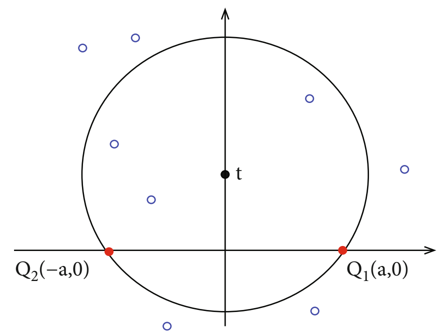

Figure 1: The constrained optimization problem of finding a circle in 2D that is closest to all points among all the circle that are constrained to pass through $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

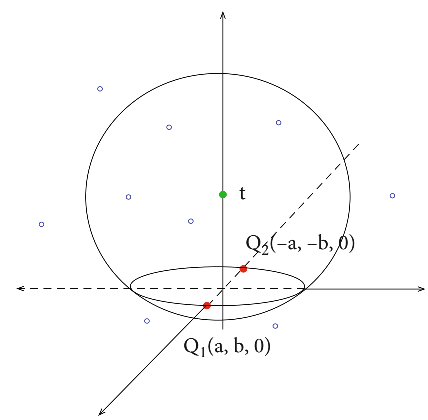

Figure 2: The constrained optimization problem of finding a sphere in 3D that is closest to all points among all the sphere that are constrained to pass through $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

For $j = 1, \cdots, n$, let $\psi_j(t)$ denote the weighted distance $\phi_j(t) = |\psi_j(t)|$, where

$$\begin{aligned} \psi_j(t) &= w_j d_j(\mathbf{X}), \\ d_j(\mathbf{X}) &= \sqrt{x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + \left(t - x_{Nj}\right)^2} \\ &\quad - \sqrt{t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2}. \end{aligned} \tag{3}$$

Since $D_w(\mathbf{X}) = (\psi_1(t), \cdots, \psi_n(t))$, problem (1) can be rewritten as finding $t^* \in \mathbb{R}$ such that

$$\phi(t^*) = \min_t \phi(t) \text{ where } \phi(t) := \max_j \phi_j(t) = J_\infty(\mathbf{X}). \tag{4}$$

2.2. Innovative Research Contribution. Our previous suggested optimized discrete data classification method (Kim method) in 2-dimensional case can be applied to the detection of a circle on the image [28, 29]. The sphere detection method (Kim method in 3-dimensional) can be applied to the 3-dimensional image with the depth intensity. And our suggested method can be applied to the optimized design of nanoscale circuits to reduce the defect rate in 2-dimensional and 3-dimensional.

Our method is very effective for big discrete data classification problems of machine learning models and algorithms [30].

As described in the beginning of the section, in this paper, we generalized to the optimization problem of finding a manifold which passes through two given points. We propose the discrete data classification methodology (Kim method) in N-dimensional to resolve this minimax problem, obtaining an exact solution algorithm which is very fast. We generalized this methodology to N-dimensional and obtained the mathematical verification of N-dimensional and physical validation with simple cases.

3. Proposed Methodology of Data Classification Method

In this section, we proposed the data classification methodology. Verification and validation are critical. We propose the methodology with mathematical verification. Validation will be shown in Section 4.

Lemma 1. *Supposes that there does not exist a manifold which passes through* $\mathbf{Q}_1$, $\mathbf{Q}_2$, *and the data points* $\boldsymbol{P}_j$, $j = 1, \cdots, n$. *A local optimum to the minimax problem is then taken at the intersection point of the graph of* $y = \phi_{k_1}(t)$ *and* $y = \phi_{k_2}(t)$ *for some* k_1 *and* k_2.

Assume that the function $\phi(t)$ has a local minimum at a point t' which is not an intersection point of any two graphs of $y = \phi_{k_1}(t)$ and $y = \phi_{k_2}(t)$ for any k_1 and k_2. Since the graph of $y = \phi(t)$ is a finite number of piecewise smooth curves, there should exist some j and a sufficiently small positive ε such that $\phi(t) = \phi_j(t)$ on $(t' - \varepsilon, t' + \varepsilon)$, where t' is a critical point of ϕ, and thus, it is a critical point of ϕ_j thereon. Such critical points should be either the zeros of the first derivative of $\psi_j(t)$ or the zero of $\phi_j(t)$. Thus, the exact critical point t' of the function $\phi_j(t)$ should be either $t_\pm = ((a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2)x_{Nj} \pm x_{Nj}(a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2)R)/(a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2 - (x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2))$, with $R = \sqrt{(x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2)}$

or

$$t_0 = \frac{\left(x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2\right) - \left(a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2\right)}{2x_{N_j}}, \tag{5}$$

which are zeros of $\psi_j'(t) = w_j(((t - x_{Nj})/\sqrt{x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + (t - x_{Nj})^2}) - (t/\sqrt{t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2}))$ or $\phi_j(t)$, respectively. To derive a contradiction from the existence of such critical points, notice that the second derivative of $\psi_j(t)$ is given as follows:

$$\psi_j''(t) = w_j\left(\frac{\sqrt{x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + \left(t - x_{Nj}\right)^2} - \left(t - x_{Nj}\right)^2 S^{(-1/2)}}{x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + \left(t - x_{Nj}\right)^2} - \frac{\sqrt{t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2} - t^2\left(t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2\right)^{(-1/2)}}{\left(t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2\right)}\right), \tag{6}$$

with $S = (x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + (t - x_{Nj})^2)$.

We treat the two cases separately.

(i) First, let us consider the points $t' = t_\pm$. Observe that $\psi_j(t_\pm)\psi_j''(t_\pm) < 0$.

This means that if $\psi_j(t_\pm)$ is greater (or less) than zero, then $\psi_j''(t_\pm)$ is less (or greater) than zero, and thus, the function $\psi_j(t)$ has a local maximum (or local minimum) at $t_\pm$. Since $\phi_j(t) = |\psi_j(t)|$, it follows that the function $\phi_j(t)$ has a local maximum at $t_\pm$

(ii) Next, consider the point $t' = t_0$. Since the function $\phi(t)$ has a local minimum at t_0 and $\phi(t) = \phi_j(t)$ on $(t' - \varepsilon, t' + \varepsilon)$, it is trivial that $\phi(t_0) = \phi_j(t_0) = 0$

Both cases (i) and (ii) lead to a contradiction to our assumption. Therefore, the local minimum of $\phi(t)$ must be taken at a point t' which is an intersection point of two graphs of $y = \phi_{k_1}(t)$ and $y = \phi_{k_2}(t)$ for some k_1 and k_2. This completes the proof.

In Figure 3, the graphs of $y = \phi_j(t)$, $y = \phi_k(t)$, and $y = \phi_{jk}(t) = \max\{\phi_j(t), \phi_k(t)\}$ are depicted. The local minima of $y = \phi_{jk}(t)$ are taken at the intersection points of $y = \phi_j(t)$ and $y = \phi_k(t)$. Figure 3 is in 2-dimensional case.

Theorem 2. *Let* t_{jk}*'s be all the intersection points of the graphs of* $y = \phi_j(t)$ *and* $y = \phi_k(t)$ *for all* $j, k = 1, \cdots, n$. *Let* t^* *be such that*

$$\phi(t^*) = \min_{j,k} \phi(t_{jk}). \tag{7}$$

If $\phi(t^*) \le \max_j w_j |y_j|$, then $\phi(t^*)$ is a global minimum for problem (2); otherwise, problem (2) does not have a global minimum solution.

Let us begin with finding the candidates of global minimum of $\phi(t)$ using the above Lemma 1. Since the equation $\phi_j(t) = \phi_k(t)$ is equivalent to the equation $\phi_j^2(t) = \phi_k^2(t)$, we can find the intersection points of ϕ_j and ϕ_k by solving

$$
\begin{aligned}
0 = \phi_j^2(t) - \phi_k^2(t) = {} & \left(w_j \sqrt{x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + \left(t - x_{Nj}\right)^2} \right. \\
& - w_k \sqrt{x_{1k}^2 + x_{2k}^2 + \cdots + x_{N-1k}^2 + (t - x_{Nk})^2} \\
& \left. - \left(w_j - w_k\right) \sqrt{t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2} \right) \\
& \times \left(w_j \sqrt{x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + \left(t - x_{Nj}\right)^2} \right. \\
& + w_k \sqrt{x_{1k}^2 + x_{2k}^2 + \cdots + x_{N-1k}^2 + (t - x_{Nk})^2} \\
& \left. - \left(w_j + w_k\right) \sqrt{t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2} \right) =: I_1^w(t) \times I_2^w(t).
\end{aligned} \tag{8}
$$

The above equation is divided into $I_1^w(t) = 0$ or $I_2^w(t) = 0$. To solve the equation $I_1^w(t) = 0$, move the radical term $(w_j - w_k)\sqrt{t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2}$ to the right side of the equation and square both sides. Then, we get

$$
\begin{aligned}
& 2 w_j w_k \sqrt{x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + \left(t - x_{Nj}\right)^2}\, T \\
& \quad = w_j^2 \left(x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + \left(t - x_{Nj}\right)^2 \right) \\
& \qquad + w_k^2 \left(x_{1k}^2 + x_{2k}^2 + \cdots + x_{N-1k}^2 + (t - x_{Nk})^2 \right) \\
& \qquad - \left(w_j - w_k\right)^2 \left(t^2 + a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2 \right),
\end{aligned} \tag{9}
$$

with $T = \sqrt{x_{1k}^2 + x_{2k}^2 + \cdots + x_{N-1k}^2 + (t - x_{Nk})^2}$.

To isolate the radical expression on the left side, move all the other terms to the right and square both sides of the equation again. Then, one gets the following cubic polynomial:

$$
p_{jk}(t) = a_0 t^3 + a_1 t^2 + a_2 t + a_3, \tag{10}
$$

where

$$
\begin{aligned}
a_0 = {} & 4 w_j^2 w_k^2 \left(x_{Nk} + x_{Nj} \right) - 2 \left(w_j{}^4 x_{Nj} + w_k{}^4 x_{Nk} \right) \\
& + 2 \left(w_j - w_k\right)^2 \left(w_j{}^2 x_{Nj} + w_k{}^2 x_{Nk} \right),
\end{aligned}
$$

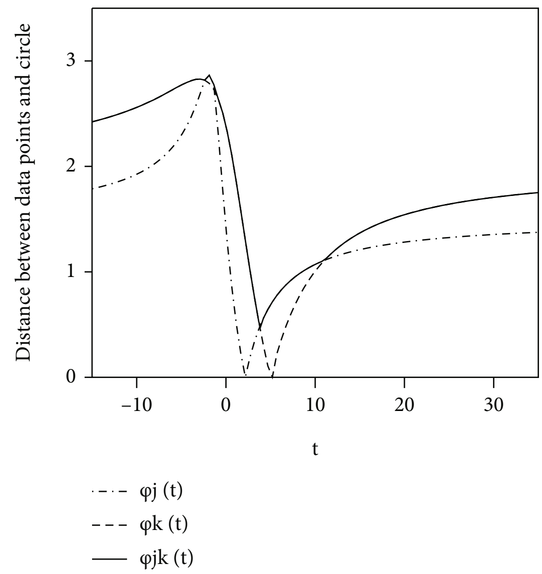

Figure 3: The 2-dimensional case graphs of $y = \phi_j(t)$, $y = \phi_k(t)$, and $y = \phi_{jk}(t) = \max\{\phi_j(t), \phi_k(t)\}$. Two local minima of $y = \phi_{jk}(t)$ are taken at the intersection points of $y = \phi_j(t)$ and $y = \phi_k(t)$.

$$
\begin{aligned}
a_1 = {} & w_j^4 \left\{ x_{Nj}^2 + 2 \left(x_{1j}{}^2 + x_{2j}{}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2 \right) \right\} \\
& + w_k^4 \left\{ x_{Nk}^2 + 2 \left(x_{1k}{}^2 + x_{2k}{}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2 \right) \right\} \\
& + 2 \left(w_j - w_k\right)^4 \left(a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2 \right) + 2 w_j^2 w_k^2 \\
& \cdot \left\{ \left(x_{1j}{}^2 + x_{2j}{}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2 \right) \right. \\
& \left. + \left(x_{1k}{}^2 + x_{2k}{}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2 \right) - 4 x_{Nj} x_{Nk} \right\} \\
& - w_k^2 \left(w_j - w_k\right)^2 \left\{ \left(x_{1k}{}^2 + x_{2k}{}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2 \right) \right. \\
& \left. + \left(a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2 \right) \right\} - w_j^2 \left(w_j - w_k\right)^2 \\
& \cdot \left\{ \left(x_{1j}{}^2 + x_{2j}{}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2 \right) + \left(a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2 \right) \right\} \\
& - 4 w_j^2 w_k^2 \left\{ \left(x_{1k}{}^2 + x_{2k}{}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2 \right) \right. \\
& \left. + \left(x_{1j}{}^2 + x_{2j}{}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2 \right) + 4 x_{Nj} x_{Nk} \right\},
\end{aligned}
$$

$$
\begin{aligned}
a_2 = {} & w_j^4 \left\{ -2 x_{Nj} \left(x_{1j}{}^2 + x_{2j}{}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2 \right) \right\} \\
& + w_k^4 \left\{ -2 x_{Nk} \left(x_{1k}{}^2 + x_{2k}{}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2 \right) \right\} \\
& + 2 \left(w_j{}^2 w_k{}^2 \right) \left\{ -2 x_{Nj} \left(x_{1j}{}^2 + x_{2j}{}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2 \right) \right. \\
& \left. - 2 x_{Nk} \left(x_{1k}{}^2 + x_{2k}{}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2 \right) \right\} \\
& - w_k^2 \left(w_j - w_k\right)^2 \left\{ -2 x_{Nk} \left(a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2 \right) \right\} \\
& - w_j^2 \left(w_j - w_k\right)^2 \left\{ -2 x_{Nj} \left(a_1{}^2 + a_2{}^2 + \cdots + a_{N-1}{}^2 \right) \right\} \\
& + 8 w_j^2 w_k^2 \left\{ x_{Nj} \left(x_{1k}{}^2 + x_{2k}{}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2 \right) \right. \\
& \left. + x_{Nk} \left(x_{1j}{}^2 + x_{2j}{}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2 \right) \right\},
\end{aligned}
$$

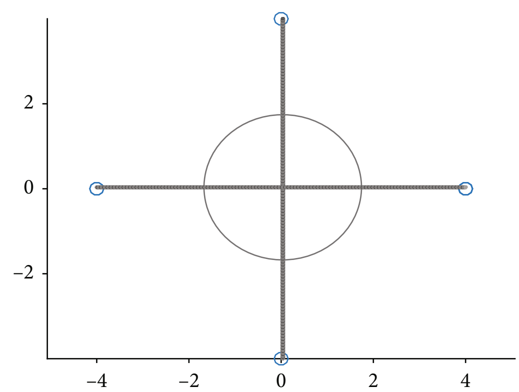

Figure 4: The validation figure of the constrained optimization problem of finding a circle in 2D that is closest to all 4 points, (3.58, 1.78), (-3.00, 2.61), (2.12, -3.45), and (-3.26, -2.34) among all the circle that are constrained to pass through (-4, -4) and (4, 4). The solution is the center (0.034375, 0.034375) with radius 1.7081.

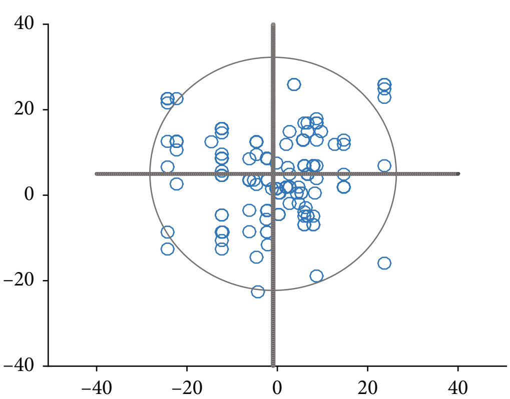

Figure 5: The constrained optimization problem of finding a circle in 2D that is closest to all 200 points among all the circle that are constrained to pass through (4, 6) and (-4, 1). The solution is the center (-0.93750, 5.0000) with radius 27.270.

$$
\begin{aligned}
a_3 = {} & w_j^4\left\{\left({x_{1j}}^2 + {x_{2j}}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2\right)^2\right\} \\
& + w_k^4\left({x_{1k}}^2 + {x_{2k}}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2\right)^2 \\
& + \left(w_j - w_k\right)^4\left({a_1}^2 + {a_2}^2 + \cdots + {a_{N-1}}^2\right)^2 + 2\left({w_j}^2 {w_k}^2\right) \\
& \cdot \left({x_{1j}}^2 + {x_{2j}}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2\right) \\
& \cdot \left({x_{1k}}^2 + {x_{2k}}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2\right) - w_k^2\left(w_j - w_k\right)^2 \\
& \cdot \left({x_{1k}}^2 + {x_{2k}}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2\right)\left({a_1}^2 + {a_2}^2 + \cdots + {a_{N-1}}^2\right) \\
& - w_j^2\left(w_j - w_k\right)^2\left({x_{1j}}^2 + {x_{2j}}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2\right) \\
& \cdot \left({a_1}^2 + {a_2}^2 + \cdots + {a_{N-1}}^2\right) - 4 w_j^2 w_k^2 \\
& \cdot \left({x_{1j}}^2 + {x_{2j}}^2 + \cdots + x_{N-1j}^2 + x_{Nj}^2\right) \\
& \cdot \left({x_{1k}}^2 + {x_{2k}}^2 + \cdots + x_{N-1k}^2 + x_{Nk}^2\right).
\end{aligned}
\tag{11}
$$

Similarly, solving the equation $I_2^w(t) = 0$ is equivalent to solving the following cubic polynomial:

$$
q_{jk}(t) = b_0 t^3 + b_1 t^2 + b_2 t + b_3. \tag{12}
$$

By using Cardano's formula, one can find all the real roots of $p_{jk}(t) = 0$ and $q_{jk}(t) = 0$. In this way, one can find the intersection points of $y = \phi_j(t)$ and $y = \phi_k(t)$ for $j, k = 1, \cdots, n$. Then, by comparing the values of the function $\phi(t)$ at these points, one can find the candidate of global minimum at t^*. Of course, as the function $\phi(t)$ is defined on $\mathbb{R}$, the global minimum may not exist. However, using the expression

$$
\phi_j(t) = \frac{w_j\left|x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + {x_N}^2 - {a_1}^2 - {a_2}^2 - \cdots - {a_{N-1}}^2 - 2x_{Nj}t\right|}{\sqrt{x_{1j}^2 + x_{2j}^2 + \cdots + x_{N-1j}^2 + \left(t - x_{Nj}\right)^2} + \sqrt{t^2 + {a_1}^2 + {a_2}^2 + \cdots + {a_{N-1}}^2}}, \tag{13}
$$

we have $\phi(t) \longrightarrow \max_j w_j |y_j|$ as $t \longrightarrow \pm\infty$. Thus, if $\phi(t^*) \le \max_j w_j |y_j|$, then $\phi(t^*)$ is a global minimum. If $\phi(t^*) > \max_j w_j |y_j|$, then the function $\phi(t)$ does not have a global minimum. This completes the proof.

Summarizing the above procedure in the proof of Theorem 2, we propose the following algorithm for solving the minimax problem (1).

Step 1. Compute the distance $2\sqrt{{a_1}^2 + {a_2}^2 + \cdots + a_{N-1}^2}$ between $\mathbf{Q}_1$ and $\mathbf{Q}_2$; choose the coordinate system such that $\mathbf{Q}_1(a_1, a_2, \cdots, a_{N-1}, 0)$ and $\mathbf{Q}_2(-a_1, -a_2, \cdots, -a_{N-1}, 0)$; by a rigid motion transform to $\mathbf{P}_j$ associate the coordinates $(x_{1j}, x_{2j}, \cdots, x_{Nj})$ for $j = 1, \cdots, n$.

Step 2. If $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{P}_j$, $j = 1, \cdots, n$ are on one specific manifold, then it is done.

Step 3. Find all intersection points t_{jk}'s of the graphs $z = \phi_j(t)$ and $z = \phi_k(t)$ for all j, k.

Step 4. For all such intersection points t_{jk}'s, evaluate $\phi_i(t_{jk})$ for all $i = 1, \cdots, n$; then, compute $\phi(t_{jk}) = \max_i \phi_i(t_{jk})$, and find the minimum $\phi(t^*) = \min_{j,k} \phi(t_{jk})$.

Step 5. If $\phi(t^*) \le \max_j w_j |z_j|$, then $\phi(t^*)$ is the global minimum. If $\phi(t^*) > \max_j w_j |z_j|$, the global minimum of the function $\phi(t)$ does not exist.

This is our optimized suggested data classification method in N-dimensional (Kim method). Our N-dimensional discrete data classification methodology is mathematically verified in this section.

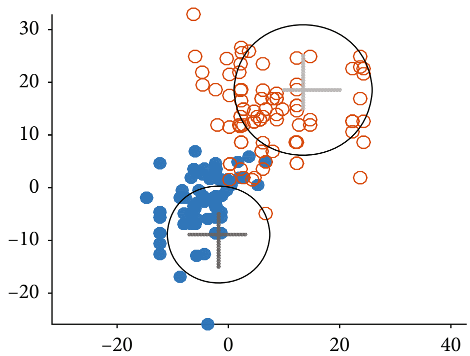

FIGURE 6: The validation figure of the constrained optimization problem of finding a circle in 2D that is closest to the different two groups; each group has 100 points. Group A (close dots) has the center (-1.7661, -8.8790) with minimax radius 9.2144, and group B (open dots) has the center (13.433, 18.536) with minimax radius 12.416.

TABLE 1: AUC values for dataset with sample size 200 with other methodology [31].

KIM	DT	k-NN	Log$R$
1.0000	0.7941	0.7683	0.6328
NB	C4.5	SVM	LC
0.7126	0.7452	0.7448	0.6408

4. Numerical Results

In this section, we show validation with simple example and results, see Figure 4. From Figure 4, we used 4 distinct points. The expected center is (0,0). Through our suggested methodology, we obtain the center $(0.034375, 0.034375)$ with the radius (1.7081). This is validated with an error $< 4.86135 \times 10^{(-2)}$. With big data (200 points), we calculate our methodology, see Figure 5. In the data classification problem, we test two groups with 200 points, see Figure 6. We can classify the scattered data with our suggested methodology. Obviously, we can obtain the two distinct circles with two groups. Through the intersection area of the two circles, we can analyze the relation of the data.

We show the comparison study with our suggested methodology and other methodology of the data classification [31], see Table 1 with comparison study of several methodology with the random data set from [31]. AUC is area under curve. Our method result of AUC (KIM) has 1.0000 in Table 1 because the two distinct circles are not overlapped. DT is a decision tree method. $k-NN$ is k -nearest neighbor method. Log$R$ is logistic regression method. NB is Naive Bayes method. C4.5 is a decision tree with divide-and-conquer. SVM is support vector machine. LC is linear classifier method [31]. Our methodology is to find the mathematical exact solution. We find the two distinct circles with centers and radii.

We have developed a minimax circle and sphere code to find the solution of the suggested minimax problem. We show a simple test case with 20 input data including the weight, see Table 2. The table shows the detailed input data with the weights (the weights are 1). $\mathbf{X}^*$ is the numerical solution from the exact solution procedure of the suggested problems. We tested many times for various random input data, and the results are very satisfactory with the robustness. The computation is executed using the Fortran compiler in a window system with the architecture Intel 11th Gen Core i7-1165G7 of 2.80 GHz. These plots show that the algorithm is very efficiently.

TABLE 2: $Q_1 = (4.00, 6.00)$ and $Q_2 = (-4.00, 1.00)$ are given with the equal weighted 20 points (value is 1). The minimax value is 15.025.

P_j	x_j	y_j
P_1	0.300	0.500
P_2	-0.200	1.53
P_3	1.97	1.90
P_4	-2.30	8.63
P_5	5.70	12.9
P_6	-2.20	3.53
P_7	7.97	6.90
P_8	-12.3	8.63
P_9	23.7	25.9
P_{10}	-24.3	22.6
P_{11}	6.7	4.90
P_{12}	-6.2	3.53
P_{13}	2.7	1.92
P_{14}	-22.3	12.6
P_{15}	8.67	16.9
P_{16}	-4.62	12.5
P_{17}	5.97	6.90
P_{18}	-12.3	4.63
P_{19}	14.7	1.90
P_{20}	-12.3	15.6
Solution	-9.8515	19.262

5. Conclusions

We have investigated the suggested minimax problem by our suggested data classification methodology (Kim method) in N-dimensional space, and this methodology is verified and validated (VV). This problem is applied to artificial intelligence classification problems (machine learning, deep learning), point data analysis problems (data science problem), the optimized design of nanoscale circuits, and the location of facility problems and circle or sphere detection problem on the image. Here, we also consider the general weighted case. And we obtain the exact solution of the minimizing manifold to maximize the distance between the manifold through two fixed points and multiple points. By finding the local optima, we find the global optimum. After we proved the exact solution of the suggested problem, we proposed a very efficient algorithm. This is the optimized data classification methodology of N-dimensional scattered data (Kim Method). As future work, the proposed methodology can be applied to the image processing. We investigate the detection algorithm of the image processing with our methodology. In the different images, we will analyze the mutual information. And our methodology can be applied to analyze supply chain problems with data.

Data Availability

All data generated or analyzed during this research are included in this paper. This data is available from this paper. If you need to have further information, then email to the author.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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