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Research Article

Hamiltonicity in Directed Toeplitz Graphs with $s_1 = 1$ and $s_3 = 4$

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A directed Toeplitz graph $T_n\langle s_1, \cdots, s_k ; t_1, \cdots, t_l\rangle$ with vertices $1, 2, \cdots, n$ is a directed graph whose adjacency matrix is a Toeplitz matrix. In this paper, we investigate the Hamiltonicity in directed Toeplitz graphs $T_n\langle s_1, \cdots, s_k ; t_1, \cdots, t_l\rangle$ with $s_1 = 1$ and $s_3 = 4$.

1. Introduction

Let G be a finite vertex-labeled graph with vertex set $V(G)$ and edge set $E(G)$. A graph H is called a subgraph of G if the vertex set and edge set of H is a subset of the vertex set and edge set of G, respectively. If $E(G) = \{(a_1, a_2), (a_2, a_3), \cdots, (a_{n-1}, a_n), (a_n, a_1)\}$, where $a_i \neq a_j$, then G is called a cycle. A cycle minus one edge is called a path. A subpath is a path making up part of a larger path. A cycle that visits each vertex of a graph H is called Hamiltonian, and H is then called a Hamiltonian graph. We consider here simple graphs as multiple edges and loops play no role in Hamiltonicity. The adjacency matrix $A = (b_{ij})_{n\times n}$ of G is the matrix in which $b_{ij} = 1$ if the vertex a_i is adjacent to the vertex a_j in G, and $b_{ij} = 0$ otherwise. The main diagonal is zero, i.e., $b_{ii} = 0$ as G has no loop.

A square matrix having constant values along all diagonals parallel to the main diagonal is called a Toeplitz matrix. A graph whose adjacency matrix is a Toeplitz matrix of order n is called a Toeplitz graph and is denoted by $T_n\langle s_1, s_2, \cdots, s_k ; t_1, t_2, \cdots, t_l\rangle$, where $s_p \geq 1 (1 \leq p \leq k)$ and $t_q \geq 1 (1 \leq q \leq l)$ are the label of the diagonal, above and below the main diagonal, respectively, containing ones such that $s_1 < s_2 < \cdots < n$ and $t_1 < t_2 < \cdots < n$, see Figure 1. We can calculate the length of an edge (a_i, a_j) by $|a_i - a_j|$. An edge (a_i, a_j) is called an increasing (decreasing) edge if $a_i < a_j$ $(a_j < a_i)$. We have both increasing and decreasing edges in $T_n\langle s_1, s_2, \cdots, s_k ; t_1, t_2, \cdots, t_l\rangle$ of length s_p and t_q, respectively, for some p and q. Note that if we reverse the direction of all edges of the graph $T_n\langle s_1, \cdots, s_k ; t_1, \cdots, t_l\rangle$, then the graph $T_n\langle t_1, \cdots, t_l ; s_1, \cdots, s_k\rangle$ is obtained. If the Toeplitz matrix is symmetric, then the corresponding Toeplitz graph is undirected and can be denoted as $T_n\langle s_1, \cdots, s_k\rangle$. Hamiltonicity results obtained in the undirected case for a Toeplitz graph have a direct impact on the directed case. Hamiltonicity of $T_n\langle s_1, s_2, \cdots, s_k\rangle$ means Hamiltonicity of $T_n\langle s_1, \cdots, s_k ; t_1, \cdots, t_l\rangle$.

Different properties of Toeplitz graphs have been studied in the literature, for example, colourability, bipartiteness, planarity, cycle discrepancy, metric dimension, decomposition, edge irregularity strength, and labeling. Hamiltonian properties of undirected Toeplitz graphs were first investigated by van Dal et al. in [1], and then these studies have been extended in [2–5]. The Hamiltonicity in directed Toeplitz graphs has also been studied in the literature.

The Hamiltonicity of the directed Toeplitz graphs $T_n\langle 1, 2 ; t\rangle$ and $T_n\langle 1, 2, 3 ; t\rangle$ were investigated in the literature, which completes the Hamiltonicity investigation in the directed Toeplitz graphs with $s_3 = 3$. In the literature, the Hamiltonicity of the directed Toeplitz graphs $T_n\langle 1, 3, 4 ; t\rangle$ was also investigated. In this paper, we first prove a conjecture regarding $T_n\langle 1, 3, 4 ; 3\rangle$ stated in [6]. Then, we investigate the Hamiltonicity of the directed Toeplitz graphs with $s_3 = 4$, $s_2 = 3$, $s_1 = 1$, and $l = 2$. We also investigate the Hamiltonicity of the directed Toeplitz graphs with $s_3 = 4$, $s_2 = 2$, and $l = 2$, i.e., $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$. Thus, in this paper, we complete the Hamiltonicity investigation of the directed Toeplitz graphs with $s_3 = 4$ and $s_1 = 1$.

Suppose that H is a Hamiltonian cycle in $T_n\langle s_1, s_2, \cdots, s_k ; t_1, t_2, \cdots, t_l\rangle$. The Hamiltonian cycle H is determined by two paths $H_{1\longrightarrow n}$ (from 1 to n) and $H_{n\longrightarrow 1}$ (from n to 1), i.e., $H = H_{1\longrightarrow n} \cup H_{n\longrightarrow 1}$. Since the path $H_{1\longrightarrow n}$ is

Hamiltonian in the subgraph of $T_n\langle s_1, s_2, \cdots, s_k ; t_1, t_2, \cdots, t_l\rangle$ induced by $V(H_{n\longrightarrow 1} \setminus \{1, n\})$, the vertices which are not covered by $H_{1\longrightarrow n}$ would be covered by $H_{n\longrightarrow 1}$.

We denote a path $P_n(a)$ of length n, from vertex a, as $P_n(a) = (a, a+1, \cdots, a+(n-1))$. By path $P_{a\longrightarrow b}$, we mean a path from vertex a to vertex b.

2. Toeplitz Graphs $T_n\langle 1, 3, 4 ; t_1, t_2\rangle$

In this section, we first prove a conjecture regarding $T_n\langle 1, 3, 4 ; 3\rangle$ stated in [6]. Then, we investigate the Hamiltonicity of the directed Toeplitz graphs with $s_3 = 4$, $s_2 = 3$, $s_1 = 1$, and $l = 2$.

Theorem 1 (see [7]). *$T_n\langle 1, 3, 4 ; 3\rangle$ is Hamiltonian for $n \in \{5, 6, 7, 9\}$.*

In Theorem 1, it was proved that $T_n\langle 1, 3, 4 ; 3\rangle$ is Hamiltonian for $n \in \{5, 6, 7, 9\}$, and in [6], it was stated as conjecture that $T_n\langle 1, 3, 4 ; 3\rangle$ is non-Hamiltonian for $n \notin \{5, 6, 7, 9\}$. Here, we prove this conjecture which allows us to restate Theorem 1 as follows:

Theorem 2. *$T_n\langle 1, 3, 4 ; 3\rangle$ is Hamiltonian if and only if $n \in \{5, 6, 7, 9\}$.*

Proof. If $n \in \{5, 6, 7, 9\}$, then Theorem 1 asserts that $T_n\langle 1, 3, 4 ; 3\rangle$ is Hamiltonian.

Conversely, we show that $T_n\langle 1, 3, 4 ; 3\rangle$ is non-Hamiltonian for $n \notin \{5, 6, 7, 9\}$. Assume to the contrary, that $T_n\langle 1, 3, 4 ; 3\rangle$ is Hamiltonian for $n \notin \{5, 6, 7, 9\}$. Let $H = H_{1\longrightarrow n} \cup H_{n\longrightarrow 1}$ be a Hamiltonian cycle in $T_n\langle 1, 3, 4 ; 3\rangle$. Since the path $H_{1\longrightarrow n}$ is Hamiltonian in the subgraph of $T_n\langle 1, 3, 4 ; 3\rangle$ induced by $V(H_{n\longrightarrow 1} \setminus \{1, n\})$, the vertices which are not covered by $H_{1\longrightarrow n}$ would be covered by $H_{n\longrightarrow 1}$. Let $V(H_{n\longrightarrow 1} \setminus \{1, n\}) = V_1 \cup V_2 \cup \cdots \cup V_k$, where each $V_{i\in\{1,2,\cdots,k\}}$ is a disjoint set of successive vertices. Since $H_{1\longrightarrow n}$ has increasing edges of length 1, 3, and 4 only, for each V_i, we have either $|V_i| = 2$ or $|V_i| = 3$. But since we have decreasing edges of length 3 only, $(n, n-3) \in E(H_{n\longrightarrow 1})$, and one can observe that there is no V_i such that $|V_i| = 3$, so $|V_i| = 2$ for all i. Since $(n, n-3) \in E(H_{n\longrightarrow 1})$, then we have either $(n-3, n-2) \in E(H_{n\longrightarrow 1})$ or $(n-3, n-6) \in E(H_{n\longrightarrow 1})$. If $(n-3, n-2) \in E(H_{n\longrightarrow 1})$, then clearly, by keeping in mind that $|V_i| = 2$ for all i, and that the decreasing edges are of length 3 only, the only possible subpath of $H_{n\longrightarrow 1}$ is $(n, n-3, n-2, n-5, n-8, n-11, n-10, n-6)$, but then it would be stuck at vertex $n-6$, see Figure 2, this is a contradiction. If $(n-3, n-6) \in E(H_{n\longrightarrow 1})$, then as in the previous case, the possible subpath of $H_{n\longrightarrow 1}$ is $(n, n-3, n-6, n-2, n-5, n-8, n-11, n-14, n-13, n-9)$ but then it would be stuck at vertex $n-9$, see Figure 3. This is a contradiction.

This completes the proof. □

Theorem 3 (see [6]). *$T_n\langle 1, 3, 4 ; 2\rangle$ is Hamiltonian if and only if $n \not\equiv 2 \bmod 6$.*

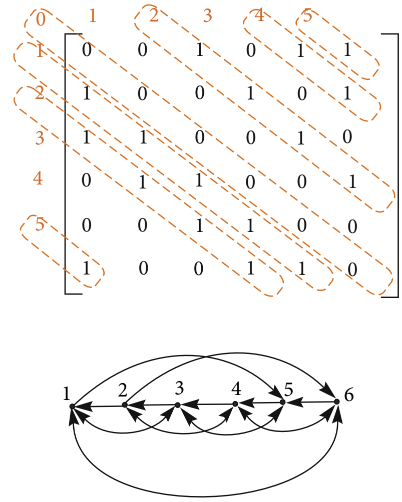

Figure 1: Toeplitz matrix and its associated Toeplitz graph $T_6\langle 2, 4, 5 ; 1, 2, 5\rangle$.

Theorem 4 (see [6]). *$T_n\langle 1, 3, 4 ; 4\rangle$ is Hamiltonian if and only if $n \notin \{6, 10, 12\}$.*

Theorem 5 (see [7]). *$T_n\langle 1, 3, 4 ; 5\rangle$ is Hamiltonian if and only if $n \neq 7$.*

Theorem 6 (see [6]). *$T_n\langle 1, 3, 4 ; 8\rangle$ is Hamiltonian if and only if $n \neq 14$.*

Note that the Hamiltonicity in $T_n\langle 1, 3, 4 ; t_1\rangle$ or $T_n\langle 1, 3, 4 ; t_2\rangle$ implies the Hamiltonicity in $T_n\langle 1, 3, 4 ; t_1, t_2\rangle$. Theorems 2 and 3 show that $T_n\langle 1, 3, 4 ; 3\rangle$ and $T_n\langle 1, 3, 4 ; 2\rangle$ are non-Hamiltonian for infinitely many n, respectively. Theorems 4, 5, and 6 show that $T_n\langle 1, 3, 4 ; 4\rangle$, $T_n\langle 1, 3, 4 ; 5\rangle$, and $T_n\langle 1, 3, 4 ; 8\rangle$ are non-Hamiltonian for only a finite number of n, respectively. Here, we study the Hamiltonicity of these cases by adding one more diagonal (containing one) below the main diagonal, say t_2, i.e., $T_n\langle 1, 3, 4 ; t_1, t_2\rangle$. Then, we will see in Theorem 11, $T_n\langle 1, 3, 4 ; t_1, t_2\rangle$ is Hamiltonian for all t_1, t_2, and n. We use the following some existing results in the proof of Theorem 11.

Theorem 7 (see [7]). *For $t \in \{6, 7\}$, $T_n\langle 1, 3, 4 ; t\rangle$ is Hamiltonian for all n.*

Theorem 8 (see [6, 7]). *For $t \geq 9$, $T_n\langle 1, 3, 4 ; t\rangle$ is Hamiltonian for all n.*

We have the following theorems in the literature.

Theorem 9. *For $t \in \{2, 4\}$, $T_n\langle 1, 3 ; 1, t\rangle$ is Hamiltonian for all n.*

Theorem 10. *For even $t \geq 8$, $T_n\langle 1, 3 ; 1, t\rangle$ is Hamiltonian if $n \cong 0, 2, 4, 6, 5, 7, \cdots, t-3 \bmod (t-1)$.*

Theorem 11. *$T_n\langle 1, 3, 4 ; t_1, t_2\rangle$ is Hamiltonian for all t_1, t_2, and n.*

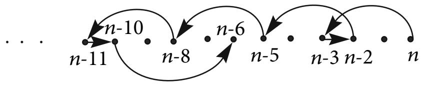

Figure 2: A subpath of $H_{n\longrightarrow 1}$ in $T_n\langle 1, 3, 4 ; 3\rangle$.

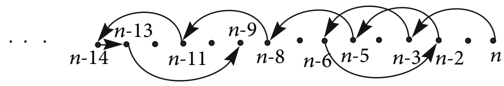

Figure 3: A subpath of $H_{n\longrightarrow 1}$ in $T_n\langle 1, 3, 4 ; 3\rangle$.

Proof. Case 1. Let $t_1 \in \{6, 7\}$ or $t_2 \in \{6, 7\}$ or $t_1 \geq 9$ or $t_2 \geq 9$.

For $t_1 \in \{6, 7\}$ or $t_2 \in \{6, 7\}$, by Theorem 7, $T_n\langle 1, 3, 4 ; t_1, t_2\rangle$ is Hamiltonian for all n. And for $t_1 \geq 9$ or $t_2 \geq 9$, by Theorem 8, $T_n\langle 1, 3, 4 ; t_1, t_2\rangle$ is Hamiltonian for all n.

Case 2. Let $t_1, t_2 < 9$ and $t_1, t_2 \notin \{6, 7\}$.

If $t_1 = 1$, then $t_2 \geq 2$. For $t_2 \in \{2, 3, 4\}$, by using Theorem 9, $T_n\langle 1, 3, 4 ; 1, t_2\rangle$ is Hamiltonian for all n, because the Hamiltonicity in $T_n\langle 1, 3 ; 1, 4\rangle$ means the Hamiltonicity in $T_n\langle 1, 4 ; 1, 3\rangle$. By using Theorem 5, $T_n\langle 1, 3, 4 ; 1, 5\rangle$ is Hamiltonian for all n different from 7. A Hamiltonian cycle in $T_7\langle 1, 3, 4 ; 1, 5\rangle$ is $(1, 5, 4, 3, 6, 7, 2, 1)$. Thus $T_n\langle 1, 3, 4 ; 2, 5\rangle$ is Hamiltonian for all n. By using Theorems 6 and 10, $T_n\langle 1, 3, 4 ; 1, 8\rangle$ is Hamiltonian for all n.

If $t_1 = 2$, then $t_2 \geq 3$. For $t_2 = 3$, by using Theorem 3, $T_n\langle 1, 3, 4 ; 2, 3\rangle$ is Hamiltonian if $n \not\cong 2 \bmod 6$. Let $n \cong 2 \bmod 6$, then the smallest such n is 8. A Hamiltonian cycle in $T_8\langle 1, 3, 4 ; 2, 3\rangle$ is $(1, 4, 7, 8, 5, 2, 6, 3, 1)$ which contains the edge $(7, 8)$, see Figure 4.

Let $T_{n=8+6r}\langle 1, 3, 4 ; 2, 3\rangle$ has a Hamiltonian cycle containing the edge $(n-1, n)$, for some nonnegative integer r. We transform this Hamiltonian cycle in $T_n\langle 1, 3, 4 ; 2, 3\rangle$ to a Hamiltonian cycle in $T_{n+6}\langle 1, 3, 4 ; 2, 3\rangle$ by replacing the edge $(n-1, n)$ with the path $(n-1, n+3, n+1, n+5, n+6, n+4, n+2, n)$. This shows that $T_{n+6}\langle 1, 3, 4 ; 2, 3\rangle$ enjoys the same property, thus $T_n\langle 1, 3, 4 ; 2, 3\rangle$ is Hamiltonian for all n. By using Theorems 3 and 4, $T_n\langle 1, 3, 4 ; 2, 4\rangle$ is Hamiltonian for all n. By using Theorems 3 and 5, $T_n\langle 1, 3, 4 ; 2, 5\rangle$ is Hamiltonian for all n. By using Theorem 6, $T_n\langle 1, 3, 4 ; 2, 8\rangle$ is Hamiltonian for all n different from 14. A Hamiltonian cycle in $T_{14}\langle 1, 3, 4 ; 2, 8\rangle$ is $(1, 4, 2, 6, 7, 11, 14, 12, 10, 8, 9, 13, 5, 3, 1)$, see Figure 5.

Thus, $T_n\langle 1, 3, 4 ; 2, 8\rangle$ is Hamiltonian for all n.

If $t_1 = 3$, then $t_2 \geq 4$. For $t_2 = 4$, by using Theorems 2 and 4, $T_n\langle 1, 3, 4 ; 3, 4\rangle$ is Hamiltonian for all $n \notin \{10, 12\}$. Hamiltonian cycles in $T_{10}\langle 1, 3, 4 ; 3, 4\rangle$ and $T_{12}\langle 1, 3, 4 ; 3, 4\rangle$ are $(1, 2, 3, 6, 9, 10, 7, 8, 4, 5, 1)$ and $(1, 2, 5, 8, 11, 12, 9, 6, 10, 7, 3, 4, 1)$, respectively, see Figure 6.

Thus, $T_n\langle 1, 3, 4 ; 3, 4\rangle$ is Hamiltonian for all n. By using Theorems 2 and 4, $T_n\langle 1, 3, 4 ; 3, 5\rangle$ is Hamiltonian for all n. By using Theorem 6, $T_n\langle 1, 3, 4 ; 3, 8\rangle$ is Hamiltonian for all $n \neq 14$. A Hamiltonian cycle in $T_{14}\langle 1, 3, 4 ; 3, 8\rangle$ is $(1, 2, 5, 8, 9, 12, 13, 14, 6, 7, 10, 11, 3, 4, 1)$, see Figure 7. Thus, $T_n\langle 1, 3, 4 ; 3, 8\rangle$ is Hamiltonian for all n.

If $t_1 = 4$, then $t_2 \geq 5$. By using Theorems 4 and 5, $T_n\langle 1, 3, 4 ; 4, 5\rangle$ is Hamiltonian for all n. By using Theorems 4 and 6, $T_n\langle 1, 3, 4 ; 4, 8\rangle$ is Hamiltonian for all n.

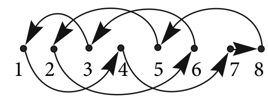

Figure 4: A Hamiltonian cycle in $T_8\langle 1, 3, 4 ; 2, 3\rangle$.

If $t_1 = 5$, then $t_2 \geq 6$. By using Theorems 4 and 6, $T_n\langle 1, 3, 4 ; 5, 8\rangle$ is Hamiltonian for all n. This completes the proof. □

3. Toeplitz Graphs $T_n\langle 1, 2, 4 ; t\rangle$

In this section, we discuss the Hamiltonicity of Toeplitz graphs still with $s_3 = 4$ but $s_2 = 2$ and $l = 1$.

We need the following lemma in the proof of Theorem 14.

Lemma 12. *For even t, $T_{t+2}\langle 1, 2, 4 ; t\rangle$ is non-Hamiltonian.*

Proof. Assume, to the contrary, that $T_{t+2}\langle 1, 2, 4 ; t\rangle$ has a Hamiltonian cycle $H = H_{1\longrightarrow t+2} \cup H_{t+2\longrightarrow 1}$. Let $V(H_{t+2\longrightarrow 1} \setminus \{1, t+2\}) = V_1 \cup V_2 \cup \cdots \cup V_k$, where each $V_{i\in\{1,2,\cdots,k\}}$ is a disjoint set of successive vertices. Since $H_{1\longrightarrow t+2}$ has no increasing edge of length 3 or of length greater than 4, for each V_i, we have either $|V_i| = 1$ or $|V_i| = 3$.

Since $d^-(t+2) = 1 = d^+(1)$ in $T_{t+2}\langle 1, 2, 4 ; t\rangle$, so $(t+2, 2), (1+t, 1) \in E(H_{t+2\longrightarrow 1})$. Then, $H_{t+2\longrightarrow 1} = (t+2, 2) \cup P_{2\longrightarrow 1+t} \cup (1+t, 1)$. But there is no path $P_{2\longrightarrow 1+t}$ because starting from vertex 2, it can only use increasing edges of length 2 or 4 (otherwise $|V_i| = 2$ for some V_i), but then this will never end up to odd $1+t$. This is a contradiction. □

Now we will discuss the Hamiltonicity of $T_n\langle 1, 2, 4 ; t\rangle$ for both even and odd t. In Theorem 13, we discuss it for even t.

Theorem 13. *For even t, $T_n\langle 1, 2, 4 ; t\rangle$ is Hamiltonian if and only if $n \neq t+4$ for $t \in \{4, 8\}$, and $n \neq t+2$.*

Proof. For even t, by using a result in the literature, $T_n\langle 1, 2, 4 ; t\rangle$ is Hamiltonian for all odd n. Now for even n, such that $n \neq t+4$ for $t \in \{4, 8\}$, and $n \neq t+2$, we prove that $T_n\langle 1, 2, 4 ; t\rangle$ is Hamiltonian.

Case 1. If $t \cong 0 \bmod 4$.

(i) Let $n \cong 0 \bmod 4$.

Assume $t \notin \{4, 8\}$. The smallest n is $t+4$. A Hamiltonian cycle in $T_{n=t+4}\langle 1, 2, 4 ; t\rangle$ is $(1, 5, 6, 10, \cdots, t-2, t-1, t+3, t+4, 4, 8, \cdots, t, t+2, 2, 3, 7, 9, \cdots, t-3, t+1, 1)$, which contains the edge $(n-1 = t+3, n = t+4)$, see Figure 8.

Let $T_{n=(t+4)+4r}\langle 1, 2, 4 ; t\rangle$ has a Hamiltonian cycle containing the edge $(n-1, n)$ for some nonnegative integer r. We transform this Hamiltonian cycle in $T_n\langle 1, 2, 4 ; t\rangle$ to a Hamiltonian cycle in $T_{n+4}\langle 1, 2, 4 ; t\rangle$ by replacing the edge $(n-1, n)$ with the path $(n-1, n+1, n+2, n+3, n+4, n)$. This shows that $T_{n+4}\langle 1, 2, 4 ; t\rangle$ enjoys the same property. Thus, for $t \cong 0 \bmod 4$ and $t \notin \{4, 8\}$, $T_n\langle 1, 2, 4 ; t\rangle$ is Hamiltonian for all $n \cong 0 \bmod 4$.

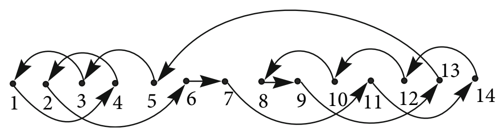

Figure 5: A Hamiltonian cycle in $T_{14}\langle 1,3,4\,;2,8\rangle$.

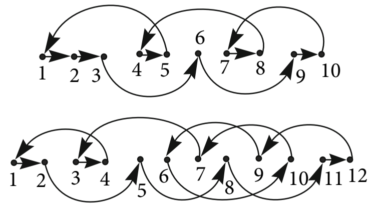

Figure 6: Hamiltonian cycles in $T_{10}\langle 1,3,4\,;3,4\rangle$ and $T_{12}\langle 1,3,4\,;3,4\rangle$.

Assume $t\in\{4,8\}$ and $n\neq t+4$. The smallest n, different from $t+4$, is $t+8$. Hamiltonian cycles in $T_{n=12}\langle 1,2,4\,;4\rangle$ and $T_{n=16}\langle 1,2,4\,;8\rangle$ are $(1,3,7,9,11,12,8,10,6,2,4,5,1)$ and $(1,3,7,11,13,15,16,8,12,14,6,10,2,4,5,9,1)$, respectively, where both cycles contain the edge $(n-1,n)$, see Figure 9.

Let $T_{n=(t+8)+4r}\langle 1,2,4\,;t\rangle$ has a Hamiltonian cycle containing the edge $(n-1,n)$ for some nonnegative integer r. We transform this Hamiltonian cycle in $T_n\langle 1,2,4\,;t\rangle$ to a Hamiltonian cycle in $T_{n+4}\langle 1,2,4\,;t\rangle$ by replacing the edge $(n-1,n)$ with the path $(n-1,n+1,n+2,n+3,n+4,n)$. This shows that $T_{n+4}\langle 1,2,4\,;t\rangle$ enjoys the same property. Thus, for $t\in\{4,8\}$ and $n\neq t+4$, $T_n\langle 1,2,4\,;t\rangle$ is Hamiltonian for all $n\cong 0 \bmod 4$.

(ii) Let $n\cong 2 \bmod 4$ and $n\neq t+2$.

The smallest n, different from $t+2$, is $t+6$. A Hamiltonian cycle in $T_{n=t+6}\langle 1,2,4\,;t\rangle$ is $(1,2,3,5,\cdots,t-1,t+3,t+5,t+6,6,10,\cdots,t+2,t+4,4,8,\cdots,t,t+1,1)$, which contains the edge $(t+5,t+6)$, see Figure 10.

By using the same technique as in the previous subcase, for $t\cong 0 \bmod 4$, $T_n\langle 1,2,4\,;t\rangle$ is Hamiltonian for all $n\cong 2 \bmod 4$ such that $n\neq t+2$.

Case 2. If $t\cong 2 \bmod 4$.

(i) Let $n\cong 0 \bmod 4$ and $n\neq t+2$.

The smallest n, different from $t+2$, is $t+6$. A Hamiltonian cycle in $T_{n=t+6}\langle 1,2,4\,;t\rangle$ is $(1,5,9,\cdots,t+3,t+5,t+6,6,10,\cdots,t+4,4,8,\cdots,t+2,2,3,7,11,\cdots,t+1,1)$, which contains the edge $(t+5,t+6)$, see Figure 11.

By using the same technique as in Case 1(i), for $t\cong 2 \bmod 4$, $T_n\langle 1,2,4\,;t\rangle$ is Hamiltonian for all $n\cong 0 \bmod 4$ such that $n\neq t+2$.

(ii) Let $n\cong 2 \bmod 4$.

The smallest n, is $t+4$. A Hamiltonian cycle in $T_{n=t+4}\langle 1,2,4\,;t\rangle$ is $(1,3,5,\cdots,t-1,t+3,t+4,4,8,\cdots,t+2,2,6,\cdots,t,t+1,1)$, which contains the edge $(t+3,t+4)$, see Figure 12.

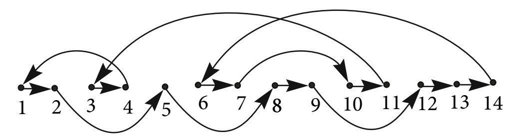

Figure 7: A Hamiltonian cycle in $T_{14}\langle 1,3,4\,;3,8\rangle$.

By using the same technique as in Case 1(i), for $t\cong 2 \bmod 4$, $T_n\langle 1,2,4\,;t\rangle$ is Hamiltonian for all $n\cong 2 \bmod 4$.

Conversely, we show that if $n=t+4$ for $t\in\{4,8\}$ and if $n=t+2$, then $T_n\langle 1,2,4\,;t\rangle$ is not Hamiltonian, i.e., $T_8\langle 1,2,4\,;4\rangle$, $T_{12}\langle 1,2,4\,;8\rangle$, and $T_{t+2}\langle 1,2,4\,;t\rangle$ are non-Hamiltonian.

Claim 1. $T_8\langle 1,2,4\,;4\rangle$ is non-Hamiltonian.

Assume, to the contrary, that $T_8\langle 1,2,4\,;4\rangle$ is Hamiltonian, and let $H=H_{1\longrightarrow 8}\cup H_{8\longrightarrow 1}$ be a Hamiltonian cycle in $T_8\langle 1,2,4\,;4\rangle$. Since $d^-(v)=1=d^+(v)$ for every vertex v in H, so $(8,4),(5,1)\in E(H_{8\longrightarrow 1})$. Then $H_{8\longrightarrow 1}=(8,4,5,1)$. Clearly, $(1,2)$ or $(1,3)\in E(H_{1\longrightarrow 8})$. If $(1,3)\in E(H_{1\longrightarrow 8})$, then $(3,7)\in E(H_{1\longrightarrow 8})$, but then $H_{1\longrightarrow 8}$ terminates at vertex 7, for otherwise vertices 2 and 6 would be missed. If $(1,2)\in E(H_{1\longrightarrow 8})$, then either $(2,6),(6,7),(7,3)\in E(H_{1\longrightarrow 8})$ or $(2,3),(3,7)\in E(H_{1\longrightarrow 8})$, but then $H_{1\longrightarrow 8}$ terminates at vertices 3 and 7, respectively. This is a contradiction.

Claim 2. $T_{12}\langle 1,2,4\,;8\rangle$ is non-Hamiltonian.

Assume, to the contrary, that $T_{12}\langle 1,2,4\,;8\rangle$ is Hamiltonian, and let $H=H_{1\longrightarrow 12}\cup H_{12\longrightarrow 1}$ be a Hamiltonian cycle in $T_{12}\langle 1,2,4\,;8\rangle$. Let $V(H_{12\longrightarrow 1}\setminus\{1,12\})=V_1\cup V_2\cup\cdots\cup V_k$, where each $V_{i\in\{1,2,\cdots,k\}}$ is a disjoint set of successive vertices. Since $H_{1\longrightarrow 12}$ has no increasing edge of length 3 or of length greater than 4. Clearly, for each V_i, we have either $|V_i|=1$ or $|V_i|=3$.

Let A be the set of all decreasing edges in $T_{12}\langle 1,2,4\,;8\rangle$, i.e., $A=\{(12,4),(11,3),(10,2),(9,1)\}$, so $|A|=4$. Let B be the set of all decreasing edges in $H_{12\longrightarrow 1}$, then clearly $B\subseteq A$. Since $d^-(1)=1=d^+(12)$ in $T_{12}\langle 1,2,4\,;8\rangle$, so $(12,4),(9,1)\in B$. But $H_{12\longrightarrow 1}$ cannot have only these two edges as its decreasing edges, because otherwise there must be a path $P_{4\longrightarrow 9}$ in $H_{12\longrightarrow 1}$, but this is not possible as otherwise, for some V_i, we have either $|V_i|=2$ or $|V_i|>3$. Thus $3\leq |B|\leq 4$.

Case 1. If $|B|=3$. Since $(12,4),(9,1)\in B$, then two subcases arise.

(i) $(11,3)\in B$. Then $H_{12\longrightarrow 1}=(12,4)\cup P_{4\longrightarrow 11}\cup(11,3)\cup P_{3\longrightarrow 9}\cup(9,1)$. Here, $P_{4\longrightarrow 11}\in\{(4,5,7,11),(4,6,10,11)\}$. If $P_{4\longrightarrow 11}=(4,5,7,11)$, but then $P_{3\longrightarrow 9}$ terminates at vertex 3. If $P_{4\longrightarrow 11}=(4,6,10,11)$, then $P_{3\longrightarrow 9}=(3,7,9)$, but then $|V_i|=2$, for some i, in $H_{12\longrightarrow 1}$, which is not possible.

(ii) $(10,2)\in B$. Then $H_{12\longrightarrow 1}=(12,4)\cup P_{4\longrightarrow 10}\cup(10,2)\cup P_{2\longrightarrow 9}\cup(9,1)$. Here, $P_{4\longrightarrow 10}\in\{(4,5,6,10),(4,6,10),(4,8,10)\}$. If $P_{4\longrightarrow 11}=(4,5,6,10)$, but then $P_{2\longrightarrow 9}$ terminates at vertex 2. If $P_{4\longrightarrow 10}=(4,6,10)$, then $P_{2\longrightarrow 9}=(2,3,7,9)$, but then $V_i=2$, for some i, in $H_{12\longrightarrow 1}$. If $P_{4\longrightarrow 11}=(4,8,10)$, but then $P_{2\longrightarrow 9}$ terminates at vertex 3 or 6, this is a contradiction.

Case 2. If $|B|=4$, then $A=B$, but then we have only two possible paths for $H_{12\longrightarrow 1}$.

(i) $H_{12\longrightarrow 1}=(12,4)\cup P_{4\longrightarrow 11}\cup(11,3)\cup P_{3\longrightarrow 10}\cup(10,2)\cup P_{2\longrightarrow 9}\cup(9,1)$, then the only possible path for $P_{4\longrightarrow 11}$ is $(4,6,11)$ but then there is no path $P_{3\longrightarrow 10}$ as otherwise $|V_i|>3$ for some i.

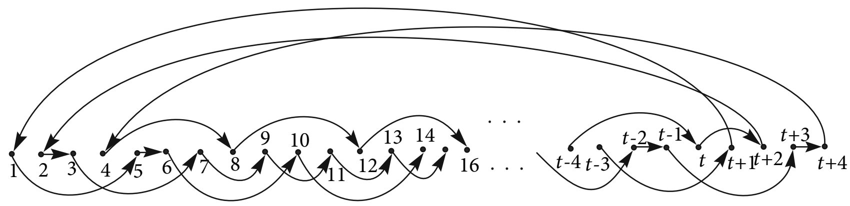

FIGURE 8: A Hamiltonian cycle in $T_{t+4}\langle 1,2,4\,;t\rangle$, $t\cong 0 \bmod 4$.

(ii) $H_{12\longrightarrow 1}=(12,4)\cup P_{4\longrightarrow 10}\cup(10,2)\cup P_{2\longrightarrow 11}\cup(11,3)\cup P_{3\longrightarrow 9}\cup(9,1)$, then the only possible path for $P_{4\longrightarrow 10}$ is $(4,6,10)$ but then there is no path $P_{2\longrightarrow 11}$ as otherwise $|V_i|>3$ for some i. So this is a contradiction.

By Lemma 12, $T_{t+2}\langle 1,2,4\,;t\rangle$ is non-Hamiltonian. This together with Claim 1 and Claim 2 shows that $T_n\langle 1,2,4\,;t\rangle$ is non-Hamiltonian if $n=t+4$ for $t\in\{4,8\}$, and if $n=t+2$.

This finishes the proof. □

Now, in Theorem 17, we will discuss the Hamiltonicity of $T_n\langle 1,2,4\,;t\rangle$ for odd t and we will be using the following known results of the literature in the proof of Theorem 17.

Theorem 14. *$T_n\langle 1,2\,;3\rangle$ is Hamiltonian if and only if $n=5$ or $n\cong 1$ mod 3.*

Theorem 15. *$T_n\langle 1,2\,;5\rangle$ is Hamiltonian if and only if $n\notin\{8,10,12,13,15,18,20,23,28\}$.*

Theorem 16. *For odd $t\geq 7$, $T_n\langle 1,2\,;t\rangle$ is Hamiltonian if and only if $n\notin\{t+3,t+5,\cdots,2t,2t+2,2t+3,2t+5,3t+5\}$.*

Theorem 17.

(1) $T_n\langle 1,2,4\,;3\rangle$ is Hamiltonian if and only if $n\cong 1$ mod 4 or $n\cong 1$ mod 3.

(2) $T_n\langle 1,2,4\,;5\rangle$ is Hamiltonian if and only if $n\notin\{8,12\}$.

(3) For odd $t\geq 7$, $T_n\langle 1,2,4\,;t\rangle$ is Hamiltonian if $n\notin\{t+3,t+5,\cdots,2t-2,2t+2\}$.

Proof.

(1) If $n\cong 1$ mod 3, then by using Theorem 14, $T_n\langle 1,2,4\,;3\rangle$ is Hamiltonian. If $n\cong 1$ mod 4, then a Hamiltonian cycle in $T_n\langle 1,2,4\,;3\rangle$ is $(1,3,7,\cdots,n-2,n,n-3,n-1,\cdots,n-4q,n-4q-3,n-4q-1,\cdots,5,2,4,1)$, where $q\in\mathbb{Z}^+$, see an example in Figure 13

Conversely, suppose $H=H_{1\longrightarrow n}\cup H_{n\longrightarrow 1}$ is a Hamiltonian cycle in $T_n\langle 1,2,4\,;3\rangle$. Since $H_{1\longrightarrow n}$ has edges of length 1, 2, and 4, only, $H_{n\longrightarrow 1}$ uses either all decreasing edges of length 3, i.e., $H_{n\longrightarrow 1}=(n,n-3,n-6,\cdots 4,1)$, or decreasing edges of length 3 along with increasing edges of length 2 (a decreasing edge of length 3, then an increasing edge of length 2, and then again an increasing edge of length 3), i.e., $H_{n\longrightarrow 1}=(n,n-3,n-1,n-4,\cdots,n-4q,n-4q-3,n-4q-1,\cdots,5,2,4,1)$, where $q\in\mathbb{Z}^+$, see Figure 14. This implies that $n-1$ is either a multiple of 3 or a multiple of 4. Thus $n\cong 1$ mod 4 or $n\cong 1$ mod 3.

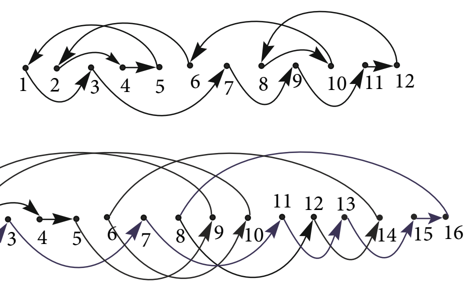

FIGURE 9: Hamiltonian cycles in $T_{12}\langle 1,2,4\,;4\rangle$ and $T_{16}\langle 1,2,4\,;8\rangle$.

(2) First, we show that $T_n\langle 1,2,4\,;5\rangle$ is Hamiltonian if $n\notin\{8,12\}$. By using Theorem 15, $T_n\langle 1,2,4\,;5\rangle$ is Hamiltonian if $n\notin\{8,10,12,13,15,18,20,23,28\}$. Now we show that $T_n\langle 1,2,4\,;5\rangle$ is Hamiltonian for $n\in\{10,13,15,18,20,23,28\}$. Let H_n be a Hamiltonian cycle in $T_n\langle 1,2,4\,;5\rangle$, then we have

$H_{10}=(1,2,3,7,8,10,5,9,4,6,1)$,
$H_{13}=(1,3,5,9,10,11,13,8,12,7,2,4,6,1)$,
$H_{15}=(1,2,3,4,5,7,8,12,13,15,10,14,9,11,6,1)$,
$H_{18}=(1,2,3,5,7,8,10,11,15,16,18,13,17,12,14,9,4,6,1)$

,

$H_{20}=(1,2,3,4,5,7,8,9,10,12,13,17,18,20,15,19,14,16,11,6,1)$

,

$H_{23}=(1,2,3,5,7,8,10,11,12,13,15,16,20,21,23,18,22,17,19,14,9,4,6,1)$

, and

$H_{28}=(1,2,3,5,7,8,10,11,12,13,15,16,17,18,20,21,25,26,28,23,27,22,24,19,14,9,4,6,1)$

, see Figure 15.

Conversely, we show that $T_n\langle 1,2,4\,;5\rangle$ is non-Hamiltonian for $n\in\{8,12\}$.

Claim 1. $T_8\langle 1,2,4\,;5\rangle$ is non-Hamiltonian.

Assume, to the contrary, that $T_8\langle 1,2,4\,;5\rangle$ is Hamiltonian and let $H=H_{1\longrightarrow 8}\cup H_{8\longrightarrow 1}$ be a Hamiltonian cycle in $T_8\langle 1,2,4\,;4\rangle$. Since $d^-(v)=1=d^+(v)$ for every vertex v in H, so $(8,3),(6,1)\in E(H_{8\longrightarrow 1})$. Then $H_{8\longrightarrow 1}$ is either $(8,3,7,2)$ or $(8,3,5,7,2)$, but in both cases, the path would be stuck at vertex 2. This is a contradiction.

Claim 2. $T_{12}\langle 1,2,4\,;5\rangle$ is non-Hamiltonian.

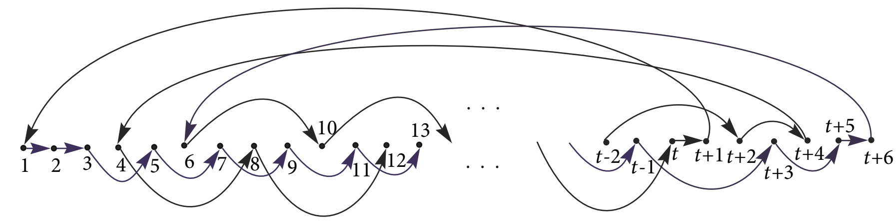

Figure 10: Hamiltonian cycles in $T_{t+6}\langle 1, 2, 4\,; t\rangle$, $t \cong 0 \bmod 4$.

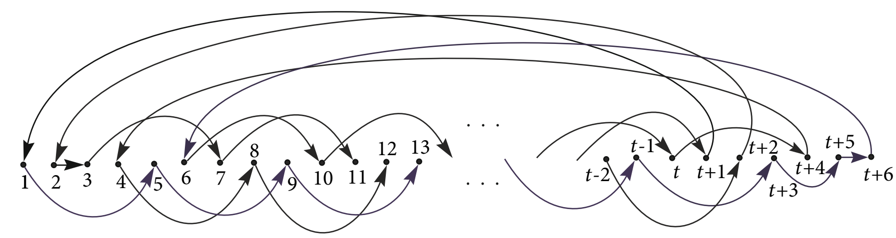

Figure 11: Hamiltonian cycles in $T_{t+6}\langle 1, 2, 4\,; t\rangle$; $t \cong 2 \bmod 4$.

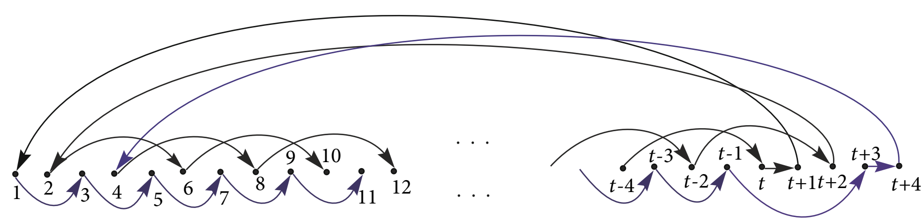

Figure 12: Hamiltonian cycles in $T_{t+4}\langle 1, 2, 4\,; t\rangle$, $t \cong 2 \bmod 4$.

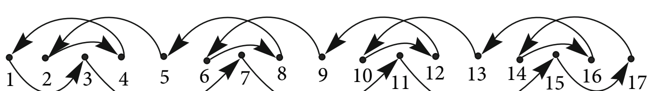

Figure 13: A Hamiltonian cycle in $T_{17}\langle 1, 2, 4\,; 3\rangle$.

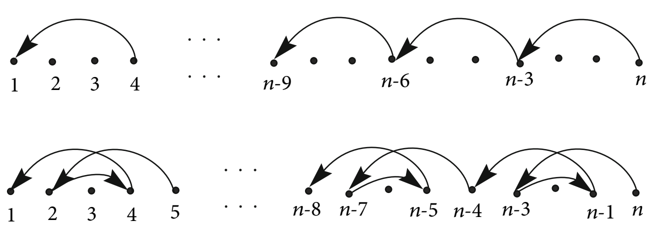

Figure 14: Possible paths $H_{n\longrightarrow 1}$ in $T_n\langle 1, 2, 4\,; 3\rangle$

Assume, to the contrary, that $T_{12}\langle 1, 2, 4\,; 5\rangle$ has a Hamiltonian cycle $H = H_{1\longrightarrow 12} \cup H_{12\longrightarrow 1}$. Let $V(H_{12\longrightarrow 1} \setminus \{1, 12\}) = V_1 \cup V_2 \cup \cdots \cup V_k$, where each $V_{i\in\{1,2,\cdots,k\}}$ is a disjoint set of successive vertices. Clearly, for each V_i, we have either $|V_i| = 1$ or $|V_i| = 3$, because $H_{1\longrightarrow 12}$ has no increasing edge of length 3 or of length greater than 4.

Let A be the set of all decreasing edges in $T_{12}\langle 1, 2, 4\,; 5\rangle$, i.e., $A = \{(12, 7), (11, 6), (10, 5), (9, 4), (8, 3), (7, 2), (6, 1)\}$, so $|A| = 7$. Let B be the set of all decreasing edges in $H_{12\longrightarrow 1}$, then clearly $B \subseteq A$. Since $d^-(1) = 1 = d^+(12)$ in $T_{12}\langle 1, 2, 4\,; 5\rangle$, so $(12, 7), (6, 1) \in B$. But $H_{12\longrightarrow 1}$ cannot have only these two edges as its decreasing edges, because otherwise, there must be a subpath $P_{6\longrightarrow 7}$ in $H_{12\longrightarrow 1}$, but this is not possible here, so $|B| \geq 3$. We also observe that $|B| \notin \{6, 7\}$, as otherwise, $|V_i| > 3$ for some V_i. Thus $3 \leq |B| \leq 5$.

Case 1. If $|B| = 3$. Since $(12, 7), (6, 1) \in B$, then $C_2^5 = 5$ subcases arise.

(i) $(11, 6) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7) \cup P_{7\longrightarrow 11} \cup (11, 6, 1)$. By keeping in mind that there is no V_i in $H_{12\longrightarrow 1}$ such that $|V_i| > 3$, so here, $P_{7\longrightarrow 11} \in \{(7, 11), (7, 9, 11), (7, 8, 10, 11)\}$. But in all of these subpaths, we have $|V_i| = 2$, for some i, in $H_{12\longrightarrow 1}$, which is not possible.

(ii) $(10, 5) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7) \cup P_{7\longrightarrow 10} \cup (10, 5) \cup P_{5\longrightarrow 6} \cup (6, 1)$. Here, $P_{7\longrightarrow 10} = (7, 9, 10)$ and $P_{5\longrightarrow 6} = (5, 6)$. But then, for some i, $|V_i| = 2$ (say $V_i = \{9, 10\}$) in $H_{12\longrightarrow 1}$, which is a contradiction.

(iii) $(9, 4) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7) \cup P_{7\longrightarrow 9} \cup (9, 4) \cup P_{4\longrightarrow 6} \cup (6, 1)$. Here, $P_{7\longrightarrow 9} = (7, 9)$ and $P_{4\longrightarrow 6} = (4, 6)$. But then, for some i, $|V_i| = 2$ (say $V_i = \{6, 7\}$) in $H_{12\longrightarrow 1}$, which is a contradiction.

(iv) $(8, 3) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7) \cup P_{7\longrightarrow 8} \cup (8, 3) \cup P_{3\longrightarrow 6} \cup (6, 1)$. Here $P_{7\longrightarrow 8} = (7, 8)$ and $P_{3\longrightarrow 6} = (3, 4, 6)$. But then, for some i, $|V_i| = 2$ (say $V_i = \{3, 4\}$) in $H_{12\longrightarrow 1}$, which is a contradiction.

(v) $(7, 2) \in B$. Then $H_{12\longrightarrow 1} = (12, 7, 2) \cup P_{2\longrightarrow 6} \cup (6, 1)$. Here, $P_{2\longrightarrow 6} \in \{(2, 6), (2, 4, 6), (2, 3, 5, 6)\}$. But then, for some i, $|V_i| = 2$ (say $V_i = \{3, 4\}$ or $V_i = \{6, 7\}$) in $H_{12\longrightarrow 1}$, which is a contradiction.

Case 2. If $|B| = 4$. Since $(12, 7), (6, 1) \in B$, then $C_2^5 = 10$ subcases arise.

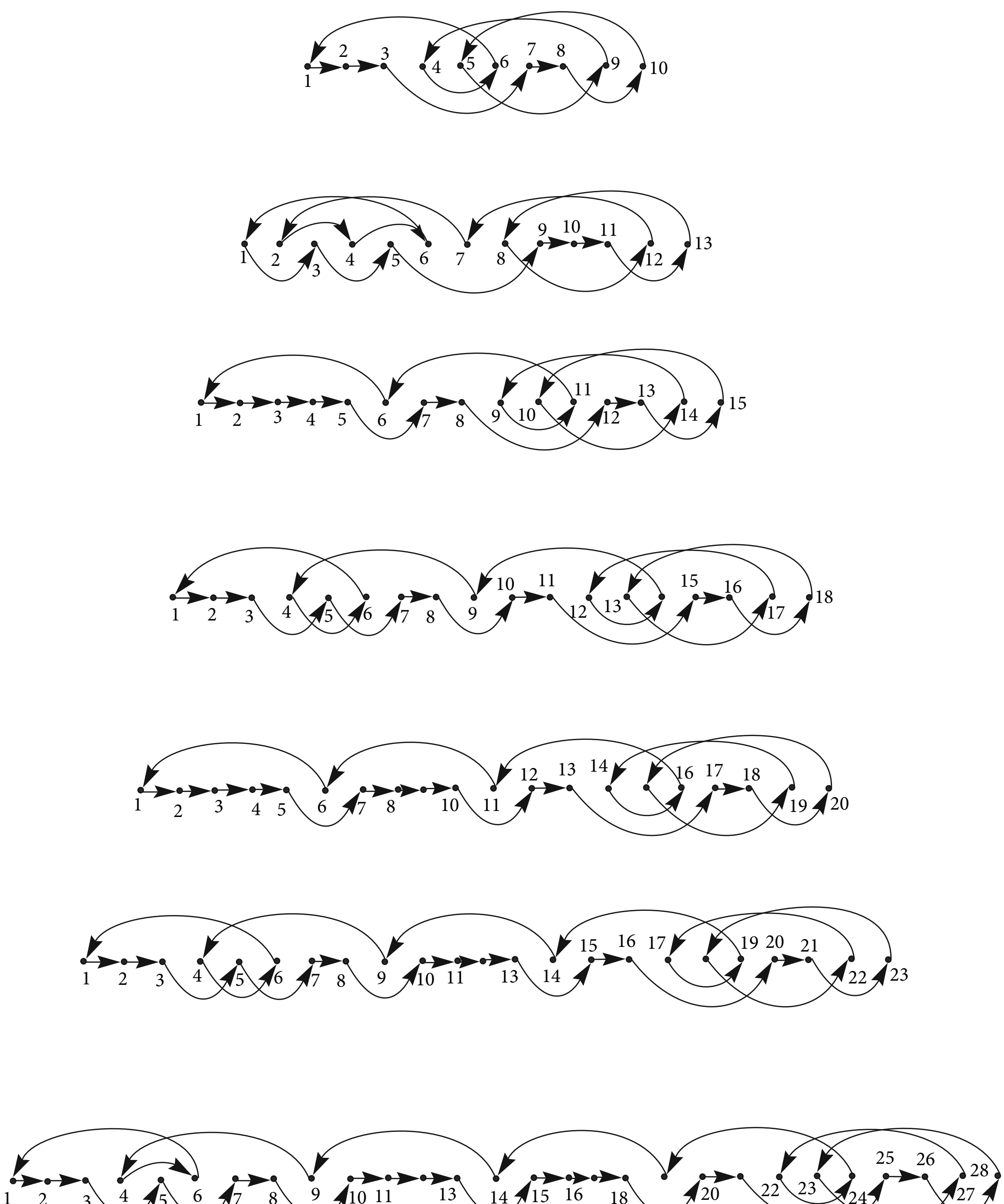

FIGURE 15: Hamiltonian cycles in $T_n\langle 1, 2, 4\,; 5\rangle$, $n \in \{10,13,15,18,20,23,28\}$.

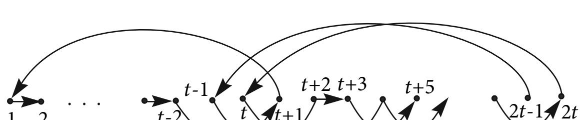

FIGURE 16: A Hamiltonian cycle in $T_{2t}\langle 1, 2, 4\,; t\rangle$.

(i) $(11, 6), (10, 5) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7) \cup P_{7\longrightarrow 10} \cup (10, 5) \cup P_{5\longrightarrow 11} \cup (11, 6, 1)$. By keeping in mind that there is no V_i in $H_{12\longrightarrow 1}$ such that $|V_i| > 3$, so $P_{7\longrightarrow 10} = (7, 10)$, but then the path $H_{5\longrightarrow 11}$ would be stuck at vertex 5.

(ii) $(11, 6), (9, 4) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7) \cup P_{7\longrightarrow 9} \cup (9, 4) \cup P_{4\longrightarrow 11} \cup (11, 6, 1)$. Due to the same reason, here $P_{7\longrightarrow 9} = (7, 9)$ but then $P_{4\longrightarrow 11}$ would be stuck at vertex 4.

(iii) $(11, 6), (8, 3) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7) \cup P_{7\longrightarrow 8} \cup (8, 3) \cup P_{3\longrightarrow 11} \cup (11, 6, 1)$. Here, $P_{7\longrightarrow 8} = (7, 8)$, but then $P_{3\longrightarrow 11}$ terminates at vertex 4.

(iv) $(11, 6), (7, 2) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7, 2) \cup P_{2\longrightarrow 11} \cup (11, 6, 1)$. Here, $P_{2\longrightarrow 11} \in$

$\{(2, 4, 8, 10, 11), (2, 3, 4, 8, 10, 11), (2, 3, 5, 9, 10, 11), (2, 3, 5, 9, 11)\}$, but in all of these subpaths, for some i, $|V_i| = 2$ (say $V_i = \{10, 11\}$ or $V_i = \{2, 3\}$) in $H_{12\longrightarrow 1}$.

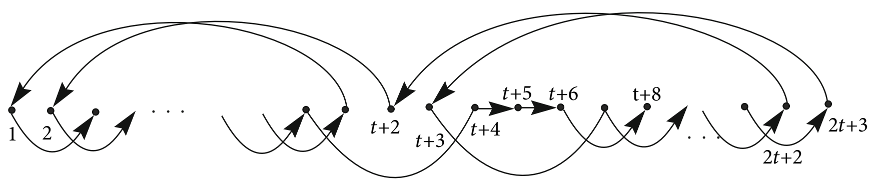

Figure 17: A Hamiltonian cycle in $T_{2t+3}\langle 1, 2, 4\,; t\rangle$.

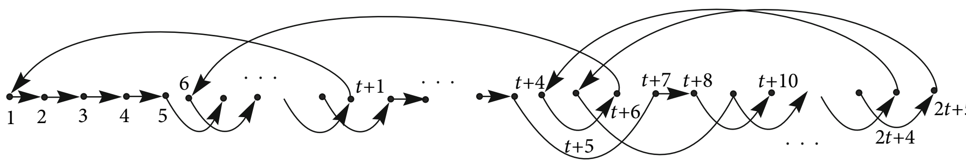

Figure 18: A Hamiltonian cycle in $T_{2t+5}\langle 1, 2, 4\,; t\rangle$.

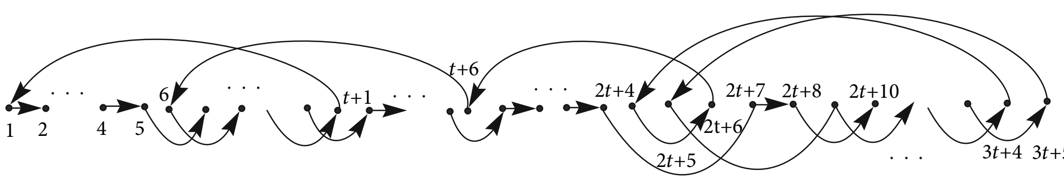

Figure 19: A Hamiltonian cycle in $T_{3t+5}\langle 1, 2, 4\,; t\rangle$.

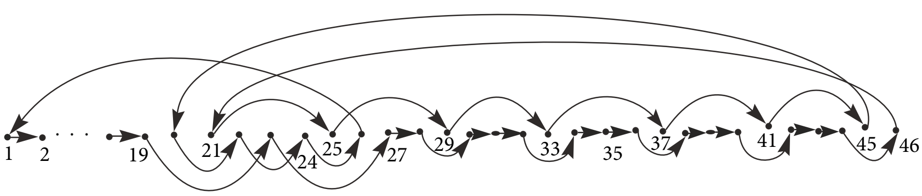

Figure 20: A Hamiltonian cycle in $T_{46}\langle 1, 2, 4\,; 25\rangle$.

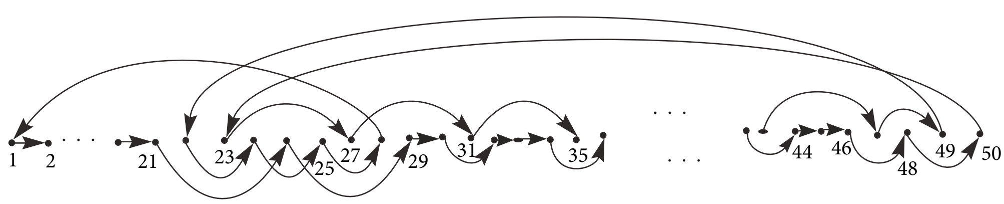

Figure 21: A Hamiltonian cycle in $T_{50}\langle 1, 2, 4\,; 27\rangle$.

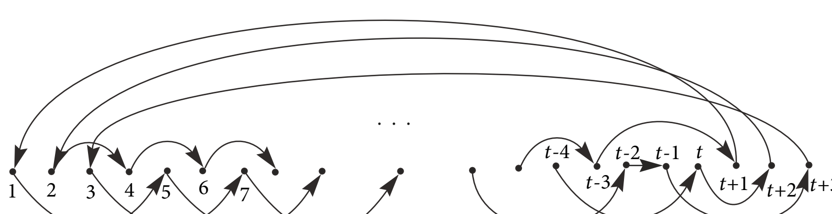

Figure 22: A Hamiltonian cycle in $T_{t+3}\langle 1, 2, 4\,; t\rangle$; $t \cong 3 \bmod 4$.

(v) $(10, 5), (7, 2) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7, 2) \cup P_{2\longrightarrow 10} \cup (10, 5) \cup P_{5\longrightarrow 6} \cup (6, 1)$. Here, $P_{5\longrightarrow 6} = (5, 6)$, but $P_{2\longrightarrow 5}$ terminates at vertex 3.

(vi) $(9, 4), (7, 2) \in B$. Then, $H_{12\longrightarrow 1} = (12, 7, 2) \cup P_{2\longrightarrow 9} \cup (9, 4) \cup P_{4\longrightarrow 6} \cup (6, 1)$. Here, $P_{4\longrightarrow 6} = (4, 6)$, but $P_{2\longrightarrow 9}$ terminates at vertex 3.

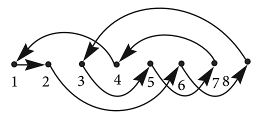

Figure 23: A Hamiltonian cycle in $T_8\langle 1,2,4;4,5\rangle$.

(vii) $(9,4),(7,2)\in B$. Then, $H_{12\longrightarrow 1}=(12,7,2)\cup P_{2\longrightarrow 8}\cup(8,3)\cup P_{3\longrightarrow 6}\cup(6,1)$. Here, $P_{3\longrightarrow 6}=(3,4,6)$, but then $P_{2\longrightarrow 8}$ would be stuck at vertex 2.

(viii)–(x) If $(10,5),(9,4)\in B$ or $(10,5),(8,3)\in B$ or $(9,4),(8,3)\in B$, then clearly $|V_i|>3$ for some i, say $V_i=\{4,5,6,7\}$, or $V_i=\{5,6,7,8\}$, or $V_i=\{6,7,8,9\}$, respectively. This is a contradiction.

Case 3. If $|B|=5$. Since $(12,4),(9,1)\in B$, then $C_3^5=10$ subcases arise.

(i) $(11,6),(10,5),(7,2)\in B$. By keeping in mind that there is no V_i in $H_{12\longrightarrow 1}$ such that $|V_i|>3$, so $H_{12\longrightarrow 1}=(12,7,2),P_{2\longrightarrow 10}\cup(10,5)\cup P_{5\longrightarrow 11}\cup(11,6,1)$. But the subpath $P_{2\longrightarrow 10}$ would be stuck at vertex 2 or 3.

(ii) $(11,6),(9,4),(7,2)\in B$. Here, $H_{12\longrightarrow 1}=(12,7,2)$, $P_{2\longrightarrow 9}\cup(9,4)\cup P_{4\longrightarrow 11}\cup(11,6,1)$, but $P_{2\longrightarrow 9}$ terminates at vertex 3.

(iii) $(11,6),(8,3),(7,2)\in B$. Here, $H_{12\longrightarrow 1}=(12,7,2)$, $P_{2\longrightarrow 8}\cup(8,3)\cup P_{3\longrightarrow 11}\cup(11,6,1)$, but $P_{2\longrightarrow 8}$ terminates at vertex 4.

(iv) $(10,5),(8,3),(7,2)\in B$. Here, $H_{12\longrightarrow 1}=(12,7,2)$, $P_{2\longrightarrow 8}\cup(8,3)\cup P_{3\longrightarrow 10}\cup(10,5,6,1)$, but $P_{2\longrightarrow 8}$ would be stuck at vertex 2.

(v)–(x) If $(11,6),(10,5),(9,4)\in B$ or $(11,6),(10,5),(8,3)\in B$ or $(11,6),(9,4),(8,3)\in B$ or $(10,5),(9,4),(8,3)\in B$ or $(10,5),(9,4),(7,2)\in B$, or $(9,4),(8,3),(7,2)\in B$, then clearly $|V_i|>3$, for some i, this is a contradiction.

(3) If $t\geq 7$. By using Theorem 16, for odd $t\geq 7$, $T_n\langle 1,2,4;t\rangle$ is Hamiltonian if $n\notin\{t+3,t+5,\cdots,2t,2t+2,2t+3,2t+5,3t+5\}$. Now we show that $T_n\langle 1,2,4;t\rangle$ is Hamiltonian for $n\in\{2t,2t+3,2t+5,3t+5\}$. A Hamiltonian cycle in $T_{2t}\langle 1,2,4;t\rangle$ is $(1,2,\cdots,t-2,t+2,t+3,t+5,\cdots,2t,t,t+4,t+6,t+8,2t-1,t-1,t+1,1)$, see Figure 16.

A Hamiltonian cycle in $T_{2t+3}\langle 1,2,4;t\rangle$ is $(1,3,\cdots,t,t+4,t+5,t+6,t+8,2t+3,t+3,t+7,t+9,\cdots,2t+2,t+2,2,4,\cdots,t+1,1)$, see Figure 17.

A Hamiltonian cycle in $T_{2t+5}\langle 1,2,4;t\rangle$ is $(1,2,3,4,5,7,\cdots,t+2,t+3,t+7,t+8,t+10,2t+5,t+5,t+9,t+11,\cdots,2t+4,t+4,t+6,6,8,\cdots,t+1,1)$, see Figure 18. And a Hamiltonian cycle in $T_{3t+5}\langle 1,2,4;t\rangle$ is $(1,2,3,4,5,7,\cdots,t+2,t+3,t+4,t+5,\cdots,t+7,t+8,\cdots,2t+3,2t+7,2t+8,2t+10,\cdots,3t+5,2t+5,2t+9,2t+11,\cdots,3t+4,2t+4,2t+6,t+6,6,8,\cdots,t+1,1)$, see Figure 19.

This finished the proof. □

In Theorem 17, we saw that for odd $t\geq 7$, $T_n\langle 1,2,4;t\rangle$ is Hamiltonian if $n\notin\{t+3,t+5,\cdots,2t-2,2t+2\}$. In the following theorem, we study the Hamiltonicity of $T_n\langle 1,2,4;t\rangle$ for $n\in\{t+3,t+5,\cdots,2t-2,2t+2\}$ and odd $t\geq 7$, with some restriction on n.

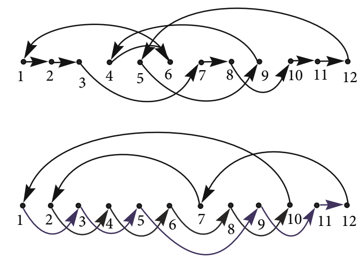

Figure 24: Hamiltonian cycles in $T_{12}\langle 1,2,4;5,7\rangle$ and $T_{12}\langle 1,2,4;5,9\rangle$.

Theorem 18. *For odd $t\geq 7$, $T_n\langle 1,2,4;t\rangle$ is Hamiltonian if $n\in\{t+3,t+5,\cdots,2t-2,2t+2\}$ and $n\cong 2 \bmod 4$.*

Proof. For odd $t\geq 7$, let $n\in\{t+3,t+5,\cdots,2t-2,2t+2\}$ and $n\cong 2 \bmod 4$. We show that $T_n\langle 1,2,4;t\rangle$ is Hamiltonian.

Case 1. If $t\cong 1 \bmod 4$, then a Hamiltonian cycle in $T_n\langle 1,2,4;t\rangle$ is $(1,2,\cdots,n-t-2,n-t+2,\cdots,t+2,t+3,P_3(t+5),P_3(t+9),\cdots,P_3(n-4),n,n-t,n-t+4,\cdots,n-1,n-1-t,n-t+1,\cdots,t+1,1)$, see an example in Figure 20.

Case 2. If $t\cong 3 \bmod 4$. For $n\neq t+3$, a Hamiltonian cycle in $T_n\langle 1,2,4;t\rangle$ is $(1,2,\cdots,n-t-2,n-t+2,\cdots,t+2,t+3,P_3(t+5),P_3(t+9),\cdots,P_3(n-6),n-2,n,n-t,n-t+4,\cdots,n-3,n-1,n-1-t,n-t+1,\cdots,t+1,1)$, see an example in Figure 21.

A Hamiltonian cycle in $T_{t+3}\langle 1,2,4;t\rangle$ is $(1,5,\cdots,t-2,t-1,t+3,3,7,\cdots,t,t+2,2,4,\cdots,t-3,t+1,1)$, see Figure 22.

This completes the proof. □

4. Toeplitz Graphs $T_n\langle 1,2,4;t_1,t_2\rangle$

Theorem 19 (see [8]). *Let $G=T_n\langle 1,2;t_1,t_2\rangle$.*

(1) If t_1 and t_2 are both even, then G is Hamiltonian if and only if n is odd.

(2) If t_1 and t_2 are of opposite parity, then G is Hamiltonian for all n.

(3) If t_1 and t_2 are both odd, and

(a) if $t_2\geq 2t_1+1$, then G is Hamiltonian for all n.

(b) if $t_2<2t_1+1$, then G is Hamiltonian if $n\notin\{t_2+3,t_2+5,\cdots,2t_1+2\}$.

In Theorem 19, the Hamiltonicity of $T_n\langle 1,2;t_1,t_2\rangle$ have been studied. Now we will see, what happens, if we add one more diagonal (containing one) above the main diagonal, say $s_3=4$. So here we discuss the Hamiltonicity of $T_n\langle 1,2,4;t_1,t_2\rangle$.

Theorem 20. *For even t_1 and even t_2, $T_n\langle 1,2,4;t_1,t_2\rangle$ is Hamiltonian if and only if n is odd. Otherwise it is Hamiltonian for all n.*

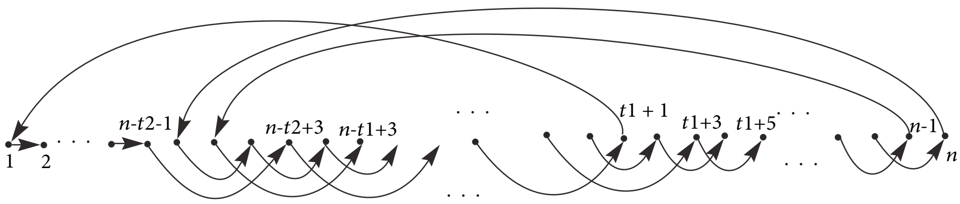

FIGURE 25: A Hamiltonian cycle in $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$; $t_2 = t_1 + 2$, odd t_1, and t_2.

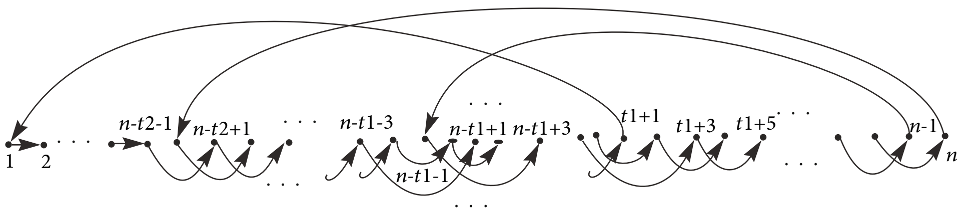

FIGURE 26: A Hamiltonian cycle in $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$; $t_2 \neq t_1 + 2$, odd t_1, and t_2.

Proof. Case 1. If t_1 and t_2 are both even, then by using Theorem 19, $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is Hamiltonian for all odd n. Conversely, suppose that $T_n\langle 1, 2 ; t_1, t_2\rangle$ has a Hamiltonian cycle $H = H_{1\longrightarrow n} \cup H_{n\longrightarrow 1}$. Now, we show that n is odd. Assume, to the contrary, that n is even. Since $H_{1\longrightarrow n}$ cannot use an increasing edge of length 3 or of length greater than 4, $H_{n\longrightarrow 1}$ cannot use any increasing edges of the types $(v, v+1)$ or $(v, v+1, v+2, v+3)$. Hence, $H_{n\longrightarrow 1}$ uses only edges of even length, and therefore vertices of the same parity, and this implies that n is odd, because otherwise $H_{n\longrightarrow 1}$ would never end up to vertex 1. This is a contradiction; thus, n is odd.

Case 2. If t_1 and t_2 are of opposite parity, then by using Theorem 19, $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is Hamiltonian for all n.

Case 3. If t_1 and t_2 are both odd and $t_2 \geq 2t_1 + 1$, then by using Theorem 19, $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is Hamiltonian for all n. Now assume $t_2 < 2t_1 + 1$, where both t_1 and t_2 are odd, then by using Theorem 19, $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is Hamiltonian for $n \notin \{t_2 + 3, t_2 + 5, \cdots, 2t_1 + 2\}$.

Now we prove that, for both odd t_1 and $t_2 < 2t_1 + 1$, $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is Hamiltonian for $n \in \{t_2 + 3, t_2 + 5, \cdots, 2t_1 + 2\}$. Clearly, here, n is an even integer such that $t_2 + 3 \leq n \leq 2t_1 + 2$ and $t_1 \geq 3$ (as if $t_1 = 1$, then $t_2 \nless 2t_1 + 1$). If $t_1 = 3$, then $t_2 = 5$, and therefore $n = 8$ (because $t_2 < 2t_1 + 1$ and $n \in \{t_2 + 3, t_2 + 5, \cdots, 2t_1 + 2\}$). A Hamiltonian cycle in $T_8\langle 1, 2, 4 ; 4, 5\rangle$ is $(1, 2, 6, 8, 3, 5, 7, 4, 1)$, see Figure 23.

If $t_1 = 5$, then $t_2 \in \{7, 9\}$ and then $n \in \{t_2 + 3, t_2 + 5, \cdots, 12\}$. By using Theorem 17, $T_n\langle 1, 2, 4 ; 5, t_2\rangle$ is Hamiltonian for $n \notin \{8, 12\}$. Since here $n \geq 10$ (because $n \geq t_2 + 3 \geq 10$, as $t_2 \geq 7$), we need to consider only $n = 12$. The Hamiltonian cycles in $T_{12}\langle 1, 2, 4 ; 5, 7\rangle$ and $T_{12}\langle 1, 2, 4 ; 5, 9\rangle$ are $(1, 2, 3, 7, 8, 10, 11, 12, 5, 9, 4, 6, 1)$ and $(1, 3, 5, 9, 11, 12, 7, 2, 4, 6, 8, 10, 1)$, respectively, see Figure 24.

If $t_1 \geq 7$, then by using Theorem 17, $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is Hamiltonian for $n = 2t_1$, and by using Theorem 18, $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is Hamiltonian for $n \in \{t_2 + 3, t_2 + 5, \cdots, 2t_1 - 2, 2t_1 + 2\}$ and $n \cong 2 \bmod 4$. Now, we need to show that $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is Hamiltonian for $n \in \{t_2 + 3, t_2 + 5, \cdots, 2t_1 - 2, 2t_1 + 2\}$ and $n \cong 0 \bmod 4$. Let $n \in \{t_2 + 3, t_2 + 5, \cdots, 2t_1 - 2, 2t_1 + 2\}$ and $n \cong 0 \bmod 4$. For $t_2 = t_1 + 2$, a Hamiltonian cycle in $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is $(1, 2, \cdots, n - t_2 - 1, n - t_2 + 3, \cdots, t_1 + 3, t_1 + 5, \cdots, n, n - t_2, n - t_2 + 2, \cdots, n - 1, n - 1 - t_1, n - t_1 + 3, \cdots, t_1 + 1, 1)$, see Figure 25. And for $t_2 \neq t_1 + 2$, a Hamiltonian cycle in $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is $(1, 2, \cdots, n - t_2 - 1, n - t_2 + 1, \cdots, n - t_1 - 3, n - t_1 + 1, \cdots, t_1 + 3, t_1 + 5, \cdots, n, n - t_2, n - t_2 + 2, \cdots, n - 1, n - 1 - t_1, n - t_1 + 3, \cdots, t_1 + 1, 1)$, see Figure 26. Thus, if t_1 and t_2 are both odd, $T_n\langle 1, 2, 4 ; t_1, t_2\rangle$ is Hamiltonian for all n.

This completes the proof. □

Conjecture: for odd $t \geq 7$, $T_n\langle 1, 2, 4 ; t\rangle$ is non-Hamiltonian if $n \in \{t + 3, t + 5, \cdots, 2t - 2, 2t + 2\}$ and $n \cong 0 \bmod 4$.

5. Concluding Remark

An affirmative resolution of the conjectures for $T_n\langle 1, 2, 4 ; t\rangle$ will complete the study of the Hamiltonicity of the Toeplitz graph with $s_1 = 1$ and $s_3 = 4$. The next task in our opinion is to investigate the Hamiltonicity of the Toeplitz graph with $s_1 = 2$ and $s_3 = 4$, which will then complete the Hamiltonicity investigation in the Toeplitz graph with $s_3 = 4$. To make this paper not very long, we have not discussed that case here.

Data Availability

No data has been used for producing the result of this paper.

Conflicts of Interest

The author declares that she has no conflicts of interest.

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