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Research Article

A Fitted Numerical Approach for Singularly Perturbed Two-Parameter Parabolic Problem with Time Delay

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This paper is aimed at constructing and analyzing a fitted approach for singularly perturbed time delay parabolic problems with two small parameters. The proposed computational scheme comprises the implicit Euler and especially finite difference method for the time and space variable discretization, respectively, on uniform step size. The stability and convergence analysis of the method is provided and is first-order parameter uniform convergent. Further, the numerical results depict that the present method is more convergent than some methods available in the literature.

1. Introduction

Consider one-dimensional unsteady singularly perturbed time delay parabolic problems with two parameters on the domain $\nabla = \nabla_s \times \nabla_t = (0, 1) \times (0, T]$ of the form:

$$\begin{cases} \$_{\varepsilon,\mu} w(s,t) \equiv \varepsilon \dfrac{\partial^2 w(s,t)}{\partial s^2} + \mu a(s,t)\dfrac{\partial w(s,t)}{\partial s} - \delta(s,t)w(s,t) - \dfrac{\partial w(s,t)}{\partial t} = -c(s,t)w(s,t-\lambda) + f(s,t), (s,t) \in \nabla_s, \\ w(s,t) = \varphi(s,t), (s,t) \in [0,1] \times [-\lambda, 0], \\ w(0,t) = \chi_0(t), t \in \bar{\nabla}_t, \\ w(1,t) = \chi_1(t), t \in \bar{\nabla}_t, \end{cases} \quad (1)$$

where $\varepsilon (0 < \varepsilon \ll 1)$ and $\mu (0 \leq \mu \ll 1)$ are two small perturbation parameters and $\lambda > 0$ is the delay parameter. For the existence and uniqueness of the solution, the functions $a(s,t)$, $\delta(s,t)$, $c(s,t)$, $f(s,t)$, $\chi_0(t)$, $\chi_1(t)$, and $\varphi(s,t)$ are sufficiently smooth and bounded with $a(s,t) \geq \alpha > 0$ and $\delta(s,t) \geq \beta > 0$. Under sufficiently smoothness and compatibility conditions on the functions $\chi_0(t)$, $\chi_1(t)$, and $\varphi(s,t)$, the IBVP (1) has a unique solution $w(s,t)$ [1, 2].

The nature of Equation (1) changes based on the values of perturbations ε and μ. If $\mu = 0$, then the problem is reaction-diffusion type [3] and boundary layers exhibit near $s = 0$ and $s = 1$ of width $O(\sqrt{(\varepsilon)})$. If $\mu = 1$, then the problem is convection-diffusion type [4, 5] and boundary layers exhibit near $s = 0$ of width $O(\varepsilon)$. These problems have several fascinating phenomena in lubrication theory [6] and chemical flow reactor theory [7, 8]. When $0 \leq \mu \leq 1$, problem (1) is modelling different phenomena in applied sciences and engineering, for instance, problems found in control theory [9], mechanical systems, population dynamics in the biosciences [10], and heat and mass transfer in chemical engineering [11].

Various parameter uniform numerical methods for singularly perturbed parabolic problems with two parameters without time lag are suggested by several scholars. For instance, the authors in [12] developed robust nonstandard finite difference method based on Micken's type discretization rule for spatial discretization and the implicit finite difference method for temporal discretization for singularly perturbed parabolic problems with two parameters. The researchers in [13, 14] suggested parameter uniform numerical methods using finite difference schemes with fitted techniques to solve singularly perturbed two parametric parabolic problems. In article [15], the authors developed a fitted mesh cubic spline in tension for singularly perturbed problems with two parameters. [16] suggested a uniformly convergent computational scheme which consists of the Crank-Nicholson method to discretize the time variable and the central difference approximation on the nonstandard methodology of Mickens for the space variable for Equation (58). Quadratic B-spline collocation method on exponentially graded mesh for two-parameter singularly perturbed problem is presented by [17]. An implicit computational method on a predefined Shishkin mesh is presented for solving two-parameter parabolic singularly perturbed boundary value problems with nonsmooth data by [18]. [19] suggested parameter uniform finite element method for two-parameter singularly perturbed parabolic reaction-diffusion problems. [20] developed nonstandard finite difference method on uniform mesh for two-parameter singular perturbation problem.

The authors in papers [21, 22] developed uniform numerical methods for a singularly perturbed reaction-convection-diffusion equation in one dimension with a discontinuous source term. In articles [23–25], the authors have developed robust numerical methods for singularly perturbed time delay parabolic problems with two parameters based on adaptive layer mesh methods. However, exponentially fitted difference (EFD) schemes have gained popularity as a powerful technique to solve boundary value problems. For instance, the authors in [26–28] suggested different EFD schemes for singularly perturbed two-point boundary value problems.

Nevertheless, the solution methodologies for singularly perturbed time delay parabolic problems with two parameters are at a primarily stage and need a lot of investigation. Therefore, to diminish the gap observed, we proposed a novel parameter uniform numerical approach formulated based on uniform mesh implicit Euler approximation for time variable and especially finite difference method for the spatial variable. The novelty of the presented method, unlike the Shishkin and Bakhvalov mesh types, does not require a priori information about the location and width of the boundary layer.

2. Properties of Continuous Solution

Lemma 1 (minimum principle). *Suppose* $w|_{\partial\nabla} \geq 0$ *and* $(\$_{\varepsilon,\mu} - \partial/\partial t)w|_{\nabla} \leq 0$, *then* $w|_{\bar{\nabla}} \geq 0$.

Proof (see [24]). An immediate consequence of minimum principle above for the solution of Equation (1) provides the next Lemma 2. □

Lemma 2 (uniform stability estimate). *Let* $w(s,t)$ *be the solution of problem (1); then, we have*

$$\|w\|_{\bar{\nabla}} \leq \max\{|\varphi(s,t)|, |\chi_0(t)|, |\chi_1(t)|\} + \beta^{-1}\|f\|. \tag{2}$$

Proof. By defining barrier functions

$$Z^{\pm}(s,t) = \max\{|\varphi(s,t)|, |\chi_0(t)|, |\chi_1(t)|\} + \beta^{-1}\|f\| \pm w(s,t), \tag{3}$$

and using Lemma 1, we get the desired bound. □

3. Construction of the Numerical Scheme

3.1. Temporal Semidiscretization. On applying the implicit Euler method to approximate the t-direction of Equation (1) with the uniform mesh, $\nabla_{\Delta t}^{M} = \{j\Delta t, j = 0, 1, 2, \cdots, M, t_M = T, \Delta t = T/M\}$ and $\nabla_t^{\theta} = \{m\Delta t, m = 0, 1, 2, \cdots, \theta, t_\theta = \lambda, \Delta t = \lambda/\theta\}$, where M is number of mesh points in t-direction in the interval $[0, T]$ and θ is the number of mesh points in $[-\lambda, 0]$. The step size Δt satisfies $\theta\Delta t = \lambda$, where θ is a natural number and $t_j = j\Delta t, j \geq -\theta$; we obtain

$$\begin{cases} W(s,-m) = \varphi(s,-t_m), \text{form} = 0, 1, \cdots, \theta, s \in \bar{\nabla}_s, \\ \left(-1 + \Delta t \$_{\varepsilon}^{M}\right) W\left(s, t_{j+1}\right) = v\left(s, t_{j+1}\right), \\ W\left(0, t_{j+1}\right) = \chi_0\left(t_{j+1}\right), W\left(1, t_{j+1}\right) = \chi_1\left(t_{j+1}\right), 0 \leq j \leq M-1, \end{cases} \tag{4}$$

where $\$_{\varepsilon}^{M} = (\varepsilon(\partial^2 W/\partial s^2) + \mu a(\partial W/\partial s) - \delta W)(s, t_{j+1})$, $v(s, t_{j+1}) = -\Delta t c(s, t_{j+1}) W(s, t_{j-\lambda+1}) + \Delta t f(s, t_{j+1}) - W(s, t_j)$.

Clearly, the operator $(-I + \Delta t \$_{\varepsilon}^{M})$ achieves the maximum principle, which confirms the stability of the semidiscrete (4).

The local truncation error (LTE) devoted in the semidiscrete scheme is the difference between the analytical solution $w(s, t_{j+1})$ and the estimate solution $W(s, t_{j+1})$ of Equation (4), i.e., $(\text{LTE})_{j+1} = w(s, t_{j+1}) - W(s, t_{j+1})$, and the global error E_{j+1} is the contribution of the local error up to the $(j+1)$th time level. The bound of error for the semidiscrete scheme is estimated as follows.

Lemma 3 (LTE). *If* $|\partial^n w(s,t)/\partial t^n| \leq C, \forall (s,t) \in \bar{\nabla}, n = 0, 1, 2,$ *then the LTE in the temporal direction gratifies*

$$\left\|(LTE)_{j+1}\right\|_{\infty} \leq C(\Delta t)^2, \tag{5}$$

where C is a positive constant independent of ε *and* Δt.

Proof. Using Taylor's series expansion for $w(s, t_{j-1})$, we have

$$w\left(s, t_{j+1}\right) = w\left(s, t_j\right) - \Delta t w_t\left(s, t_j\right) + O\left((\Delta t)^2\right). \tag{6}$$

This implies

$$\frac{w\left(s, t_{j+1}\right) - w\left(s, t_j\right)}{\Delta t} = w_t\left(s, t_j + O((\Delta t)\right). \tag{7}$$

Substituting Equation (1) into Equation (7), we have

$$\begin{aligned}\frac{w\left(s, t_{j+1}\right) - w\left(s, t_j\right)}{\Delta t} = & \left(\left(\varepsilon \frac{\partial^2 w}{\partial s^2} + a \frac{\partial w}{\partial s} - \delta w\right)\left(s, t_j\right)\right) \\ & + c\left(s, t_j\right) w\left(s, t_{j-\lambda}\right) - f\left(s, t_j\right) + O((\Delta t)) \\ \Longrightarrow & \left(-1 + \Delta t \$_\varepsilon^M\right) w\left(s, t_{j+1}\right) + \Delta t c\left(s, t_j\right) w\left(s, t_{j-\lambda}\right) \\ & - f\left(s, t_j\right) + \Delta t w\left(t, t_j\right) = O\left((\Delta t)^2\right).\end{aligned} \tag{8}$$

Subtracting Equation (4) from Equation (8), the local truncation error $(\mathrm{LTE})_{j+1} = w(s, t_{j+1}) - W(s, t_{j+1})$ at $(j+1)$ th is the solution of a boundary problem

$$\left(-1 + \Delta t \$_\varepsilon^M\right)(\mathrm{LTE})_{j+1} = O\left((\Delta t)^2\right), (\mathrm{LTE})_{j+1}(0) = 0 = (\mathrm{LTE})_j(1), \tag{9}$$

where $W(s, t_j)$ is the solution of the boundary value problem (4).

Hence, using the maximum principle on the operator provides

$$\left\|(\mathrm{LTE})_{j+1}\right\|_\infty \leq C(\Delta t)^2. \tag{10}$$

□

Lemma 4 (global error estimate (GEE)). *Under the hypothesis of Lemma 3, the GEE in the temporal direction is given by*

$$\left\|E_{j+1}\right\|_\infty \leq C(\Delta t), \forall j \leq \frac{T}{\Delta t}. \tag{11}$$

Proof. Using Lemma 3 at $(j+1)$th time step, we have

$$\begin{aligned}\left\|E_{j+1}\right\|_\infty = & \left\|\sum_{i=1}^{j} (\mathrm{LTE})_i\right\|_\infty, j \leq \frac{T}{\Delta t} \leq \left\|(\mathrm{LTE})_1\right\|_\infty + \left\|(\mathrm{LTE})_2\right\|_\infty \\ & + \left\|(\mathrm{LTE})_3\right\|_\infty + \cdots + \left\|(\mathrm{LTE})_j\right\|_\infty \leq c_1 j (\Delta t)^2 \leq c_1 (j) \Delta t)(\Delta t) \\ & \leq c_1 T(\Delta t)(j\Delta t \leq T) \leq C(\Delta t),\end{aligned} \tag{12}$$

where c_1 and C are the positive constants independent of ε and Δt.

Rewrite Equation (4) as

$$\begin{cases} W(s,-m) = \varphi(s,-t_m), \text{form} = 0, 1, \cdots, \theta, s \in \bar{\mathfrak{J}}_s, \\ \varepsilon \dfrac{d^2 W^{j+1}(s)}{ds^2} + a(s) \dfrac{dY^{j+1}(s)}{ds} - q(s) W^{j+1}(s) = p^{j+1}(s), 0 \leq s \leq 1, \\ W^{j+1}(0) = \chi_0^{j+1}, W^{j+1}(1) = \chi_1^{j+1}, 0 < j < M-1, \end{cases} \tag{13}$$

where

$$\begin{aligned} q(s) &= \left(\delta(s) + \frac{1}{\Delta t}\right), \\ p^{j+1}(s) &= \left(\nu^{j+1}(s) - \frac{W^j(s)}{\Delta t}\right). \end{aligned} \tag{14}$$

□

3.2. Spatial Semidiscretization. For right boundary layer problem from the theory of singular perturbation in [29], the asymptotic solution of the zero-order approximation of Equation (13) is written as

$$W^{j+1}(s) \approx W_0^{j+1}(s) + \frac{a(1)}{a(s)}\left(\chi_1^{j+1} - W_0^{j+1}(s)\right) \exp\left(-a(s)\frac{1-s}{\varepsilon}\right) + O(\varepsilon), \tag{15}$$

where $W_0^{j+1}(s)$ is the solution of reduced problem

$$a(s)\frac{dW_0^{j+1}(s)}{ds} - q(s) Y_0^{j+1}(s) = p^{j+1}(s), \text{with } W_0^{j+1}(1) = \chi_1^{j+1}. \tag{16}$$

Taking Taylor's series expansion for $a(s)$ about the point "1" and restricting to their first terms, Equation (15) becomes

$$W^{j+1}(s) \approx W_0^{j+1}(s) + \left(\chi_1^{j+1} - W_0^{j+1}(s)\right) \exp\left(-a(1)\frac{1-s}{\varepsilon}\right) + O(\varepsilon). \tag{17}$$

Let us subdivide the domain $[0, 1]$ into N uniform meshes $\Delta s = 1/N$ as $\nabla_s^N = \{0 = s_0, s_1, s_2, \cdots, s_N = 1\}$, and the mesh can be written as $s_i = i\Delta s$, $i = 0, 1, 2, \cdots, N$. By considering problem (17) at $s_i = i\Delta s$ as $\Delta s \longrightarrow 0$, we have

$$\begin{aligned}\lim_{\Delta s \longrightarrow 0} W^{j+1}(i\Delta s) \approx & W_0^{j+1}(0) + \left(\chi_1^{j+1} - W_0^{j+1}(1)\right) \exp \\ & \cdot \left(-a(1)\left(\frac{1}{\varepsilon} - i\rho\right)\right) + O(\varepsilon),\end{aligned} \tag{18}$$

where $\rho = \Delta s/\varepsilon$.

Assume that $W^{j+1}(s)$ is a smooth function in the domain $\bar{\nabla}_s^N$. Then, by employing Taylor's series, we obtain

$$W^{j+1}(s_{i+1}) \approx W_{i+1}^{j+1} \approx W_i^{j+1} + \Delta s \frac{dW_i^{j+1}}{ds} + \frac{\Delta s^2}{2!}\frac{d^2W_i^{j+1}}{ds^2} + \frac{\Delta s^3}{3!}\frac{d^3W_i^{j+1}}{ds^3} + \frac{\Delta s^4}{4!}\frac{d^4W_i^{j+1}}{ds^4} + \frac{\Delta s^5}{5!}\frac{d^5W_i^{j+1}}{ds^5} + \frac{\Delta s^6}{2!}\frac{d^6W_i^{j+1}}{ds^6} + \frac{\Delta s^7}{7!}\frac{d^7W_i^{j+1}}{ds^7} + \frac{\Delta s^8}{8!}\frac{d^8W_i^{j+1}}{ds^8} + O(\Delta s^9), \tag{19}$$

$$W^{j+1}(s_{i-1}) \approx W_{i-1}^{j+1} \approx W_i^{j+1} - \Delta s \frac{dW_i^{j+1}}{ds} + \frac{\Delta s^2}{2!}\frac{d^2W_i^{j+1}}{ds^2} - \frac{\Delta s^3}{3!}\frac{d^3W_i^{j+1}}{ds^3} + \frac{\Delta s^4}{4!}\frac{d^4W_i^{j+1}}{ds^4} - \frac{\Delta s^5}{5!}\frac{d^5W_i^{j+1}}{ds^5} + \frac{\Delta s^6}{2!}\frac{d^6W_i^{j+1}}{ds^6} - \frac{\Delta s^7}{7!}\frac{d^7W_i^{j+1}}{ds^7} + \frac{\Delta s^8}{8!}\frac{d^8W_i^{j+1}}{ds^8} - O(\Delta s^9). \tag{20}$$

Adding Equation (19) and Equation (20), we get

$$W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1} = \frac{2\Delta s^2}{2!}\frac{d^2W_i^{j+1}}{ds^2} + \frac{2\Delta s^4}{4!}\frac{d^4W_i^{j+1}}{ds^4} + \frac{2\Delta s^6}{2!}\frac{d^6W_i^{j+1}}{ds^6} + \frac{2\Delta s^8}{8!}\frac{d^8W_i^{j+1}}{ds^8} + O(\Delta s^{10}), \tag{21}$$

$$\frac{d^2W_{i-1}^{j+1}}{ds^2} - \frac{d^2W_i^{j+1}}{ds^2} + \frac{d^2W_{i+1}^{j+1}}{ds^2} = \frac{2\Delta s^2}{2!}\frac{d^4W_i^{j+1}}{ds^4} + \frac{2\Delta s^4}{4!}\frac{d^6W_i^{j+1}}{ds^6} + \frac{2\Delta s^6}{6!}\frac{d^8W_i^{j+1}}{ds^8} + \frac{2\ell^8}{8!}\frac{d^{10}W_i^{j+1}}{ds^{10}} + O(\Delta s^{12}). \tag{22}$$

Plugging $(\Delta s^4/12)(d^6W_i^{j+1}/ds^6)$ from Equation (22) into Equation (21), we get

$$W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1} = \frac{\Delta s^2}{30}\left(\frac{d^2W_{i-1}^{j+1}}{ds^2} + 28\frac{d^2W_i^{j+1}}{ds^2} + \frac{d^2W_{i+1}^{j+1}}{ds^2}\right) + R, \tag{23}$$

where $R = (\Delta s^4/20)(d^4W_i^{j+1}/ds^4) - (13\Delta s^6/302400)(d^8W_i^{j+1}/ds^8) + O(\Delta s^{(10)})$.

Equation (13) at $s_{i,i\pm1}$ can be written as

$$\begin{cases} \varepsilon \dfrac{d^2W_{i-1}^{j+1}}{ds^2} = -\mu a_{i-1}\dfrac{dW_{i-1}^{j+1}}{ds} + q_{i-1}W_{i-1}^{j+1} + p_{i-1}^{j+1}, \\ \varepsilon \dfrac{d^2W_i^{j+1}}{ds^2} = -\mu a_i\dfrac{dW_i^{j+1}}{ds} + q_iW_i^{j+1} + p_i^{j+1}, \\ \varepsilon \dfrac{d^2W_{i+1}^{j+1}}{ds^2} = -\mu a_{i+1}\dfrac{dW_{i+1}^{j+1}}{ds} + q_{i+1}W_{i+1}^{j+1} + p_{i+1}^{j+1}, \end{cases} \tag{24}$$

where we approximate dW_{i-1}^{j+1}/ds, dW_i^{j+1}/ds, and dW_{i+1}^{j+1}/ds using nonsymmetric finite differences [30]:

$$\begin{cases} \dfrac{dW_{i-1}^{j+1}}{ds} \approx \dfrac{-W_{i+1}^{j+1} + 4W_i^{j+1} - 3W_{i-1}^{j+1}}{2\Delta s} + \Delta s\dfrac{d^2W_i^{j+1}}{ds^2} + O(\Delta s^2), \\ \dfrac{dW_i^{j+1}}{ds} \approx \dfrac{W_{i+1}^{j+1} - W_{i-1}^{j+1}}{2\Delta s} + O(\Delta s^2), \\ \dfrac{dW_{i+1}^{j+1}}{ds} \approx \dfrac{3W_{i+1}^{j+1} - 4W_i^{j+1} + W_{i-1}^{j+1}}{2\Delta s} - \ell\dfrac{d^2W_i^{j+1}}{ds^2} + O(\Delta s^2). \end{cases} \tag{25}$$

Taking Equation (25) into Equation (24), we get

$$\begin{cases} \varepsilon \dfrac{d^2W_{i-1}^{j+1}}{ds^2} = -\mu a_{i-1}\left(\dfrac{-W_{i+1}^{j+1} + 4W_i^{j+1} - 3W_{i-1}^{j+1}}{2\Delta s} + \Delta s\dfrac{d^2W_i^{j+1}}{ds^2}\right) + q_{i-1}W_{i-1}^{j+1} + p_{i-1}^{j+1}, \\ \varepsilon \dfrac{d^2W_i^{j+1}}{ds^2} = -\mu a_i\left(\dfrac{W_{i+1}^{j+1} - W_{i-1}^{j+1}}{2\Delta s}\right) + q_iY_i^{j+1} + p_i^{j+1}, \\ \varepsilon \dfrac{d^2W_{i+1}^{j+1}}{ds^2} = -\mu a_{i+1}\left(\dfrac{3W_{i+1}^{j+1} - 4W_i^{j+1} + W_{i-1}^{j+1}}{2\Delta s} - \Delta s\dfrac{d^2W_i^{j+1}}{ds^2}\right) + q_{i+1}Y_{i+1}^{j+1} + p_{i+1}^{j+1}. \end{cases} \tag{26}$$

Substitute Equation (26) into Equation (23) and rearrange the result as

$$\begin{aligned} &\left(\varepsilon - \frac{\mu a_{i+1}\Delta s}{30} + \frac{\mu a_{i-1}\Delta s}{30}\right)\left(\frac{W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}}{\Delta s^2}\right) \\ &= \left(\frac{\mu a_{i-1}}{20\Delta s} + \frac{q_{i-1}}{30} + \frac{7\mu a_i}{15\Delta s} - \frac{\mu a_{i+1}}{60\Delta s}\right)W_{i-1}^{j+1} \\ &+ \left(\frac{\mu a_{i-1}}{15\Delta s} - \frac{14q_i}{15\Delta s} + \frac{\mu a_{i+1}}{15\Delta s}\right)W_i^{j+1} \\ &+ \left(\frac{\mu a_{i-1}}{60\Delta s} - \frac{7\mu a_i}{15\Delta s} - \frac{a_{i+1}}{20\Delta s} + \frac{q_{i+1}}{30}\right)W_{i+1}^{j+1} \\ &+ \frac{1}{30}\left(p_{i-1}^{j+1} + 28p_i^{j+1} + p_{i+1}^{j+1}\right). \end{aligned} \tag{27}$$

To handle the effect of the perturbation parameter, exponential fitting factor $\sigma(\rho)$ is multiplied (Equation (27)) on the term containing the perturbation parameter as

$$\begin{aligned} &\left(\sigma(\rho)\varepsilon - \frac{\mu a_{i+1}\Delta s}{30} + \frac{\mu a_{i-1}\Delta s}{30}\right)\left(\frac{W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}}{\Delta s^2}\right) \\ &= \left(\frac{\mu a_{i-1}}{20\Delta s} + \frac{q_{i-1}}{30} + \frac{7\mu a_i}{15\Delta s} - \frac{\mu a_{i+1}}{60\Delta s}\right)W_{i-1}^{j+1} \\ &+ \left(-\frac{\mu a_{i-1}}{15\Delta s} + \frac{14q_i}{15\Delta s} + \frac{\mu a_{i+1}}{15\Delta s}\right)W_i^{j+1} \\ &+ \left(\frac{\mu a_{i-1}}{60\Delta s} - \frac{7\mu a_i}{15\Delta s} - \frac{\mu a_{i+1}}{20\Delta s} + \frac{q_{i+1}}{30}\right)W_{i+1}^{j+1} \\ &+ \frac{1}{30}\left(p_{i-1}^{j+1} + 28p_i^{j+1} + p_{i+1}^{j+1}\right). \end{aligned} \tag{28}$$

Multiplying (28) by ℓ and taking the limit as $\Delta s \longrightarrow 0$, we get

$$\lim_{\Delta s \longrightarrow 0} \sigma(\rho)\left(\frac{W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}}{\rho}\right) = \frac{\mu a(0)}{2}\left(W_{i+1}^{j+1} - W_{i-1}^{j+1}\right). \tag{29}$$

Using Equation (18), we have

$$\begin{cases} \dfrac{\sigma(\rho)}{\rho} \lim\limits_{\Delta s \longrightarrow 0}\left(W^{j+1}(i\Delta s - \Delta s) - 2W^{j+1}(i\Delta s) + W^{j+1}(i\Delta s + \Delta s)\right) \approx \left(\chi_1^{j+1} - W_0^{j+1}(1)\right) \exp\left(-\mu a(1)\left(\dfrac{1}{\varepsilon} - i\rho\right)\right)(\exp(\mu a(1)\rho) - 2 + \exp(-\mu a(1)\rho)), \\ \dfrac{\sigma(\rho)}{\rho} \lim\limits_{\Delta s \longrightarrow 0}\left(W^{j+1}(i\Delta s + \Delta s) - W^{j+1}(i\Delta s - \Delta s)\right) \approx \left(\chi_1^{j+1} - W_0^{j+1}(1)\right) \exp\left(-\mu a(1)\left(\dfrac{1}{\varepsilon} - i\rho\right)\right)(\exp(-\mu a(1)\rho) - \exp(\mu a(1)\rho)). \end{cases} \tag{30}$$

Substituting Equation (30) in Equation (29), we obtain

$$\frac{\sigma(\rho)}{\rho}\left(e^{\mu a(1)\rho} - 2 + e^{-\mu a(1)\rho}\right) = \frac{\mu a(0)}{2}\left(e^{\mu a(1)\rho} - e^{-\mu a(1)\rho}\right). \tag{31}$$

Simplifying Equation (31), we get

$$\sigma(\rho) = \frac{\mu a(0)\rho}{2} \coth\left(\frac{\mu a(1)\rho}{2}\right), \tag{32}$$

which is the required value of the fitting factor $\sigma(\rho)$. Finally, from Equation (28) and Equation (32), we obtain

$$\$^{N,M} W_i^{j+1} = \begin{cases} W_i^{-m} = \varphi(s_i, -t_m), form = 0, 1, \cdots, \theta, i = 1, 2, \cdots, N-1, \\ \vartheta_i^- W_{i-1}^{j+1} - \vartheta_i^c W_i^{j+1} + \vartheta_i^+ W_{i+1}^{j+1} = \vartheta_i^{j+1}, i = 1, 2, \cdots, N-1, j = 0, 1, \cdots, M-1, \\ W^{j+1}(0) = \chi_0^{j+1}, W^{j+1}(1) = \chi_1^{j+1}, 0 < j < M-1, \end{cases} \tag{33}$$

where

$$\begin{cases} \vartheta_i^- = \dfrac{1}{\Delta s^2}\left(\sigma(\rho)\varepsilon - \dfrac{\mu a_{i+1}\Delta s}{30} + \dfrac{\mu a_{i-1}\Delta s}{30}\right) + \left(-\dfrac{\mu a_{i-1}}{20\Delta s} - \dfrac{q_{i-1}}{30} - \dfrac{7\mu a_i}{15\Delta s} + \dfrac{\mu a_{i+1}}{60\Delta s}\right), \\ \vartheta_i^c = \dfrac{2}{\Delta s^2}\left(\sigma(\rho)\varepsilon - \dfrac{\mu a_{i+1}\Delta s}{30} + \dfrac{\mu a_{i-1}\Delta s}{30}\right) - \left(\dfrac{\mu a_{i-1}}{15\Delta s} - \dfrac{14 q_i}{15\Delta s} - \dfrac{\mu a_{i+1}}{15\Delta s}\right), \\ \vartheta_i^+ = \dfrac{1}{\Delta s^2}\left(\sigma(\rho)\varepsilon - \dfrac{\mu a_{i+1}\Delta s}{30} + \dfrac{\mu a_{i-1}\Delta s}{30}\right) + \left(-\dfrac{\mu a_{i-1}}{60\Delta s} + \dfrac{7\mu a_i}{15\Delta s} + \dfrac{\mu a_{i+1}}{20\Delta s} - \dfrac{q_{i+1}}{30}\right), \\ \xi_i^{j+1} = \dfrac{1}{30}\left(\vartheta_{i-1}^{j+1} + 28\vartheta_i^{j+1} + \vartheta_{i+1}^{j+1}\right). \end{cases} \tag{34}$$

For small mesh sizes, the above matrix is $|\vartheta_i^c| \geq |\vartheta_i^c| + |\vartheta_i^+|$ (i.e., the matrix is diagonally dominant) and nonsingular. Hence, by [31], the matrix ϑ is M-matrix and the system of equations can be solved by matrix inverse with the given boundary conditions.

4. Convergence Analysis

Lemma 5. *The matrix associated with the discrete scheme (33) is M-matrix.*

Proof. By assuming that $a(s) = A$ and $q(s) = B$ are constant functions in $[0, 1]$, where A and B are arbitrary constants, one can easily see that the inequalities $\vartheta_i^- > 0$, $\vartheta_i^+ > 0$, $\vartheta_i^c > 0$, $\vartheta_i^c > \vartheta_i^- + \vartheta_i\vartheta^+$, and $|\vartheta_i^-| \leq |\vartheta_i^+|$ are satisfied under the assumptions that $(a(s) = A) > 0$, $(q(s) = B) > 0$, and $(A/2\Delta s + B/30) < \sigma(\rho)\varepsilon/\Delta s^2$. Therefore, the matrix associated with the discrete scheme (33) is M-matrix. □

Lemma 6 (discrete maximum principle). *Assume that the discrete function Π_i^{j+1} gratifies $\Pi_i^{j+1} \geq 0$ on $i = 0, N$. Then, $\$^{N,M}\Pi_i^{j+1} \geq 0$ on $\bar{\nabla}^{N,M}$ implies that $\Pi_i^{j+1} \geq 0$ at each point of $\bar{\nabla}^{N,M}$.*

Lemma 7. *The solution W_i^{j+1} of the discrete scheme in (33) on $\bar{\nabla}^{N,M}$ gratifies the following estimate:*

$$\left\| W_i^{j+1} \right\| \leq \max\left\{ \left| W_0^{j+1} \right|, \left| W_N^{j+1} \right| \right\} + \frac{\left\| \$^{N,M} \right\|}{q^*}, \tag{35}$$

where $q(s_i) \geq q^* > 0$.

Hence, Lemma 7 depicts that the scheme in Equation (33) is stable in supremum norm.

Lemma 8. *If $W \in C^3(I)$, then the LTE in space discretization is written as*

$$|T_i| \leq \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{28\mu a \Delta s^2}{180} \left| \frac{d^3 W^{j+1}(s)}{ds^3} \right| \right\} + O\left(\Delta s^3\right), i = 1, 2, \cdots, N-1. \tag{36}$$

Proof. By definition

$$\begin{aligned} T_i &= \sigma\varepsilon \left\{ \frac{W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}}{\Delta s^2} - \frac{d^2 W_i^{j+1}}{ds^2} \right\} \\ &+ \frac{\mu a_{i-1}}{30} \left\{ \left(\frac{-3W_{i-1}^{j+1} + 4W_i^{j+1} - W_{i-1}^{j+1}}{2\Delta s} + \Delta s \frac{d^2 W_i^{j+1}}{ds^2} \right) - \frac{dW_{i-1}^{j+1}}{ds} \right\} \\ &+ \frac{28\mu a_i}{30} \left\{ \frac{W_{i+1}^{j+1} - W_{i-1}^{j+1}}{2\Delta s} - \frac{dW_i^{j+1}}{ds} \right\} \\ &+ \frac{\mu a_{i+1}}{30} \left\{ \left(\frac{W_{i+1}^{j+1} - 4W_i^{j+1} + 3W_{i-1}^{j+1}}{2\Delta s} - \Delta s \frac{d^2 W_i^{j+1}}{ds^2} \right) - \frac{dW_{i+1}^{j+1}}{ds} \right\}, i = 1(1)N-1, \end{aligned}$$

$$\begin{aligned} \Longrightarrow T_i &= \sigma\varepsilon \left\{ \frac{\Delta s^2}{12} \frac{d^4 W_i^{j+1}}{ds^4} + \frac{\Delta s^4}{360} \frac{d^6 W_i^{j+1}}{ds^6} + \cdots \right\} \\ &+ \frac{\mu a_{i-1}}{30} \left\{ \Delta s \frac{d^2 W_i^{j+1}}{ds^2} - \frac{2\Delta s^2}{3} \frac{d^3 W_i^{j+1}}{ds^3} + \right\} \\ &+ \frac{28\mu a_i}{30} \left\{ \frac{\Delta s^2}{6} \frac{d^3 W_i^{j+1}}{ds^3} + \frac{\Delta s^4}{120} \frac{d^5 W_i^{j+1}}{ds^5} + \cdots \right\} \\ &+ \frac{\mu a_{i+1}}{30} \left\{ -\Delta s \frac{d^2 W_i^{j+1}}{ds^2} - \frac{2\Delta s^2}{3} \frac{d^3 W_i^{j+1}}{ds^3} + \cdots \right\}, \end{aligned}$$

$$\Longrightarrow |T_i| \leq \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{\sigma \Delta s^2 \varepsilon}{12} \left| \frac{d^4 W^{j+1}(s)}{ds^4} \right| \right\} + \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{28}{180} \mu a \Delta s^2 \left| \frac{d^3 W^{j+1}(s)}{ds^3} \right| \right\}. \tag{37}$$

Using the relation (33) with $W = (\mu a(0)/2) \coth (\mu a(1) \rho/2)$ we get,

$$\begin{aligned} &\Longrightarrow |T_i| \leq \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{W\Delta s^3}{12} \left| \frac{d^4 W^{j+1}(s)}{ds^4} \right| \right\} \\ &+ \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{28}{180} \mu a \Delta s^2 \left| \frac{d^3 W^{j+1}(s)}{ds^3} \right| \right\} \\ &\Longrightarrow |T_i| \leq \max_{s_{i-1} \leq s \leq s_{i+1}} \left\{ \frac{28}{180} \mu a \Delta s^2 \left| \frac{d^3 W^{j+1}(s)}{ds^3} \right| \right\} + O\left(\Delta s^3\right) \\ &\Longrightarrow |T_i| \leq O\left(\Delta s^2\right), i = 1, 2, \cdots, N-1. \end{aligned} \tag{38}$$

Thus, the desired result is obtained. □

Lemma 9. *Let $W(s_i, t_{j+1})$ be the solution of problem (13) and W_i^{j+1} be the solution of the discrete problem (33). Then, the following estimate is obtained:*

$$\left| W\left(s_i, t_{j+1}\right) - W_i^{j+1} \right| \leq O\left(\Delta s^2\right). \tag{39}$$

Proof. Rewrite Equation (33) in matrix vector form as

$$YW = H, \tag{40}$$

where $Y = (\vartheta_{i,j}), 0 \leq j \leq M-1, 1 \leq i \leq N-1$ is a tridiagonal matrix with

$$\begin{aligned} \vartheta_{i-1,j+1} &= \frac{\sigma(\rho)\varepsilon}{\Delta s^2} - \frac{\mu a_{i-1}}{60\Delta s} - \frac{28\mu a_i}{60\Delta s} - \frac{\mu a_{i+1}}{60\Delta s} - \frac{q_{i-1}}{30}, \\ \vartheta_{i,j+1} &= \frac{-2\sigma(\rho)\varepsilon}{\Delta s^2} - \frac{28 q_i}{30}, \\ \vartheta_{i+1,j+1} &= \frac{\sigma(\rho)\varepsilon}{\Delta s^2} + \frac{\mu a_{i-1}}{60\Delta s} - \frac{\mu a_{i+1}}{60\Delta s} + \frac{2\mu a_i}{60\Delta s} - \frac{q_{i+1}}{30}, \end{aligned} \tag{41}$$

and $H = (\xi_i^{j+1})$ is a column vector with $(\xi_i^{j+1}) = (1/30)(p_{i-1}^{j+1} + 28p_i^{j+1} + p_{i+1}^{j+1})$ for $i = 1, 2, \cdots, N-1$, with local truncation error e_i:

$$|T_i| \le C\left(\Delta s^2\right). \tag{42}$$

We also have

$$Y\bar{W} - T(\Delta s) = H, \tag{43}$$

where $\bar{W} = (\bar{W}_0, \bar{W}_1, ,\cdots,\bar{W}_N)^t$ and $T(\Delta s) = (T_1(\Delta s), T_2(\Delta s), ,T_2(\Delta s),\cdots,T_N(\Delta s))^t$ denote the actual solution and the local truncation error, respectively.

From Equations (40) and (43), we get

$$Y\left(\bar{W} - W\right) = T(\Delta s). \tag{44}$$

Thus, the error equation is

$$YE = T(\Delta s), \tag{45}$$

where $E = \bar{W} - W = (T_0, T_1, T_2,\cdots,T_N)^t$. Let S be the sum of elements of the ith row of Y; then, we have

$$\begin{aligned}
\Omega_1 &= \sum_{j=1}^{N-1} \vartheta_{1,j} = \frac{\sigma\varepsilon}{\Delta s^2} + \frac{\mu a_{i+1}}{60\Delta s} + \frac{\mu a_{i-1}}{60\Delta s} + \frac{28q_i}{30} \\
&\quad + \frac{q_{i+1}}{30} + \frac{28\mu a_i}{60\Delta s}, \\
\Omega_{N-1} &= \sum_{j=1}^{N-1} \vartheta_{N-1,j} = \frac{\sigma\varepsilon}{\Delta s^2} - \frac{\mu a_{i+1}}{60\Delta s} - \frac{\mu a_{i-1}}{60\Delta s} + \frac{28q_i}{30} \\
&\quad + \frac{q_{i-1}}{30} - \frac{28\mu a_i}{60\Delta s}, \\
\Omega_i &= \sum_{j=1}^{N-1} \vartheta_{i,j} = \frac{1}{30}\left(p_{i-1}^{j+1} + 28p_i^{j+1} + p_{i+1}^{j+1}\right) = \Omega_i + O\left(\Delta s^2\right) \\
&= B_{i0}, i = 2(1)N-2,
\end{aligned} \tag{46}$$

where $B_{i0} = \Omega_i = (1/30)(p_{i-1}^{j+1} + 28p_i^{j+1} + p_{i+1}^{j+1})$.

Since $0 < \varepsilon \ll 1$, for sufficiently small Δs, the matrix W is irreducible and monotone. Then, it follows that Y^{-1} exists, and its elements are nonnegative [32]. Hence, from Equation (45), we obtain

$$E = Y^{-1}T(\Delta s), \tag{47}$$

$$\|E\| \le \left\|Y^{-1}\right\|\|T(\Delta s)\|. \tag{48}$$

Let $\bar{\vartheta}_{ki}$ be the (ki)th elements of Y^{-1}. Since $\bar{\vartheta}_{ki} \ge 0$ by the definition of multiplication of matrices with its inverses, we have

$$\sum_{i=1}^{N-1} \bar{\vartheta}_{ki}\Omega_i = 1, k = 1, 2, \cdots, N-1. \tag{49}$$

Therefore, it follows that

$$\sum_{i=1}^{N-1} \bar{\vartheta}_{ki} \le \frac{1}{\min\limits_{0\le i\le N-1} \Omega_i} = \frac{1}{B_{i,0}} \le \frac{1}{|B_{i0}|}, \tag{50}$$

for some $i0$ between 1 and $N-1$, and $B_{i0} = \Omega_i$ From Equations (40), (48), and (50), we obtain

$$E_i = \sum_{i=1}^{N-1} \bar{\vartheta}_{ki}T(\Delta s), i = 1(1)N-1, \tag{51}$$

which implies

$$E_i \le \frac{C\left(\Delta s^2\right)}{|\Omega_i|} i = 1(1)N-1. \tag{52}$$

Therefore,

$$\|E\| \le C\left(\Delta s^2\right). \tag{53}$$

□

Theorem 10. *Let $w(s,t)$ be the solution of the problem (1) and W_i^j be the numerical solution obtained by the proposed scheme (33). Then, for sufficiently small Δs, the error estimate for the totally discrete scheme is given by*

$$\sup_{0<\varepsilon\ll 1} \max_{s_i,t_j} \left|w\left(s_i, t_j\right) - W_i^j\right| \le C\left(\Delta t + (\Delta s)^2\right). \tag{54}$$

Proof. By combining the result of Lemma 4 and Lemma 9, the required bound is obtained. □

5. Numerical Examples, Results, and Discussions

Some numerical examples are presented to show the applicability of the proposed numerical scheme. Since the analytical solutions of the considered problems are not available, we used double mesh principle to compute the maximum principle given in [33]

$$E_{\varepsilon,\mu}^{N,M} = \max_{\left(s_i,t_{j+1}\right)\in\nabla^{N,M}} \left|W^{N,M}\left(s_i, t_{j+1}\right) - W^{2N,2M}\left(s_i, t_{j+1}\right)\right|, \tag{55}$$

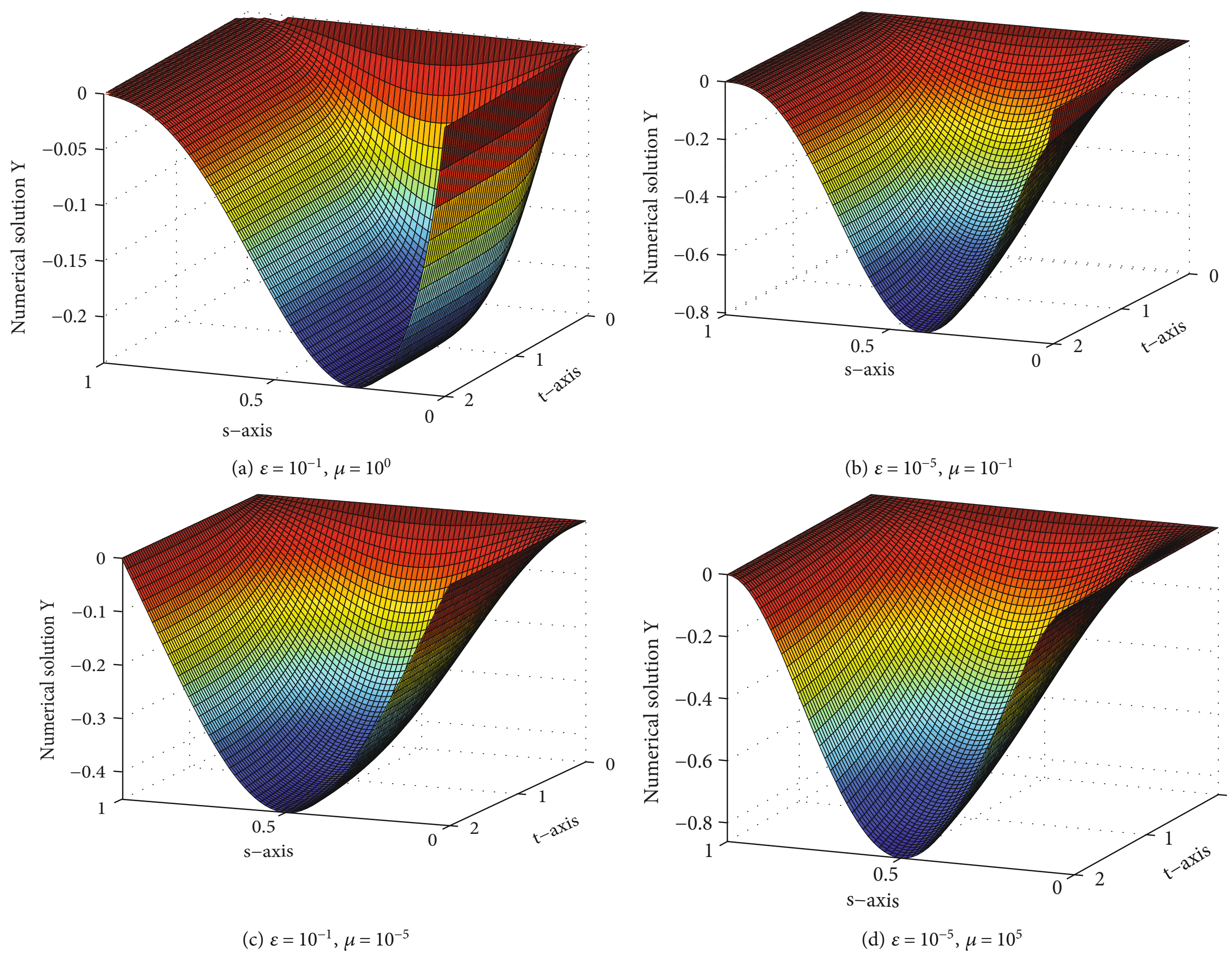
where $W^{N,M}(s_i)$ are computed numerical solutions obtained on the mesh $\nabla^{N,M} = \nabla_s^N \times \nabla_t^M$ with N and M mesh intervals in the spatial and temporal directions, respectively, whereas $W^{2N,2M}(s_i, t_{j+1})$ are computed numerical solutions on the mesh $\nabla^{2N,2M} = \nabla_s^{2N} \times \nabla_t^{2M}$ by adding the midpoint $s_{i+1/2} =$

Table 1: $E_{\varepsilon,\mu}^{N,M}$, $r_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 1 with $\mu = 10^{-3}$.

$\varepsilon N\longrightarrow$ $\downarrow M\longrightarrow$	2^5 2^3	2^6 2^4	2^7 2^5	2^8 2^6	2^9 2^7	2^{10} 2^8
Proposed scheme						
10^{-0}	7.3516e-03	5.7683e-03	3.6428e-03	2.1296e-03	1.1630e-03	6.0953e-04
	3.4991e-01	6.6310e-01	7.7447e-01	8.7273e-01	9.3208e-01	
10^{-2}	1.9343e-02	1.0465e-02	5.4599e-03	2.7902e-03	1.4107e-03	7.0935e-04
	8.8624e-01	9.3863e-01	9.6851e-01	9.8396e-01	9.9184e-01	
10^{-4}	1.9854e-02	1.0658e-02	5.5313e-03	2.8189e-03	1.4231e-03	7.1502e-04
	8.9749e-01	9.4625e-01	9.7249e-01	9.8610e-01	9.9298e-01	
10^{-6}	1.9860e-02	1.0660e-02	5.5318e-03	2.8190e-03	1.4231e-03	7.1503e-04
	8.9766e-01	9.4639e-01	9.7257e-01	9.8615e-01	9.9296e-01	
10^{-8}	1.9861e-02	1.0661e-02	5.5327e-03	2.8195e-03	1.4234e-03	7.1518e-04
	8.9760e-01	9.4629e-01	9.7254e-01	9.8610e-01	9.9296e-01	
10^{-10}	1.9861e-02	1.0661e-02	5.5327e-03	2.8195e-03	1.4234e-03	7.1518e-04
	8.9760e-01	9.4629e-01	9.7254e-01	9.8610e-01	9.9296e-01	
10^{-12}	1.9861e-02	1.0661e-02	5.5327e-03	2.8195e-03	1.4234e-03	7.1518e-04
	8.9760e-01	9.4629e-01	9.7254e-01	9.8610e-01	9.9296e-01	
$E^{N,M}$	1.9861e-02	1.0661e-02	5.5327e-03	2.8195e-03	1.4234e-03	7.1518e-04
$r^{N,M}$	8.9760e-01	9.4629e-01	9.7254e-01	9.8610e-01	9.9296e-01	
Results in [24]						
$E^{N,M}$	4.3705e-2	1.6704e-2	7.3802e-3	3.7406e-3	1.8967e-3	9.5511e-4
$r^{N,M}$	1.3876	1.1785	9.803e-01	9.797e-01	9.898e-01	

Table 2: $E_{\varepsilon,\mu}^{N,M}$, $r_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ for Example 2 with $\mu = 10^{-3}$.

$\varepsilon N\longrightarrow$ $\downarrow M\longrightarrow$	2^5 2^3	2^6 2^4	2^7 2^5	2^8 2^6	2^9 2^7	2^{10} 2^8
Proposed scheme						
10^{-0}	3.9899e-04	2.0538e-04	1.0431e-04	5.2577e-05	2.6397e-05	1.3226e-05
	9.5806e-01	9.7742e-01	9.8837e-01	9.9406e-01	9.9700e-01	
10^{-2}	3.7603e-03	1.9594e-03	9.9849e-04	5.0406e-04	2.5324e-04	1.2692e-04
	9.4044e-01	9.7259e-01	9.8615e-01	9.9309e-01	9.9659e-01	
10^{-4}	4.5642e-03	2.3635e-03	1.2012e-03	6.0550e-04	3.0394e-04	1.5227e-04
	9.4944e-01	9.7645e-01	9.8828e-01	9.9434e-01	9.9715e-01	
10^{-6}	4.5778e-03	2.3685e-03	1.2040e-03	6.0687e-04	3.0463e-04	1.5261e-04
	9.5068e-01	9.7614e-01	9.8838e-01	9.9433e-01	9.9721e-01	
10^{-8}	4.5773e-03	2.3682e-03	1.2038e-03	6.0679e-04	3.0458e-04	1.5259e-04
	9.5071e-01	9.7620e-01	9.8833e-01	9.9438e-01	9.9716e-01	
10^{-10}	4.5773e-03	2.3682e-03	1.2038e-03	6.0679e-04	3.0458e-04	1.5259e-04
	9.5071e-01	9.7620e-01	9.8833e-01	9.9438e-01	9.9716e-01	
10^{-12}	4.5773e-03	2.3682e-03	1.2038e-03	6.0679e-04	3.0458e-04	1.5259e-04
	9.5071e-01	9.7620e-01	9.8833e-01	9.9438e-01	9.9716e-01	
$E^{N,M}$	4.5778e-03	2.3685e-03	1.2040e-03	6.0687e-04	3.0463e-04	1.5261e-04
$r^{N,M}$	9.5068e-01	9.7614e-01	9.8838e-01	9.9433e-01	9.9721e-01	
Results in [24]						
$E^{N,M}$	1.1161e-2	5.1087e-3	2.4749e-3	1.2214e-3	6.0706e-4	3.0264e-4
$r^{N,M}$	1.1274	1.0455	1.0188	1.0086	1.0042	



(a) $\varepsilon = 10^{-1}$, $\mu = 10^{0}$

(b) $\varepsilon = 10^{-5}$, $\mu = 10^{-1}$

(c) $\varepsilon = 10^{-1}$, $\mu = 10^{-5}$

(d) $\varepsilon = 10^{-5}$, $\mu = 10^{5}$

Figure 1: Numerical solution of Example 1 for $N = M = 64$.

$(s_{i+1} + s_i)/2$ and $t_{j+1/2} = (t_{j+1} + t_j)/2$ into the mesh points. The corresponding rate of convergence for the proposed scheme is determined by

$$r_{\varepsilon,\mu}^{N,M} = \log_2\left(\frac{E_{\varepsilon,\mu}^{N,M}}{E_{\varepsilon,\mu}^{2N,2M}}\right). \tag{56}$$

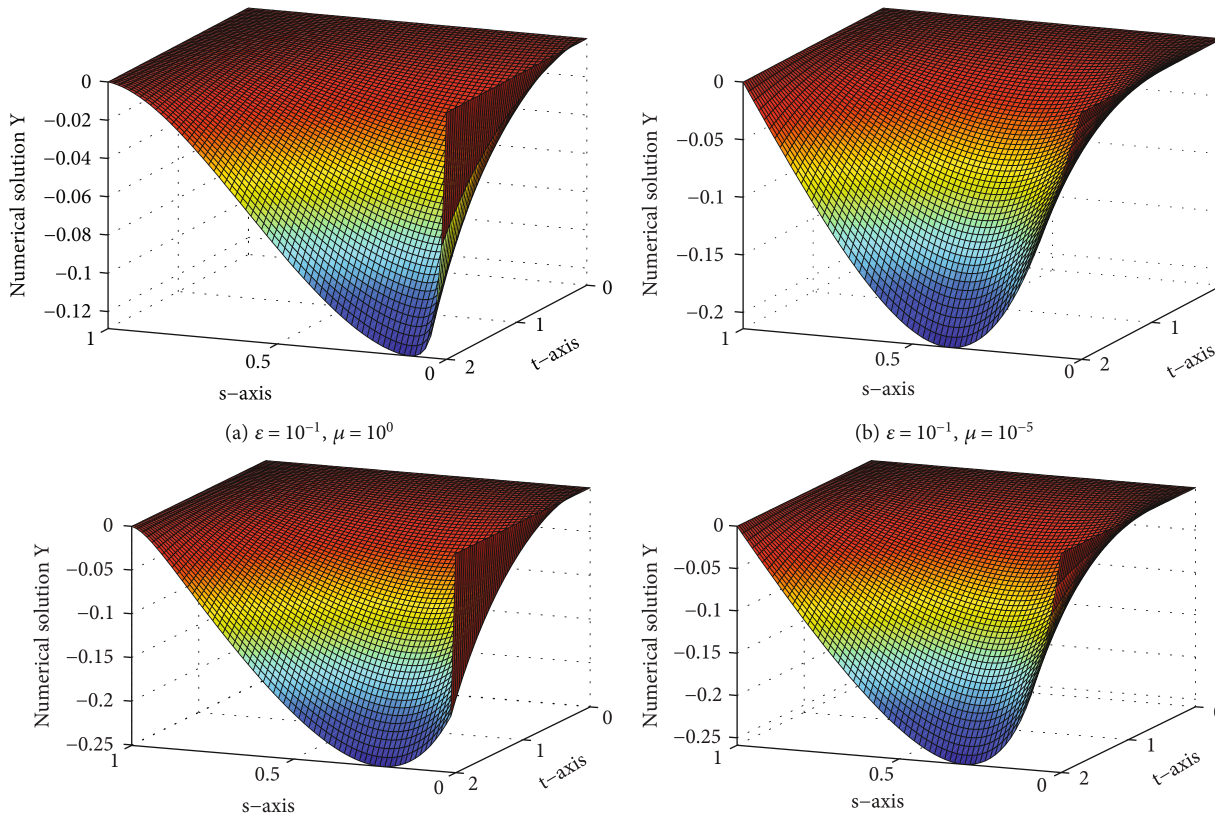
The parameter uniform maximum absolute error $(E^{N,M})$ and uniform order of convergence $(r^{N,M})$ are calculated using

$$\begin{aligned} E^{N,M} &= \max_{\varepsilon,\mu}\left\{E_{\varepsilon,\mu}^{N,M}\right\}, \\ r^{N,M} &= \frac{\log\left(E^{N,M}\right) - \log\left(E^{2N,2M}\right)}{\log(2)}, \end{aligned} \tag{57}$$

respectively.

Example 1 (see [24]). Consider

$$\begin{cases} \varepsilon \dfrac{\partial^2 w}{\partial s^2} + \mu(1+s)\dfrac{\partial w}{\partial s} - w(s,t) - \dfrac{\partial w}{\partial t} = -w(s,t-1) + 16s^2(1-s)^2, (s,t) \in (0,1)\times(0,2], \\ w(s,t) = 0, (s,t) \in (0,1)\times(-1,0], \\ w(0,t) = 0, w(1,t) = 0, t \in (0,2]. \end{cases} \tag{58}$$



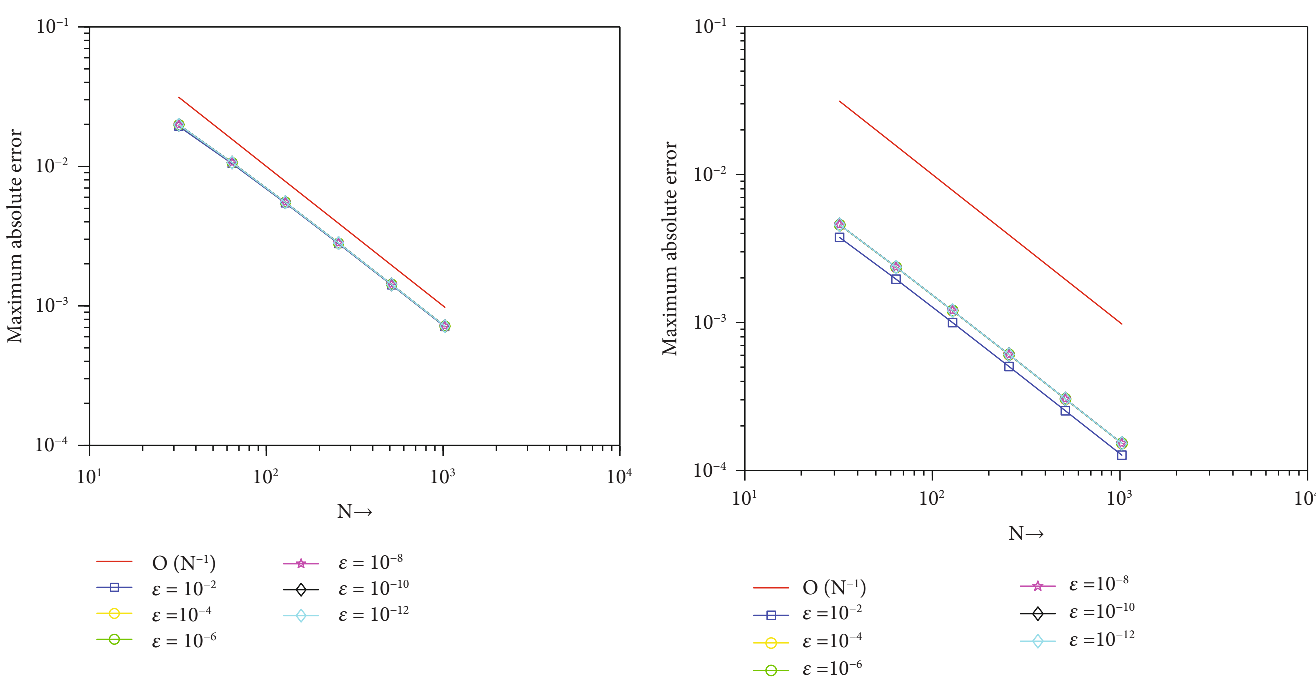
(a) $\varepsilon = 10^{-1}$, $\mu = 10^{0}$

(b) $\varepsilon = 10^{-1}$, $\mu = 10^{-5}$

(c) $\varepsilon = 10^{-5}$, $\mu = 10^{-1}$

(d) $\varepsilon = 10^{-5}$, $\mu = 10^{5}$

Figure 2: Numerical solution of Example 2 for $N = M = 64$.



(a) Example 1

(b) Example 2

Figure 3: Log-log plots for maximum absolute error vs. N at $\mu = 10^{-3}$.

Example 2 (see [24]). Consider

$$\begin{cases} \varepsilon \dfrac{\partial^2 w}{\partial s^2} + \mu\left(1 + s(1-s) + t^2\right)\dfrac{\partial w}{\partial s} - (1 + 5st)w(s,t) - \dfrac{\partial w}{\partial t} = -w(s, t-1) + s(1-s)\left(e^t - 1\right), \\ (s,t) \in (0,1) \times (0,2], \\ w(s,t) = 0, (s,t) \in (0,1) \times (-1,0], \\ w(0,t) = 0, w(1,t) = 0, t \in (0,2]. \end{cases} \quad (59)$$

The validation of the theoretical findings is carried out by considering two examples whose $E_{\varepsilon,\mu}^{N,M}$, $r_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$, and $r^{N,M}$ are plotted in Tables 1 and 2. From these tables, one can see that as $\varepsilon \longrightarrow 0$ and $\mu = 10^{-3}$, the error goes constant. Moreover, as the mesh size decreases, the order of convergence goes to one, and the maximum absolute error decreases. These reveal that the solution of the proposed method converges parameter uniformly with the order of convergence in good agreement with the theoretical findings.

For several values of ε and μ, the numerical solutions for Examples 1 and 2 are plotted in Figures 1 and 2, respectively. These figures depict that the solution to the problem under consideration exhibits a parabolic type boundary layer. To show the relationship between the space variable and the solution, we have used the log-log plot in Figure 3 which is a straight line. This shows that the solution changes as a power of the space variable and confirms the first-order uniform convergence of the proposed method.

6. Conclusion

A parameter uniform convergent numerical scheme for singularly perturbed two parametric parabolic problem with time lag is presented. The proposed numerical scheme comprises the implicit Euler method and novel finite difference method in the time and space directions, respectively. Parameter uniform convergence analysis of the scheme is investigated theoretically as well as numerically. The obtained results show that the presented scheme gives better accuracy than the existing schemes. The presented scheme is accurate and convergent with the order of convergence $O(\Delta t + \Delta s^2)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

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