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Research Article

Employing a Modified Sumudu with a Modified Iteration Method to Solve the System of Nonlinear Partial Differential Equations

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The Sumudu transform is presented in this paper in a modified form which is aimed at improving its performance and employing it along with a modified iteration method in order to determine the solution to a system of nonlinear partial differential equations. This includes a theoretical analysis of the associated modified Sumudu transform. It also includes an explanation of the mathematical method for utilizing the transform in conjunction with the modified iteration technique. The iteration method is employed to determine the nonlinear terms of the equations. The research is valuable in the sense that it allows approximate and exact solution configurations to be determined by combining the modified Sumudu transform with a modified iteration method. As another benefit, the modified Sumudu transform can be developed and enhanced to be applicable to a wide range of equations, making it an effective solution tool. By combining techniques, a final advantage is that the solutions can be derived quickly and easily as a result of the combined approach. Finally, an old transformation which has been modified from the Sumudu transform is combined with the modified iteration method to examine its capability of yielding convergent solutions by incorporating the modified iteration method into it.

1. Introduction

Some phenomena occurring in nature can be modeled using nonlinear partial differential equations (NPDEs) which is possible in the vast majority of cases. They can be used to model physical phenomena such as nonlinear fibre optics and wave propagation in the Kerr media [1–3]. Systems of NPDEs have wide-ranging engineering and science applications in many areas, such as quantum field theory and fluid mechanics [4, 5]. Solutions of these equations can be used to predict the behavior of a system over time, allowing us to gain a deeper understanding of the underlying physical phenomena. By utilizing this knowledge, one can design effective solutions to problems that are difficult to solve [6]. In the areas mentioned above, these solutions are extremely valuable, and they can help us optimize the design of a system. In short, NPDEs have a wide range of applications and are essential tools for understanding the world around us [1, 2].

The NPDEs are notoriously difficult to solve and can require advanced mathematical techniques, their solutions are often not unique. However, numerical methods can be used to approximate the solutions, allowing us to gain insight into the behavior of the system being modeled. This is because the fact that these equations can represent intricate systems with a large number of interacting components provides a more sophisticated description of physical systems than linear equations [4, 5].

Different methods have been developed by mathematicians and physicists to solve systems of partial differential equations (PDEs). In this regard, the Sumudu transform (ST) can be regarded as one of these methods. The ST belongs to the class of integral transforms such as the Laplace, Kamal, and Sawi transforms [7–9]. It is easy to implement and an efficient, reliable technique and powerful tool for solving a wide range of mathematical problems, among them PDEs [10, 11]. Furthermore, by utilizing this transform, not only will time be saved, but also precise and reliable results can be obtained in a fraction of the time by using its capabilities [12–14].

The ST has applications in various fields, including physics, engineering, and mathematics. It is widely used in physics applications, such as signal processing, heat transfer, and fluid dynamics [11, 15–17]. The ST can be used to transform a given signal into a different domain in which the original signal can be more easily analyzed. This is particularly useful for solving differential equations and integral equations, which are often encountered in physics and engineering [7, 10, 18, 19].

It is worthwhile to mention that the use of the ST in conjunction with other methods is useful for solving PDEs with nonlinear terms [20]. Combining these techniques can be used to simplify the complexity of equations and make the solution process easier. Additionally, it can be applied for the precise determination of various systems [21–25].

Combining modified iteration (MI) and the ST is one of the possibilities. This combination allows for the accurate solution of many NPDEs that were previously difficult to solve [22, 26]. In connection with the MI method, it is a numerical technique which involves repeatedly applying a function to the current approximation of the solution in order to obtain a better approximation. It starts with a guess of the solution and then uses steps of calculations to improve the accuracy of the guess. Each iteration adds more details and refines the solution until it converges to a precise answer. As a result of this method, it is often used when solving a problem for which there is no analytical approach. It uses a numerical approach to approximate the solution of the equation. This approach is simpler than analytical methods and can be used to quickly find approximate solutions. The accuracy of the solution depends on the number of iterations used [27].

The ST may allow more efficient and appropriate equation-solving methods to be developed by understanding their underlying principles [28, 29]. Thus, this would enable researchers to explore more effective approaches to solving equations. By doing so, researchers could unlock new solutions to complicated mathematical problems that could impact many areas of study, enabling major advances in different fields, and paving the way for more challenging and effective equation-solving applications [30].

Specifically, and as a follow-up to the ST, in this article, the aim of this study is to modify the ST and then to combine it with the MI method, which becomes a new technique that can be used to solve a system of NPDEs in order to achieve the desired results. We will demonstrate the use of this combination by solving three systems of NPDEs as examples. We will also present a clear example of a NPDE and demonstrate how to combine the old transformations from [31] with the MI method to obtain convergent solutions to the NPDE.

Based on the results of our study, it is worth mentioning that the modified Sumudu transform (MST) is an effective technique for determining various equations that involve derivatives. The MST is able to accurately calculate derivatives of functions. It is based on the idea that a function can be represented in terms of its derivatives. By using this method, we can find the value of a function at a certain point and its derivatives at that point. Additionally, it can be used to reduce the equations to simpler forms, making them easier to solve. This could lead to faster problem-solving and a better comprehension of the mathematics involved. This allows for more factual solutions to equations, as well as a better understanding of the mathematics involved.

As a matter of fact, each integral transform has its own properties and structure typical of a mathematical method, and their aim is to find the solution to the differential equation based on their properties [7–9]. In spite of this, these techniques face dilemmas if they stop at a certain number of iterations. For instance, stopping on the third or fourth iteration would indicate that there is an error associated with the iteration. In order to avoid this dilemma, the MST has been designed with a condition in its mathematical method structure to avoid this dilemma from ever occurring (see Section 3).

The article is structured in the following way: In Section 2, a fundamental definition of MST is presented, followed by proofs of its theorems regarding certain functions. Examples that we prove include constants, polynomials, trigonometric, and exponential functions which are the main examples that we use in this section. In Section 3, a mathematical technique for using the MST in combination with the MI method to solve systems of NPDEs is explained which includes the design of a combination technique and illustrating how to apply the new approach. In Section 4, we will present a theorem concerning the convergence of series solution. Section 5 presents three different examples of NPDE systems solved using an appropriate MST method combined with an MI approach, deriving the solutions of the corresponding NPDE system that evaluates the effectiveness of the modified Sumudu-modified iteration (MSMI) method. At last, Section 6 demonstrates how the transformation proposed in [31] can be combined with the MI method to solve a NPDE.

2. Fundamental Definition and Theorems

It is the primary objective of this section to establish a fundamental understanding of the MST and to prove the theorems of our transform that lead to results for some functions, and this explores how the MST can be applied to a range of different functions, demonstrating its potential for a variety of applications. Furthermore, this section will provide an analysis of the various properties of the MST that distinguish it from the traditional ST.

2.1. Modified Sumudu Transform. The MST of the function (t) is specified as

$$\mathscr{M}\mathscr{S}[f(t)] = \frac{\log(a)}{p}\int_0^\infty f(t)a^{-(t/p)}\mathrm{d}t = G(p), \quad p \in (\eta_1, \eta_2). \tag{1}$$

Here, G represents a function with respect to p and $a \in (0,\infty) \setminus \{1\}$, η_1 and η_2 may be finite or infinite, p is a real number, $f(t)$ is a piece-wise continuous function, and the abbreviation "$\mathcal{MS}$" denoted to the integral transform of the modified Sumudu that is studied on the collection of functions:

$$\mathcal{A} = \left\{ f(t) | \exists M, M > 0, \eta_1, \eta_2 > 0, \text{s.t.} |f(t)| < M a^{|t|/\eta_g}, \text{if } t \in (-1)^g \times [0,\infty) \right\}. \tag{2}$$

Here, $g = 1, 2$ and the constant M must be finite numbers, and the definition above is established with its applications on the system of NPDEs. The following section is devoted to using the above theorems with the aid of the MI method to determine an appropriate solution to the NPDEs.

Theorem 1. *For $t \in (0,\infty)$, $\mathcal{MS}[f(t)] = \mathcal{G}(p)$, then,*

$$\begin{aligned} \mathcal{MS}\left[f^{(j)}(t)\right] &= \left(\frac{\log(a)}{p}\right)^j \mathcal{MS}[f(t)] \\ &\quad - \sum_{k=0}^{j-1} \left(\frac{\log(a)}{p}\right)^{j-k} f(0), \quad \text{for} \quad j \geq 0. \end{aligned} \tag{3}$$

Here, $(f^{(j)})$ represents the jth derivative. In regards to $j = 1$, it gives

$$\mathcal{MS}\left[f'(t)\right] = \frac{\log(a)}{p} \mathcal{MS}[f(t)] - \frac{\log(a)}{p} f(0). \tag{4}$$

Proof. According to the definition (1) of the $\mathcal{MS}(f)$,

$$\mathcal{MS}\left[f'(t)\right] = \frac{\log(a)}{p} \int_0^\infty f'(t) a^{-(t/p)} dt. \tag{5}$$

Using integration by parts results in

$$\begin{aligned} \mathcal{MS}\left[f'(t)\right] &= \frac{\log(a)}{p} \left[a^{-(t/p)} f(t) \Big|_0^\infty - \int_0^\infty - \frac{\log(a)}{p} a^{-(t/p)} f(t) \mathrm{d}t \right] \\ &= = -\frac{\log(a)}{p} f(0) + \frac{\log(a)}{p} \mathcal{MS}[f(t)]. \end{aligned} \tag{6}$$

□

Theorem 2. *Let s be a constant and $s \geq 0$, then*

$$\mathcal{MS}[s] = s. \tag{7}$$

Proof. We have $(t) = s$, and using definition (1) of the MST ($\mathcal{MS}$), we obtain

$$\begin{aligned} \mathcal{MS}[f(t)] &= \frac{\log(a)}{p} \int_0^\infty f(t) a^{-(t/p)} dt \\ &= \frac{\log(a)}{p} \int_0^\infty s\, a^{-(t/p)} dt \\ &= -s\, a^{-(t/p)} \Big|_0^\infty = s. \end{aligned} \tag{8}$$

□

Theorem 3. *For an integer number $j \geq 1$, the $\mathcal{MS}[t^j]$ is*

$$\mathcal{MS}\left[t^j\right] = j! \frac{p^j}{\log^j(a)}. \tag{9}$$

Let us take $j = 1$:

$$\mathcal{MS}[t] = \frac{p}{\log(a)}. \tag{10}$$

Proof. When $j = 1$, $f(t) = t$, according to definition (1) of the MST, we obtain

$$\begin{aligned} \mathcal{MS}[f(t)] &= \frac{\log(a)}{p} \int_0^\infty f(t) a^{-(t/p)} \mathrm{d}t, \\ \mathcal{MS}[t] &= \frac{\log(a)}{p} \int_0^\infty t\, a^{-(t/p)} dt. \end{aligned} \tag{11}$$

Using integration by parts results in

$$\mathcal{MS}[t] = 0 - \frac{p}{\log(a)} a^{-(t/p)} \Big|_0^\infty = \frac{p}{\log(a)}. \tag{12}$$

For $j = 2$

$$\mathcal{MS}\left[t^2\right] = 2 \frac{p^2}{\log^2(a)}. \tag{13}$$

□

Proof. When $j = 1$, $f(t) = t$, according to definition (1) of the MST, we obtain

$$\begin{aligned} \mathcal{MS}[f(t)] &= \frac{\log(a)}{p} \int_0^\infty f(t) a^{-(t/p)} \mathrm{d}t, \\ \mathcal{MS}\left[t^2\right] &= \frac{\log(a)}{p} \int_0^\infty t^2 a^{-(t/p)} dt. \end{aligned} \tag{14}$$

Using integration by parts results in

$$\mathscr{M}\mathcal{S}\left[t^2\right] = t^2 a^{-(t/p)}\Big|_0^\infty - \frac{2\,t\,p}{\log^2(a)} a^{-(t/p)}\Bigg|_0^\infty - 2\frac{p^2}{\log^2(a)} a^{-(t/p)}\Bigg|_0^\infty = 0 - 0 - 0 + \frac{2p^2}{\log^2(a)} = \frac{2p^2}{\log^2(a)}, \tag{15}$$

and so on for $j = 3, 4, \cdots$. □

Remark 4. The MST of some other functions

$$\begin{aligned} \mathscr{M}\mathcal{S}\left[e^{mt}\right] &= \frac{\log(a)}{\log(a) - pm}, \quad m < \frac{\log(a)}{p}, \\ \mathscr{M}\mathcal{S}[\sin(mt)] &= \frac{mp\log(a)}{\log^2(a) - p^2m^2}, \\ \mathscr{M}\mathcal{S}[\cos(mt)] &= \frac{mp\log^2(a)}{\log^2(a) - p^2m^2}. \end{aligned} \tag{16}$$

3. Mathematical Method

In this section, we exhibit our mathematical technique for solving NPDEs. This technique involves incorporating MST into a MI method so that it can enhance precision and adequacy.

3.1. Modified Sumudu-Modified Iteration Method for Solving Partial Differential Equation. The combination method corresponding to MSMI will be shown in the following mathematical steps. We begin by considering NPDEs in the following equation as a starting point for our technique:

$$L[\gamma(r,t)] + \mathrm{LN}[\gamma(r,t)] + B(r,t) = 0. \tag{17}$$

Here, $\gamma(r,t)$ is a function with respect to the scaled spatial variable r and time variable t, a linear operator is L of the first order, functions in LN are linear and nonlinear, the initial condition(s) (IC(s)) of equation (17) is $\gamma(r,0) = V(r)$, and the functions $V(r)$ and $B(r,t)$ are known. As part of the next step, we first apply the MST to eq. (17):

$$\mathscr{M}\mathcal{S}[L[\gamma(r,t)]] + \mathscr{M}\mathcal{S}[\mathrm{LN}[\gamma(r,t)]] + \mathscr{M}\mathcal{S}[\mathrm{B}(r,t)] = 0, \tag{18}$$

Theorems 1 and >2 can now be used along with the IC on eq. (18) as follows:

$$\bar{\gamma}(r,v) = \gamma(r,0) - \frac{p}{\log(a)}\mathscr{M}\mathcal{S}[\mathrm{LN}[\gamma(r,t)] + \mathrm{B}(r,t)]. \tag{19}$$

According to Theorem 1, we get

$$\mathscr{M}\mathcal{S}[L[\gamma(r,t)]] = \frac{\log(a)}{p}(\bar{\gamma}(r,v) - \gamma(r,0)). \tag{20}$$

Now, taking the inverse of MST $(\mathscr{M}\mathcal{S})^{-1}$ of eq. (19), we can then produce the value of $\gamma(r,t)$ as follows:

$$\gamma(r,t) = \gamma(r,0) - \mathscr{M}\mathcal{S}^{-1}\left[\frac{p}{\log(a)}\mathscr{M}\mathcal{S}[\mathrm{LN}[\gamma(r,t)] + \mathrm{B}(r,t)]\right]. \tag{21}$$

In the second stage, employing the MI method, eq. (17) can be solved. In eq. (21), assume $\gamma(r,0) = \gamma_0(r,t)$, which would be the first approximate solution, so eq. (21) would become

$$\gamma_{n+1}(r,t) = \gamma_0(r,0) - \mathscr{M}\mathcal{S}^{-1} \cdot \left[\frac{p}{\log(a)}\mathscr{M}\mathcal{S}[LN[\gamma_n(r,t)] + \mathrm{B}_n(r,t)]\right], \quad n = 0,1,2,\cdots. \tag{22}$$

Here, γ_{+1n} represents the number of approximate solution, and γ_0 is defined as a known function.

Remark 5. In the classical iteration method, some terms are generated during the calculation process. As a consequence, we have to eliminate unnecessary terms from our calculations in order to avoid any complications that disturb our calculation [3]. This can be achieved by ensuring that a condition is defined in the MI procedure as follows:

$$\mathrm{LN}[\gamma_n(r,t)] = \mathrm{Q}_n(r,t) + \mathcal{O}\left(t^{n+1}\right). \tag{23}$$

Here, $\mathcal{O}(t^{n+1})$ represents terms of higher order that should be disregarded. In this case, we can write eq. (22) as follows:

$$\gamma_{n+1}(r,t) = \gamma_0(r,0) - \mathscr{M}\mathcal{S}^{-1}\left[\frac{p}{\log(a)}\mathscr{M}\mathcal{S}[\mathrm{Q}_n(r,t) + B_n(r,t)]\right]. \tag{24}$$

Assume that $B_n(r,t)$ is taken as smooth enough so that Taylor's expansion can be found in the variable t,

$$\mathrm{B}_n(r,t) = \mathscr{B}_n(r,t) + \mathcal{O}\left(t^{n+1}\right), \tag{25}$$

where

$$\begin{aligned} \mathscr{B}_n &= \sum_{k=0}^{n+1} S_k t^k = S_0 + S_1 t + S_2 t^2 + \cdots + S_{n+1} t^{n+1}, \\ S_k &= \frac{1}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathscr{B}_n(r,t)\right]. \end{aligned} \tag{26}$$

So, eq. (24) becomes

$$\gamma_{n+1}(r,t) = \gamma_0(r,0) - \mathscr{M}\mathcal{S}^{-1}\left[\frac{p}{\log(a)}\mathscr{M}\mathcal{S}[\mathrm{Q}_n(r,t) + B_n(r,t)]\right]. \tag{27}$$

In contrast, it is possible to write eq. (27) as follows:

$$\gamma_{n+1}(r,t)=\gamma_0(r,0)-\mathscr{M}\mathscr{S}^{-1}\cdot\left[\frac{p}{\log(a)}\mathscr{M}\mathscr{S}[Q_{n-1}+B_{n-1}]\right]--\mathscr{M}\mathscr{S}^{-1}\cdot\left[\frac{p}{\log(a)}\mathscr{M}\mathscr{S}[(Q_n-Q_{n-1})+(B_n-B_{n-1})]\right]. \quad (28)$$

Put $n=n-1$ in (27), and after that, we employ it to eq. (28), which gives us

$$\gamma_{n+1}(r,t)=\gamma_n(r,0)-\mathscr{M}\mathscr{S}^{-1}\cdot\left[\frac{p}{\log(a)}\mathscr{M}\mathscr{S}[(Q_n-Q_{n-1})+(B_n-B_{n-1})]\right], \quad (29)$$

where the value $Q_{-1}=B_{-1}=0$. Next, assume $T[\gamma_n(r,t)]$ be defined as follows:

$$T[\gamma_n(r,t)]=-\mathscr{M}\mathscr{S}^{-1}\left[\frac{p}{\log(a)}\mathscr{M}\mathscr{S}[(Q_n-Q_{n-1})+(B_n-B_{n-1})]\right]. \quad (30)$$

Assuming $\xi_{n+1}=T[\xi_0+\xi_1+\xi_2+\cdots+\xi_n]$, such that

$$\begin{aligned}\xi_0&=\gamma_0,\\ \xi_1&=T[\xi_0],\\ \xi_2&=T[\xi_0+\xi_1]=T[\gamma_1],\\ &\vdots\\ \xi_{n+1}&=T[\xi_0+\xi_1+\xi_2+\cdots+\xi_n],\end{aligned} \quad (31)$$

eq. (29) will be

$$\gamma_{n+1}(r,t)=\gamma_n(r,0)+T[\gamma_n(r,t)], \quad (32)$$

where

$$\begin{aligned}\gamma_1&=\gamma_0+T[\gamma_0]=\xi_0+\xi_1,\\ \gamma_2&=\gamma_1+T[\gamma_1]=\xi_0+\xi_1+T[\xi_0+\xi_1]=\xi_0+\xi_1+\xi_2,\\ &\vdots\\ \gamma_n&=\xi_0+\xi_1+\xi_2+\cdots+\xi_n.\end{aligned} \quad (33)$$

Considering iterative solutions to the problem, an approximate solution $\gamma_{n+1}(r,t)$ has been determined which represents the sum of the solutions that can be obtained through a series of solution:

$$\gamma(r,t)=\lim_{n\longrightarrow\infty}\gamma_n(r,t)=\sum_{k=0}^{\infty}\xi_k. \quad (34)$$

4. Convergence of Series Solution

The series approaches a definite limit as the number of terms increases, indicating that the series converges to a particular value. This is usually seen as the series approaching a single value as the number of terms increases. There is no doubt that infinite series is one of the most widely used concepts in math and science. These series are useful for approximating trigonometrical and logarithmic functions, evaluating difficult differential equations, and defining new functions. In this study, it will be useful to refer to the following theorem.

Theorem 6. *Assuming that the series starts with λ, and each term is the result of multiplying the preceding term by μ, which is known as a geometric series, and has the following form*

$$\sum_{c=0}^{\infty}\lambda\mu^c=\lambda+\lambda\mu+\lambda\mu^2+\lambda\mu^3+\cdots,\quad(\lambda\neq 0), \quad (35)$$

then the series (35) tends to converge if $|\mu|<1$ and tends to diverge if $|\mu|\geq 1$ [32]. If convergence occurs then, the sum is

$$\sum_{c=0}^{\infty}\lambda\mu^c=\frac{\lambda}{1-\mu}. \quad (36)$$

5. Numerical Applications

Here, we illustrate how the MSMI method is effective for solving certain systems of NPDEs, and the semianalytic (numerical with analytic) MSMI method is employed for homogeneous and nonhomogeneous systems. In order to attain the desired solution, we start with a first approximate solution $\gamma_0(r,h,t)$, based on the IC $\gamma(r,h,0)$, and then apply the MSMI method onto the system of NPDEs.

Example 7. Take into consideration the system of homogeneous NPDEs,

$$\begin{aligned}\phi_t(r,h,t)&=\phi^2\psi-2\phi+\frac{1}{4}[\phi_{rr}+\phi_{hh}],\\ \psi_t(r,h,t)&=\phi-\phi^2\psi+\frac{1}{4}[\psi_{rr}+\psi_{hh}],\end{aligned} \quad (37)$$

where ϕ and ψ are functions with respect to the scaled spatial variables r and h and time variable t. The ICs of eq. (37) are

$$\begin{aligned}\phi(r,h,0)&=\phi_0=e^{-r-h},\\ \psi(r,h,0)&=\psi_0=e^{r+h}.\end{aligned} \quad (38)$$

In comparison with eq. (17), it is evident that $B(r,h,t)=0$ in eq. (37), and

$$\begin{aligned}\phi_t &= \frac{\partial\phi(r,h,t)}{\partial t},\\ \mathrm{LN}[\phi(r,h,t)] &= -\left(\phi^2\psi - 2\phi + \frac{1}{4}[\phi_{rr}+\phi_{hh}]\right),\\ \psi_t &= \frac{\partial\psi(r,h,t)}{\partial t},\\ \mathrm{LN}[\psi(r,h,t)] &= -\left(\phi - \phi^2\psi + \frac{1}{4}[\psi_{rr}+\psi_{hh}]\right).\end{aligned} \tag{39}$$

As outlined in Section 3, by using the MST and applying the steps from eq. (18) to eq. (29), we have

$$\begin{aligned}\gamma_{\{n+1,\phi\}}(r,h,t) &= e^{-r-h} - \mathscr{MS}^{-1}\left[\frac{p}{\log(a)}\mathscr{MS}\left[Q_{\{n,\phi\}} - Q_{\{n-1,\phi\}}\right]\right],\\ \gamma_{\{n+1,\psi\}}(r,h,t) &= e^{r+h} - \mathscr{MS}^{-1}\left[\frac{p}{\log(a)}\mathscr{MS}\left[Q_{\{n,\psi\}} - Q_{\{n-1,\psi\}}\right]\right],\end{aligned} \tag{40}$$

where $\gamma_{\{0,\phi\}}(r,h,t)=e^{-r-h}$ and $\gamma_{\{0,\psi\}}(r,h,t)=e^{r+h}$ are first approximate solutions. Significantly from eq. (23), we can find $Q_{\{n,\phi\}}$ and $Q_{\{n,\psi\}}$ in eq. (40) from the following relations:

$$\begin{aligned}Q_{\{n,\phi\}} &= \mathrm{LN}[\phi_n(r,h,t)] = -\phi_n^2\psi_n + 2\phi_n - \frac{1}{4}\left[\phi_{\{n,rr\}} + \phi_{\{n,hh\}}\right],\\ Q_{\{n,\psi\}} &= \mathrm{LN}[\psi_n(r,h,t)] = -\phi_n + \phi_n^2\psi_n - \frac{1}{4}\left[\psi_{\{n,rr\}} + \psi_{\{n,hh\}}\right].\end{aligned} \tag{41}$$

Next, we need to follow the steps from eq. (30) to relation (34). When $n=0$, eq. (30) becomes

$$\begin{aligned}\xi_{\{1,\psi\}} &= T\left[\gamma_{\{0,\psi\}}\right] = -\mathscr{MS}^{-1}\cdot\left[\frac{p}{\log(a)}\mathscr{MS}\left[\left(Q_{\{0,\psi\}} - Q_{\{0-1,\psi\}}\right)\right]\right], \text{ where } Q_{\{-1,\psi\}}=0,\\ \xi_{\{1,\phi\}} &= T\left[\gamma_{\{0,\phi\}}\right] = -\mathscr{MS}^{-1}\cdot\left[\frac{p}{\log(a)}\mathscr{MS}\left[\left(Q_{\{0,\phi\}} - Q_{\{0-1,\phi\}}\right)\right]\right], \text{ where } Q_{\{-1,\phi\}}=0,\end{aligned} \tag{42}$$

$$\begin{aligned}\xi_{\{0,\phi\}} &= \gamma_{\{0,\phi\}}(r,h,t) = e^{-r-h},\\ \xi_{\{0,\psi,\}} &= \gamma_{\{0,\psi\}}(r,h,t) = e^{r+h}.\end{aligned} \tag{43}$$

As $n=0$, eq. (41) becomes

$$\begin{aligned}Q_{\{0,\psi\}} &= \phi_0^2\psi_0 - 2\phi_0 + \frac{1}{4}\left[\phi_{\{0,rr\}} + \phi_{\{0,hh\}}\right],\\ Q_{\{0,\phi\}} &= \phi_0 - \phi_0^2\psi_0 + \frac{1}{4}\left[\psi_{\{0,rr\}} + \psi_{\{0,hh\}}\right],\end{aligned} \tag{44}$$

by substituting relation (43) in eq. (44) and neglecting the coefficients of t for which the indicators are greater or equal to 1 ($\mathcal{O}(t)$). Then

$$\begin{aligned}Q_{\{0,\phi\}} &= \frac{1}{2}e^{-r-h},\\ Q_{\{0,\psi\}} &= -\frac{1}{2}e^{r+h}.\end{aligned} \tag{45}$$

Here, $Q_{\{0,\psi\}}$ and $Q_{\{0,\phi\}}$ in relation (45) are time independent, so, substituting the values of $Q_{\{0,\psi\}}$ and $Q_{\{0,\phi\}}$ in eq. (42), we get

$$\begin{aligned}\xi_{\{1,\phi\}} &= -\mathscr{MS}^{-1}\left[\frac{p}{\log(a)}\mathscr{MS}\left[-\frac{1}{2}e^{-r-h}\right]\right],\\ \xi_{\{1,\psi\}} &= -\mathscr{MS}^{-1}\left[\frac{p}{\log(a)}\mathscr{MS}\left[\frac{1}{2}e^{r+h}\right]\right],\end{aligned} \tag{46}$$

Since $-(1/2)e^{-r-h}$ and $(1/2)e^{r+h}$ are time independent, which mean that they are constants with respect to t, then MST for a constant is the same constant (see Theorem 2):

$$\begin{aligned}\xi_{\{1,\phi\}} &= -\mathscr{MS}^{-1}\left[\frac{p}{\log(a)}\left(\frac{1}{2}e^{-r-h}\right)\right],\\ \xi_{\{1,\psi\}} &= -\mathscr{MS}^{-1}\left[\frac{p}{\log(a)}\left(-\frac{1}{2}e^{r+h}\right)\right],\end{aligned} \tag{47}$$

and $\mathscr{MS}^{-1}[p/\log(a)]$ is t (see Theorem 3). So, we have

$$\begin{aligned}\xi_{\{1,\phi\}} &= -\frac{t}{2}e^{-r-h},\\ \xi_{\{1,\psi\}} &= \frac{t}{2}e^{r+h}.\end{aligned} \tag{48}$$

Then, the first iterations $\gamma_{\{1,\psi\}}$ and $\gamma_{\{1,\phi\}}$ are

$$\gamma_{\{1,\phi\}}(r,h,t) = \xi_{\{0,\phi\}} + \xi_{\{1,\phi\}} = e^{-r-h} - \frac{t}{2}e^{-r-h}, \tag{49}$$

$$\gamma_{\{1,\psi\}}(r,h,t) = \xi_{\{0,\psi\}} + \xi_{\{1,\psi\}} = e^{r+h} + \frac{t}{2}e^{r+h}, \tag{50}$$

When $n=1$, we substitute $\gamma_{\{1,\psi\}}$ and $\gamma_{\{1,\phi\}}$ in eq. (41) to find $Q_{\{1,\psi\}}$ and $Q_{\{1,\phi\}}$. Our approach in this case is to neglect the coefficient of t for which the indicators are greater or equal to 2 ($\mathcal{O}(t^2)$). In this regard, we can observe that $Q_{\{1,\psi\}}$ and $Q_{\{1,\phi\}}$ are t independent:

$$\begin{aligned}Q_{\{1,\phi\}} &= \frac{1}{2}e^{-r-h} - \frac{t}{4}e^{-r-h} + \mathcal{O}\left(t^2\right),\\ Q_{\{1,\psi\}} &= -\frac{1}{2}e^{r+h} - \frac{t}{4}e^{r+h} + \mathcal{O}\left(t^2\right),\end{aligned} \tag{51}$$

substituting the values of $Q_{\{1,\psi\}}$ and $Q_{\{1,\phi\}}$ in eq. (42) to find $\xi_{\{2,\psi\}}$ and $\xi_{\{2,\phi\}}$. So, the second iterations $\gamma_{\{2,\psi\}}$ and $\gamma_{\{2,\phi\}}$ are

$$\begin{aligned}\gamma_{\{2,\phi\}}(r,h,t) &= e^{-r-h} - \frac{t}{2}e^{-r-h} + \frac{t^2}{8}e^{-r-h},\\ \gamma_{\{2,\psi\}}(r,h,t) &= e^{r+h} + \frac{1}{2}e^{r+h} + \frac{t^2}{8}e^{r+h},\end{aligned} \tag{52}$$

and so on, and in the case of $n \longrightarrow \infty$, an approximate solution of the NPDE system can be obtained by nth iterations as shown below:

$$\begin{aligned}\gamma_{\{\phi\}}(r,h,t) &= \lim_{n\longrightarrow\infty}\sum_{k=0}^{n}\xi_{\{k,\phi\}}\\ &= \mathrm{e}^{-r-h} - \frac{\mathrm{e}^{-r-h}t}{2} + \frac{\mathrm{e}^{-r-h}t^2}{8} - \frac{\mathrm{e}^{-r-h}t^3}{48}\\ &\quad + \frac{\mathrm{e}^{-r-h}t^4}{384} - \frac{\mathrm{e}^{-r-h}t^5}{3840} + \cdots,\end{aligned} \tag{53a}$$

$$\begin{aligned}\gamma_{\{\psi\}}(r,h,t) &= \lim_{n\longrightarrow\infty}\sum_{k=0}^{n}\xi_{\{k,\psi\}}\\ &= \mathrm{e}^{r+h} + \frac{\mathrm{e}^{r+h}t}{2} + \frac{\mathrm{e}^{r+h}t^2}{8} + \frac{\mathrm{e}^{r+h}t^3}{48}\\ &\quad + \frac{\mathrm{e}^{r+h}t^4}{384} + \frac{\mathrm{e}^{r+h}t^5}{3840} + \cdots.\end{aligned} \tag{53b}$$

As a result of Theorem 6, we can verify that the series (53a) converges, where $\lambda = e^{-r-h}$ and $\mu = -t/2$. Now, when $|t/2| < 1$, the resulting sum is

$$\sum_{c=0}^{\infty} e^{-r-h}\left(-\frac{t}{2}\right)^{c} = \frac{e^{-r-h}}{1+(t/2)}, \tag{54}$$

and the series in t of $\gamma\{\phi\}$ will give

$$\gamma_{\{\phi\}}(r,h,t) = e^{-r-h-(1/2)t}. \tag{55a}$$

The same discussion can be made about (53b), where the series in t of $\{\psi\}$ gives

$$\gamma_{\{\psi\}}(r,h,t) = e^{r+h+(1/2)t}. \tag{55b}$$

From relations (55a) and (55b), the convergent Taylor's series yield the solutions shown in Figures 1 and 2, Figure 3 illustrates the exact and approximate solutions of (37) at $t = 1$ related to (55a) and (53a), respectively, and Figure 4 illustrates the exact and approximate solutions of (37) at $t = 1$ related to (55b) and (53b), respectively. The results from Figures 3 and 4 with the least square error are shown in Table 1.

Example 8. Take into consideration the system of nonhomogeneous NPDEs:

$$\begin{aligned}\phi_t(r,h,t) - \psi_r\chi_h &= 1,\\ \psi_t(r,h,t) - \chi_r\phi_h &= 5,\\ \chi_t(r,h,t) - \phi_r\psi_h &= 5,\end{aligned} \tag{56}$$

with the ICs

$$\begin{aligned}\phi(r,h,0) &= r + 2h,\\ \psi(r,h,0) &= r - 2h,\\ \chi(r,h,0) &= -r + 2h.\end{aligned} \tag{57}$$

Compering eq. (56) to eq. (17)

$$\begin{aligned}\phi_t &= \frac{\partial\phi(r,h,t)}{\partial t},\\ \mathrm{LN}[\phi(r,h,t)] &= -\psi_r\chi_h,\\ \psi_t &= \frac{\partial\psi(r,h,t)}{\partial t},\\ \mathrm{LN}[\psi(r,h,t)] &= -\chi_r\phi_h,\\ \chi_t &= \frac{\partial\chi(r,h,t)}{\partial t},\\ \mathrm{LN}[\chi(r,h,t)] &= -\phi_r\psi_h,\end{aligned} \tag{58}$$

$$\begin{aligned}B_\phi(r,h,t) &= -1,\\ B_\psi(r,h,t) &= -5,\\ B_\chi(r,h,t) &= -5.\end{aligned} \tag{59}$$

As outlined in Section 3, by using the MS method and applying the steps from eq. (18) to eq. (29), we have

$$\begin{aligned}\gamma_{\{n+1,\phi\}}(r,h,t) &= t + r + 2h - \mathscr{MS}^{-1}\\ &\quad\cdot\left[\frac{p}{\log(a)}\mathscr{MS}\left[\left(Q_{\{n,\phi\}} - Q_{\{n-1,\phi\}}\right)\right.\right.\\ &\quad\left.\left. + \left(B_{\{n,\phi\}} - B_{\{n-1,\phi\}}\right)\right]\right],\\ \gamma_{\{n+1,\psi\}}(r,h,t) &= 5t + r - 2h - \mathscr{MS}^{-1}\\ &\quad\cdot\left[\frac{p}{\log(a)}\mathscr{MS}\left[\left(Q_{\{n,\psi\}} - Q_{\{n-1,\psi\}}\right)\right.\right.\\ &\quad\left.\left. + \left(B_{\{n,\psi\}} - B_{\{n-1,\psi\}}\right)\right]\right],\\ \gamma_{\{n+1,\chi\}}(r,h,t) &= 5t - r + 2h - \mathscr{MS}^{-1}\\ &\quad\cdot\left[\frac{p}{\log(a)}\mathscr{MS}\left[\left(Q_{\{n,\chi\}} - Q_{\{n-1,\chi\}}\right)\right.\right.\\ &\quad\left.\left. + \left(B_{\{n,\chi\}} - B_{\{n-1,\chi\}}\right)\right]\right],\end{aligned} \tag{60}$$

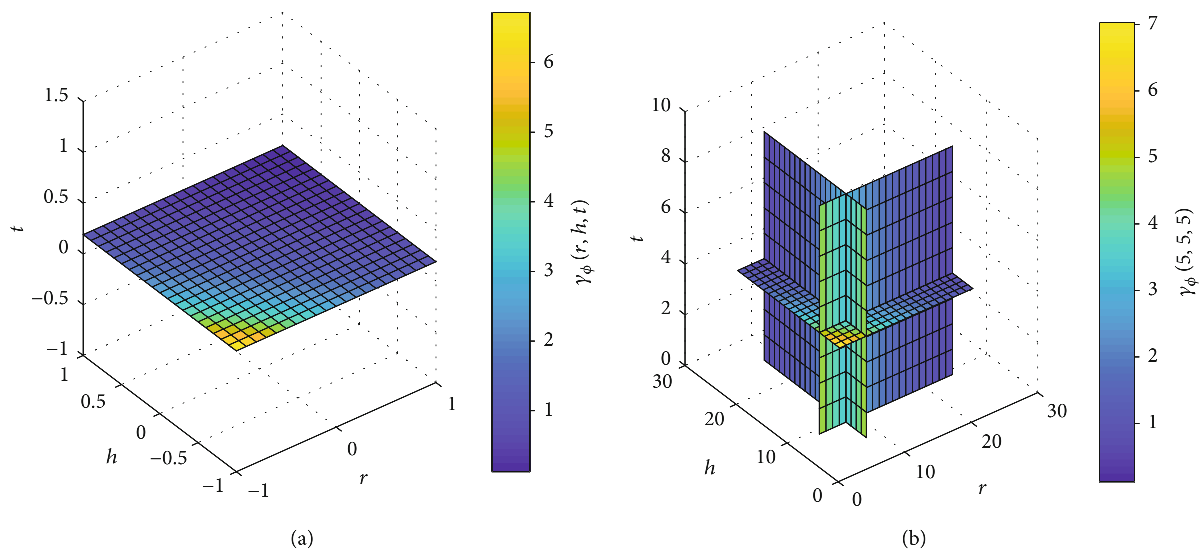

FIGURE 1: The plot that shows the solution of (37): (a) the function $\gamma_{\{\phi\}}(r,h,t) = e^{-r-h-(1/2)t}$ and (b) the function $\gamma_{\{\phi\}}(r,h,t)$ at $r = 5$, $h = 5$, $t = 5$.

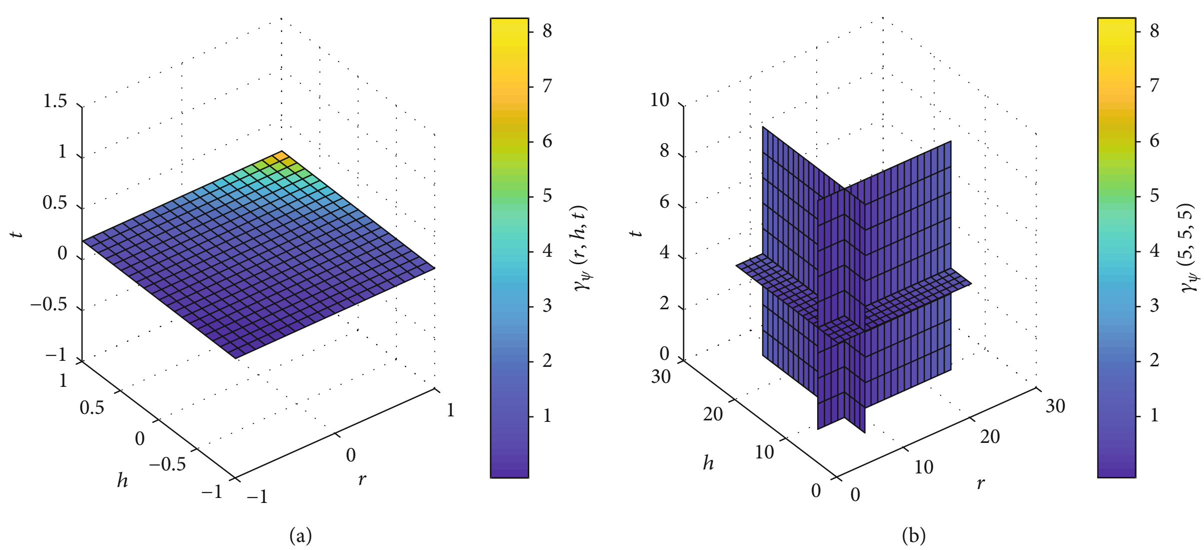

FIGURE 2: The plot that shows the solution of (37): (a) the function $\gamma_{\{\psi\}}(r,h,t) = e^{r+h+(1/2)t}$ and (b) the function $\gamma_{\{\psi\}}(r,h,t)$ at $r = 5$, $h = 5$, $t = 5$.

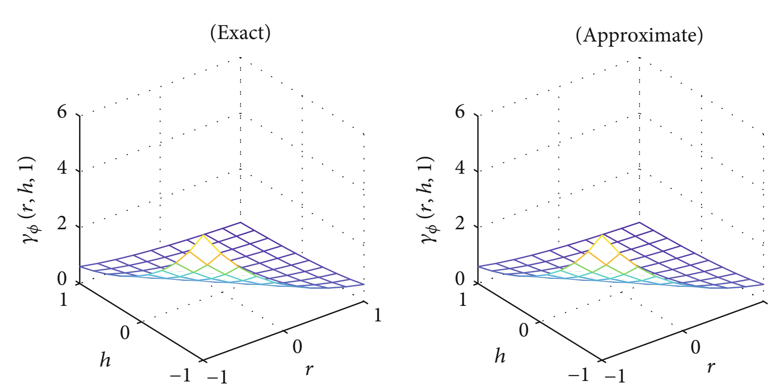

FIGURE 3: The plot that shows the exact solution (55a) and approximate solution (53a) of (37) at $t = 1$.

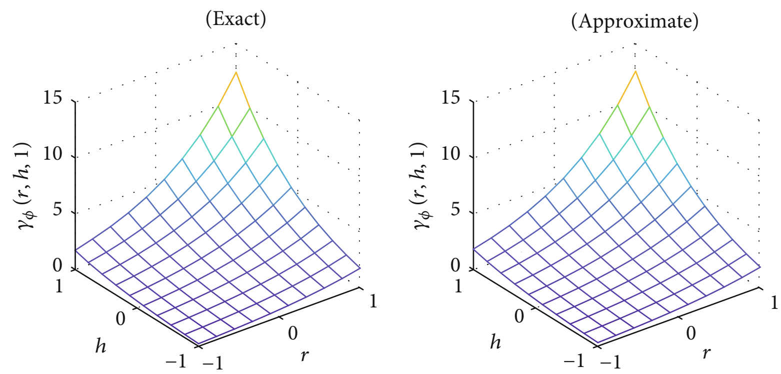

FIGURE 4: The plot that shows the exact solution (55b) and approximate solution (53b) of (37) at $t = 1$.

TABLE 1: Least square error of the solutions for Example 7.

The points	The results from Figure 3 and the errors			The solutions from Figure 4 and the errors		
r	Exact	Approximate	Error	Exact	Approximate	Error
-1	4.48169	4.48154	$2.2373e-08$	0.22313	0.22313	$9.9895e-12$
-0.77778	3.58866	3.58854	$3.6719e-08$	0.27866	0.27865	$2.5569e-11$
-0.55556	2.873571	2.87348	$4.5917e-08$	0.34800	0.34799	$4.9868e-11$
-0.33333	2.30098	2.30089	$5.1814e-08$	0.43460	0.43459	$8.7765e-11$
-0.11111	1.84248	1.84242	$5.5595e-08$	0.54275	0.54274	$1.4687e-10$
0.11111	1.47534	1.47529	$5.8020e-08$	0.67781	0.67780	$2.3905e-10$
0.33333	1.18136	1.18132	$5.9575e-08$	0.84648	0.84647	$3.8282e-10$
0.55556	0.94596	0.94593	$6.0571e-08$	1.05713	1.05711	$6.0705e-10$
0.77778	0.75747	0.75744	$6.1210e-08$	1.32019	1.32017	$9.5675e-10$
1	0.60653	0.60651	$6.1620e-08$	1.64872	1.64870	$1.5022e-09$
Least square error			$4.5179e-07$	Least square error		$2.5057e-09$

where

$$\begin{aligned}\gamma_{\{0,\phi\}}(r,h,t) &= t + r + 2h,\\ \gamma_{\{0,\psi\}}(r,h,t) &= 5t + r - 2h,\\ \gamma_{\{0,\chi\}}(r,h,t) &= 5t - r + 2h,\end{aligned} \tag{61}$$

are first approximate solutions. Significantly from eq. (23), we can find $Q_{\{n,\phi\}}$ and $Q_{\{n,\psi\}}$ in eq. (60) from the following relations:

$$\begin{aligned}Q_{\{n,\phi\}} &= -\psi_{\{n,r\}}\chi_{\{n,h\}},\\ Q_{\{n,\psi\}} &= -\chi_{\{n,r\}}\phi_{\{n,h\}},\\ Q_{\{n,\chi\}} &= -\phi_{\{n,r\}}\psi_{\{n,h\}}.\end{aligned} \tag{62}$$

Next, we need to follow the steps from eq. (30) to relation (34). When $n = 0$, eq. (30) becomes

$$\begin{aligned}\xi_{\{1,\phi\}}(r,h,t) &= T\left[\gamma_{\{1,\phi\}}\right]\\ &= -\mathscr{M}\mathscr{S}^{-1}\left[\frac{p}{\log(a)}\mathscr{M}\mathscr{S}\left[\left(Q_{\{0,\phi\}} - Q_{\{-1,\phi\}}\right)\right.\right.\\ &\quad \left.\left.+\left(B_{\{0,\phi\}} - B_{\{-1,\phi\}}\right)\right]\right],\\ \xi_{\{1,\psi\}}(r,h,t) &= T\left[\gamma_{\{1,\psi\}}\right]\\ &= -\mathscr{M}\mathscr{S}^{-1}\left[\frac{p}{\log(a)}\mathscr{M}\mathscr{S}\left[\left(Q_{\{0,\psi\}} - Q_{\{-1,\psi\}}\right)\right.\right.\\ &\quad \left.\left.+\left(B_{\{0,\psi\}} - B_{\{-1,\psi\}}\right)\right]\right],\\ \xi_{\{1,\chi\}}(r,h,t) &= T\left[\gamma_{\{1,\chi\}}\right]\\ &= -\mathscr{M}\mathscr{S}^{-1}\left[\frac{p}{\log(a)}\mathscr{M}\mathscr{S}\left[\left(Q_{\{0,\chi\}} - Q_{\{-1,\chi\}}\right)\right.\right.\\ &\quad \left.\left.+\left(B_{\{0,\chi\}} - B_{\{-1,\chi\}}\right)\right]\right].\end{aligned} \tag{63}$$

The values $Q_{-1} = 0$ and $B_{-1} = 0$, and

$$\begin{aligned} \xi_{\{0,\phi\}} &= \gamma_{\{0,\phi\}}(r, h, t) = t + r + 2h, \\ \xi_{\{0,\psi,\}} &= \gamma_{\{0,\psi\}}(r, h, t) = 5t + r - 2h, \\ \xi_{\{0,\chi,\}} &= \gamma_{\{0,\chi\}}(r, h, t) = 5t - r + 2h. \end{aligned} \tag{64}$$

As $n = 0$, relations (59) and (62) become

$$\begin{aligned} Q_{\{0,\phi\}} &= -\psi_{\{0,r\}}\chi_{\{0,h\}} = -2, \quad B_{\{0,\phi\}} = -1, \\ Q_{\{0,\psi\}} &= -\chi_{\{0,r\}}\phi_{\{0,h\}} = 2, \quad B_{\{0,\psi\}} = -5, \\ Q_{\{0,\chi\}} &= -\phi_{\{0,r\}}\psi_{\{0,h\}} = 2, \quad B_{\{0,\chi\}} = -5. \end{aligned} \tag{65}$$

Here, $Q_{\{0,\phi\}}, Q_{\{0,\psi\}}, Q_{\{0,\chi\}}, B_{\{0,\phi\}}, B_{\{0,\psi\}}$, and $B_{\{0,\chi\}}$ in (65) are time independent. So, substitute relation (65) in eq. (63) to find $\xi_{\{1,\psi\}}, \xi_{\{1,\phi\}}$ and $\xi_{\{1,\chi\}}$:

$$\begin{aligned} \xi_{\{1,\phi\}} &= -\mathcal{MS}^{-1}\left[\frac{p}{\log(a)}\mathcal{MS}[-3]\right], \\ \xi_{\{1,\psi\}} &= -\mathcal{MS}^{-1}\left[\frac{p}{\log(a)}\mathcal{MS}[-3]\right], \\ \xi_{\{1,\chi\}} &= -\mathcal{MS}^{-1}\left[\frac{p}{\log(a)}\mathcal{MS}[-3]\right]. \end{aligned} \tag{66}$$

Since 3 is constant, then $\mathcal{MS}$ of a constant is the same constant (see Theorem 2):

$$\begin{aligned} \xi_{\{1,\phi\}} &= \mathcal{MS}^{-1}\left[\frac{3p}{\log(a)}\right], \\ \xi_{\{1,\psi\}} &= \mathcal{MS}^{-1}\left[\frac{3p}{\log(a)}\right], \\ \xi_{\{1,\chi\}} &= \mathcal{MS}^{-1}\left[\frac{3p}{\log(a)}\right], \end{aligned} \tag{67}$$

and $\mathcal{MS}^{-1}[p/\log(a)]$ is t (see Theorem 3), so we get

$$\begin{aligned} \xi_{\{1,\phi\}} &= 3t, \\ \xi_{\{1,\psi\}} &= 3t, \\ \xi_{\{1,\chi\}} &= 3t, \end{aligned} \tag{68}$$

and then, the first iterations $\gamma_{\{1,\psi\}}, \gamma_{\{1,\phi\}}$ and $\gamma_{\{1,\chi\}}$ are

$$\begin{aligned} \gamma_{\{1,\phi\}}(r, h, t) &= \xi_{\{0,\phi\}} + \xi_{\{1,\phi\}} = r + 2h + 3t, \\ \gamma_{\{1,\psi\}}(r, h, t) &= \xi_{\{0,\psi\}} + \xi_{\{1,\psi\}} = r - 2h + 3t, \\ \gamma_{\{1,\chi\}}(r, h, t) &= \xi_{\{0,\chi\}} + \xi_{\{1,\chi\}} = -r + 2h + 3t. \end{aligned} \tag{69}$$

When $n = 1$, to find $Q_{\{1,\psi\}}, Q_{\{1,\phi\}}$ and $Q_{\{1,\chi\}}$ and $B_{\{1,\psi\}}$, $B_{\{1,\phi\}}$ and $B_{\{1,\chi\}}$, we substitute $\gamma_{\{1,\psi\}}$ and $\gamma_{\{1,\phi\}}$ in relation (62). Clearly, we do not have any neglected terms ($\mathcal{O}(t^2)$). In this regard, we can observe that $Q_{\{1,\psi\}}, Q_{\{1,\phi\}}$ and $Q_{\{1,\chi\}}$ are t independent,

$$\begin{aligned} Q_{\{1,\phi\}} &= -2, B_{\{1,\phi\}} = -1, \\ Q_{\{1,\psi\}} &= 2, B_{\{1,\psi\}} = -5, \\ Q_{\{1,\chi\}} &= 2, B_{\{1,\chi\}} = -5, \end{aligned} \tag{70}$$

and in order to find $\xi_{\{2,\psi\}}, \xi_{\{2,\phi\}}$ and $\xi_{\{2,\chi\}}$, substitute the values of relation (70) in eq. (30):

$$\begin{aligned} \xi_{\{2,\phi\}}(r, h, t) &= -\mathcal{MS}^{-1}\left[\frac{p}{\log(a)}\mathcal{MS}\left[\left(Q_{\{1,\phi\}} - Q_{\{0,\phi\}}\right) + \left(B_{\{1,\phi\}} - B_{\{0,\phi\}}\right)\right]\right] = 0, \\ \xi_{\{2,\psi\}}(r, h, t) &= -\mathcal{MS}^{-1}\left[\frac{p}{\log(a)}\mathcal{MS}\left[\left(Q_{\{1,\psi\}} - Q_{\{0,\psi\}}\right) + \left(B_{\{1,\psi\}} - B_{\{0,\psi\}}\right)\right]\right] = 0, \\ \xi_{\{2,\chi\}}(r, h, t) &= -\mathcal{MS}^{-1}\left[\frac{p}{\log(a)}\mathcal{MS}\left[\left(Q_{\{1,\chi\}} - Q_{\{0,\chi\}}\right) + \left(B_{\{1,\chi\}} - B_{\{0,\chi\}}\right)\right]\right] = 0. \end{aligned} \tag{71}$$

So, the second iterations $\gamma_{\{2,\psi\}}, \gamma_{\{2,\phi\}}$ and $\gamma_{\{2,\chi\}}$ are

$$\begin{aligned} \gamma_{\{1,\phi\}}(r, h, t) &= \xi_{\{0,\phi\}} + \xi_{\{1,\phi\}} + \xi_{\{2,\phi\}} = r + 2h + 3t + 0, \\ \gamma_{\{1,\psi\}}(r, h, t) &= \xi_{\{0,\psi\}} + \xi_{\{1,\psi\}} + \xi_{\{2,\psi\}} = r - 2h + 3t + 0, \\ \gamma_{\{1,\chi\}}(r, h, t) &= \xi_{\{0,\chi\}} + \xi_{\{1,\chi\}} + \xi_{\{2,\chi\}} = -r + 2h + 3t + 0. \end{aligned} \tag{72}$$

Therefore, the series solutions (72) of the system (56) are ended in the second iterations. As a result, the solution of the system (56) is

$$\gamma_{\{1,\phi\}}(r, h, t) = r + 2h + 3t, \tag{73a}$$

$$\gamma_{\{1,\psi\}}(r, h, t) = r - 2h + 3t, \tag{73b}$$

$$\gamma_{\{1,\chi\}}(r, h, t) = -r + 2h + 3t. \tag{73c}$$

From (73a), (73b), and (73c), the convergent Taylor's series yield the solutions shown in Figure 5, and Figures 6–8 illustrate the exact solutions of (56) at $t = 1$ related to (73a), (73b), and (73c), respectively. Based on the results of (71), the third iteration is equal to the second

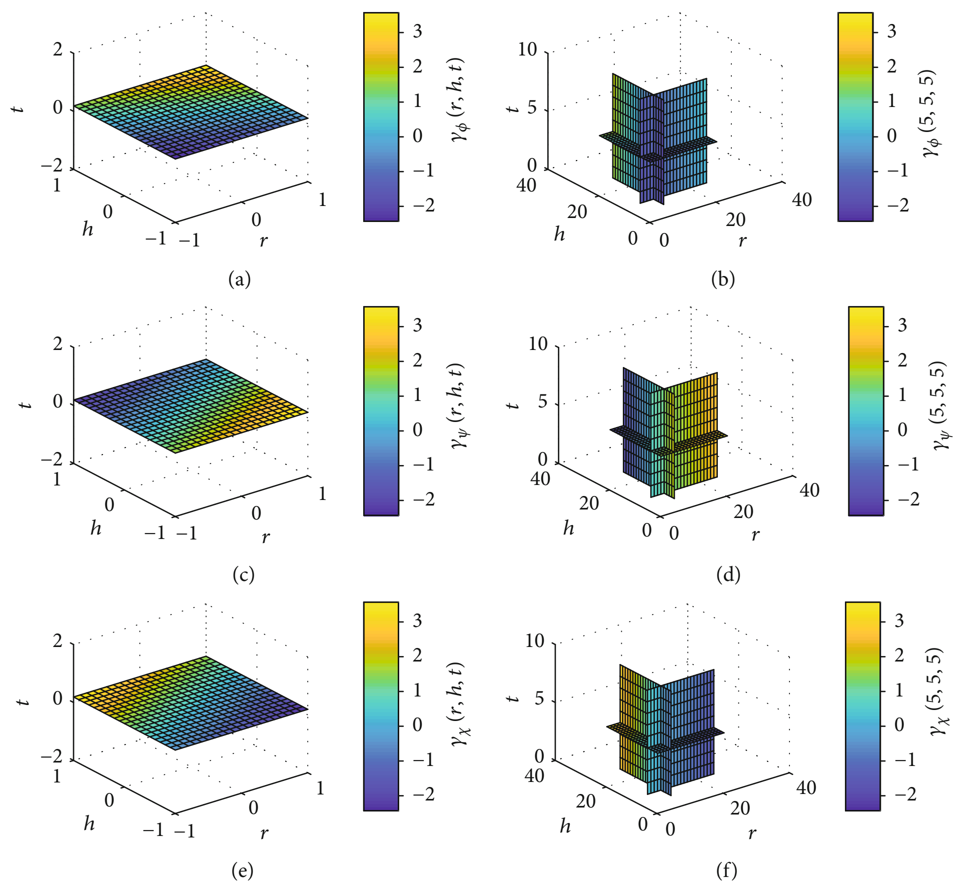

Figure 5: The plot that shows the solution of (56). (a) The function $\gamma_{\{1,\phi\}}(r,h,t)=r+2h+3t$. (b) The function $\gamma_{\{1,\phi\}}(r,h,t)=r+2h+3t$ at $r=5,h=5,t=5$. (c) The function $\gamma_{\{\psi\}}(r,h,t)=r-2h+3t$. (d) The function $\gamma_{\{\psi\}}(r,h,t)=r-2h+3t$ at $r=5,h=5,t=5$. (e) The function $\gamma_{\{\chi\}}(r,h,t)=-r+2h+3t$. (f) The function $\gamma_{\{\chi\}}(r,h,t)=-r+2h+3t$ at $r=5,h=5,t=5$.

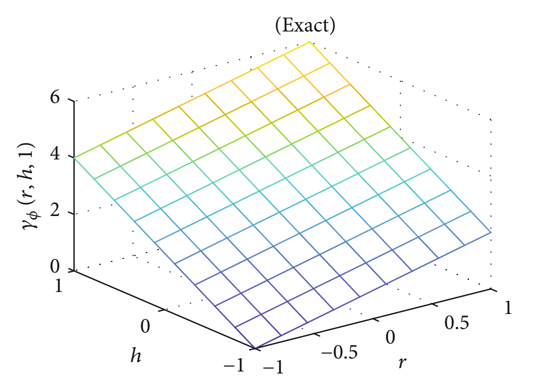

Figure 6: The plot that shows the exact solution (73a) of (56) at $t=1$.

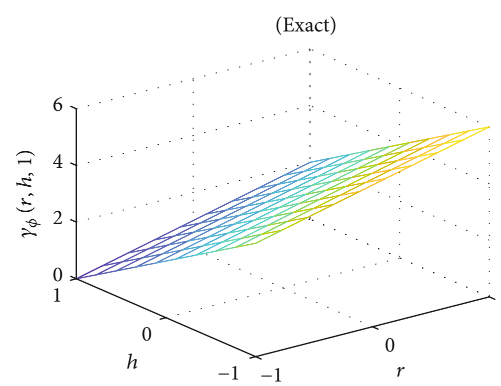

Figure 7: The plot that shows the exact solution (73b) of (56) at $t=1$.

iteration. So, the functions (73a), (73b), and (73c) consider the final (exact) solutions.

Example 9. Take into consideration the system of homogeneous NPDEs:

$$\begin{aligned}\phi_t(r,h,t)+\phi\phi_r+\psi\phi_h-\phi_{rr}-\phi_{hh}&=0,\\ \psi_t(r,h,t)+\phi\psi_r+\psi\psi_h-\psi_{rr}-\psi_{hh}&=0,\end{aligned} \tag{74}$$

with the ICs

$$\begin{aligned}\phi(r,h,0)&=r+h,\\ \psi(r,h,0)&=r-h.\end{aligned} \tag{75}$$

As in Example 7, it is evident that $B(r,h,t)=0$, and we need to follow the process in Section 3, and apply the steps from (18) to (29), we get

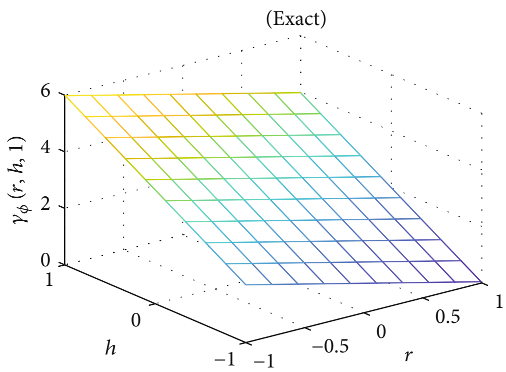

Figure 8: The plot that shows the exact solution (73c) of (56) at $t = 1$.

$$\gamma_{\{n+1,\phi\}}(r,h,t) = r + h - \mathcal{MS}^{-1}\left[\frac{p}{\log(a)}\mathcal{MS}\left[Q_{\{n,\phi\}} - Q_{\{n-1,\phi\}}\right]\right],$$
$$\gamma_{\{n+1,\psi\}}(r,h,t) = r - h - \mathcal{MS}^{-1}\left[\frac{p}{\log(a)}\mathcal{MS}\left[Q_{\{n,\psi\}} - Q_{\{n-1,\psi\}}\right]\right]. \quad (76)$$

So, when $n = 0$, we get

$$\begin{aligned}\xi_{\{0,\phi\}} &= \gamma_{\{0,\phi\}}(r,h,t) = r + h,\\ \xi_{\{0,\psi,\}} &= \gamma_{\{0,\psi\}}(r,h,t) = r - h,\\ \xi_{\{1,\phi\}} &= -2rt,\\ \xi_{\{1,\psi\}} &= -2ht.\end{aligned} \quad (77)$$

Then, the first iterations $\gamma_{\{1,\psi\}}$ and $\gamma_{\{1,\phi\}}$ are

$$\begin{aligned}\gamma_{\{1,\phi\}}(r,h,t) &= \xi_{\{0,\phi\}} + \xi_{\{1,\phi\}} = r + h - 2rt,\\ \gamma_{\{1,\psi\}}(r,h,t) &= \xi_{\{0,\psi\}} + \xi_{\{1,\psi\}} = r - h - 2ht,\end{aligned} \quad (78)$$

when $n = 1$, we get

$$\begin{aligned}\xi_{\{2,\phi\}} &= 2ht^2 + 2rt^2,\\ \xi_{\{2,\psi\}} &= 2rt^2 - 2ht^2.\end{aligned} \quad (79)$$

Then, the second iterations will be

$$\begin{aligned}\gamma_{\{2,\phi\}}(r,h,t) &= r + h - 2rt + 2ht^2 + 2rt^2,\\ \gamma_{\{2,\psi\}}(r,h,t) &= r - h - 2ht + 2rt^2 - 2ht^2,\end{aligned} \quad (80)$$

and so on, and when $n \longrightarrow \infty$, the nth iterations will be

$$\begin{aligned}\gamma_{\{\phi\}}(r,h,t) &= \lim_{n\longrightarrow\infty}\sum_{k=0}^{n}\xi_{\{k,\phi\}} = r + h - 2rt + 2ht^2 + 2rt^2 - 4rt^3 + \cdots,\\ \gamma_{\{\psi\}}(r,h,t) &= \lim_{n\longrightarrow\infty}\sum_{k=0}^{n}\xi_{\{k,\psi\}} = r - h - 2ht + 2rt^2 - 2ht^2 - 4ht^3 + \cdots.\end{aligned} \quad (81)$$

more clearly,

$$\begin{aligned}\gamma_{\{\phi\}}(r,h,t) &= \lim_{n\longrightarrow\infty}\sum_{k=0}^{n}\xi_{\{k,\phi\}}\\ &= r + h - 2rt + (2r + 2h)t^2 - 4rt^3\\ &\quad + (4r + 4h)t^4 - 8rt^5 + \cdots,\end{aligned} \quad (82a)$$

$$\begin{aligned}\gamma_{\{\psi\}}(r,h,t) &= \lim_{n\longrightarrow\infty}\sum_{k=0}^{n}\xi_{\{k,\psi\}}\\ &= r - h - 2rt + (2r - 2h)t^2 - 4rt^3\\ &\quad + (4r - 4h)t^4 - 8rt^5 + \cdots.\end{aligned} \quad (82b)$$

As a result of Theorem 6, we can verify that the series (82a) converges, where $\lambda = 2t^2$ and $\mu = -2t$. Now, when $|2t| < 1$, the resulting sum is

$$\sum_{c=2}^{\infty} 2t^2(-2t)^c = \frac{2t^2}{1 + 2t}, \quad (83)$$

and the series in t of $\gamma\{\phi\}$ will give

$$\gamma_{\{\phi\}}(r,h,t) = \frac{r + h - 2rt}{1 - 2t^2}. \quad (84a)$$

The same discussion can be made about (82b), where the series in t of $\{\psi\}$ gives

$$\gamma_{\{\psi\}}(r,h,t) = \frac{r - h - 2ht}{1 - 2t^2}. \quad (84b)$$

From (84a) and (84b), the convergent Taylor's series yield the solutions shown in Figures 9 and 10, and in Figure 11, we can see the exact and approximate solutions to the system (74) at $t = 0.1$, which relate to (84a) and (82a), respectively, while in Figure 12, we can see the exact and approximate solutions to the system (74) at $t = 0.1$, which relate to (84b) and (82b), respectively. Table 2 shows the results from Figures 11 and 12 as well as the least square error for the two figures.

6. Combination of the Old Modified Sumudu Transform with the MI Method

According to article [31], the author introduced a transform called "the modified Sumudu transform", which was applied to a number of functions, and its properties were studied. Nevertheless, despite its presented properties, it is possible to investigate whether a convergent series solution can be achieved when it is combined with the MI method for

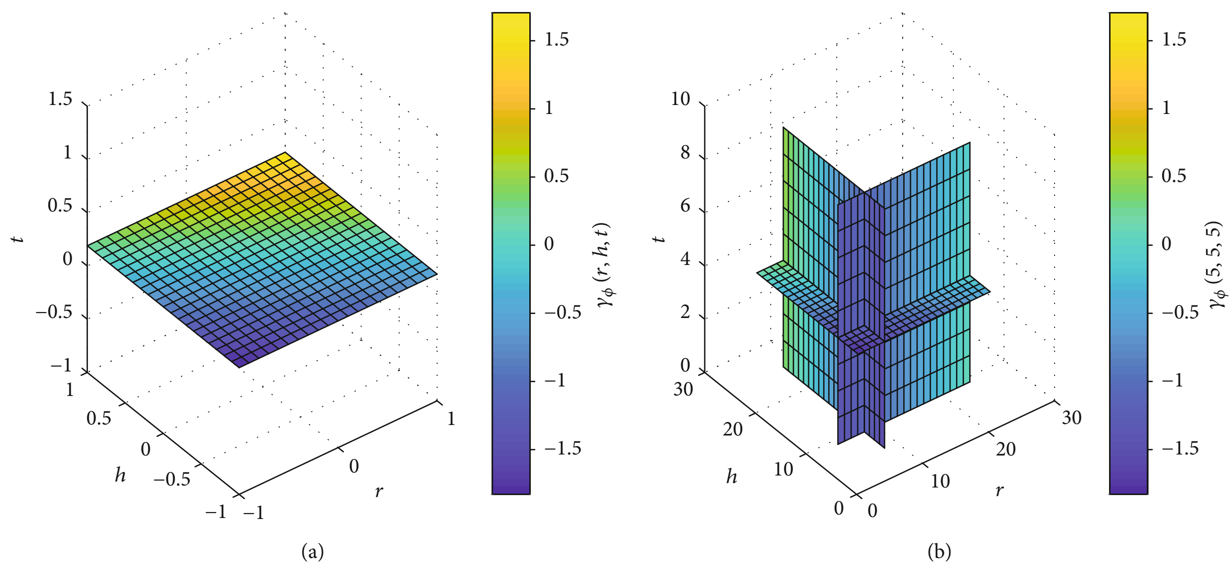

FIGURE 9: The plot that shows the solution of (74): (a) the function $\gamma_{\{\phi\}}(r, h, t) = (r + h - 2rt)/(1 - 2t^2)$ and (b) the function $\gamma_{\{\phi\}}(r, h, t) = (r + h - 2rt)/(1 - 2t^2)$ at $r = 5, h = 5, t = 5$.

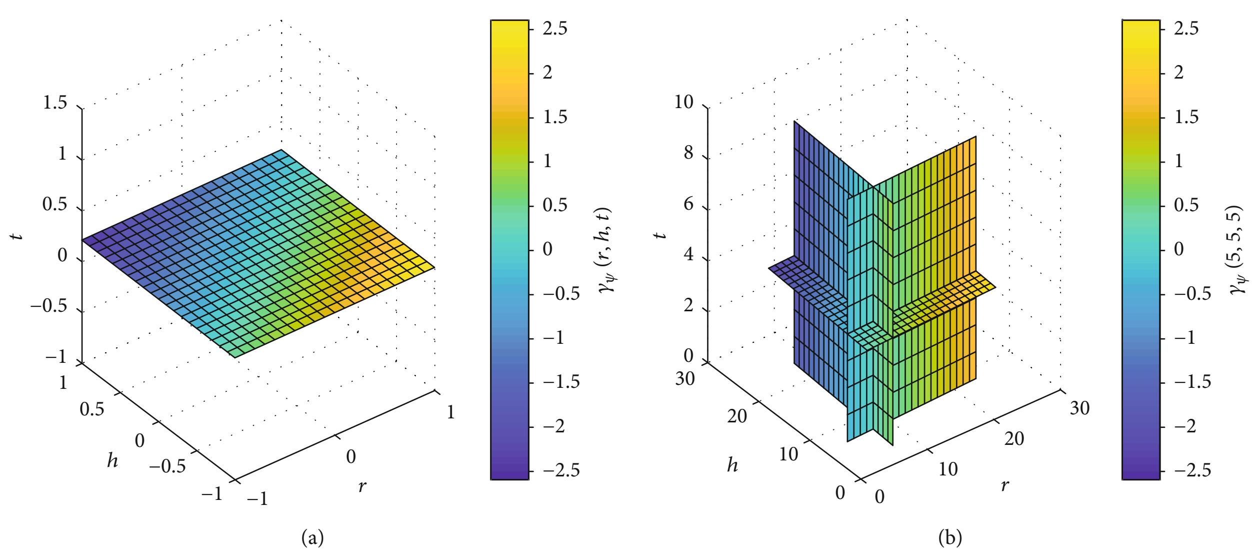

FIGURE 10: The plot that shows the solution of (74): (a) the function $\gamma_{\{\psi\}}(r, h, t) = (r - h - 2ht)/(1 - 2t^2)$ and (b) the function $\gamma_{\{\psi\}}(r, h, t) = (r - h - 2ht)/(1 - 2t^2)$ at $r = 5, h = 5, t = 5$.

solving NPDE. The following example illustrates that the combination of these techniques can produce an approximate solution when used in conjunction with the MI method and that it can be used effectively.

Example 10. Take into consideration the NPDE:

$$\phi_t + \phi\phi_r(r, t) = 0, \tag{85}$$

with the ICs

$$\phi(r, 0) = r, \phi(0, t) = 0, \tag{86}$$

and using the transformation mentioned in [31] to eq. (85) with utilizing (86), we obtain

$$\begin{gathered}
\frac{1}{v}[\log(a)\Phi(v) - r] + \mathcal{S}_a[\phi\phi_r] = 0, \\
\log(a)\Phi(v) - r + v\mathcal{S}_a[\phi\phi_r] = 0, \\
\log(a)\Phi(v) = r - v\mathcal{S}_a[\phi\phi_r], \\
\Phi(v) = \frac{r}{\log(a)} - \frac{1}{\log(a)} v\mathcal{S}_a[\phi\phi_r].
\end{gathered} \tag{87}$$

Now, both sides of eq. (87) need to be transformed in the inverse direction as follows:

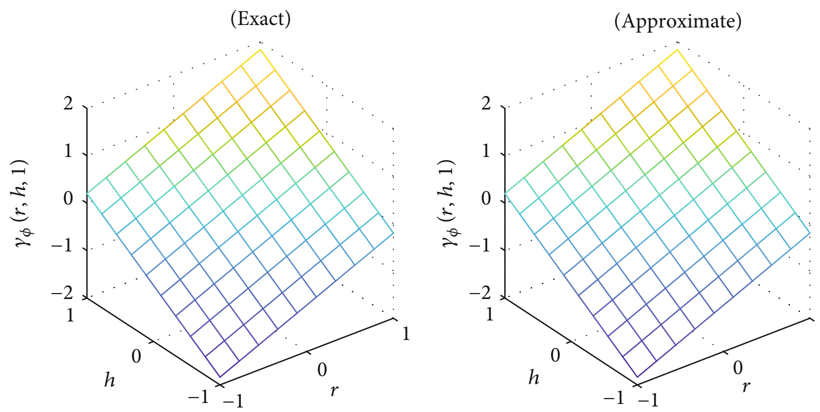

Figure 11: The plot that shows the exact solution (84a) and approximate solution (82a) of (74) at $t = 0.1$.

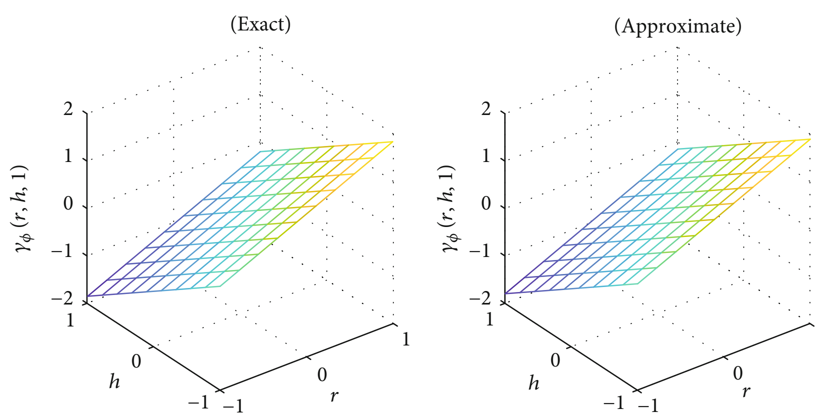

Figure 12: The plot that shows the exact solution (84b) and approximate solution (82b) of (74) at $t = 0.1$.

Table 2: Least square error of the solutions for Example 9.

The points	The results from Figure 11 and the errors			The results from Figure 12 and the errors		
r	Exact	Approximate	Error	Exact	Approximate	Error
-1	-1.83673	-1.83672	$2.1591e-10$	0.20408	0.20408	$2.6656e-12$
-0.77778	-1.60998	-1.60996	$3.8180e-10$	-0.02268	-0.02268	$2.6985e-12$
-0.55556	-1.38322	-1.38321	$5.0425e-10$	-0.24943	-0.24943	$6.6803e-12$
-0.33333	-1.15646	-1.15645	$5.8984e-10$	-0.47619	-0.47619	$2.1193e-11$
-0.11111	-0.92971	-0.92970	$6.4516e-10$	-0.70295	-0.70294	$5.2817e-11$
0.11111	-0.70295	-0.70294	$6.7679e-10$	-0.92971	-0.92970	$1.0814e-10$
0.33333	-0.47619	-0.47619	$6.9130e-10$	-1.15646	-1.15645	$1.9373e-10$
0.55556	-0.24943	-0.24943	$6.9528e-10$	-1.38322	-1.38321	$3.1618e-10$
0.77778	-0.02268	-0.02268	$6.9532e-10$	-1.60998	-1.60996	$4.8207e-10$
1	0.20408	0.20408	$6.9798e-10$	-1.83673	-1.83672	$6.9798e-10$
Least square error			$5.0956e-9$	Least square error		$1.1861e-9$

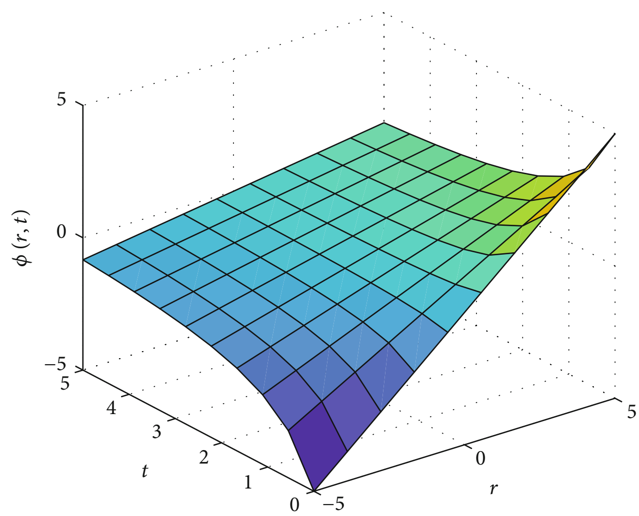

Figure 13: The plot that shows the solution $\phi(r,t) = r/(1+t)$ of eq. (85).

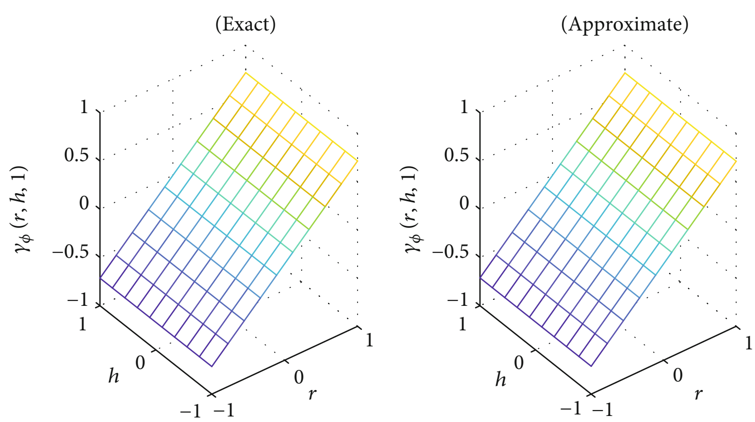

Figure 14: The plot that shows the exact solution (94) and approximate solution (93) of (85) at $t = 0.4$.

$$\phi(r,t) = r - \mathcal{S}_a^{-1}\left[\frac{1}{\log(a)} v\mathcal{S}_a[\phi\phi_r]\right], = r - \mathcal{S}_a^{-1}\left[\frac{1}{\log(a)} v\mathcal{S}_a[\phi\phi_r]\right]. \tag{88}$$

Then, when

$$\phi(r,t) = \sum_{n=0}^{\infty} \phi_n(r,t), \tag{89}$$

then,

$$\phi_{n+1}(r,t) = r - \mathcal{S}_a^{-1}\left[\frac{1}{\log(a)} v\mathcal{S}_a[\phi\phi_r]\right]. \tag{90}$$

Now, the zero iteration is

$$\phi_0(r,t) = r, \tag{91}$$

and using the MI method, we get the first iteration,

$$\phi_1(r,t) = -\mathcal{S}_a^{-1}\left[\frac{1}{\log(a)} v\mathcal{S}_a\left[\phi_0\phi_{\{0,r\}}\right]\right], = -rt, \tag{92}$$

and so on,

$$\phi(r,t) = -rt^5 + rt^4 - rt^3 + rt^2 - rt + r + \cdots. \tag{93}$$

In this case, according to Theorem 6, eq. (93) converges when $|t| < 1$ and the sum is

$$\sum_{c=0}^{\infty} r(-t)^c = \frac{r}{1+t}. \tag{94}$$

The function (93) is a convergent series solution in which the sum of a convergent series is $\phi(r,t) = r/(1+t)$ (see Figure 13). Consequently, this result has implications for the understanding that the combination of the old MST presented in [31] with the MI method is able to provide a solution for eq. (85) that cannot be achieved by using the transformation only. This means that, by leveraging the MI

Table 3: The results and least square error of the solutions for Example 10.

r	Exact	Approximate	Error
-1	-0.71429	-0.71136	$8.5598e-06$
-0.77778	-0.55556	-0.55328	$1.7120e-05$
-0.55556	-0.39683	-0.39520	$2.5679e-05$
-0.33333	-0.23810	-0.23712	$3.4239e-05$
-0.11111	-0.07937	-0.07904	$4.2799e-05$
0.11111	0.07937	0.07904	$5.1359e-05$
0.33333	0.23810	0.23712	$5.9919e-05$
0.55556	0.39683	0.39520	$6.8478e-05$
0.77778	0.55556	0.55328	$7.7038e-05$
1	0.71429	0.71136	$8.5598e-05$
Least square error			$3.8519e-04$

method, we can combine the old transformation with the MI method in order to solve the NPDE and have certainty that the solution will be accurate. In Figure 14, we can see the exact and approximate solutions of (85) as a result of using the combination of the old MST with the MI method. Also, the values of solutions and the least square error are introduced in Table 3.

7. Conclusion

The major results of our research have been summarized briefly. As a first step, a new transformation corresponding to the ST has been formulated. This has been implemented by studying transform theory and achieving MST. To get a better understanding of MST, we began the first part of the investigation by giving a preliminary essential definition about the functions associated with MST. In the second step, we proceeded to formulate the theorems of the MST, in which we proved that the MST for various functions, as well as the unknown function derivatives, which were utilized in the problems, have achieved results. The transforms of these functions have enabled us to determine the solutions to equation systems.

In the second part of our study, we have presented a mathematical method that employs the newly discovered MST in conjunction with the MI method to determine systems of NPDEs whose nonlinear terms are evaluated by the MI method. Based on this combination, the MSMI method has produced a convergent series that is highly accurate and reliable. Consequently, the combination of these two methods has been effective in solving a variety of NPDE systems. As shown in Examples 7 and 9, the approximate solutions have been found for the systems of homogeneous NPDEs, and according to Theorem 6, the exact solutions have been derived (see Figures 1, 2, 9, and 10), whereas in Example 8, an exact solution has been found to the system of nonhomogeneous NPDEs (see Figure 5). Figures 3, 4, 11, and 12 exhibit to the exact and approximate solutions. Additionally, Figures 6–8 only exhibit the exact solution since the third iteration equals zero, which is the second iteration.

After that, it has been demonstrated that the old MST referred in [31] can be used in conjunction with the MI method to determine a NPDE, and this combination has given a guaranteed convergent series solution, which led to appropriate results being obtained. Accordingly, Figure 13 has shown the convergent solution of (85) using the method in [31] in conjunction with the MI method which reveals through Example 10 that it gives an increasingly convergent series. Consequently, we can now confidently conclude that this combination provided us the proper solution. It is evident that this method leads to the solutions of (85) as shown in Figure 14, and the values of the results are displayed in Table 3.

Finally, our method has the advantage of being efficient and reliable, as it resulted in precise calculations and results, and being able to solve NPDEs of any order. Consequently, it is an important source of research for solving a variety of mathematical physics problems, and one of the most widely used tools in the field, making it one of the most sought-after tools for researchers.

Data Availability

Data is available upon request.

Conflicts of Interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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