Hindawi
Computational and Mathematical Methods
Volume 2023, Article ID 9431476, 24 pages
https://doi.org/10.1155/2023/9431476

WILEY | Hindawi

Research Article

A Bit-Parallel Tabu Search Algorithm for Finding $E(s^2)$-Optimal and Minimax-Optimal Supersaturated Designs

Luis B. Morales[1] **and Dursun A. Bulutoglu**[2]

[1]*Unidad Acad, IIMAS Estado de Yucatán, Universidad Nacional Autónoma de México, Mérida, Yuc, Mexico*
[2]*Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, USA*

Correspondence should be addressed to Dursun A. Bulutoglu; dursun.bulutoglu@gmail.com

Received 16 January 2023; Revised 1 May 2023; Accepted 5 May 2023; Published 6 June 2023

Academic Editor: Suresh Rasappan

We prove the equivalence of two-symbol supersaturated designs (SSDs) with N (even) rows, m columns, and $s_{\max} = 4t + i$, where $i \in \{0, 2\}$ and $t \in \mathbb{Z}^{\geq 0}$ and resolvable incomplete block designs (RIBDs) whose any two blocks intersect in at most $(N + 4t + i)/4$ points. Using this equivalence, we formulate the search for two-symbol $E(s^2)$-optimal and minimax-optimal SSDs with $s_{\max} \in \{2, 4, 6\}$ as a search for RIBDs whose blocks intersect accordingly. This allows developing a bit-parallel tabu search (TS) algorithm. The TS algorithm found $E(s^2)$-optimal and minimax-optimal SSDs achieving the sharpest known $E(s^2)$ lower bound with $s_{\max} \in \{2, 4, 6\}$ of sizes $(N, m) = (16, 25)$, $(16, 26)$, $(16, 27)$, $(18, 23)$, $(18, 24)$, $(18, 25)$, $(18, 26)$, $(18, 27)$, $(18, 28)$, $(18, 29)$, $(20, 21)$, $(22, 22)$, $(22, 23)$, $(24, 24)$, and $(24, 25)$. In each of these cases, no such SSD could previously be found.

1. Introduction

Two-symbol supersaturated designs (SSDs) are two symbol arrays in which the number of rows is less than or equal to the number of columns. Throughout this paper, an SSD refers to a two-symbol SSD. An SSD with N rows and m columns is represented by an $N \times m$ matrix and will be denoted by $\mathbf{D}(N, m)$ or simply by $\mathbf{D}$. Each entry of $\mathbf{D}$ is ± 1, and the frequencies of $+1$ and -1 are the same in each column. Moreover, $\mathbf{D}$ has no two columns such that $\mathbf{d}_i = \mathbf{d}_j$ or $\mathbf{d}_i = -\mathbf{d}_j$.

The $E(s^2)$-optimality criterion was defined in [1], for comparing two-symbol SSDs. The $E(s^2)$ criterion compares two $\{-1, 1\}$ arrays of the same size by picking the one that minimize

$$E(s^2) = \sum_{i<j} \frac{s_{ij}^2}{\binom{m}{2}}, \quad (1)$$

where s_{ij} is the (i, j)th entry of the matrix $\mathbf{H}^T\mathbf{H}$ for a $\{-1, 1\}$ -array $\mathbf{H}$. The term s_{ij} in (1) measures the degree of non-orthogonality between the ith and jth columns. An SSD is called $E(s^2)$-optimal if no SSD with the same number of rows and columns having a smaller $E(s^2)$ value exists. For $\{-1, 1\}$ arrays, let $s_{\max} = \max_{i<j}|s_{ij}|$ and $f_{s_{\max}}$ be the frequency of $s_{\max}$ in $\{|s_{ij}|\}_{i<j}$. The minimax criterion proposed in [1] minimizes $s_{\max}$ first and $f_{s_{\max}}$ second. An SSD is called *minimax-optimal* if no other SSD with the same size has a lower $s_{\max}$ or the same $s_{\max}$ with a smaller $f_{s_{\max}}$ (see [2]). In [2], a search algorithm generalizing the exchange algorithm of [3] was provided to construct $E(s^2)$-optimal and minimax-optimal SSDs. An $E(s^2)$-optimal and minimax-optimal SSD with 16 rows and 60 columns was found by [4]. In [5], a simulated annealing algorithm was used for finding $E(s^2)$-optimal and minimax-optimal cyclic SSDs. In [6], tabu search (TS) was used to construct $E(s^2)$-optimal SSDs with good properties by constructing supplementary difference sets. In [7], a TS procedure for constructing $E(s^2)$-optimal

and minimax-optimal k-circulant SSDs was implemented. Recently, all isomorphism classes of $E(s^2)$-optimal and minimax-optimal k-circulant SSDs with $N = 6,10,14,18,22,26$ rows, $m = k(N-1)$ columns, and $s_{max} \in \{2, 6\}$ were classified in a computer search by [8]. They also classified all isomorphism classes of $E(s^2)$-optimal and minimax-optimal k-circulant SSDs with $N \equiv 0 \pmod 4$ and $s_{max} = 4$. For a comprehensive review of SSDs, see [9].

In Section 2, we provide some background material on $E(s^2)$ lower bounds and the $E(s^2)$ and minimax optimality of SSDs. In Section 3, we prove an equivalence between SSDs with N (even) rows, m columns, and $s_{max} = 4t + i$, where $i \in \{0, 2\}$ and $t \in \mathbb{Z}^{\geq 0}$ and resolvable incomplete block designs (RIBDs) such that any two distinct blocks intersect in at most $(N + 4t + i)/4$ points. In Section 4, using this equivalence, we formulate the problem of constructing $E(s^2)$-optimal and minimax-optimal SSDs with $s_{max} \in \{2, 6\}$ as a problem to find RIBDs whose blocks intersect in at most $(N + 4t + i)/4$ points for $(t, i) \in \{(0, 2), (1, 0), (1, 2)\}$. We formulate the construction of such RIBDs as an optimization problem. Unlike many optimization problems where a good approximate solution is sufficient, in the construction of such resolvable designs (as in the construction of other combinatorial designs), the main goal is to find an optimum solution. For this purpose, an algorithm based on TS [10] is developed in Section 5. Sets (blocks) with elements from a set V of cardinality N are stored as bit strings where the number of bits is equal to N. Each bit corresponds to exactly one element of V. Thus, a set is represented by a bit string in which the bits corresponding to the elements of that set are 1 and all others bits are 0. (For example, in $V = \{0, 1, 2, 3, 4, 5\}$, bit string $BB = 010011$ represents the set $B = \{1, 4, 5\}$.) Our data structure for storing sets (blocks) is the same as that in [11]. This allows us to exploit bit-parallelism as in [11] for computing the intersections of the sets (blocks) by using bitwise operations. Thus, all our computations are made using bit-parallel Boolean instructions, which in praxis (on a x86-64 CPU) implies that 64 bits of data are processed at once. This improves the overall performance by a factor of $\min(N, 64) = N$ (as N is less than 64 in all the cases we studied). In Section 5, we also provide the computational complexity analysis of our algorithm. The implementation details of our algorithm are discussed in Section 6. We end the paper with concluding remarks in Section 7.

The bit-parallel TS algorithm was able to construct fifteen previously unknown $E(s^2)$-optimal and minimax-optimal SSDs of sizes $(N, m) = (16, 25)$, (16, 26), (16, 27), (18, 23), (18, 24), (18, 25), (18, 26), (18, 27), (18, 28), (18, 29), (20, 21), (22, 22), (22, 23), (24, 24), and (24, 25). All these SSDs, their $E(s^2)$ values, s_{max}s, and $f_{s_{max}}$s, as well as their Gram matrices, are provided in Tables 1–30. The newly found $E(s^2)$-optimal and minimax-optimal SSDs could not have been found by the NOA$_p$ algorithms in [2] for $p = 2$, 4, 8 despite running these algorithms for a very long time. So, the TS algorithm in this paper outperforms the NOA$_p$ algorithms at least for the SSD cases searched in this paper. The bit-parallel TS algorithm also found all $E(s^2)$-optimal and minimax-optimal SSDs obtained by the NOA$_p$ algorithms in [2]. We made a comparison of TS with the NOA$_p$ algorithms. This is because in TS, NOA$_4$, and NOA$_8$, the objective function was chosen with respect to both the $E(s^2)$ and the minimax criteria. In addition, NOA$_p$ algorithms were successful in locating all $E(s^2)$-optimal SSDs achieving the Ryan and Bulutoglu $E(s^2)$ lower bound [2] for $N \leq 16$ except the 14-row, 16-column case, where the Ryan and Bulutoglu $E(s^2)$ lower bound for this case was recently improved [12]. No other previous algorithm had solved so many cases.

SSDs are used in computer experiments; software testing; medical, industrial, and engineering experiments; chromatography (separation science); and in biometric applications. In [3], an SSD with $N = 28$ runs (rows) and $m = 54$ factors (columns) for a crash test experiment on a planned new four-wheel drive range was discussed, where the objective was to find the best possible subset of safety features among the 54 proposed. In [13], an SSD with $N = 16$ runs (rows) and $m = 18$ factors (columns) to fine-tune 16 potential factors affecting the thermal performance of project homes was proposed, where two additional columns were used as blocking factors. For designing a multistage axial compressor (turbine engine), the design engineer has selected 27 potentially important factors [14]. In [14], SSDs with $m = 27$ factors (columns) and $N = 16$, $N = 20$, and $N = 12$ runs (rows) from [3] and [15], respectively, were compared by using data from a computer experiment for the design of a multistage axial compressor. Our newly found $N = 16$ run (row), $m = 27$ factor (column) $E(s^2)$-optimal, and minimax-optimal SSD could have also been used in this study. In [16], an SSD with $N = 12$ runs (rows) and $m = 24$ factors (columns) was used in composite sampling for monitoring pesticide residues in water. Our newly found $N = 18$ run (row), $m = 24$ factor (column) $E(s^2)$-optimal, and minimax-optimal SSD could also have been used for the same purpose. In [17], an $N = 14$ run (row), $m = 23$ factor (column) SSD was proposed in place of a Plackett-Burman design that had been previously used in [18] for developing an epoxide adhesive system for bonding a polyester cord. We propose our newly found $N = 18$ run (row), $m = 23$ factor (column) $E(s^2)$-optimal, and minimax-optimal SSD for the same purpose.

2. Lower Bounds for $E(s^2)$ and $E(s^2)$ and Minimax Optimality of SSDs

In [3, 19], it was independently shown that

$$E(s^2) \geq \frac{N^2(m - N + 1)}{(m - 1)(N - 1)}. \quad (2)$$

Bound (2) can be achieved only if $m = q(N-1)$ and $N \equiv 0 \pmod 4$, or if $m = 2q(N-1)$ and $N \equiv 2 \pmod 4$ for some

Table 1: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 16$, $m = 25$, and $E(s^2) = 7.68000$.

-1	1	-1	-1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	1	1	1	1	-1	-1	-1	-1	-1	1
1	-1	1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1
-1	1	1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1	1	-1	1	1	-1	-1	1	1	-1	1	1
1	-1	1	-1	1	1	1	-1	1	1	1	1	-1	-1	1	1	-1	1	1	1	1	1	1	1	1
-1	1	-1	1	-1	1	1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	-1
1	1	-1	1	-1	-1	-1	1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	1
-1	1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	-1	-1	1	1	1	1
-1	-1	1	-1	-1	-1	1	1	1	1	1	1	1	-1	1	-1	-1	-1	-1	1	-1	1	-1	-1	-1
1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	-1	1	-1	1	1	1	-1	1	1	-1	-1	-1	1	1
-1	1	1	1	-1	1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	1	1	1	-1	1	1	-1
1	-1	-1	1	1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	-1	1	1
1	-1	1	1	1	-1	-1	-1	-1	1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	-1	-1
-1	-1	-1	-1	1	1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	1	-1	-1
-1	-1	1	1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	1	-1	1	1	-1	-1	1	-1	-1	-1
1	1	-1	-1	1	-1	1	1	1	-1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	1	-1
1	1	1	1	1	1	1	-1	-1	-1	-1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	-1	1

Table 2: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 4$, and $f_{s_{\max}} = 144$.

16	-4	0	0	4	-4	0	0	-4	4	-4	0	0	0	-4	4	0	-4	0	4	0	0	0	4	4
-4	16	-4	4	-4	0	4	0	0	-4	-4	0	0	0	-4	0	0	4	-4	0	0	0	4	0	4
0	-4	16	4	4	0	4	-4	-4	0	0	4	0	-4	0	-4	0	4	0	0	0	4	-4	0	-4
0	4	4	16	4	0	-4	0	-4	0	0	0	4	-4	-4	0	0	0	0	-4	4	-4	0	-4	0
4	-4	4	4	16	0	-4	-4	4	-4	0	0	0	0	0	0	-4	4	-4	0	0	0	0	0	0
-4	0	0	0	0	16	0	0	-4	-4	0	0	-4	-4	4	0	-4	0	4	4	0	-4	4	0	0
0	4	4	-4	-4	0	16	-4	4	0	0	4	0	0	-4	0	-4	0	4	4	0	0	0	0	-4
0	0	-4	0	-4	0	-4	16	-4	0	4	0	4	0	-4	0	-4	0	0	4	-4	4	0	4	0
-4	0	-4	-4	4	-4	4	-4	16	0	4	0	4	0	4	0	-4	0	0	0	4	0	0	0	0
4	-4	0	0	-4	-4	0	0	0	16	0	4	0	0	4	0	-4	-4	0	-4	4	4	4	0	0
-4	-4	0	0	0	0	0	4	4	0	16	4	-4	0	-4	0	0	0	4	4	4	0	0	-4	0
0	0	4	0	0	0	4	0	0	4	4	16	0	4	0	-4	-4	4	0	0	-4	-4	0	0	4
0	0	0	4	0	-4	0	4	4	0	-4	0	16	0	4	0	0	0	0	0	0	-4	0	4	-4
0	0	-4	-4	0	-4	0	0	0	0	0	4	0	16	-4	-4	4	4	0	-4	0	-4	4	4	-4
-4	-4	0	-4	0	4	-4	-4	4	4	-4	0	4	-4	16	0	0	0	-4	0	0	0	4	0	0
4	0	-4	0	0	0	0	0	0	0	0	-4	0	-4	0	16	0	4	4	0	-4	0	4	-4	0
0	0	0	0	-4	-4	-4	-4	-4	-4	0	-4	0	4	0	0	16	0	0	0	4	-4	0	0	0
-4	4	4	0	4	0	0	0	0	-4	0	4	0	4	0	4	0	16	0	-4	-4	4	4	4	0
0	-4	0	0	-4	4	4	0	0	0	4	0	0	0	-4	4	0	0	16	-4	4	-4	-4	4	0
4	0	0	-4	0	4	4	4	0	-4	4	0	0	-4	0	0	0	-4	-4	16	0	0	4	0	0
0	0	0	4	0	0	0	-4	4	4	4	-4	0	0	0	-4	4	-4	4	0	16	0	4	4	0
0	0	4	-4	0	-4	0	4	0	4	0	-4	-4	-4	0	0	-4	4	-4	0	0	16	0	4	0
0	4	-4	0	0	4	0	0	0	4	0	0	0	4	4	4	0	4	-4	4	4	0	16	0	-4
4	0	0	-4	0	0	0	4	0	0	-4	0	4	4	0	-4	0	4	4	0	4	4	0	16	4
4	4	-4	0	0	0	-4	0	0	0	0	4	-4	-4	0	0	0	0	0	0	0	0	-4	4	16

Table 3: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 16$, $m = 26$, and $E(s^2) = 7.87692$.

1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	1	-1	1	1	-1	-1	-1	-1	-1	1	1	-1
1	-1	1	1	-1	-1	1	-1	1	-1	1	1	1	-1	-1	-1	1	1	-1	1	1	1	-1	-1	1	-1
-1	1	1	1	1	1	-1	1	1	1	1	-1	1	1	-1	-1	-1	1	-1	1	-1	-1	1	1	1	-1
1	1	-1	1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	1	-1	-1	-1	1
1	-1	1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	-1	1	1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	1	-1	-1	1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	1	1	1	-1	1	1	-1	-1	1
-1	1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	-1	1	1	1	1	-1	1	1	1
-1	1	1	1	1	-1	-1	1	-1	-1	-1	1	-1	1	-1	1	1	-1	-1	-1	-1	1	-1	-1	-1	-1
1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1	1
1	1	1	1	-1	-1	1	1	-1	1	1	1	1	1	1	1	1	1	1	-1	1	-1	1	1	-1	1
-1	1	1	-1	-1	1	1	-1	1	1	-1	1	1	1	1	-1	-1	-1	1	-1	-1	1	-1	-1	1	1
-1	-1	-1	1	1	-1	1	1	1	-1	-1	1	-1	-1	1	-1	-1	-1	1	-1	1	-1	1	1	1	-1
-1	-1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	1	-1	1	1	1	1	1	1	-1	-1	1	-1	1
1	1	1	-1	-1	1	1	-1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	1	1	-1	-1
-1	-1	1	-1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	-1	1	-1	-1	1
1	1	-1	-1	1	-1	-1	-1	-1	-1	1	1	1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	1	-1

Table 4: $\mathbf{D}^T\mathbf{D}$, $s_{max} = 4$, and $f_{s_{max}} = 160$.

16	0	0	0	0	0	4	-4	-4	4	0	0	0	-4	4	4	0	4	-4	0	0	0	0	0	0	-4
0	16	4	0	0	0	-4	4	0	4	0	4	0	4	4	4	-4	-4	0	4	0	4	0	0	0	0
0	4	16	0	-4	4	4	4	4	0	4	0	4	4	0	0	0	0	-4	-4	0	0	0	-4	-4	-4
0	0	0	16	4	-4	4	4	-4	0	0	4	0	0	-4	-4	0	4	-4	0	0	4	4	0	0	0
0	0	-4	4	16	4	-4	4	0	0	-4	4	0	-4	-4	4	-4	0	0	0	-4	-4	0	0	0	-4
0	0	4	-4	4	16	0	0	4	4	-4	-4	4	0	0	0	-4	0	-4	0	0	0	0	0	-4	4
4	-4	4	4	-4	0	16	-4	4	0	0	0	0	0	0	0	-4	0	4	-4	0	4	4	0	0	0
-4	4	4	4	4	0	-4	16	0	4	4	0	-4	-4	0	0	0	0	0	-4	4	-4	0	0	-4	0
-4	0	4	-4	0	4	4	0	16	0	0	0	0	0	-4	0	-4	0	4	4	4	0	-4	4	4	-4
4	4	0	0	0	4	0	4	0	16	0	-4	0	0	0	0	0	4	0	-4	-4	0	-4	4	4	4
0	0	4	0	-4	-4	0	4	0	0	16	-4	4	-4	-4	0	0	4	4	4	0	-4	4	-4	0	0
0	4	0	4	4	-4	0	0	0	-4	-4	16	4	0	4	0	0	0	4	0	4	0	-4	-4	0	0
0	0	4	0	0	4	0	-4	0	0	4	4	16	0	0	0	4	0	0	0	0	-4	4	0	4	4
-4	4	4	0	-4	0	0	-4	0	0	-4	0	0	16	0	0	0	4	0	0	-4	0	0	4	-4	0
4	4	0	-4	-4	0	0	0	-4	0	-4	4	0	0	16	-4	-4	-4	0	-4	4	-4	4	0	0	0
4	4	0	-4	4	0	0	0	0	0	0	0	0	0	-4	16	4	-4	4	0	0	0	0	4	-4	0
0	-4	0	0	-4	-4	-4	0	-4	0	0	0	4	0	-4	4	16	0	-4	-4	4	0	-4	4	0	4
4	-4	0	4	0	0	0	0	0	4	4	0	0	4	-4	-4	0	16	0	4	0	-4	-4	0	-4	0
-4	0	-4	-4	0	-4	4	0	4	0	4	4	0	0	0	4	-4	0	16	0	0	-4	0	0	0	4
0	4	-4	0	0	0	-4	-4	4	-4	4	0	0	0	-4	0	-4	4	0	16	4	4	0	0	0	0
0	0	0	0	-4	0	0	4	4	-4	0	4	0	-4	4	0	4	0	0	4	16	0	0	4	-4	4
0	4	0	4	-4	0	4	-4	0	0	-4	0	-4	0	-4	0	0	-4	-4	4	0	16	-4	-4	0	4
0	0	0	4	0	0	4	0	-4	-4	4	-4	4	0	4	0	-4	-4	0	0	0	-4	16	4	0	0
0	0	-4	0	0	0	0	0	4	4	-4	-4	0	4	0	4	4	0	0	0	4	-4	4	16	4	0
0	0	-4	0	0	-4	0	-4	4	4	0	0	4	-4	0	-4	0	-4	0	0	-4	0	0	4	16	-4
-4	0	-4	0	-4	4	0	0	-4	4	0	0	4	0	0	0	4	0	4	0	4	4	0	0	-4	16

Table 5: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 16$, $m = 27$, and $E(s^2) = 8.38746$.

-1	-1	-1	1	-1	1	-1	1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	-1	1	-1	-1
-1	1	1	-1	1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	1	1	-1	1	1	-1	1	-1	1	1	1	-1
1	1	-1	1	1	1	1	-1	-1	1	1	1	-1	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	1	-1
-1	-1	-1	-1	1	-1	1	1	1	1	1	-1	1	-1	1	1	1	1	1	-1	1	1	-1	-1	-1	1	-1
-1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	1	1	1	-1
-1	1	-1	-1	-1	-1	1	1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	1
-1	1	1	1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	1
1	-1	1	1	-1	-1	-1	1	-1	1	-1	-1	-1	1	1	1	-1	1	-1	1	1	1	-1	1	-1	-1	-1
1	-1	1	-1	1	-1	-1	-1	1	-1	-1	1	-1	1	1	1	1	1	-1	-1	-1	-1	1	-1	-1	-1	1
1	-1	1	-1	1	1	1	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	-1	1	1	-1	1	-1
1	1	-1	1	-1	1	-1	1	1	-1	-1	1	1	1	1	-1	1	-1	1	-1	1	1	1	1	-1	1	1
1	1	-1	-1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1	-1	1	1	1	-1	1	1	1	1	-1	1
1	1	1	1	-1	1	1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	-1
1	1	1	-1	-1	1	-1	-1	-1	1	1	-1	1	-1	-1	1	1	1	-1	-1	1	-1	-1	1	1	1	1
-1	-1	-1	1	1	-1	-1	-1	1	-1	1	-1	-1	1	-1	-1	-1	-1	-1	-1	1	1	-1	-1	1	1	1
-1	-1	1	1	1	-1	1	-1	-1	1	1	1	1	1	1	-1	1	-1	1	1	1	-1	1	1	1	-1	1

Table 6: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 4$, and $f_{s_{\max}} = 184$.

16	4	4	0	-4	4	0	-4	4	0	0	0	-4	4	0	4	0	4	0	0	0	4	4	4	-4	0	0
4	16	0	0	-4	4	0	4	-4	0	4	4	0	-4	-4	0	0	0	4	0	-4	4	0	0	0	0	4
4	0	16	0	4	0	-4	-4	-4	0	0	0	-4	-4	4	4	0	0	4	4	0	-4	-4	4	0	-4	0
0	0	0	16	0	4	-4	0	0	4	4	0	0	4	4	-4	0	-4	0	4	0	4	0	-4	0	-4	0
-4	-4	4	0	16	-4	0	-4	0	-4	4	0	-4	0	0	-4	4	0	4	4	0	0	0	-4	-4	4	0
4	4	0	4	-4	16	0	-4	0	0	0	0	4	-4	0	-4	0	0	0	0	-4	-4	0	0	0	4	-4
0	0	-4	-4	0	0	16	-4	0	4	4	4	0	0	0	-4	-4	0	4	0	0	0	4	0	0	0	-4
-4	4	-4	0	-4	-4	-4	16	0	0	-4	-4	4	-4	0	4	0	0	4	4	0	4	0	0	-4	-4	0
4	-4	-4	0	0	0	0	0	16	-4	0	-4	0	0	0	4	0	-4	4	-4	0	4	4	-4	0	0	0
0	0	0	4	-4	0	4	0	-4	16	4	0	0	-4	0	4	4	0	-4	0	4	0	0	-4	0	-4	-4
0	4	0	4	4	0	4	-4	0	4	16	-4	4	0	0	0	0	4	4	0	0	4	-4	-4	4	0	4
0	4	0	0	0	0	4	-4	-4	0	-4	16	-4	4	4	-4	4	-4	0	-4	-4	0	4	0	0	0	0
-4	0	-4	0	-4	4	0	4	0	0	4	-4	16	0	4	0	4	4	4	0	0	-4	0	4	4	0	4
4	-4	-4	4	0	-4	0	-4	0	-4	0	4	0	16	4	-4	0	4	-4	0	0	4	4	4	0	0	4
0	-4	4	4	0	0	0	0	0	0	0	4	4	4	16	0	0	4	4	-4	0	0	-4	0	-4	-4	0
4	0	4	-4	-4	-4	-4	4	4	4	0	-4	0	-4	0	16	4	4	0	0	-4	4	-4	0	4	-4	-4
0	0	0	0	4	0	-4	0	0	4	0	4	4	0	0	4	16	0	0	0	0	0	4	0	0	4	0
4	0	0	-4	0	0	0	0	-4	0	4	-4	4	4	4	4	0	16	-4	0	-4	0	-4	0	-4	0	0
0	4	4	0	4	0	4	4	4	-4	4	0	4	-4	4	0	0	-4	16	4	0	4	0	4	0	0	0
0	0	4	4	4	0	0	4	-4	0	0	-4	0	0	-4	0	0	0	4	16	-4	0	4	4	0	-4	-4
0	-4	0	0	0	-4	0	0	0	4	0	-4	0	0	0	-4	0	-4	0	-4	16	0	0	4	-4	4	4
4	4	-4	4	0	-4	0	4	4	0	4	0	-4	4	0	4	0	0	4	0	0	16	-4	0	0	4	-4
4	0	-4	0	0	0	4	0	4	0	-4	4	0	4	-4	-4	4	-4	0	4	0	-4	16	0	-4	-4	4
4	0	4	-4	-4	0	0	0	-4	-4	-4	0	4	4	0	0	0	0	4	4	4	0	0	16	4	4	0
-4	0	0	0	-4	0	0	-4	0	0	4	0	4	0	-4	4	0	-4	0	0	-4	0	-4	4	16	0	0
0	0	-4	-4	4	4	0	-4	0	-4	0	0	0	0	-4	-4	4	0	0	-4	4	4	-4	4	0	16	-4
0	4	0	0	0	-4	-4	0	0	-4	4	0	4	4	0	-4	0	0	0	-4	4	-4	4	0	0	-4	16

Table 7: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 18$, $m = 23$, and $E(s^2) = 6.15020$.

-1	1	1	-1	1	-1	-1	1	-1	1	-1	-1	1	1	1	1	-1	1	1	1	1	1	-1
-1	-1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	-1	1
1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	1	1	-1	1	-1	1	-1	-1
-1	1	-1	-1	-1	-1	1	-1	1	1	1	1	1	-1	1	-1	1	-1	1	1	1	1	1
1	-1	1	1	1	1	-1	1	1	1	1	1	-1	-1	1	-1	-1	-1	1	1	1	-1	1
-1	-1	-1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	1	-1	1
-1	1	-1	1	-1	-1	-1	1	-1	1	1	1	-1	1	-1	1	1	-1	-1	-1	1	-1	-1
-1	1	1	1	-1	1	-1	-1	1	-1	-1	1	1	1	1	-1	1	1	1	-1	-1	-1	-1
1	1	-1	1	1	-1	1	-1	1	-1	-1	-1	-1	-1	1	1	1	1	-1	1	1	-1	-1
1	1	1	-1	1	1	1	1	-1	1	1	1	-1	-1	1	1	1	1	-1	-1	-1	1	1
1	1	1	-1	1	-1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	1	-1	-1	1
1	1	1	1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	1	1
1	-1	-1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1	1	-1
1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	1	1	1	1	1	1	1	1
-1	-1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	-1	-1	1	1	1
1	-1	-1	-1	-1	1	-1	1	1	-1	-1	1	-1	1	1	1	-1	-1	-1	1	-1	1	-1
-1	-1	1	1	1	1	1	-1	1	1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1
-1	1	-1	-1	-1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1

Table 8: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 6$, and $f_{s_{\max}} = 17$.

18	-2	2	2	2	2	-2	-2	-2	-2	2	2	2	-6	2	6	-2	-2	-2	2	-2	2	2
-2	18	2	-2	-2	-6	2	2	-2	-2	2	2	2	-2	2	2	2	2	2	2	-2	-2	-2
2	2	18	-2	2	2	-2	2	2	2	-2	-2	2	-2	-2	-2	2	-2	2	-2	-2	2	2
2	-2	-2	18	2	2	-2	-2	2	-2	6	-2	2	6	2	-2	2	-2	-2	-2	2	-2	-2
2	-2	2	2	18	2	2	-2	-2	2	-2	2	-2	2	2	2	-2	2	-2	6	2	-2	6
2	-6	2	2	2	18	2	2	-2	-2	2	2	-2	2	2	-2	2	-2	2	2	-2	-2	-2
-2	2	-2	-2	2	2	18	-2	2	2	2	-6	-2	-2	-2	2	2	-2	2	-2	-2	-2	2
-2	2	2	-2	-2	2	-2	18	-2	-2	2	-2	-6	2	2	2	-6	-2	-2	-2	2	2	2
-2	-2	2	2	-2	-2	2	-2	18	-2	-2	2	-6	2	2	-2	2	-2	2	2	-2	2	2
-2	-2	2	-2	2	-2	2	-2	-2	18	6	2	-2	-2	2	2	2	-2	2	-2	2	2	-2
2	2	-2	6	-2	2	2	2	-2	6	18	2	-2	-2	-2	-2	2	-2	2	2	-2	6	2
2	2	-2	-2	2	2	-6	-2	2	2	2	18	-2	2	2	2	2	-2	2	2	-2	-2	6
2	2	2	2	-2	-2	-2	-6	-6	-2	-2	-2	18	2	2	-2	-2	-2	6	-2	2	2	2
-6	-2	-2	6	2	2	-2	2	2	-2	-2	2	2	18	-2	6	-2	-2	2	-2	-2	2	-2
2	2	-2	2	2	2	-2	2	2	2	-2	2	2	-2	18	-2	-2	2	-2	2	2	2	-2
6	2	-2	-2	2	-2	2	2	-2	2	-2	2	-2	6	-2	18	2	2	2	-2	2	2	-2
-2	2	2	2	-2	2	2	-6	2	2	2	2	-2	-2	-2	2	18	2	-2	-2	6	2	-2
-2	2	-2	-2	2	-2	-2	-2	-2	-2	-2	-2	-2	-2	2	2	2	18	2	-2	-2	2	-2
-2	2	2	-2	-2	2	2	-2	2	2	2	2	6	2	-2	2	-2	2	18	2	2	-2	2
2	2	-2	-2	6	2	-2	-2	2	-2	2	2	-2	-2	2	-2	-2	-2	2	18	2	2	-2
-2	-2	-2	2	2	-2	-2	2	-2	2	-2	-2	2	-2	2	2	6	-2	2	2	18	-2	2
2	-2	2	-2	-2	-2	-2	2	2	2	6	-2	2	2	2	2	2	2	-2	2	-2	18	2
2	-2	2	-2	6	-2	2	2	2	-2	2	6	2	-2	-2	-2	-2	-2	2	-2	2	2	18

Table 9: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 18$, $m = 24$, and $E(s^2) = 6.66667$.

1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	1	-1	1
-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	1	1	-1	-1	1	1	-1	-1	-1	1
1	-1	1	-1	-1	1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1	1	1	-1	-1	-1	1	-1
-1	1	1	1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	-1	-1	1	1	1	1	-1	-1	-1
-1	1	1	1	1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	1
-1	1	-1	-1	-1	-1	1	1	-1	1	1	1	-1	1	1	1	1	1	-1	-1	1	1	-1	-1
1	1	1	-1	1	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	-1	1	1	1	1	-1	-1
1	-1	-1	1	1	-1	1	1	1	-1	1	-1	1	-1	-1	-1	1	1	-1	1	-1	-1	-1	-1
-1	-1	-1	-1	1	1	1	-1	1	1	1	1	-1	1	-1	-1	1	1	1	1	-1	1	1	1
1	-1	1	-1	1	-1	-1	-1	-1	-1	1	1	1	-1	1	-1	-1	1	-1	-1	1	1	1	1
1	1	1	-1	-1	1	-1	1	-1	-1	1	-1	1	1	1	-1	1	-1	1	1	-1	1	1	-1
1	1	-1	1	1	1	1	-1	-1	-1	-1	1	-1	1	1	1	-1	-1	-1	1	-1	-1	-1	1
1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1
-1	1	-1	-1	1	1	-1	1	1	-1	1	-1	-1	-1	-1	1	1	-1	-1	-1	1	-1	1	1
-1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	1	1	-1
1	1	-1	1	-1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1	1	-1	1	1	-1	1
-1	-1	1	1	-1	-1	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	1	-1	1	-1	-1	1
-1	-1	1	1	1	1	1	1	1	1	-1	1	1	1	1	-1	1	-1	-1	-1	1	-1	1	-1

Table 10: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 6$, and $f_{s_{\max}} = 23$.

18	-2	2	-2	-2	2	-2	-2	-2	-6	-6	2	2	-2	-2	-2	-2	2	-2	2	-2	2	-2	-2
-2	18	-2	2	2	2	-2	2	-2	-2	-2	-2	-2	2	2	2	-2	-2	2	2	2	6	-2	-2
2	-2	18	2	2	2	-2	-2	-6	2	-2	-2	6	-2	2	2	2	2	2	-2	2	2	2	-2
-2	2	2	18	2	2	2	-2	2	-2	-6	-6	2	2	2	-2	-2	2	-2	-2	-2	-2	-6	2
-2	2	2	2	18	2	2	-2	2	2	2	2	2	-2	-2	-2	2	2	-6	6	2	-2	2	2
2	2	2	2	2	18	-2	2	2	-2	-2	2	-6	2	2	-2	2	2	2	-2	-2	-2	2	2
-2	-2	-2	2	2	-2	18	-2	2	-2	-2	6	-2	2	-2	2	6	-2	2	-2	-2	-2	-2	-6
-2	2	-2	-2	-2	2	-2	18	2	-2	2	2	6	2	-2	2	2	2	-2	-2	-2	-2	2	-2
-2	-2	-6	2	2	2	2	2	18	2	-2	2	2	-6	-2	-2	6	-2	2	-2	-2	2	-2	2
-6	-2	2	-2	2	-2	-2	-2	2	18	-2	2	-2	2	2	2	2	2	-2	6	2	2	-2	-2
-6	-2	-2	-6	2	-2	-2	2	-2	-2	18	-2	2	-2	2	-2	6	2	2	2	2	-2	-2	2
2	-2	-2	-6	2	2	6	2	2	2	-2	18	2	2	2	2	-2	2	2	-2	2	2	-2	2
2	-2	6	2	2	-6	-2	6	2	-2	2	2	18	2	-2	-2	2	-2	2	2	2	2	-2	2
-2	2	-2	2	-2	2	2	2	-6	2	-2	2	2	18	-2	-2	2	-2	-2	2	2	2	2	2
-2	2	2	2	-2	2	-2	-2	-2	2	2	2	-2	-2	18	-2	-2	-2	-2	-2	-2	2	-2	-2
-2	2	2	-2	-2	-2	2	2	-2	2	-2	2	-2	-2	-2	18	2	-2	-2	-2	-2	-2	-6	6
-2	-2	2	-2	2	2	6	2	6	2	6	-2	2	2	-2	2	18	-2	-2	-2	2	2	-2	-2
2	-2	2	2	2	2	-2	2	-2	2	2	2	-2	-2	-2	-2	-2	18	-2	-2	-2	6	-2	2
-2	2	2	-2	-6	2	2	-2	2	-2	2	2	2	-2	-2	-2	-2	-2	18	2	-2	2	-2	-2
2	2	-2	-2	6	-2	-2	-2	-2	6	2	-2	2	2	-2	-2	-2	-2	2	18	-6	-2	-2	-2
-2	2	2	-2	2	-2	-2	-2	-2	2	2	2	2	2	-2	-2	2	-2	-2	-6	18	-2	-2	-2
2	6	2	-2	-2	-2	-2	-2	2	2	-2	2	2	2	2	-2	2	6	2	-2	-2	18	2	2
-2	-2	2	-6	2	2	-2	2	-2	-2	-2	-2	-2	2	-2	-6	-2	-2	-2	-2	-2	2	18	-2
-2	-2	-2	2	2	2	-6	-2	2	-2	2	2	2	2	-2	6	-2	2	-2	-2	-2	2	-2	18

Table 11: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 18$, $m = 25$, and $E(s^2) = 7.20000$.

1	1	-1	1	1	-1	-1	-1	1	1	1	-1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	-1
1	-1	1	1	-1	1	1	-1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	1	1	1	-1	1
1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	1	1	1	1	1	1	1	1	1	-1	1	1
1	-1	1	1	-1	-1	1	1	1	1	1	1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	1
1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	1	-1	1	1	-1	-1	-1	-1	1	1	1	1	1	-1
1	1	1	-1	-1	1	-1	1	1	1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	-1
-1	-1	-1	1	1	1	-1	1	1	-1	1	1	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	-1
-1	1	1	-1	-1	1	-1	1	-1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	-1	1	-1
-1	1	-1	1	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	1	1	1	1
-1	-1	1	-1	1	1	1	-1	1	1	-1	-1	-1	1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
-1	1	-1	1	-1	-1	1	-1	-1	1	1	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1	1	1	1	-1	1	1	-1	-1	1	1	1	-1
-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	1
-1	1	1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	-1	1	1	1	1	-1	-1	1
1	1	-1	1	1	1	1	1	-1	-1	1	-1	-1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1
-1	-1	1	-1	1	-1	-1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	1	-1	1	1	1
1	1	1	-1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	-1
1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	1	1	-1	1	-1	-1	-1	-1	-1	1

Table 12: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 6$, and $f_{s_{\max}} = 30$.

18	-2	-2	2	-2	2	-2	2	-2	-2	-6	2	2	6	-2	2	2	-2	-6	-2	-6	2	2	-2	2
-2	18	2	2	2	2	2	2	-2	2	2	2	-2	-2	-2	2	-2	-2	-2	2	-6	2	-6	-2	-2
-2	2	18	-6	-2	2	2	2	-2	2	-2	-2	2	-2	-6	-2	-2	2	-6	-2	2	-2	-2	2	2
2	2	-6	18	2	2	2	2	2	2	2	-2	-2	-2	2	-2	2	2	-2	-2	2	2	2	-2	6
-2	2	-2	2	18	2	-2	-2	2	-2	-2	-2	-2	2	-2	-2	2	-2	-2	2	2	-2	2	2	-2
2	2	2	2	2	18	2	2	-2	2	-6	-2	2	-2	6	-2	-2	2	6	2	-2	-2	2	-2	-2
-2	2	2	2	-2	2	18	-6	-2	-2	2	2	-2	6	2	-2	-2	2	2	2	-2	-2	2	-2	2
2	2	2	2	-2	2	-6	18	-2	-2	2	2	-2	2	-2	-2	-2	2	2	2	2	2	2	2	2
-2	-2	-2	2	2	-2	-2	-2	18	2	-2	2	-2	-2	-6	-2	2	6	2	-6	-6	2	2	-2	-2
-2	2	2	2	-2	2	-2	-2	2	18	2	-2	-2	-2	-2	-2	2	2	-2	6	-2	-2	-2	2	-2
-6	2	-2	2	-2	-6	2	2	-2	2	18	-2	2	2	-2	-2	2	-2	2	-2	-2	-6	2	-2	-2
2	2	-2	-2	-2	-2	2	2	2	-2	-2	18	-2	-2	2	-2	6	2	-2	2	-2	-2	2	-2	-2
2	-2	2	-2	-2	2	-2	-2	-2	-2	2	-2	18	2	2	2	6	2	-2	-2	2	6	2	-2	-2
6	-2	-2	-2	2	-2	6	2	-2	-2	2	-2	2	18	-2	-2	2	2	2	2	-2	2	-2	2	-2
-2	-2	-6	2	-2	6	2	-2	-6	-2	-2	2	2	-2	18	-2	-2	2	2	-2	2	2	2	6	-6
2	2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	2	-2	-2	18	-2	6	-2	2	-2	-2	-2	2	2
2	-2	-2	2	2	-2	-2	-2	2	2	2	6	6	2	-2	-2	18	-2	2	-2	-2	-2	-2	2	6
-2	-2	2	2	-2	2	2	2	6	2	-2	2	2	2	2	6	-2	18	-2	-2	2	-2	-2	2	-6
-6	-2	-6	-2	-2	6	2	2	2	-2	2	-2	-2	2	2	-2	2	-2	18	2	-2	-2	-2	-2	2
-2	2	-2	-2	2	2	2	2	-6	6	-2	2	-2	2	-2	2	-2	-2	2	18	6	2	2	-2	2
-6	-6	2	2	2	-2	-2	2	-6	-2	-2	-2	2	-2	2	-2	-2	2	-2	6	18	2	-2	-2	2
2	2	-2	2	-2	-2	-2	2	2	-2	-6	-2	6	2	2	-2	-2	-2	-2	2	2	18	2	2	2
2	-6	-2	2	2	2	2	2	2	-2	2	2	2	-2	2	-2	-2	-2	-2	2	-2	2	18	2	2
-2	-2	2	-2	2	-2	-2	2	-2	2	-2	-2	-2	2	6	2	2	2	-2	-2	-2	2	2	18	2
2	-2	2	6	-2	-2	2	2	-2	-2	-2	-2	-2	-2	-6	2	6	-6	2	2	2	2	2	2	18

Table 13: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 18$, $m = 26$, and $E(s^2) = 7.64308$.

1	1	-1	-1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	1	1	-1	1	-1	-1	-1	-1	-1
-1	-1	1	1	1	1	1	1	-1	-1	-1	1	1	-1	1	1	-1	1	1	-1	-1	1	1	-1	-1	1
-1	1	-1	1	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	1	1
1	1	1	-1	1	1	-1	1	-1	1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	1	1	-1	1	-1	1
1	1	-1	1	-1	-1	1	1	-1	1	1	1	1	1	1	1	-1	1	-1	1	1	1	1	1	1	-1
-1	1	-1	1	1	1	-1	1	1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	1	-1	-1
1	-1	1	-1	-1	1	1	-1	1	1	1	-1	1	-1	1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	-1
-1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	-1	1	-1	-1	1	-1
1	-1	-1	1	-1	-1	-1	-1	1	-1	1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1
-1	1	1	-1	1	-1	1	1	1	-1	1	1	1	1	1	-1	1	-1	-1	-1	-1	-1	-1	1	1	1
1	-1	1	1	1	-1	1	1	-1	1	1	-1	1	-1	-1	-1	1	-1	1	1	1	-1	-1	-1	1	1
-1	-1	1	-1	-1	-1	-1	1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1
-1	-1	1	1	1	-1	-1	-1	1	1	-1	-1	1	1	1	1	1	1	-1	1	-1	1	-1	1	1	-1
1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	1	-1	1	1	-1	-1	-1	-1	1	-1	-1	-1	-1	1	1
1	1	1	1	-1	-1	1	-1	1	-1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	1	1	-1	1
-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	1	1	1	-1	-1	1	-1	-1	1
-1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	-1	1	-1	-1	1	-1	-1	1	1	-1	-1
1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	1	1	1	-1

Table 14: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 6$, and $f_{s_{\max}} = 37$.

18	-2	-2	2	-2	-2	2	-6	-6	-2	2	-2	2	2	-2	-2	-2	-2	-2	-2	6	2	-2	2	-2	2
-2	18	-2	-2	2	-2	2	2	2	-2	2	2	-6	-2	2	2	2	6	-2	2	6	2	-2	2	-2	2
-2	-2	18	-2	2	-2	6	2	2	6	-2	-2	2	2	2	2	2	-2	-2	-2	2	2	-2	-2	-2	6
2	-2	-2	18	2	-2	-2	-2	2	-2	2	2	2	-2	2	2	-2	2	-2	6	2	2	2	2	2	6
-2	2	2	2	18	2	-2	2	-2	-2	2	-2	2	-2	2	2	6	2	2	-6	2	2	-6	2	2	2
-2	-2	-2	-2	2	18	-2	-2	-6	2	2	-2	-6	-2	2	-2	-2	-2	-2	-2	-2	2	2	-2	-6	2
2	2	6	-2	-2	-2	18	2	-2	-2	6	-2	2	-2	2	2	2	-2	2	2	-2	6	2	-2	2	2
-6	2	2	-2	2	-2	2	18	-2	2	-2	2	2	-2	-2	-2	-6	-2	2	2	2	-2	-2	2	2	-2
-6	2	2	2	-2	-6	-2	-2	18	-2	2	-2	-2	-2	-2	-2	-2	-2	-2	-2	2	2	-2	2	2	-2
-2	-2	6	-2	-2	2	-2	2	-2	18	2	-2	2	2	-2	-2	2	2	-6	2	6	2	2	-2	2	-2
2	2	-2	2	2	2	6	-2	2	2	18	-6	2	2	2	-2	2	-2	2	-2	2	-2	2	2	2	2
-2	2	-2	2	-2	-2	-2	2	-2	-2	-6	18	2	-2	2	2	2	-2	-6	-2	2	-2	6	-2	2	2
2	-6	2	2	2	-6	2	2	-2	2	2	2	18	-2	6	-2	2	2	-2	-2	-2	-2	-2	2	-2	-2
2	-2	2	-2	-2	-2	-2	-2	-2	2	2	-2	-2	18	2	2	-2	2	-2	-2	-6	-2	2	6	6	2
-2	2	2	2	2	2	2	-2	-2	-2	2	2	6	2	18	6	-2	2	-2	2	-2	-2	-6	-2	-2	-2
-2	2	2	2	2	-2	2	-2	-2	-2	-2	2	-2	2	6	18	2	2	6	2	2	2	6	2	-2	-6
-2	2	2	-2	6	-2	2	-6	-2	2	2	2	2	-2	-2	2	18	-2	-2	2	-2	-2	2	2	2	2
-2	6	-2	2	2	-2	-2	-2	-2	2	-2	-2	2	2	2	2	-2	18	2	-2	-2	6	2	-2	-2	2
-2	-2	-2	-2	2	-2	2	2	-2	-6	2	-6	-2	-2	-2	6	-2	2	18	-2	-2	-6	2	-6	-2	-2
-2	2	-2	6	-6	-2	2	2	-2	2	-2	-2	-2	-2	2	2	2	-2	-2	18	-2	-2	-2	2	2	-2
6	6	2	2	2	-2	-2	2	2	6	2	2	-2	-6	-2	2	-2	-2	-2	-2	18	2	-2	-2	-2	-2
2	2	2	2	2	2	6	-2	2	2	-2	-2	-2	-2	-2	2	-2	6	-6	-2	2	18	2	2	2	-2
-2	-2	-2	2	-6	2	2	-2	-2	2	2	6	-2	2	-6	6	2	2	2	-2	-2	2	18	2	-2	2
2	2	-2	2	2	-2	-2	2	2	-2	2	-2	2	6	-2	2	2	-2	-6	2	-2	2	2	18	-2	-2
-2	-2	-2	2	2	-6	2	2	2	2	2	2	-2	6	-2	-2	2	-2	-2	2	-2	2	-2	-2	18	-2
2	2	6	6	2	2	2	-2	-2	-2	2	2	-2	2	-2	-6	2	2	-2	-2	-2	-2	2	-2	-2	18

Table 15: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 18$, $m = 27$, and $E(s^2) = 8.10256$.

1	-1	-1	-1	1	-1	1	1	1	-1	-1	1	1	1	-1	1	1	-1	1	1	1	-1	1	-1	-1	1	-1
1	1	1	-1	1	-1	1	-1	-1	1	1	1	-1	-1	1	1	-1	1	-1	-1	-1	1	1	-1	-1	1	-1
-1	1	-1	1	-1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	1	1	-1	1	-1	1
-1	-1	1	1	1	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	1	1	1	1	-1	-1	1	-1	1	1
-1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1	1	1	-1	-1	-1
-1	-1	1	-1	1	1	1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1
1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	-1	-1	-1	1	-1	1	-1	-1	1	-1	1	-1	-1
1	-1	1	-1	-1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	1	-1	1	1	-1	1	1	1	-1
1	-1	-1	-1	1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1
1	1	-1	-1	-1	1	1	-1	-1	1	-1	-1	-1	1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1	1
1	-1	-1	1	-1	-1	1	-1	1	1	1	-1	1	1	1	-1	1	1	-1	1	-1	-1	1	1	-1	1	1
-1	1	-1	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1
1	1	-1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	1	1	1	-1	1	1	-1	1	1	-1	-1
-1	1	1	1	1	1	1	-1	1	-1	-1	-1	-1	-1	1	-1	1	-1	-1	1	-1	-1	-1	-1	1	1	-1
-1	1	1	-1	-1	-1	1	1	1	1	-1	1	-1	1	1	-1	-1	-1	1	1	-1	1	-1	1	-1	-1	1
-1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	-1	1	1	-1	-1	1	-1	1	-1	1	1	1	1
-1	-1	1	1	-1	-1	-1	1	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	-1	1	1	-1
1	1	1	1	-1	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	-1	1	1	1	1	1	-1	-1	-1	1

Table 16: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 6$, and $f_{s_{\max}} = 45$.

18	-2	-6	-6	-2	-2	2	-2	2	6	2	-2	-2	-2	-2	-2	2	2	2	-2	2	-2	2	-2	2	2	-2
-2	18	2	2	-2	2	6	-6	-2	2	-2	-2	-6	-2	-2	2	2	-2	2	-6	-2	2	2	2	-2	-6	-2
-6	2	18	-2	-2	2	-2	-2	2	2	2	2	-2	-2	6	-2	-2	-2	-2	2	6	6	-2	-2	-6	2	-2
-6	2	-2	18	2	2	-2	-2	2	2	-2	-6	2	-2	2	6	6	2	2	2	-2	2	2	2	2	2	2
-2	-2	-2	2	18	-2	2	2	-2	-6	2	-2	2	-6	-2	2	-2	2	2	2	-2	-2	-6	-2	-2	2	-2
-2	2	2	2	-2	18	-2	-2	-2	-2	-2	-2	-6	2	-2	2	-2	6	2	2	2	-2	-2	-2	2	-2	2
2	6	-2	-2	2	-2	18	-2	-2	2	-6	-2	2	6	2	-2	2	2	-2	2	-2	2	-2	-2	-2	2	2
-2	-6	-2	-2	2	-2	-2	18	2	-2	-2	2	-2	6	-6	-2	-2	2	2	2	2	6	-2	2	2	-2	-6
2	-2	2	2	-2	-2	-2	2	18	-2	2	-6	-2	6	6	-2	2	-6	6	2	-2	-2	2	-2	-2	2	-2
6	2	2	2	-6	-2	2	-2	-2	18	-6	-2	-2	-2	2	2	-6	2	-2	2	2	2	2	2	-2	2	2
2	-2	2	-2	2	-2	-6	-2	2	-6	18	-2	-2	-2	-2	-2	2	2	-2	-2	-2	2	2	2	-6	-2	2
-2	-2	2	-6	-2	-2	-2	2	-6	-2	-2	18	-2	-2	2	2	2	2	2	2	-2	2	2	2	-2	2	-2
-2	-6	-2	2	2	-6	2	-2	-2	-2	-2	-2	18	2	2	-2	2	2	2	-2	6	-2	2	-2	-2	-2	2
-2	-2	-2	-2	-6	2	6	6	6	-2	-2	-2	2	18	2	2	-2	2	2	-2	2	2	2	2	-2	2	2
-2	-2	6	2	-2	-2	2	-6	6	2	-2	2	2	2	18	-2	-2	2	2	-2	-6	2	-2	-2	2	6	2
-2	2	-2	6	2	2	-2	-2	-2	2	-2	2	-2	2	-2	18	-2	-2	2	-2	2	2	6	-2	-2	6	2
2	2	-2	6	-2	-2	2	-2	2	-6	2	2	2	-2	-2	-2	18	-2	2	2	2	2	2	2	2	2	-2
2	-2	-2	2	2	6	2	2	-6	2	2	2	2	2	2	-2	-2	18	2	-2	-2	2	2	2	-2	-2	-2
2	2	-2	2	2	2	-2	2	6	-2	-2	2	2	2	2	2	2	2	18	-2	2	2	-2	2	-2	-6	2
-2	-6	2	2	2	2	2	2	2	2	-2	2	-2	-2	-2	-2	2	-2	-2	18	-2	-2	-2	-2	-6	2	6
2	-2	6	-2	-2	2	-2	2	-2	2	-2	-2	6	2	-6	2	2	-2	2	-2	18	2	-2	2	-2	2	-2
-2	2	6	2	-2	-2	2	6	-2	2	2	2	-2	2	2	2	2	2	2	-2	2	18	-2	-2	2	-2	2
2	2	-2	2	-6	-2	-2	-2	2	2	2	2	2	2	-2	6	2	2	-2	-2	-2	-2	18	-6	-6	-2	-6
-2	2	-2	2	-2	-2	-2	2	-2	2	2	2	-2	2	-2	-2	2	2	2	-2	2	-2	-6	18	-2	2	2
2	-2	-6	2	-2	2	-2	2	-2	-2	-6	-2	-2	-2	2	-2	2	-2	-2	-6	-2	2	-6	-2	18	2	-2
2	-6	2	2	2	-2	2	-2	2	2	-2	2	-2	2	6	6	2	-2	-6	2	2	-2	-2	2	2	18	-2
-2	-2	-2	2	-2	2	2	-6	-2	2	2	-2	2	2	2	2	-2	-2	2	6	-2	2	-6	2	-2	-2	18

Table 17: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 18$, $m = 28$, and $E(s^2) = 8.48677$.

-1	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1	1	1	-1	1	1	-1	-1	-1	1	-1	1	1	1	1	1
1	1	1	-1	1	1	-1	-1	-1	-1	1	1	-1	1	1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
-1	-1	-1	1	1	-1	1	1	-1	-1	1	1	1	-1	1	1	-1	1	1	1	-1	-1	-1	1	-1	-1	1	-1
1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	1	-1	1
1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	1	-1	1	-1
1	1	-1	1	1	1	1	-1	1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	-1	1
-1	1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	1	1
-1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1	1	-1	-1	-1	1	1	-1	1	-1	-1	-1	-1	1	-1	-1
-1	1	-1	-1	1	1	-1	-1	-1	1	-1	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	1	1	1	-1	-1
1	-1	1	-1	-1	1	1	1	-1	-1	-1	1	1	1	1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1
1	1	1	-1	-1	1	-1	1	1	1	1	-1	1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	-1	1	-1
-1	-1	1	1	1	1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1	1
-1	-1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	1
-1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	1	-1	-1	1	1	1	-1	-1	-1
1	1	1	1	-1	1	1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1	1	1
-1	-1	-1	-1	1	1	1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	1	1	-1	1	1	-1	-1	1	1	1
1	-1	-1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	1	1	1	-1	-1	1	-1	1	-1
1	1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	1	1	1	1	-1	1	1	1	-1	1	-1	-1	-1	-1	-1

Table 18: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 6$, and $f_{s_{\max}} = 53$.

18	2	2	2	-2	6	-2	-6	2	2	-2	2	2	2	2	2	2	-2	-2	6	2	2	2	-2	-2	-2	2	-2
2	18	-2	-2	-2	2	-2	-2	-2	2	2	2	-2	2	-2	2	6	2	-2	-2	2	-6	2	6	2	2	2	2
2	-2	18	-6	-2	2	2	-2	2	-2	-2	-2	-2	2	2	-2	-2	2	-2	-2	-2	-6	2	2	2	-2	-2	-2
2	-2	-6	18	-2	-2	2	-2	-2	2	-2	-2	2	-2	6	-2	-2	2	-6	-2	2	-2	-2	-2	-6	-2	-2	2
-2	-2	-2	-2	18	2	-2	2	-2	2	2	2	-2	2	2	2	2	6	6	-2	-2	2	2	-2	-2	-6	-2	2
6	2	2	-2	2	18	-2	-2	-2	2	2	2	2	2	-2	-2	-6	2	-2	2	2	2	2	2	-2	6	-2	6
-2	-2	2	2	-2	-2	18	-2	2	-2	-2	-2	2	2	-2	-6	-2	2	2	-2	-6	2	6	6	-2	2	2	-2
-6	-2	-2	-2	2	-2	-2	18	-2	2	2	-2	6	6	6	2	-2	-2	2	2	2	-2	2	2	-2	-2	2	2
2	-2	2	-2	-2	-2	2	-2	18	2	2	2	-2	2	-6	2	-2	-2	-2	-6	2	2	2	-2	-6	-2	6	2
2	2	-2	2	2	2	-2	2	2	18	-2	-6	-2	-2	2	2	-6	2	-2	-2	2	2	2	-2	2	-2	6	-2
-2	2	-2	-2	2	2	-2	2	2	-2	18	2	2	-2	-2	-6	-2	2	-2	6	2	-6	2	-2	2	-6	6	2
2	2	-2	-2	2	2	-2	-2	2	-6	2	18	-2	-2	6	2	-2	-2	2	-2	2	2	-2	6	2	2	2	2
2	-2	-2	2	-2	2	2	6	-2	-2	2	-2	18	-2	-2	2	2	6	-2	-2	2	2	2	2	2	-2	2	2
2	2	2	-2	2	2	2	6	2	-2	-2	-2	-2	18	2	-6	2	2	2	2	2	2	-2	2	-6	2	-2	2
2	-2	2	6	2	-2	-2	6	-6	2	-2	6	-2	2	18	-2	-2	-2	-2	2	-2	-2	-2	2	2	-2	2	2
2	2	-2	-2	2	-2	-6	2	2	2	-6	2	2	-6	-2	18	2	-2	2	-2	-2	-2	-2	2	-6	-2	2	-2
2	6	-2	-2	2	-6	-2	-2	-2	-6	-2	-2	2	2	-2	2	18	-2	-2	-2	-6	2	2	-2	2	-2	-2	6
-2	2	2	2	6	2	2	-2	-2	2	2	-2	6	2	-2	-2	-2	18	2	-6	2	-2	-6	2	2	-2	2	-2
-2	-2	-2	-6	6	-2	2	2	-2	-2	-2	2	-2	2	-2	2	-2	2	18	2	2	-2	2	-6	2	6	2	-6
6	-2	-2	-2	-2	2	-2	2	-6	-2	6	-2	-2	2	2	-2	-2	-6	2	18	-2	-2	-2	-2	-2	-2	2	-6
2	2	-2	2	-2	2	-6	2	2	2	2	2	2	2	-2	-2	-6	2	2	-2	18	-2	2	-2	2	-2	-6	-6
2	-6	-6	-2	2	2	2	-2	2	2	-6	2	2	2	-2	-2	2	-2	-2	-2	-2	18	-2	2	2	2	-2	2
2	2	2	-2	2	2	6	2	2	2	2	-2	2	-2	-2	-2	2	-6	2	-2	2	-2	18	-2	2	-2	-2	2
-2	6	2	-2	-2	2	6	2	-2	-2	-2	6	2	2	2	2	-2	2	-6	-2	-2	2	-2	18	-2	-2	-2	-2
-2	2	2	-6	-2	-2	-2	-2	-6	2	2	2	2	-6	2	-6	2	2	2	-2	2	2	2	-2	18	2	2	-2
-2	2	-2	-2	-6	6	2	-2	-2	-2	-6	2	-2	2	-2	-2	-2	-2	6	-2	-2	2	-2	-2	2	18	2	6
2	2	-2	-2	-2	-2	2	2	6	6	6	2	2	-2	2	2	-2	2	2	2	-6	-2	-2	-2	2	2	18	2
-2	2	-2	2	2	6	-2	2	2	-2	2	2	2	2	2	-2	6	-2	-6	-6	-6	2	2	-2	-2	6	2	18

Table 19: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 18$, $m = 29$, and $E(s^2) = 8.72906$.

1	-1	1	1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	1
-1	1	-1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	-1
-1	-1	-1	-1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	-1	-1	-1
1	-1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	1	-1	-1	1	1	1	-1	1	-1	1	1	1	1	1
-1	1	1	1	1	1	1	1	-1	1	-1	-1	1	-1	-1	1	1	1	-1	-1	1	-1	1	1	-1	1	-1	1	-1
1	-1	-1	-1	-1	1	-1	1	1	1	1	1	1	-1	1	1	-1	1	1	-1	1	1	1	-1	1	-1	-1	1	-1
1	1	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	-1	1
1	1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	1	-1	-1	-1	1	1	1	-1	1	1	-1	1	1	1	1	-1	1
-1	1	-1	1	-1	1	-1	1	1	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	1	1	1	1	1	1
1	-1	-1	-1	1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	1	-1	1	1	-1	-1	-1	1	-1	-1	-1	1	1
1	1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1	1	1	1	1	1	1	1	1	1	-1	-1	-1	1	-1	-1	-1
-1	-1	1	-1	1	1	-1	-1	-1	1	1	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	1	1	-1	1	1
-1	-1	1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	-1	-1	-1	1	1	1	-1
1	1	1	-1	-1	-1	-1	1	-1	1	1	1	-1	1	-1	-1	-1	1	-1	1	-1	-1	1	1	1	-1	1	1	-1
1	1	1	1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	1
-1	-1	1	1	1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	1	-1	1	-1	-1
-1	1	-1	-1	1	1	1	-1	1	-1	1	-1	-1	1	1	1	-1	1	1	1	1	-1	1	1	-1	-1	1	-1	-1
-1	-1	-1	1	-1	1	1	-1	1	-1	1	1	-1	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	1	-1	1	1

Table 20: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 6$, and $f_{s_{\max}} = 60$.

18	2	2	-2	-2	-6	-2	2	-2	-2	-2	6	2	2	-2	6	-2	2	2	-2	2	2	2	-2	2	-2	2	-2	6
2	18	2	2	-2	-2	-2	2	2	2	2	-2	2	2	-2	2	2	6	-6	-2	6	-2	-2	2	-2	2	6	-6	-2
2	2	18	2	2	2	-2	6	-6	-2	-2	2	-2	2	2	-2	2	-2	-2	-6	-2	-2	2	-2	-2	2	2	-2	-2
-2	2	2	18	-2	6	-2	-2	-2	-2	-6	-2	-2	-2	-6	2	2	-2	-6	2	2	2	2	-6	2	2	-2	-2	2
-2	-2	2	-2	18	2	2	-2	-2	2	-2	2	6	-2	-2	2	2	-2	2	2	2	-6	2	2	-6	-2	2	-2	2
-6	-2	2	6	2	18	-2	-2	2	-2	2	-2	2	2	2	-2	-2	2	-2	-2	6	2	6	2	2	-2	-2	2	2
-2	-2	-2	-2	2	-2	18	-6	-2	-2	-2	-2	-2	-2	2	2	-2	2	-2	-2	-2	2	6	2	-6	6	2	-2	2
2	2	6	-2	-2	-2	-6	18	2	2	-2	-2	6	2	2	2	2	-2	2	2	-2	2	2	-2	-2	2	2	6	-6
-2	2	-6	-2	-2	2	-2	2	18	-2	2	-2	2	2	2	2	-2	-2	-2	-2	-2	-2	-2	-6	-6	-2	2	2	2
-2	2	-2	-2	2	-2	-2	2	-2	18	2	6	2	-2	2	2	-2	-2	-6	-2	2	-2	2	2	6	2	-2	6	-2
-2	2	-2	-6	-2	2	-2	-2	2	2	18	2	-2	2	-2	-2	6	2	2	-2	-2	2	6	-2	6	-2	2	-2	-2
6	-2	2	-2	2	-2	-2	-2	-2	6	2	18	-2	2	2	-2	-2	6	-2	2	2	2	2	-6	-2	2	-2	2	2
2	2	-2	-2	6	2	-2	6	2	2	-2	-2	18	-6	-2	-2	2	2	-2	-2	2	6	-2	2	-2	-2	2	-2	2
2	2	2	-2	-2	2	-2	2	2	-2	2	2	-6	18	6	2	2	-2	-2	6	-2	2	2	6	-2	2	-2	2	6
-2	-2	2	-6	-2	2	2	2	2	2	-2	2	-2	6	18	2	-6	-2	2	-2	2	6	-2	2	-2	2	-2	-2	-2
6	2	-2	2	2	-2	2	2	2	2	-2	-2	-2	2	2	18	2	-2	2	-2	2	-2	6	-2	-2	-2	-6	-2	-2
-2	2	2	2	2	-2	-2	2	-2	-2	6	-2	2	2	-6	2	18	-2	2	-2	-2	2	-2	-2	-2	6	-6	-2	2
2	6	-2	-2	-2	2	2	-2	-2	-2	2	6	2	-2	-2	-2	-2	18	-2	2	6	2	2	2	-6	-2	-2	-2	-6
2	-6	-2	-6	2	-2	-2	2	-2	-6	2	-2	-2	-2	2	2	2	-2	18	2	6	-2	-2	-2	2	2	2	2	-2
-2	-2	-6	2	2	-2	-2	2	-2	-2	-2	2	-2	6	-2	-2	-2	2	2	18	-2	2	2	2	-2	-2	2	2	-2
2	6	-2	2	2	6	-2	-2	-2	2	-2	2	2	-2	2	2	-2	6	6	-2	18	-2	-2	2	2	6	2	2	2
2	-2	-2	2	-6	2	2	2	-2	-2	2	2	6	2	6	-2	2	2	-2	2	-2	18	2	-2	2	2	-2	-6	2
2	-2	2	2	2	6	6	2	-2	2	6	2	-2	2	-2	6	-2	2	-2	2	-2	2	18	-2	2	-2	2	2	-2
-2	2	-2	-6	2	2	2	-2	-6	2	-2	-6	2	6	2	-2	-2	2	-2	2	2	-2	-2	18	2	-2	-2	2	2
2	-2	-2	2	-6	2	-6	-2	-6	6	6	-2	-2	-2	-2	-2	-2	-6	2	-2	2	2	2	2	18	-2	2	2	2
-2	2	2	2	-2	-2	6	2	-2	2	-2	2	-2	2	2	-2	6	-2	2	-2	6	2	-2	-2	-2	18	2	6	6
2	6	2	-2	2	-2	2	2	2	-2	2	-2	2	-2	-2	-6	-6	-2	2	2	2	-2	2	-2	2	2	18	-2	2
-2	-6	-2	-2	-2	2	-2	6	2	6	-2	2	-2	2	-2	-2	-2	-2	2	2	2	-6	2	2	2	6	-2	18	2
6	-2	-2	2	2	2	2	-6	2	-2	-2	2	2	6	-2	-2	2	-6	-2	-2	2	2	-2	2	2	6	2	2	18

Table 21: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 20$, $m = 21$, and $E(s^2) = 3.80952$.

1	-1	1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1	-1	-1	-1	1	1	1
-1	-1	-1	1	1	-1	-1	1	-1	1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1
1	-1	1	1	1	1	1	-1	1	-1	1	1	-1	1	1	1	1	-1	1	-1	1
-1	-1	1	-1	1	-1	1	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	-1	1
-1	1	-1	-1	-1	-1	1	1	1	-1	1	1	1	-1	1	1	1	1	1	1	-1
1	1	1	-1	-1	1	-1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	1	-1	-1
-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1
1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	1
1	1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1
-1	-1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	-1
1	-1	-1	1	1	1	-1	1	1	1	1	-1	1	1	-1	-1	1	1	1	1	-1
1	1	-1	1	-1	1	1	-1	1	1	-1	-1	-1	-1	1	-1	1	1	-1	-1	-1
1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1
-1	1	1	1	-1	1	1	1	1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	-1
-1	1	1	-1	1	-1	-1	-1	1	1	-1	1	1	1	1	-1	-1	1	1	-1	-1
-1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	1	-1	-1	1	-1	1	-1	1
-1	-1	1	1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	1	1
-1	1	-1	1	1	-1	1	-1	-1	1	1	-1	1	1	1	1	-1	-1	-1	1	1
1	-1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1	1	1	1	1	1

Table 22: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 4$, and $f_{s_{\max}} = 50$.

20	0	0	0	-4	4	-4	0	0	0	0	0	0	0	0	0	4	0	4	0	4
0	20	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	20	0	-4	0	0	0	0	0	0	0	0	0	0	0	-4	0	0	0	4
0	0	0	20	4	0	4	0	0	0	0	0	0	0	0	0	4	0	-4	0	0
-4	0	-4	4	20	0	0	0	0	0	4	0	-4	0	0	0	-4	0	4	-4	4
4	0	0	0	0	20	4	-4	0	0	0	0	-4	0	-4	-4	0	-4	4	0	-4
-4	4	0	4	0	4	20	4	0	0	0	4	0	0	0	4	0	0	0	-4	4
0	0	0	0	0	-4	4	20	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	20	0	0	0	0	0	0	0	0	0	-4	0	-4
0	0	0	0	0	0	0	0	0	20	0	0	0	0	0	0	0	0	0	0	-4
0	0	0	0	4	0	0	0	0	0	20	0	0	0	0	0	4	0	-4	0	0
0	0	0	0	0	0	4	0	0	0	0	20	0	0	0	0	0	0	4	0	0
0	0	0	0	-4	-4	0	0	0	0	0	0	20	0	0	0	0	0	4	0	-4
0	0	0	0	0	0	0	0	0	0	0	0	0	20	0	0	0	0	0	0	0
0	0	0	0	0	-4	0	0	0	0	0	0	0	0	20	0	-4	0	0	0	-4
0	0	0	0	0	-4	4	0	0	0	0	0	0	0	0	20	-4	0	4	0	0
4	0	-4	4	-4	0	0	0	0	0	4	0	0	0	-4	-4	20	0	4	-4	0
0	0	0	0	0	-4	0	0	0	0	0	0	0	0	0	0	0	20	4	0	-4
4	0	0	-4	4	4	0	0	-4	0	-4	4	4	0	0	4	4	4	20	0	0
0	0	0	0	-4	0	-4	0	0	0	0	0	0	0	0	0	-4	0	0	20	4
4	0	4	0	4	-4	4	0	-4	-4	0	0	-4	0	-4	0	0	-4	0	4	20

Table 23: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 22$, $m = 22$, and $E(s^2) = 4.00000$.

-1	1	1	-1	1	-1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	1	1	-1	1	1
-1	1	-1	-1	-1	-1	-1	1	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	1
1	1	-1	-1	-1	-1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	-1	-1	-1	1	-1
-1	-1	1	-1	1	1	-1	-1	-1	1	1	1	1	1	-1	-1	1	-1	-1	1	1	1
1	-1	1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	1	1
1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1	-1	-1	1	-1	1	1	-1	1	-1	-1
1	1	1	1	-1	1	-1	1	-1	-1	1	1	1	-1	1	1	1	-1	-1	-1	1	1
-1	-1	-1	1	1	-1	-1	1	-1	-1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	-1
1	1	-1	-1	-1	1	1	1	-1	1	1	-1	1	1	-1	1	-1	1	1	-1	-1	1
-1	-1	-1	-1	1	1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	1	-1	1	-1	1	1	1	-1	-1	-1	-1	-1	-1	1	-1	1	-1	-1	1
1	1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	1	1	1
-1	1	1	1	-1	-1	-1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	-1	1	1	-1	-1
1	1	1	-1	1	-1	1	1	1	-1	-1	1	1	1	1	-1	1	-1	1	1	-1	1
-1	-1	-1	1	-1	-1	1	-1	1	1	1	-1	1	1	1	1	1	1	-1	1	-1	1
1	-1	1	1	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	1	1	1	-1	1	-1
-1	1	-1	1	1	-1	1	1	-1	1	1	-1	-1	-1	1	1	1	-1	1	1	1	-1
1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	1	1	1	1	-1	-1
1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	1	1	1	1	-1	-1	1	1	1	-1
1	-1	1	1	1	-1	1	-1	-1	1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1
-1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	-1	1	1	1	1	1	1	-1	1	-1
-1	1	1	1	1	1	1	1	1	1	-1	1	-1	1	-1	1	-1	-1	-1	-1	-1	-1

Table 24: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 2$, and $f_{s_{\max}} = 231$.

22	2	2	-2	-2	2	2	2	-2	-2	-2	2	2	-2	-2	-2	2	2	2	2	2	2
2	22	-2	-2	2	-2	2	2	-2	-2	2	2	-2	2	-2	2	-2	-2	2	-2	2	2
2	-2	22	-2	2	-2	-2	2	-2	-2	-2	2	-2	2	2	-2	2	-2	2	-2	2	2
-2	-2	-2	22	2	-2	-2	2	-2	2	2	-2	-2	2	-2	2	-2	-2	2	2	-2	-2
-2	2	2	2	22	-2	2	-2	2	-2	2	2	2	-2	-2	-2	-2	2	2	2	-2	-2
2	-2	-2	-2	-2	22	-2	-2	-2	2	-2	2	-2	2	-2	2	2	-2	-2	-2	-2	-2
2	2	-2	-2	2	-2	22	-2	-2	2	-2	-2	-2	2	2	2	-2	-2	-2	-2	-2	2
2	2	2	2	-2	-2	-2	22	2	-2	-2	-2	-2	-2	-2	2	2	-2	-2	-2	-2	2
-2	-2	-2	-2	2	-2	-2	2	22	-2	-2	-2	2	2	2	2	-2	-2	2	-2	-2	2
-2	-2	-2	2	-2	2	2	-2	-2	22	-2	-2	2	-2	2	2	2	-2	2	-2	-2	2
-2	2	-2	2	2	-2	-2	-2	-2	-2	22	-2	2	2	2	-2	2	2	2	-2	-2	2
2	2	2	-2	2	2	-2	-2	-2	-2	-2	22	2	2	2	2	2	-2	-2	-2	2	-2
2	-2	-2	-2	2	-2	-2	-2	2	2	2	2	22	2	2	2	2	-2	-2	-2	2	2
-2	2	2	2	-2	2	2	-2	2	-2	2	2	2	22	-2	-2	-2	2	2	2	2	2
-2	-2	2	-2	-2	-2	2	-2	2	2	2	2	2	-2	22	2	2	2	2	-2	2	2
-2	2	-2	2	-2	2	2	2	2	2	-2	2	2	-2	2	22	-2	2	2	2	2	-2
2	-2	2	-2	-2	2	-2	2	-2	2	2	2	2	-2	2	-2	22	-2	2	2	2	-2
2	-2	-2	-2	2	-2	-2	-2	-2	-2	2	-2	-2	2	2	2	-2	22	2	-2	-2	2
2	2	2	2	2	-2	-2	-2	2	2	2	-2	-2	2	2	2	2	2	22	-2	-2	-2
2	-2	-2	2	2	-2	-2	-2	-2	-2	-2	-2	-2	2	-2	2	2	-2	-2	22	-2	-2
2	2	2	-2	-2	-2	-2	-2	-2	-2	-2	2	2	2	2	2	2	-2	-2	-2	22	2
2	2	2	-2	-2	-2	2	2	2	2	2	-2	2	2	2	-2	-2	2	-2	-2	2	22

Table 25: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 22$, $m = 23$, and $E(s^2) = 4.00000$.

-1	1	1	1	-1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	1	-1	1	-1	1	1
1	1	1	1	-1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	-1	1	-1	-1
-1	1	1	-1	1	-1	1	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	-1	1	1	1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	-1	1	1	-1	-1
-1	-1	1	-1	-1	1	1	1	1	1	1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	-1
-1	1	-1	1	1	1	1	-1	1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	1
1	1	-1	-1	-1	-1	1	1	-1	1	-1	1	-1	-1	1	1	1	1	1	1	-1	-1	1
1	1	1	1	-1	-1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	-1	-1	1	-1	1	-1
-1	-1	1	1	1	1	-1	-1	-1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	-1	-1	-1
1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	1	1	1	-1	-1	1	-1	1	-1
-1	1	-1	-1	-1	1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	-1
-1	1	-1	1	1	1	1	-1	-1	1	-1	-1	1	1	1	-1	1	1	1	1	1	1	-1
-1	-1	1	1	-1	-1	-1	-1	1	1	-1	1	-1	1	1	1	-1	1	1	-1	1	-1	-1
1	-1	1	-1	1	1	-1	-1	-1	1	1	1	-1	-1	1	-1	1	-1	1	-1	1	1	1
1	-1	1	-1	1	1	-1	1	1	1	-1	-1	1	1	-1	1	1	1	-1	1	1	-1	1
-1	-1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	1
-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	-1	-1	-1	1	1	1	-1	1
1	-1	-1	1	-1	1	1	1	-1	-1	1	1	1	1	-1	1	-1	-1	1	1	1	1	1
1	1	-1	-1	1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1	1	1	-1	-1	1	-1
1	-1	-1	1	1	1	1	1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	-1
1	1	-1	-1	-1	1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1
1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	1	-1	-1	-1	1	1	1	-1	-1	1	1

Table 26: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 2$, and $f_{s_{\max}} = 253$.

22	-2	-2	-2	-2	-2	2	2	-2	2	-2	2	2	2	2	-2	2	-2	-2	-2	-2	2	2
-2	22	-2	-2	-2	-2	2	2	-2	2	-2	2	2	2	2	-2	2	-2	-2	-2	-2	2	2
-2	-2	22	2	-2	-2	-2	-2	2	2	2	-2	-2	2	2	2	-2	2	2	-2	2	-2	2
-2	-2	2	22	-2	2	2	-2	-2	2	-2	2	-2	2	-2	-2	-2	2	-2	-2	-2	2	-2
-2	-2	-2	-2	22	2	2	-2	2	2	2	-2	-2	2	-2	-2	-2	2	-2	-2	2	2	2
-2	-2	-2	2	2	22	-2	2	2	-2	2	-2	2	2	2	2	2	-2	2	-2	2	-2	2
2	2	-2	2	2	-2	22	2	2	-2	2	-2	2	-2	2	2	2	-2	2	2	2	-2	2
2	2	-2	-2	-2	2	2	22	2	2	2	-2	-2	-2	-2	2	-2	2	-2	2	2	2	2
-2	-2	2	-2	2	2	2	2	22	-2	-2	2	-2	2	-2	2	2	2	2	-2	-2	-2	-2
2	2	2	2	2	-2	-2	2	-2	22	-2	2	2	-2	2	2	2	-2	2	2	-2	-2	-2
-2	-2	2	-2	2	2	2	2	-2	-2	22	2	2	2	-2	2	2	-2	-2	-2	-2	2	-2
2	2	-2	2	-2	-2	-2	-2	2	2	2	22	-2	-2	2	-2	-2	2	2	2	2	-2	2
2	2	-2	-2	-2	2	2	-2	-2	2	2	-2	22	-2	-2	2	-2	2	-2	2	-2	2	2
2	2	2	2	2	2	-2	-2	2	-2	2	-2	-2	22	-2	-2	-2	2	2	2	2	-2	-2
2	2	2	-2	-2	2	2	-2	-2	2	-2	2	-2	-2	22	2	-2	2	-2	2	-2	2	-2
-2	-2	2	-2	-2	2	2	2	2	2	2	-2	2	-2	2	22	-2	-2	2	2	-2	-2	-2
2	2	-2	-2	-2	2	2	-2	2	2	2	-2	-2	-2	-2	-2	22	2	-2	2	2	2	2
-2	-2	2	2	2	-2	-2	2	2	-2	-2	2	2	2	2	-2	2	22	2	-2	2	-2	-2
-2	-2	2	-2	-2	2	2	-2	2	2	-2	2	-2	2	-2	2	-2	2	22	-2	2	2	-2
-2	-2	-2	-2	-2	-2	2	2	-2	2	-2	2	2	2	2	2	2	-2	-2	22	-2	2	2
-2	-2	2	-2	2	2	2	2	-2	-2	-2	2	-2	2	-2	-2	2	2	2	-2	22	-2	-2
2	2	-2	2	2	-2	-2	2	-2	-2	2	-2	2	-2	2	-2	2	-2	2	2	-2	22	2
2	2	2	-2	2	2	2	2	-2	-2	-2	2	2	-2	-2	-2	2	-2	-2	2	-2	2	22

Table 27: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 24$, $m = 24$, and $E(s^2) = 2.08696$.

-1	1	1	1	1	-1	1	1	-1	1	-1	1	-1	1	1	1	-1	1	1	-1	1	1	1	1
1	1	-1	1	-1	-1	1	1	1	-1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	-1	1	-1	1
-1	-1	-1	-1	1	-1	1	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1
-1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	1	-1	-1	1	-1	-1
1	1	1	-1	-1	1	-1	1	-1	-1	-1	1	-1	-1	1	1	-1	1	-1	-1	-1	-1	-1	-1
1	-1	-1	1	-1	-1	-1	-1	1	-1	1	1	-1	1	-1	1	-1	1	1	1	1	-1	-1	1
-1	1	1	-1	-1	1	1	1	1	-1	1	-1	1	1	-1	-1	-1	1	1	-1	1	-1	1	-1
1	1	1	-1	-1	1	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	-1	1	1	1	1	1
1	-1	1	1	1	1	1	-1	-1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	-1	-1
1	1	1	-1	1	-1	-1	1	1	1	1	-1	-1	-1	1	-1	1	-1	1	1	1	-1	-1	1
-1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	-1	-1	1	1	1	-1	1	1	-1	-1	1	-1
-1	1	1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	1	1	-1	1	-1	1
-1	-1	-1	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	1
-1	1	-1	-1	-1	-1	1	-1	-1	1	-1	1	1	1	1	-1	1	1	-1	1	1	-1	-1	-1
-1	-1	1	1	1	1	-1	1	1	1	-1	1	1	1	-1	1	1	-1	-1	-1	1	-1	-1	-1
1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	-1
-1	-1	1	-1	-1	1	-1	-1	1	1	1	1	-1	1	1	-1	1	1	-1	-1	-1	1	1	1
-1	-1	-1	-1	-1	1	1	1	-1	-1	1	1	1	-1	1	1	1	-1	1	-1	1	1	-1	1
1	1	-1	-1	1	1	1	-1	-1	1	1	1	-1	1	-1	1	1	-1	1	-1	-1	-1	1	-1
-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	1	1	1	-1	-1	-1	1	-1	-1	1	1
1	-1	-1	1	1	1	1	-1	1	1	-1	-1	1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1
1	-1	1	1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1	1	-1	1	-1	-1	1	1
1	-1	-1	-1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	-1	1	1	-1	1	1	-1
1	1	-1	1	1	-1	-1	-1	1	-1	1	-1	1	-1	1	1	1	1	-1	-1	1	1	1	-1

positive integer q. Bound (2) was independently improved in [20, 21]. These improved lower bounds are equal when they both apply.

Let $\lfloor x \rfloor^+ = \max\{0, \lfloor x \rfloor\}$ and $\lceil x \rceil^+ = \max\{0, \lceil x \rceil\}$, where $\lfloor . \rfloor$ and $\lceil . \rceil$ are the floor and ceiling functions. The following theorem by [2] provides an improved version of the [20] lower bound.

Theorem 1 (see [2]). *Let m be a positive integer such that $m > N - 1$. Then, there exists a unique nonnegative integer q (which depends on N and m) such that $-2N + 2 < m - q(N - 1) < 2N - 2$ and $(m + q) \equiv 2 \pmod 4$. Define $g = (m + q)^2 N - q^2 N^2 - mN^2$.*

(a) If $N \equiv 0 \pmod 4$, then

$$E(s^2) \geq \begin{cases} \dfrac{g + 2N^2 - 4N}{m(m-1)} & \text{if } |m - q(N-1)| < N - 1, \\ \dfrac{g - 2N^2 + 4N + 4N|m - q(N-1)|}{m(m-1)} & \text{if } N - 1 < |m - q(N-1)| \leq \dfrac{3}{2}N - 2, \\ \dfrac{g + 4N^2 - 4N}{m(m-1)} & \text{if } |m - q(N-1)| > \dfrac{3}{2}N - 2 \end{cases} \tag{3}$$

Table 28: $\mathbf{D}^T\mathbf{D}$, $s_{max} = 4$, and $f_{s_{max}} = 36$.

24	0	-4	4	4	0	0	0	4	-4	0	0	-4	-4	0	-4	0	0	0	4	0	0	0	0
0	24	4	-4	-4	0	0	0	-4	4	0	0	4	4	0	4	0	0	0	-4	0	0	0	0
-4	4	24	0	0	4	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0
4	-4	0	24	0	-4	0	0	0	0	0	-4	0	0	0	0	0	0	0	0	0	0	0	0
4	-4	0	0	24	-4	0	0	0	0	0	-4	0	0	0	0	0	0	0	0	0	0	0	0
0	0	4	-4	-4	24	0	0	-4	4	0	0	4	4	0	4	0	0	0	-4	0	0	0	0
0	0	0	0	0	0	24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	-4	0	0	0	-4	0	0	24	0	0	-4	0	0	0	0	0	0	0	0	0	0	0	0
-4	4	0	0	0	4	0	0	0	24	0	4	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	24	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	4	-4	-4	0	0	0	-4	4	0	24	4	4	0	4	0	0	0	-4	0	0	0	0
-4	4	0	0	0	4	0	0	0	0	0	4	24	0	0	0	0	0	0	0	0	0	0	0
-4	4	0	0	0	4	0	0	0	0	0	4	0	24	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0	0	0	0	0	0	0	0	0
-4	4	0	0	0	4	0	0	0	0	0	4	0	0	0	24	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0	0	0	0	0
4	-4	0	0	0	-4	0	0	0	0	0	-4	0	0	0	0	0	0	0	24	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24

Table 29: An $E(s^2)$-optimal, minimax-optimal SSD of size $N = 24$, $m = 25$, and $E(s^2) = 3.84000$.

-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	1	-1	1	-1	-1	-1	-1	1	-1	-1	-1	-1	1
-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	1	-1	1	1	1	1	1	1	-1	-1	1	1
-1	1	1	1	1	-1	1	-1	-1	-1	-1	1	1	1	1	1	-1	1	1	1	-1	1	1	1	1
1	1	1	-1	1	1	-1	1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	-1	1	1	1	1	1
-1	-1	1	-1	-1	1	1	1	-1	1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	-1	-1
1	-1	1	-1	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	-1	1	1	1	1
1	-1	-1	1	-1	1	-1	-1	1	-1	1	1	1	-1	1	1	1	1	1	-1	1	1	1	-1	1
1	1	-1	1	-1	1	1	1	1	-1	-1	1	-1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	1
1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1
-1	1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	-1	-1	-1	1	-1	1	-1	-1	1	-1	1	-1
1	-1	1	1	1	-1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	1	1	-1	-1	-1	-1	1	-1
-1	1	1	1	-1	-1	1	1	1	-1	-1	1	-1	-1	1	-1	1	1	-1	1	1	1	-1	-1	-1
-1	1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1
1	1	1	1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1
-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	1	1	-1	1	1
-1	-1	-1	1	-1	1	-1	-1	-1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	-1	1	-1
-1	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1
1	-1	-1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	1	1	1	1	-1	-1	-1	1
1	-1	1	1	1	1	1	1	1	1	1	-1	1	1	-1	-1	-1	-1	1	1	1	1	1	-1	-1
1	-1	-1	1	-1	-1	-1	1	-1	1	-1	1	1	-1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
-1	1	-1	-1	1	-1	-1	1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	1	-1	1	1	-1
1	-1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	-1
1	1	-1	1	1	-1	-1	1	1	1	-1	-1	1	1	-1	1	1	1	-1	-1	-1	1	-1	-1	1
-1	1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	-1	1	-1	1	-1	-1	-1	1	-1	1	-1	1

(b) *If $N \equiv 2 \pmod 4$, then*

$$E(s^2) \geq \frac{4m(m-1) + 64\lceil m(m-1)(h-4)/64 \rceil^{+}}{m(m-1)}, \quad (4)$$

where for even q

$$h = \begin{cases} \dfrac{g + 2N^2 - 4N + 8}{m(m-1)} & \text{if } |m - q(N-1)| < N-1, \\ \dfrac{g - 2N^2 + 20N + (4N-8)|m - q(N-1)| - 24}{m(m-1)} & \text{if } N-1 < |m - q(N-1)| \leq \dfrac{3}{2}N - 3, \\ \dfrac{g + 4N^2 - 4N}{m(m-1)} & \text{if } |m - q(N-1)| > \dfrac{3}{2}N - 3, \end{cases} \quad (5)$$

and for odd q

$$h = \begin{cases} \dfrac{g + 2N^2 - 4N}{m(m-1)} & \text{if } |m - q(N-1)| < N-1, \\ \dfrac{g - 2N^2 + 4N + 4N|m - q(N-1)|}{m(m-1)} & \text{if } N-1 < |m - q(N-1)| \leq \dfrac{3}{2}N - 1, \\ \dfrac{g + 4N^2 - 12N + 8|m - q(N-1)| + 8}{m(m-1)} & \text{if } |m - q(N-1)| > \dfrac{3}{2}N - 1. \end{cases} \quad (6)$$

In this paper, we search for $E(s^2)$-optimal SSDs achieving the lower bound defined by (3), (4), (5), and (6) in Theorem 1 with $s_{\max} \leq 6$. The following theorem shows that $E(s^2)$-optimality is a sufficient condition for minimax optimality if $s_{\max} \leq 6$.

Theorem 2 (see [2]). *Let $\mathbf{D}(N, m)$ be an $E(s^2)$-optimal SSD.*

(a) *If $N \equiv 0 \pmod 4$ and $s_{\max} = 4$, then $\mathbf{D}(N, m)$ is minimax-optimal*

(b) *If $N \equiv 2 \pmod 4$ and $s_{\max} \in \{2, 6\}$, then $\mathbf{D}(N, m)$ is minimax-optimal*

3. The Equivalence between SSDs and RIBDs

An incomplete block design (IBD) with parameters (v, b, r, h), denoted by IBD (v, b, r, h), is a pair $(V, \mathscr{B})$ where V is a v-set of points and $\mathscr{B}$ is a collection of bh-subsets (*blocks*) of V, $h < v$. The parameters must satisfy the condition

$$vr = bh. \quad (7)$$

An IBD $(V, \mathscr{B})$ with parameters (v, b, r, h) satisfying (7) is called a *resolvable incomplete block design*, denoted by RIBD, if the collection $\mathscr{B}$ of blocks can be partitioned into r subsets called *parallel classes* of size $q = b/r$, each of which partitions the point set.

Henceforth, N will denote a positive even integer greater than or equal to 8. Let $Q = (B_1, B_2)$ and $Q' = (B_1', B_2')$ be two different parallel classes on N points. `1Define their *parallel class intersection matrix* (PCIM) as the 2×2 matrix $\mathbf{A}(Q, Q')$ with entries defined by $a_{ij} = |B_i \cap B_j'|$ (see [22]). An IBD $(N, 2m, m, N/2)$ has m parallel classes. Thus, for an arbitrary fixed parallel class Q, there are $m-1$ PCIMs of the form $\mathbf{A}(Q, Q')$. Since each point belongs to exactly one block of a parallel class, both the column and row sums in a PCIM are $N/2$. Then, by relabeling the blocks in parallel classes if necessary, each of the $m-1$ PCIMs associated with Q can be assumed to be one of

$$\mathscr{T}_{i,t} = \begin{bmatrix} \dfrac{N-i}{4} - t & \dfrac{N+i}{4} + t \\ \dfrac{N+i}{4} + t & \dfrac{N-i}{4} - t \end{bmatrix}, \ t = 0, \cdots, \frac{N-i}{4}, \text{ for } N \equiv i \pmod 4, \quad (8)$$

for $i \in \{0, 2\}$. For any 2×2 matrix $\mathbf{A} = [a_{ij}]$, let $S(\mathbf{A}) = |a_{11} - a_{12} - a_{21} + a_{22}|$; hence, $S(\mathscr{T}_{i,t}) = 4t + i$.

Theorem 3. *An RIBD with parameters $(N, 2m, m, N/2)$ such that for any two distinct parallel classes Q and Q', $S(\mathbf{A}(Q, Q')) \leq 4t + i$ exists if and only if a N row, m column, $\{-1, 1\}$-array $\mathbf{H}(N, m)$ with each column orthogonal to the all 1s column and $s_{\max} = 4t + i$, where $i \in \{0, 2\}$ exists.*

Table 30: $\mathbf{D}^T\mathbf{D}$, $s_{\max} = 4$, and $f_{s_{\max}} = 72$.

24	-4	-4	4	0	0	0	0	0	0	4	0	0	0	-4	-4	0	4	4	-4	-4	4	4	0	0
-4	24	0	4	4	0	0	0	0	-4	-4	0	-4	-4	0	0	-4	4	-4	-4	0	0	4	0	0
-4	0	24	0	-4	0	0	0	0	4	0	0	0	0	0	0	-4	0	0	0	0	0	0	0	0
4	4	0	24	-4	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0
0	4	-4	-4	24	0	0	0	4	0	4	-4	0	0	4	-4	0	4	4	0	-4	4	0	0	0
0	0	0	0	0	24	0	0	0	4	0	0	0	0	0	0	-4	0	0	0	0	0	0	0	0
0	0	0	0	0	0	24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	4	0	0	0	24	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0
0	-4	4	0	0	4	0	0	0	24	-4	-4	0	0	-4	4	0	-4	-4	0	4	-4	4	0	0
4	-4	0	0	4	0	0	0	0	-4	24	0	-4	-4	-4	4	4	0	0	0	4	0	0	0	0
0	0	0	0	-4	0	0	0	0	-4	0	24	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-4	0	0	0	0	0	0	0	0	-4	0	24	0	-4	0	0	4	0	0	0	0	0	0	0
0	-4	0	0	0	0	0	0	0	0	-4	0	0	24	-4	0	0	4	0	0	0	0	0	0	0
-4	0	0	4	4	0	0	0	0	-4	-4	0	-4	-4	24	0	-4	4	-4	-4	0	0	4	0	0
-4	0	0	0	-4	0	0	0	0	4	4	0	0	0	0	24	-4	4	0	0	0	0	0	0	0
0	-4	-4	0	0	-4	0	0	4	0	4	0	0	0	-4	-4	24	4	-4	-4	-4	4	0	0	0
4	4	0	0	4	0	0	0	0	-4	0	0	4	4	4	4	4	24	0	0	4	0	0	0	0
4	-4	0	0	4	0	0	0	0	-4	0	0	0	0	-4	0	-4	0	24	0	0	0	0	0	0
-4	-4	0	0	0	0	0	0	0	0	0	0	0	0	-4	0	-4	0	0	24	0	0	0	0	0
-4	0	0	0	-4	0	0	0	0	4	4	0	0	0	0	0	-4	4	0	0	24	0	0	0	0
4	0	0	0	4	0	0	0	0	-4	0	0	0	0	0	0	4	0	0	0	0	24	0	0	0
4	4	0	0	0	0	0	0	0	4	0	0	0	0	4	0	0	0	0	0	0	0	24	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24

Proof. Suppose that $(B_{1,1}, B_{1,2}), \cdots, (B_{m,1}, B_{m,2})$ are the parallel classes of the RIBD $(N, 2m, m, N/2)$. Then, for each parallel class $(B_{\ell,1}, B_{\ell,2})$ $(1 \le \ell \le m)$, we define the column vector $\mathbf{h}_\ell$ as follows:

$$h_{\ell,p} = \begin{cases} 1 & \text{if } p \in B_{\ell,1}, \\ -1 & \text{if } p \in B_{\ell,2}, \end{cases} \tag{9}$$

for each $1 \le p \le N$, where $h_{\ell,p}$ is the pth entry of $\mathbf{h}_\ell$. Note that the frequencies of +1 and −1 are the same in each column constructed from the RIBD. Hence, these m columns form a $\{-1, 1\}$-array $\mathbf{H}(N, m)$ with each column orthogonal to the all 1s column. For any two columns $\mathbf{h}_\ell$ and $\mathbf{h}_j$ of $\mathbf{H}(N, m)$ defined by the parallel classes Q_ℓ and Q_j, we have $|s_{\ell,j}| = S(\mathbf{A}(Q_\ell, Q_j)) \le 4t + i$ by (8) and (9). This implies that $s_{\max} = 4t + i$, where $i \in \{0, 2\}$.

Conversely, suppose that $\mathbf{H}(N, m)$ is a $\{-1, 1\}$ array with each column orthogonal to the all 1s column. For each column $\mathbf{h}_\ell$ $(1 \le \ell \le m)$ of the array $\mathbf{H}(N, m)$, there exist two blocks $B_{\ell,1}$ and $B_{\ell,2}$ that partition $V = \{1, 2, \cdots, N\}$, such that if the pth entry of $\mathbf{h}_\ell$ is 1, then p is contained in the block $B_{\ell,2}$; otherwise, p is contained in $B_{\ell,2}$. Clearly, these two blocks form a parallel class. Since the frequencies of +1 and −1 in each column are both $N/2$, each block has size $N/2$. Now, $s_{\max} = 4t + i$, where $i \in \{0, 2\}$ implies that $S(\mathbf{A}(Q_\ell, Q_j)) \le 4t + i$ for any two parallel classes Q_ℓ and Q_j defined by the ℓth and jth columns. □

Next, we provide an example for Theorem 3, where $(B_{1,1}, B_{1,2}), \cdots, (B_{8,1}, B_{8,2})$ is an RIBD (6,16,8, 3) corresponding to a 6 row, 8 column, $\{-1, 1\}$ array with each column orthogonal to the all 1s column and $s_{\max} = 6$.

Example 1.

$$\mathbf{H}(6, 8) = \begin{bmatrix} -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}. \tag{10}$$

$B_{1,1} = \{2, 5, 6\}$, $B_{1,2} = \{1, 3, 4\}$, $B_{2,1} = \{1, 4, 5\}$, $B_{2,2} = \{2, 3, 6\}$, $B_{3,1} = \{3, 5, 6\}$, $B_{3,2} = \{1, 2, 4\}$, $B_{4,1} = \{2, 4, 5\}$, $B_{4,2} = \{1, 3, 6\}$, $B_{5,1} = \{2, 4, 6\}$, $B_{5,2} = \{1, 3, 5\}$, $B_{6,1} = \{1, 2, 4\}$, $B_{6,2} = \{3, 5, 6\}$, $B_{7,1} = \{1, 5, 6\}$, $B_{7,2} = \{2, 3, 4\}$, $B_{8,1} = \{1, 2, 3\}$, and $B_{8,2} = \{4, 5, 6\}$.

4. The Optimization Problem

In this section, using the equivalence given in Theorem 3, we formulate the problem of constructing an $E(s^2)$-optimal and minimax-optimal SSD with N (even) rows, m columns, and $s_{max} = 4t + i$ for $(t, i) \in \{(0, 2), (1, 0), (1, 2)\}$ (i.e., $s_{\max} \in \{2, 4, 6\}$) as a discrete optimization problem. The following theorem is used to define the objective function for this optimization problem.

Theorem 4. *Let* **D** *be an SSD with N rows and m columns and* $(B_{1,1}, B_{1,2}), \cdots, (B_{m,1}, B_{m,2})$ *be the parallel classes of the RIBD defined by the columns of* **D** *according to Theorem 3. Let* $1 \le h,\ p \le 2$, *and* ℓ=j; *then, the following hold:*

(a) $|s_{\ell,j}| = |4|B_{\ell,p} \cap B_{j,h}| - N|$

(b) $E(s^2) = (1/\binom{m}{2}) \sum_{\ell<j} (4|B_{\ell,h} \cap B_{j,p}| - N)^2$

(c) $s_{\max} = 4t + i \Leftrightarrow ((N - i)/4) - t \le |B_{\ell,h} \cap B_{j,p}| \le ((N + i)/4) + t$, *where* $i \in \{0, 2\}$

Proof. By the proof of Theorem 3 and (8), we have $|B_{\ell,p} \cap B_{j,h}| = ((N - i)/4) - t$ (or $((N + i)/4) + t$) for some $1 \le t \le (N - i)/4$. Then, $|4|B_{\ell,p} \cap B_{j,h}| - N| = |\pm(i + 4t)| = 4t + i = |s_{\ell,j}|$. Statements (b) and (c) follow from (a). □

By Theorem 3, a feasible solution to our optimization problem is a RIBD with parameters $(N, 2m, m, N/2)$. However, since each parallel class of the RIBD is uniquely determined by one of its blocks, a feasible solution reduces to a set $\mathscr{B}$ of m blocks, $B_1, \cdots, B_m$ each of size $N/2$. Then, based on Theorem 4, we define the objective function as

$$g(\mathscr{B}) = \sum_{\ell<j} \frac{\left(4|B_\ell \cap B_j| - N\right)^2}{\binom{m}{2}}. \tag{11}$$

Computing the intersections of blocks is the bottleneck in computing $g(\mathscr{B})$. We used the bit array data structure to store each set (block) as in [11]. This allowed speeding up the calculation of intersections of blocks by using bit-parallel Boolean instructions. To determine the number of points of intersection of two blocks, we used the SSE4.2 SIMD instruction, _mm_popcnt_u64, included in the recent general-purpose processors. It counts the number of bits set to 1 in a word of 64 bits. Thus, in the C language, $|B_\ell \cap B_j|$ is calculated by _mm_popcnt_u64(BB_ℓ &BB_j), where BB_h is the binary representation of the block B_h with $N \le 64$. These instructions increase the speed by a factor of min $(N, 64) = N$.

To construct $E(s^2)$-optimal and minimax-optimal SSDs based on Theorem 2, it is necessary to require that $s_{\max} = 4t + i$ for $(t, i) \in \{(0, 2), (1, 0), (1, 2)\}$ (i.e., $s_{\max} \in \{2, 4, 6\}$). For this purpose, we define

$$w(\ell, j) = \begin{cases} 1 & \text{if } \dfrac{N - i}{4} - 1 \le |B_\ell \cap B_j| \le \dfrac{N + i}{4} + 1, \\ b(N, m) & \text{otherwise,} \end{cases} \tag{12}$$

for $1 \le \ell < j \le m$, where $b(N, m)$ is the lower bound given in Theorem 1. Then, we modify the objective function (11) to

$$f(\mathscr{B}) = \sum_{\ell<j} w(\ell, j) \frac{\left(4|B_\ell \cap B_j| - N\right)^2}{\binom{m}{2}}. \tag{13}$$

It follows from Theorem 4 (c) and (12) that if the objective function (13) reaches the value $b(N, m)$, then we have $s_{\max} = 4t + i$ for $(t, i) \in \{(0, 2), (1, 0), (1, 2)\}$. Hence, Theorems 2, 3, and 4 (a) and (b) imply that an $E(s^2)$-optimal and minimax-optimal SSD with N (even) rows, m columns, and $s_{\max} = 4t + i$ for $(t, i) \in \{(0, 2), (1, 0), (1, 2)\}$ is found whenever the objective function (13) achieves $b(N, m)$ for a RIBD.

5. Tabu Search for SSDs

The TS algorithm introduced by [10] is an iterative metaheuristic technique used to search for a solution that minimizes an objective function f over a set of feasible solutions X. TS has been used successfully to construct D-optimal designs, constant weight codes, 1-rotational resolvable balanced incomplete block designs, covering designs, and $E(s^2)$-optimal and minimax-optimal k-circulant SSDs (see [7, 23–26]).

TS is based on a neighborhood search (NS). In NS, each feasible solution x has an associated set of neighbors, $N(x) \subset X$, called the *neighborhood* of x. It starts with a given initial feasible solution and searches the set X by moving from one solution to another in its neighborhood. At each iteration, a move from the current solution x to a best one x' in $N(x)$ regardless of whether $f(x') \le f(x)$ is made. If more than one solution has the same minimum value, the tie is broken randomly. However, the main shortcoming of NS is cycling through a set of solutions, i.e., keeping on revisiting the same set of solutions. To prevent cycling, TS maintains a list called the *tabu list* T of *length* $|T| = M$. Each move in T is removed after M iterations.

Sometimes, the tabu list may forbid certain desirable moves, such as those that lead to a better solution than the best one found so far. An *aspiration criterion* asp$F$ is introduced to cancel the tabu status of a move when this move is judged useful.

TS stops when the objective function reaches the lower bound $b(N, m)$. However, there is no guarantee of reaching the lower bound, and the search process is stopped if the number of iterations used without improving the best solution exceeds a preset nitmax limit.

Two IBDs with parameters $(N, 2m, m, N/2)$ are defined as *neighbors* if they are identical for every parallel class but one, and in that parallel class, there are exactly two points that switch blocks. A swap move is entirely determined by the vector (ℓ, u, w), where points u and w are switched in the parallel class ℓ. The definitions of the neighborhood and the objective function in Section 4 allow calculating the change in the objective function (13) value without recomputing the objective function (13). The only blocks that change after the move (ℓ, u, w) are $B_{\ell,1}$ and $B_{\ell,2}$.

Whenever the points u and w switch blocks in the parallel class ℓ, the tabu list forbids the exchange of the points u and w at the parallel class ℓ in the subsequent M iterations. Formally, the tabu list consists of vectors (ℓ, u, w), where the points u and w were forbidden to be exchanged during the preceding M iterations, in the parallel class ℓ. The tabu list length M was adjusted experimentally. For the problem instances in this paper, the best M seems to be some integer between 6 and 8. The pseudocode of our TS algorithm is presented in Algorithm 1.

In the above described algorithm, the computation time is mainly spent on iterations. Hence, we next provide the complexity analysis of each iteration. Let $\mathscr{B} = \{B_1, \cdots, B_\ell, B_{\ell+1}, \cdots, B_m\}$ be an RIBD (a feasible solution to our optimization problem). In Algorithm 1, for each block B_ℓ that changes to $B_\ell{}'$, there are $m-1$ blocks, $\{B_1, \cdots, B_{\ell-1}, B_{\ell+1}, \cdots, B_m\}$ that do not change. Let $\mathscr{B}' = \mathscr{B} - \{B_\ell\} \cup \{B'_\ell\}$. Then, the objective function (13) is updated according to

$$f\left(\mathscr{B}'\right) = f(\mathscr{B}) + \frac{1}{\binom{m}{2}} \sum_{j=1, j\neq \ell}^{m} w(\ell, j) \cdot \left[\left(4\left|B'_\ell \cap B_j\right| - N\right)^2 - \left(4\left|B_\ell \cap B_j\right| - N\right)^2\right]. \quad (14)$$

Since the intersection of two blocks is performed in $\lceil N/64 \rceil$ bitwise operations, the complexity to update the objective function after a move is $\mathcal{O}(mN)$. The only blocks that change after the move (ℓ, u, w) are $B_{\ell,1}$ and $B_{\ell,2}$ $(1 \leq \ell \leq m)$, and there are $(N/2)^2$ possible changes to $(B_{\ell,1}, B_{\ell,2})$. Hence, the size of the neighborhood of any RIBD is $m(N/2)^2$. Then, the overall time spent for each iteration of this algorithm is

$$m \times \frac{N^2}{4} \times (m-1) \times \left\lceil \frac{N}{64} \right\rceil. \quad (15)$$

Let $I(m, N)$ denote the expected number of iterations of the algorithm for the m column and N row case. Then, for $N \leq 64$, the expected time complexity for each run of this algorithm is $I(m, N) \times \mathcal{O}(m^2N^2)$. For the most difficult cases, the algorithm was run for 4,000,000 times. So, the overall expected running time of the algorithm was $4{,}000{,}000 \times I(m, N) \times \mathcal{O}(m^2N^2)$. If we had not used bit-parallelism, then the expected time complexity for each run would have been $I(m, N) \times \mathcal{O}(m^2N^3)$.

6. Implementation Details

The TS algorithm described above was programmed in C, and all computations were carried out on a 2.67 GHz or 2.4 GHz processor. The source code of the algorithm can be requested by sending an email to the first author.

The TS algorithm was used to construct fifteen $E(s^2)$-optimal and minimax-optimal SSDs achieving the $E(s^2)$ lower bound of [2] with $s_{max} \leq 6$ for $N = 16,18,20,22,24$ and $N \leq m \leq 30$. The existence question for each of these SSDs was previously unknown.

Initial computational experiments showed that the strategy of running the algorithm a larger number of times with a smaller nitmaxwas better than running the algorithm a smaller number of times with a larger nitmax. For example, for $N = 18$ and $m = 23$, 8,000 runs of the algorithm with nitmax = 200 found 3 optimum solutions, whereas only 1 optimum solution was found by 2,000 runs with nitmax = 800. Then, the TS procedure was carried out at most 80,000 times using nitmax = 300 on each instance tested. With these values, the TS algorithm did not produce any optimum solutions for $(N, m) = (16, 27), (18, 24), (18, 29), (24, 25)$. However, the best SSDs found for these cases had $s_{max} = 4t + i$ for $(t, i) \in \{(1, 0), (1, 2)\}$ and an $E(s^2)$ value equal to $b(N, m) + 32/(m(m-1))$ and $b(N, m) + 64/(m(m-1))$, where $f_{s_{max}}$ is larger by one than the $f_{s_{max}}$ of a hypothetical $E(s^2)$-optimal and minimax-optimal SSD achieving the $E(s^2)$ lower bound of [2] with $s_{max} \leq 6$. Then, for these cases, the TS algorithm was carried out at most 4,000,000 times with nitmax = 450. Since the TS algorithm runs are completely independent and no information is exchanged, an independent-thread parallelization strategy was used. Different random number seed values were used to avoid an overlapping search.

The bit-parallel TS was able to construct fifteen previously unknown $E(s^2)$-optimal and minimax-optimal SSDs. Table 31 lists all cases where the best obtained $E(s^2)$-optimal SSD is minimax-optimal. Each row of this table corresponds to an SSD. The first column shows the number of rows and columns for the SSDs. Column nRUNs gives the number of runs it took the TS algorithm to find an $E(s^2)$-optimal and minimax-optimal SSD. The last column gives the CPU time. The $N = 16$ row, $m = 27$ column case was solved in 585.4 CPU hours. However, with the independent-thread parallelization approach, this search took only 16.1 hours, using a cluster with 40 threads.

In Table 31, we do not observe that the CPU time always increases with the number of columns or rows. There are also big increases on the CPU times, just by the addition of one column. There are two reasons for these observations. Firstly, the geometry of the problem can change as the number of columns of the sought after SSD increases. In particular, let $\rho(N, m)$ and $\rho'(N, m)$ be the ratio of the number of all N row, m column, $E(s^2)$-optimal and minimax-optimal SSDs to the number of all and all locally optimum N row, m column SSDs. It is possible that for fixed N, $\rho(N, m)$ and/or $\rho'(N, m)$ is not a nonincreasing function of m. This is mainly because not every $E(s^2)$-optimal and minimax-optimal, $m+1$ column, N row

```
Input N, m, nitmax, b(N, m), M.
Generate an initial RIBD (N, 2m, m, N/2) (solution) 𝓑_0 randomly;
Set 𝓑_best := 𝓑_0, T := ∅, r := rbest := 1, fbest := aspF := f(𝓑_best);
while (r − rbest ≤ nitmax & fbest > b(N, m)) do
  Set min = ∞;
  for 𝓑′ ∈ N(𝓑_0) do
    Set s := move from 𝓑_0 to 𝓑′;
    if (f(𝓑′) ≤ min & (s ∉ T or f(𝓑′) < aspF)) then
      if (f(𝓑′) == min) then
          Set 𝓑′′ := 𝓑′ with 50% probability;
       else if
          Set 𝓑′′ := 𝓑′, min = f(𝓑′);
       end if
     end if
   end for
   if (min < fbest) then
     Update 𝓑_best := 𝓑′′;
     Set fbest := aspF := min, rbest := r;
   end if
   Update T := T ∪ {move from 𝓑′′ to 𝓑_0};
   Update 𝓑_0 := 𝓑′′;
   if (|T| > M) then
     remove oldest move from T
   end if
   Set r := r + 1;
end while
Output 𝓑_best, fbest.
```

Algorithm 1: The TS algorithm.

Table 31: $E(s^2)$-optimal and minimax-optimal SSDs obtained by the TS algorithm.

N	m	nRUNs	CPU time
16	25	95,989	211.5 minutes
	26	190,049	1.1 days
	27	3,954,798	24.4 days
18	23	1,232	3.5 minutes
	24	880,559	2.7 days
	25	40,058	241.6 minutes
	26	12,516	74.2 minutes
	27	26,689	174 minutes
	28	14,243	96.9 minutes
	29	392,285	3.1 days
20	21	19	0.1 minutes
22	22	6	0.05 minutes
	23	7	1.0 minutes
24	24	509	4.0 minutes
	25	112,976	1.6 days

SSD can be obtained by adding a column to an $E(s^2)$-optimal and minimax-optimal, m column, N row SSD. Then, the probability of finding an $E(s^2)$-optimal and minimax-optimal SSD in one iteration of the TS algorithm may actually increase going from m columns to $m+1$ columns. Secondly, TS is not a deterministic algorithm and the CPU times are random with potentially large variances. Large variances may easily blur an increasing pattern. In fact, this is more of an issue for the previously unsolved difficult cases.

The only other algorithm that is competitive with the TS algorithm is the NOA$_p$ algorithm. For the NOA$_p$ algorithm, each new random starting SSD is independently picked from the previous random starting SSDs. So, each trial of the NOA$_p$ algorithm with a new random starting SSD can be thought as a Bernoulli trial with a success probability of p of finding an SSD that achieves the best known $E(s^2)$ and minimax lower bounds. Then, we can use $Y =$ the number of trials before finding an SSD that achieves the $E(s^2)$ and minimax lower bounds in [2] as a surrogate for the time it takes to find an SSD that achieves the best known $E(s^2)$ and minimax lower bounds. Now, $Y \sim \text{Geometric}(p)$ where

$$\begin{aligned} E(Y) &= \frac{1-p}{p}, \\ \operatorname{Var}(Y) &= \frac{1-p}{p^2}. \end{aligned} \tag{16}$$

For the most difficult cases, p is very small, making Var (Y) very large. So, it is possible to get very lucky and find a solution very quickly or be very unlucky and not be able to find a solution after a very long time. Hence, for the most difficult cases, we do not gain as much information by comparing CPU times as in the case of exact algorithms. When TS is used, we do not know the distribution of the random variable Y as the independence of trials is no longer a valid assumption. Finding an empirical distribution for Y would require repeating the computational experiments in the paper a large number of times. This is neither feasible due to resource and time constraints nor worthed as all cases solved by the TS algorithm have no corresponding CPU times based on the NOA_p algorithm (or any other algorithm). No corresponding CPU times exist because the NO A_p algorithm failed to solve them despite being run for a very long time.

7. Concluding Remarks

In this paper, we developed a heuristic algorithm for finding $E(s^2)$-optimal and minimax-optimal SSDs that is more effective than the previously known most effective algorithm for the same purpose. Our algorithm brings fifteen cases of $E(s^2)$-optimal and minimax-optimal SSDs within computational reach by taking advantage of the equivalence between an SSD and an RIBD described in the proof of Theorem 3 and bit-parallelism from [11].

Data Availability

All the data is in the paper.

Disclosure

The views expressed in this article are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the US Government. This paper is published on Arxiv [27].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors thank the High Performance Computing Lab at IIMAS-UNAM for providing computing resources.

References

[1] K. H. V. Booth and D. R. Cox, "Some systematic supersaturated designs," *Technometrics*, vol. 4, no. 4, pp. 489–495, 1962.

[2] K. J. Ryan and D. A. Bulutoglu, "$E(s^2)$-optimal supersaturated designs with good minimax properties," *Journal of Statistical Planning and Inference*, vol. 137, no. 7, pp. 2250–2262, 2007.

[3] N.-K. Nguyen, "An algorithmic approach to constructing supersaturated designs," *Technometrics*, vol. 38, no. 1, pp. 69–73, 1996.

[4] N. A. Butler, "Minimax 16-run supersaturated designs," *Statistics & Probability Letters*, vol. 73, no. 2, pp. 139–145, 2005.

[5] C. Koukouvinos, K. Mylona, and D. E. Simos, "$E(s^2)$-optimal and minimax-optimal cyclic supersaturated designs via multi-objective simulated annealing," *Journal of Statistical Planning and Inference*, vol. 138, no. 6, pp. 1639–1646, 2008.

[6] C. Koukouvinos, K. Mylona, and D. E. Simos, "An algorithmic construction of $E(s^2)$-optimal supersaturated designs," *Journal of Statistical Theory and Practice*, vol. 5, no. 2, pp. 357–367, 2011.

[7] S. Gupta and L. B. Morales, "Constructing $E(s^2)$-optimal and minimax-optimal k-circulant supersaturated designs via multi-objective tabu search," *Journal of Statistical Planning and Inference*, vol. 142, no. 6, pp. 1415–1420, 2012.

[8] L. B. Morales and G. Vega, "On the enumeration of $E(s^2)$ -optimal and minimax-optimal k-circulant supersaturated designs," *Journal of Combinatorial Designs*, vol. 22, no. 4, pp. 149–160, 2014.

[9] S. Georgiou, "Supersaturated designs: a review of their construction and analysis," *Journal of Statistical Planning and Inference*, vol. 144, pp. 92–109, 2014.

[10] F. Glover, "Tabu Search—Part I," *ORSA Journal on Computing*, vol. 1, no. 3, pp. 190–206, 1989.

[11] P. S. Segundo, D. R. Losada, and A. Jiménez, "An exact bit-parallel algorithm for the maximum clique problem," *Computers & Operations Research*, vol. 38, no. 2, pp. 571–581, 2011.

[12] L. B. Morales, D. A. Bulutoglu, and K. T. Arasu, "The maximum number of columns in supersaturated designs with $s_{\max} = 2$," *Journal of Combinatorial Designs*, vol. 27, no. 7, pp. 448–472, 2019.

[13] N.-K. Nguyen and C. S. Cheng, "New $E(s^2)$-optimal supersaturated designs constructed from incomplete block designs," *Technometrics*, vol. 50, no. 1, pp. 26–31, 2008.

[14] D. R. Holcomb, D. C. Montgomery, and W. M. Carlyle, "The use of supersaturated experiments in turbine engine development," *Quality Engineering*, vol. 19, no. 1, pp. 17–27, 2007.

[15] C. F. J. Wu, "Construction of supersaturated designs through partially aliased interactions," *Biometrika*, vol. 80, no. 3, pp. 661–669, 1993.

[16] R. Rodil, E. Mart'ınez, A. M. Carro, R. A. Lorenzo, and R. Cela, "Applying supersaturated experimental designs to the study of composite sampling for monitoring pesticide residues in water," *LCGC North America*, vol. 22, pp. 272–286, 2004.

[17] C. F. J. Wu and M. Hamada, *Experiments Planning, Analysis, and Parameter Design Optimization*, Wiley, New York, NY, USA, 2000.

[18] K. R. Williams, "Designed experiments," *Rubber Age*, vol. 100, pp. 65–71, 1968.

[19] B. Tang and C. F. J. Wu, "A method for constructing supersaturated designs and $E(s^2)$ optimality," *Canadian Journal of Statistics*, vol. 25, no. 2, pp. 191–201, 1997.

[20] D. A. Bulutoglu and C. S. Cheng, "Construction of $E(s^2)$ -optimal supersaturated designs," *The Annals of Statistics*, vol. 32, no. 4, pp. 1662–1678, 2004.

[21] N. A. Butler, R. Mead, K. M. Eskridge, and S. G. Gilmour, "A general method of constructing $E(s^2)$-optimal supersaturated designs," *Journal of the Royal Statistical Society Series B: Statistical Methodology*, vol. 63, no. 3, pp. 621–632, 2001.

[22] L. B. Morales and C. Velarde, "Enumeration of resolvable 2-(10, 5, 16) and 3-(10, 5, 6) designs," *Journal of Combinatorial Designs*, vol. 13, no. 2, pp. 108–119, 2005.

[23] C. Dai, B. Li, and M. Toulouse, "A multilevel cooperative tabu search algorithm for the covering design problem," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 68, pp. 33–65, 2009.

[24] S. J. Joo and J. Y. Bong, "Construction of exact D-optimal designs by tabu search," *Computational Statistics and Data Analysis*, vol. 21, no. 2, pp. 181–191, 1996.

[25] L. B. Morales, "Two new 1-rotational (36, 9, 8) and (40, 10, 9) RBIBDs," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 36, pp. 119–126, 2001.

[26] K. J. Nurmela, M. K. Kaikkonen, and P. R. J. Östergård, "New constant weight codes from linear permutation groups," *IEEE Transactions on Information Theory*, vol. 43, no. 5, pp. 1623–1630, 1997.

[27] L. B. Morales and D. A. Bulutoglu, "A bit-parallel tabu search algorithm for finding $E(s^2)$-optimal and minimax-optimal supersaturated designs," Arxiv, 2023, https://arxiv.org/abs/2303.09104.