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Research Article

A New Fractional Representation of the Higher Order Taylor Scheme

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In this work, we suggest a new numerical scheme called the fractional higher order Taylor method (FHOTM) to solve fractional differential equations (FDEs). Using the generalized Taylor's theorem is the fundamental concept of this approach. Then, the local truncation error generated by the suggested FHOTM is estimated by proving suitable theoretical results. At last, several numerical applications are given to demonstrate the applicability of the suggested approach in relation to their exact solutions.

Keywords: Caputo fractional derivative operator; generalized Taylor's theorem; higher order Taylor method; local truncation error

1. Introduction

Fractional calculus is regarded as a powerful tool of mathematical analysis and shows the integration or differentiation in fractional order [1, 2]. Numerous researchers have expressed interest in this field, with some focusing on the analytical side of solving fractional differential equations (FDEs), which involves uniqueness, existence, stability, and other aspects; for example, refer to the study works [3–5]. Furthermore, a significant amount of research has been done with an applied focus since many fractional problems are challenging to solve analytically.

Nowadays, the majority of studies address this area since it is less expensive to generate a numerical approximation for a given nonlinear fractional problem than to get an analytical estimate. In this context, a variety of numerical methods have been suggested and applied to solve FDEs (see, e.g., [6–11]). Among these methods, we cite the following: homotopy perturbation and the matrix approach methods [12, 13], Adomian decomposition method [14], neural networks [15], fractional difference method [16], fractional Euler method (FEM) [17], modified fractional Euler method (MFEM) [18], and others (see [19–22]).

The applied method that caught our interest and that we used in our research is that of Taylor, which is an algorithm utilized to estimate the solution of the classical IVP given as follows:

$$\begin{cases} \phi'(t) = g(t, \phi(t)), \quad t \in [t_0, T] \\ \phi(t_0) = \phi_0, \end{cases} \tag{1}$$

where g is a continuous function. Subsequently, numerous authors have investigated Taylor series methods and created mathematical methods for solving fractional differential problems. These authors include those listed in [23–27] and the references therein.

For the fractional case, this method is witnessing remarkable development by researchers. For example, in [28], the authors created a novel method for nonlinear fractional partial differential equations: a combination of DTM and generalized Taylor's formula. Then, in [29], Huang et al. presented an approximate solution of fractional integro-differential equations by the Taylor expansion method. The author in [30] developed the general fractional Euler and Taylor methods, and those methods are applied to different FDEs of first order. In general, various numerical methods are improved to provide better performance in accordance with particular needs. In this context, this paper presents a new approach for giving an approximate numerical solution for FDEs. It is called the fractional higher order Taylor method (FHOTM). One characteristic of the suggested algorithm is that it improves the FEM in terms of efficiency and accuracy, meaning that this approach's accuracy increases with increasing order.

The present work is organized as follows: The fundamental concepts of our research are introduced in Section 2, along with some definitions and characteristics of fractional calculus theory. The fractional initial value problem (FIVP) is solved by the FHOTM, which is established in Section 3. Next, we determine a relevant theoretical result and use it to analyze and estimate the local truncation error produced by the FHOTM. Several numerical applications demonstrating the accuracy of the suggested method are provided in Section 4. For completeness, a brief summary of the study's findings is provided at the conclusion.

2. Some Preliminary Results

The purpose of this section is to recall the following key result.

Definition 1 (see [31]). The fractional Riemann-Liouville integrator for a continuous function ϕ on $[0, b]$ is defined by

$$J^{\beta}\phi(t) = \begin{cases} \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1}\phi(s)ds, & \beta > 0, 0 < t \le b \\ \phi(t), & \beta = 0, 0 < t \le b \end{cases}$$

where $\Gamma(\beta) \coloneqq \int_0^{+\infty} e^{-t} t^{\beta-1} dt$.

Note that for $\alpha, \beta \ge 0$, we have

$$J^{\beta} t^{\delta} = \frac{\Gamma(\delta+1) t^{\beta+\delta}}{\Gamma(\beta+\delta+1)} \delta > -1 \quad (2)$$

$$J^{\alpha} J^{\beta}\phi(t) = J^{\alpha+\beta}\phi(t) = J^{\beta} J^{\alpha}\phi(t). \quad (3)$$

Definition 2 (see [31]). Let $\phi \in C^m([0, b], \mathbb{R})$, $m \in \mathbb{N}^*$, and $m-1 < \beta \le m$; the Caputo fractional derivative is given by

$$D^{\beta}\phi(t) = J^{m-\beta} D^m \phi(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t (t-s)^{m-\beta-1} \phi^{(m)}(s) ds.$$

The Caputo fractional differentiator satisfies the following properties [31]:

$$D^{\beta}\mathscr{C} = 0, \quad \text{where} \quad \mathscr{C} \in \mathbb{R} \quad (4)$$

$$D^{\beta} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta+1)} t^{\gamma-\beta}, \quad \text{where} \quad \gamma > \beta - 1 \quad (5)$$

$$D^{\beta}(\lambda g(t) + \omega h(t)) = \lambda D^{\beta}(g(t)) + \omega D^{\beta}(h(t)) \quad (6)$$

where $\lambda, \omega \in \mathbb{R}$. Additionally, we present below some other properties that are connected to the combination of the preceding two operators [31]:

$$D^{\beta} J^{\beta}\phi(t) = \phi(t) \quad (7)$$

$$J^{\beta} D^{\beta}\phi(t) = \phi(t) - \sum_{k=1}^{m-1} \phi^k(0^+) \frac{t^k}{k!} \quad (8)$$

where $t > 0$ and $m - 1 < \beta \le m$ such that $m \in \mathbb{N}^*$.

Theorem 1 (generalized Taylor's formula) [17]. *Suppose that $D^{j\beta}\phi(t)$ is a continuous function on $(0, b]$, for $j = 0, 1, 2, \cdots, m+1$, where $0 < \beta \le 1$. Then, we can extend the function ϕ for the node t_0 in the subsequent way:*

$$\phi(t) = \sum_{i=0}^{m} \frac{(t-t_0)^{i\beta}}{\Gamma(i\beta+1)} D^{i\beta}\phi(t_0) + \frac{(t-t_0)^{(m+1)\beta}}{\Gamma((m+1)\beta+1)} D^{(m+1)\beta}\phi(\zeta) \quad (9)$$

with $0 < \zeta \le t$, for all $t \in (0, b]$.

To provide more clarity, we can formulate the above expression of ϕ by

$$\begin{aligned} \phi(t) = \phi(t_0) &+ \frac{(t-t_0)^{\beta}}{\Gamma(\beta+1)} D^{\beta}\phi(t_0) \\ &+ \frac{(t-t_0)^{2\beta}}{\Gamma(2\beta+1)} D^{2\beta}\phi(t_0) + \cdots + \frac{(t-t_0)^{m\beta}}{\Gamma(m\beta+1)} D^{m\beta}\phi(t_0) \\ &+ \frac{(t-t_0)^{(m+1)\beta}}{\Gamma((m+1)\beta+1)} D^{(m+1)\beta}\phi(\zeta). \end{aligned} \quad (10)$$

3. Results and Discussion

Our objective in this part is to propose a FHOTM. The generalized Taylor's method will be used to accomplish this. Furthermore, we will present a theoretical result concerning the estimation of the local truncation error of our scheme-developed approach.

3.1. Taylor's Higher Order Fractional Approach. In regard to the Caputo fractional derivative, this technique can be used to find an approximate solution for FIVPs. The form of such a problem is

$$D^{\beta}\phi(t) = g(t, \phi(t)) \tag{11}$$

dependent on the initial condition

$$\phi(0) = \phi_0 \tag{12}$$

where $t \in I = [0, b]$, $0 < \beta \leq 1$, D^{β} denotes the differential operator in the sense of Caputo, and $g : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. In order to solve problems (11) and (12), we divide $[0, b]$ into discrete parts as $0 = t_0 < t_1 = t_0 + h < \cdots < t_m = t_0 + mh = b$, where $h = b/m$ is the step size corresponds to the mesh points $t_i = t_0 + ih$, $i = 1, \cdots, m$. Now, for $j = 0, \cdots, m + 1$ and $m \in \mathbb{N}$, we assume that $\phi(t)$ and $D^{j\beta}\phi(t)$ fulfill the conditions of the generalized Taylor's theorem provided in Theorem 1. Then, by extending the solution ϕ with regard to Theorem 1, where $t_0 = t_i$, we obtain

$$\begin{aligned}\phi(t) = \phi(t_i) &+ \frac{D^{\beta}\phi(t_i)}{\Gamma(\beta+1)}(t - t_i)^{\beta} + \frac{D^{2\beta}\phi(t_i)}{\Gamma(2\beta+1)}(t - t_i)^{2\beta} \\ &+ \cdots + \frac{D^{m\beta}\phi(t_i)}{\Gamma(m\beta+1)}(t - t_i)^{m\beta} \\ &+ \frac{D^{(m+1)\beta}\phi(\zeta)}{\Gamma((m+1)\beta+1)}(t - t_i)^{(m+1)\beta}.\end{aligned}$$

After that, we change t to t_{i+1} in the previous equality to get the following equation:

$$\begin{aligned}\phi(t_{i+1}) = \phi(t_i) &+ \frac{h^{\beta}}{\Gamma(\beta+1)}D^{\beta}\phi(t_i) \\ &+ \frac{h^{2\beta}}{\Gamma(2\beta+1)}D^{2\beta}\phi(t_i) + \cdots + \frac{h^{m\beta}}{\Gamma(m\beta+1)}D^{m\beta}\phi(t_i) \\ &+ \frac{h^{(m+1)\beta}}{\Gamma((m+1)\beta+1)}D^{(m+1)\beta}\phi(\zeta)\end{aligned} \tag{13}$$

where $\zeta \in (t_i, t_{i+1})$. According to

$$\begin{aligned}D^{\beta}\phi(t) &= g(t, \phi(t)) \\ D^{2\beta}\phi(t) &= D^{\beta}g(t, \phi(t)) \\ &\vdots \\ D^{m\beta}\phi(t) &= D^{(m-1)\beta}g(t, \phi(t)),\end{aligned}$$

then the above expression (13) becomes of the following manner:

$$\begin{aligned}\phi(t_{i+1}) = \phi(t_i) &+ \left[\frac{h^{\beta}}{\Gamma(\beta+1)}g(t, \phi(t)) + \frac{h^{2\beta}}{\Gamma(2\beta+1)}D^{\beta}g(t, \phi(t)) \right. \\ &\left. + \cdots + \frac{h^{m\beta}}{\Gamma(m\beta+1)}D^{(m-1)\beta}g(t, \phi(t))\right] \\ &+ \frac{h^{(m+1)\beta}}{\Gamma((m+1)\beta+1)}D^{m\beta}g(\zeta, \phi(\zeta)).\end{aligned} \tag{14}$$

Indeed, Formula (14) can be approximately defined in the form that follows:

$$\begin{aligned}\varkappa_0 &= \phi_0 \\ \varkappa_{i+1} &\approx \varkappa_i + h^{\beta}\mathcal{T}(t_i, \varkappa_i)\end{aligned} \tag{15}$$

where

$$\begin{aligned}\mathcal{T}(t_i, \varkappa_i) = &\frac{1}{\Gamma(\beta+1)}g(t_i, \varkappa_i) + \frac{h^{\beta}}{\Gamma(2\beta+1)}D^{\beta}g(t_i, \varkappa_i) \\ &+ \cdots + \frac{h^{(m-1)\beta}}{\Gamma(m\beta+1)}D^{(m-1)\beta}g(t_i, \varkappa_i)\end{aligned} \tag{16}$$

for $i = 0, 1, \cdots, m - 1$, such that w_i represents the approximate values of the exact solution y at t_i. However, the steps of solutions of problem (11) can be given in the form of Algorithm 1.

3.2. Estimation of Local Truncation Error. The object of this section is to estimate the local truncation error from the suggested method developed in Equation (15). For this purpose, we present the following theorem.

Theorem 2. *Let us assume that the FIVP (11) and (12) can be approximated using the proposed FHOTM with a step size h. We additionally suppose that $D^{j\beta}\phi(t) \in C[0, b]$, for $j = 0, 1, 2, \cdots, m + 1$, $m \in \mathbb{N}$, where $0 < \beta \leq 1$. Then, in this case, $\mathcal{O}(h^{m\beta})$ is the local truncation error.*

Proof 1. Equation (14) gives us the ability to derive the following:

$$\begin{aligned}\phi(t_{i+1}) - \phi(t_i) &- \frac{h^{\beta}}{\Gamma(\beta+1)}g(t, \phi(t)) - \frac{h^{2\beta}}{\Gamma(2\beta+1)}D^{\beta}g(t, \phi(t)) \\ &- \cdots - \frac{h^{m\beta}}{\Gamma(m\beta+1)}D^{(m-1)\beta}g(t, \phi(t)) \\ &= \frac{h^{(m+1)\beta}}{\Gamma((m+1)\beta+1)}D^{m\beta}g(\zeta, \phi(\zeta))\end{aligned}$$

for $\zeta \in (t_i, t_{i+1})$. This consequently gives

$$\phi(t_{i+1}) - \phi(t_i) - h^{\beta}\mathcal{T}(t_i, \phi(t_i)) = \frac{h^{(m+1)\beta}}{\Gamma((m+1)\beta+1)}D^{m\beta}g(\zeta, \phi(\zeta))$$

```
Start
Define function g(t, y)
Read initial values of (t_0 and y_0), the value of (β), number of steps (n) and calculation point (t_n)
Calculate step size h = (t_n − t_0)/n
Set i = 0
Loop
wprime = g(t(i), y(i))/Γ(β + 1) + (h^β D^β g(t(i), y(i)))/Γ(2β + 1)
y_{n+1} = y_n + h^β * wprime
i = i + 1
 while i < n do
     Display y_n as result
 end while
 Stop
```

Algorithm 1: 2β-FHOTM.

where

$$\mathcal{T}(t_i, \phi(t_i)) = \frac{1}{\Gamma(\beta+1)} g(t_i, \phi(t_i)) + \frac{h^\beta}{\Gamma(2\beta+1)} D^\beta g(t_i, \phi(t_i)) + \cdots + \frac{h^{(m-1)\beta}}{\Gamma(m\beta+1)} D^{(m-1)\beta} g(t_i, \phi(t_i))$$

for $i = 0, 1, 2, \cdots, m-1$. Consequently, we can define the local truncation error at step $i+1$ in the following form:

$$E_{i+1}^{\mathcal{T}}(h) = \frac{\phi(t_{i+1}) - \phi(t_i)}{h^\beta} - \mathcal{T}(t_i, \phi(t_i))$$

such that

$$E_{i+1}^{\mathcal{T}}(h) = \frac{h^{m\beta}}{\Gamma((m+1)\beta+1)} D^{m\beta} g(\zeta, \phi(\zeta)).$$

As a result of $D^{j\beta}\phi(t) \in C[0, b]$, for $j = 0, 1, 2, \cdots, m+1$, then we get

$$D^{(m+1)\beta}\phi(t) = D^{m\beta} g(t, \phi(t))$$

which has bounds on the interval $[0, b]$. Therefore, we have

$$E_{i+1}^{\mathcal{T}}(h) = \mathcal{O}\left(h^{m\beta}\right).$$

□

4. Illustrative Applications

In this part, we explore two different examples of FDEs to validate our novel approach. We present the results obtained from each example using comparative figures and tables.

Example 1. Let as assess the following problem:

$$\begin{cases} D^\beta \phi(\mathbf{t}) = \phi(\mathbf{t}) - \mathbf{t}, & \mathbf{t} \in [0, 1]. \\ \phi(0) = 0.5. \end{cases} \tag{17}$$

At this point, it is necessary to point out that the exact solution to the given problem (17) when $\beta = 1$ is presented by

$$\phi(\mathbf{t}) = \mathbf{t} + 1 - \frac{1}{2} e^{\mathbf{t}}.$$

In this case, we put $m = 10$, and therefore, $h = 0.1$. Thus, we assume that

$$g(\mathbf{t}, \phi(\mathbf{t})) = \phi(\mathbf{t}) - \mathbf{t}.$$

Furthermore, let us suppose that we desire to apply on the FHOTM of order 2β. In order to achieve this, we calculate

$$D^\beta g(\mathbf{t}, \phi(\mathbf{t})) = \phi(\mathbf{t}) - \mathbf{t} - \frac{1}{\Gamma(2-\beta)} \mathbf{t}^{1-\beta}.$$

Consequently, by using Equation (16), $\mathcal{T}(\mathbf{t}_i, \varkappa_i)$ can be found as follows:

$$\mathcal{T}(\mathbf{t}_i, \varkappa_i) = \frac{1}{\Gamma(\beta+1)}(\varkappa_i - \mathbf{t}_i) + \frac{h^\beta}{\Gamma(2\beta+1)}\left(\varkappa_i - \mathbf{t}_i - \frac{1}{\Gamma(2-\beta)} \mathbf{t}_i^{1-\beta}\right) \tag{18}$$

where $\varkappa_1$ indicate estimates for $\phi(\mathbf{t}_i)$, such that $i = 0, 1, 2, \cdots, 9$. Thanks to $\mathbf{t}_i = 0.1i$, Equation (18) can be reformulated as

$$\mathcal{T}(\mathbf{t}_i, \varkappa_i) = \frac{1}{\Gamma(\beta+1)}(\varkappa_i - 0.1i) + \frac{(0.1)^\beta}{\Gamma(2\beta+1)}\left(\varkappa_i - 0.1i - \frac{1}{\Gamma(2-\beta)}(0.1i)^{1-\beta}\right). \tag{19}$$

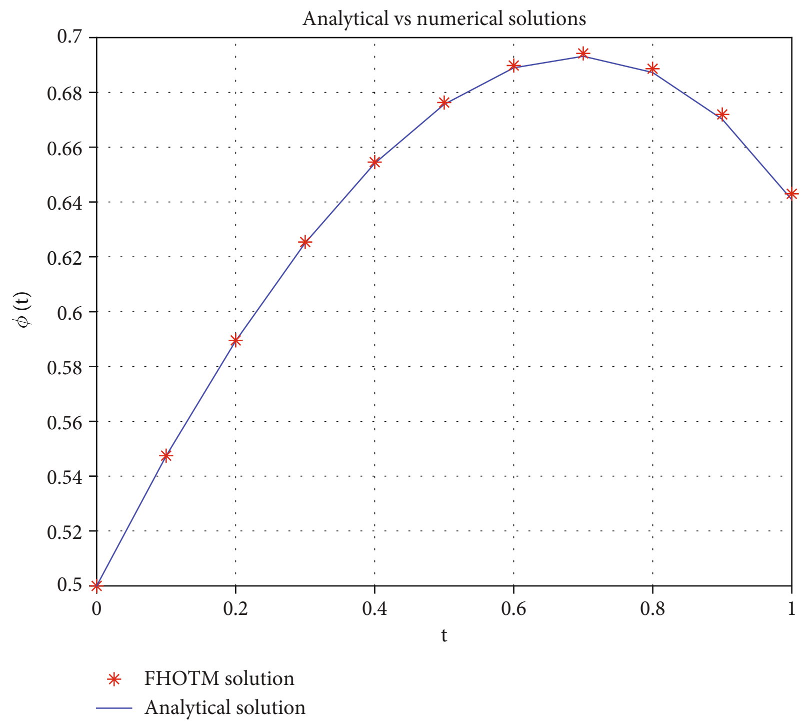

FIGURE 1: A graphical comparison of Example 1: an approximate solution of order 2β for $\beta = 1$ vs. the analytical solution.

Thus, the 2β order of the FHOTM can be expressed, based on Equation (19), as follows:

$$\begin{aligned} \varkappa_0 &= 0.5 \\ \varkappa_{i+1} &= \varkappa_i + (0.1)^\beta \left[\frac{1}{\Gamma(\beta+1)} (\varkappa_i - 0.1i) \right. \\ &\quad \left. + \frac{(0.1)^\beta}{\Gamma(2\beta+1)} \left(\varkappa_i - 0.1i - \frac{1}{\Gamma(2-\beta)} (0.1i)^{1-\beta} \right) \right], \end{aligned} \tag{20}$$

for $i = 0, 1, 2, \cdots, 9$. We have the capability to simulate an approximate solution of problem (17) for the FHOTM of 2β order in regard to Equation (20), which is shown in Figures 1 and 2, respectively, when $\beta = 1$ and for various values of β.

To analyze the absolute error values of the numerical solution produced by the suggested FHOTM of order 2β, we plot Figure 3.

Example 2. The next problem studied corresponds to the following FIVP:

$$\begin{cases} D^\beta \phi(\mathbf{t}) = \phi(\mathbf{t}) - \mathbf{t}^2 + 1, & \mathbf{t} \in [0, 2] \\ \phi(0) = 0.5. \end{cases} \tag{21}$$

where $0 \le \beta \le 1$. We notice that the problem defined in Equation (21) with $\beta = 1$ has the following exact solution:

$$\phi(\mathbf{t}) = (\mathbf{t}+1)^2 - \frac{1}{2} e^{\mathbf{t}}.$$

Now, we choose $m = 10$ in order to apply the suggested FHOTM, and thus, $h = 0.2$. In the next step, we are looking to apply the FHOTM of order 2β and 3β, respectively. In order to achieve this, we suppose that

$$g(\mathbf{t}, \phi(\mathbf{t})) = \phi(\mathbf{t}) - \mathbf{t}^2 + 1.$$

This therefore indicates

$$D^\beta g(\mathbf{t}, \phi(\mathbf{t})) = D^\beta \phi(\mathbf{t}) - \frac{2}{\Gamma(3-\beta)} \mathbf{t}^{2-\beta}.$$

Thus,

$$D^\beta g(t, \phi(\mathbf{t})) = \phi(\mathbf{t}) - \mathbf{t}^2 + 1 - \frac{2}{\Gamma(3-\beta)} \mathbf{t}^{2-\beta}.$$

Analogously, we can obtain

$$D^{2\beta} g(\mathbf{t}, \phi(\mathbf{t})) = D^\beta \phi(\mathbf{t}) - \frac{2}{\Gamma(3-\beta)} \mathbf{t}^{2-\beta} - \frac{2}{\Gamma(3-2\beta)} \mathbf{t}^{2-2\beta}$$

which implies that

$$D^{2\beta} g(\mathbf{t}, \phi(\mathbf{t})) = \phi(\mathbf{t}) - \mathbf{t}^2 + 1 - \frac{2}{\Gamma(3-\beta)} \mathbf{t}^{2-\beta} - \frac{2}{\Gamma(3-2\beta)} \mathbf{t}^{2-2\beta}.$$

Applying Equation (16) allows us to express $\mathcal{T}(\mathbf{t}_i, \varkappa_i)$ in the following way:

$$\begin{aligned} \mathcal{T}(\mathbf{t}_i, \varkappa_i) &= \frac{1}{\Gamma(\beta+1)} (\varkappa_i - \mathbf{t}_i^2 + 1) \\ &\quad + \frac{h^\beta}{\Gamma(2\beta+1)} \left(\varkappa_i - \mathbf{t}_i^2 + 1 - \frac{2}{\Gamma(3-\beta)} \mathbf{t}_i^{2-\beta} \right) + \frac{h^{2\beta}}{\Gamma(3\beta+1)} \\ &\quad \cdot \left(\varkappa_i - \mathbf{t}_i^2 + 1 - \frac{2}{\Gamma(3-\beta)} \mathbf{t}_i^{2-\beta} - \frac{2}{\Gamma(3-2\beta)} \mathbf{t}_i^{2-2\beta} \right). \end{aligned} \tag{22}$$

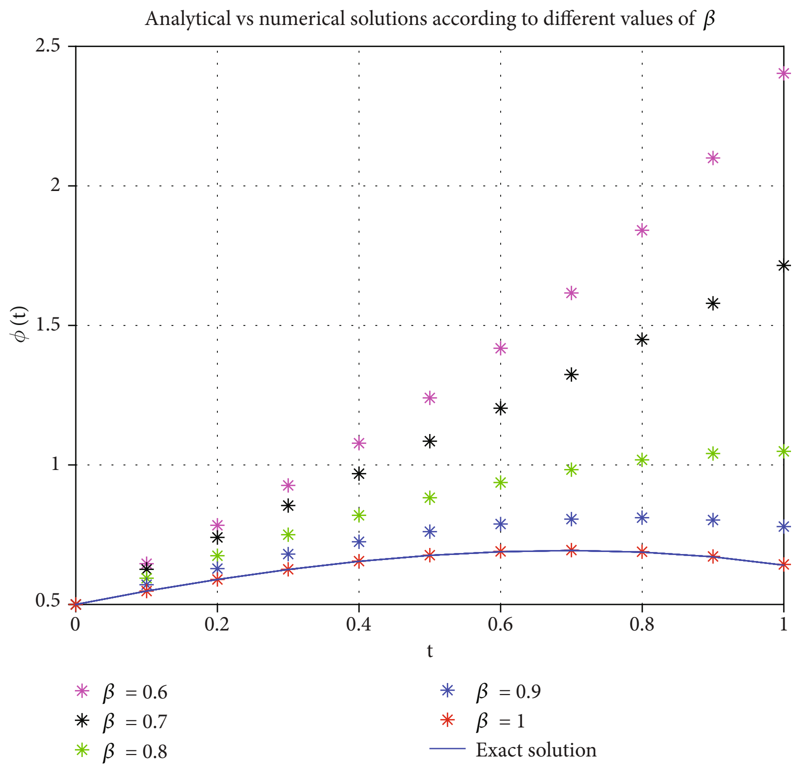

FIGURE 2: Comparisons between the exact solution of Example 1 and the numerical solution of ϕ according to different values of β.

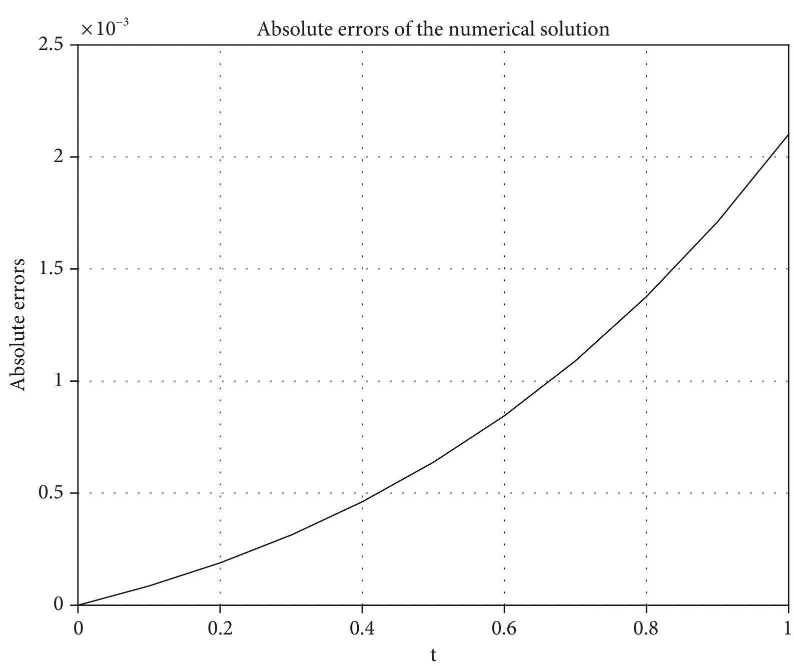

FIGURE 3: A graphical representation of Example 1 for the absolute errors of the numerical solutions of order 2β for $\beta = 1$.

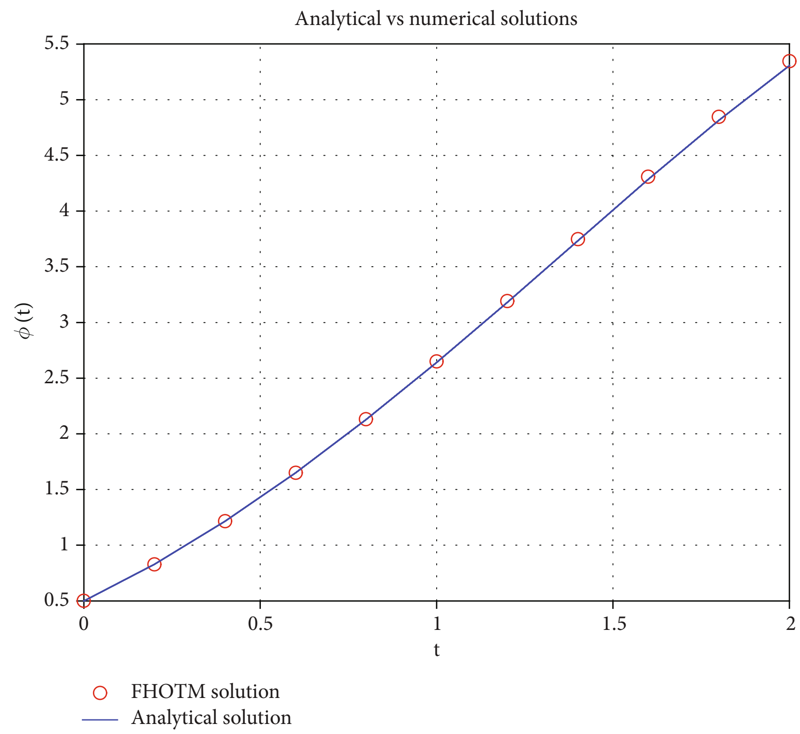

Figure 4: A graphical comparison of Example 2: an approximate solution of order 2β for $\beta = 1$ and the analytical solution.

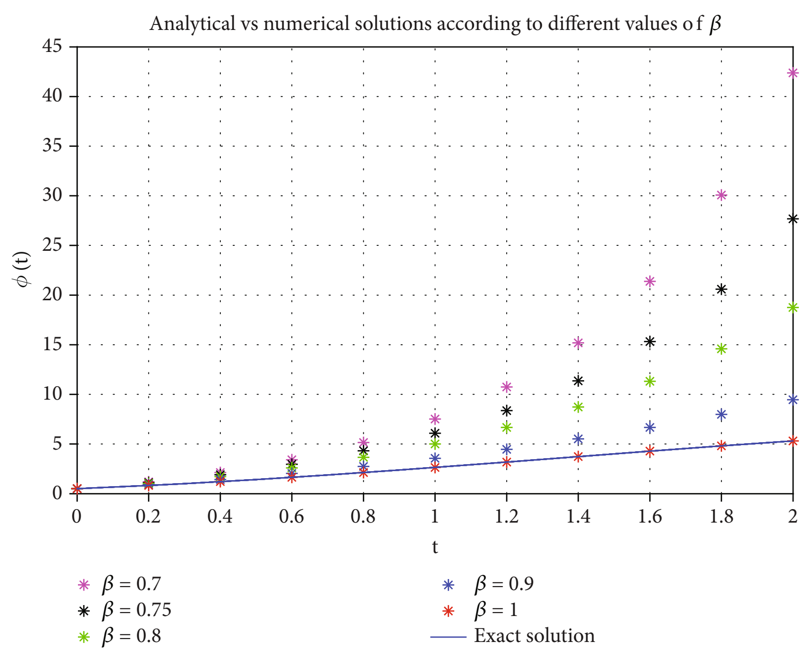

Figure 5: Comparisons between the exact solution of Example 2 and the numerical solution of ϕ according to different values of β.

For $i = 0, 1, 2, \cdots, 9$, Equation (22) can be reformulated, since $t_i = 0.2i$, as follows:

$$\begin{aligned}\mathcal{T}(t_i, \varkappa_i) = {} & \frac{1}{\Gamma(\beta+1)}\left(\varkappa_i - 0.04i^2 + 1\right) + \frac{(0.2)^{\beta}}{\Gamma(2\beta+1)} \\ & \cdot \left(\varkappa_i - 0.04i^2 + 1 - \frac{2}{\Gamma(3-\beta)}(0.2i)^{2-\beta}\right) + \frac{(0.2)^{2\beta}}{\Gamma(3\beta+1)} \\ & \cdot \left(\varkappa_i - 0.04i^2 + 1 - \frac{2}{\Gamma(3-\beta)}(0.2i)^{2-\beta} - \frac{2}{\Gamma(3-2\beta)}(0.2i)^{2-2\beta}\right).\end{aligned} \tag{23}$$

Thus, with the main expression (19), the FHOTM of order 2β and 3β is presented for $i = 0, 1, 2, \cdots, 9$ as

$$\begin{aligned}\varkappa_0 &= 0.5 \\ \varkappa_{i+1} &= \varkappa_i + (0.2)^{\beta}\left[\frac{1}{\Gamma(\beta+1)}\left(\varkappa_i - 0.04i^2 + 1\right) \right. \\ &\quad \left. + \frac{(0.2)^{\beta}}{\Gamma(2\beta+1)}\left(\varkappa_i - 0.04i^2 + 1 - \frac{2}{\Gamma(3-\beta)}(0.2i)^{2-\beta}\right)\right]\end{aligned} \tag{24}$$

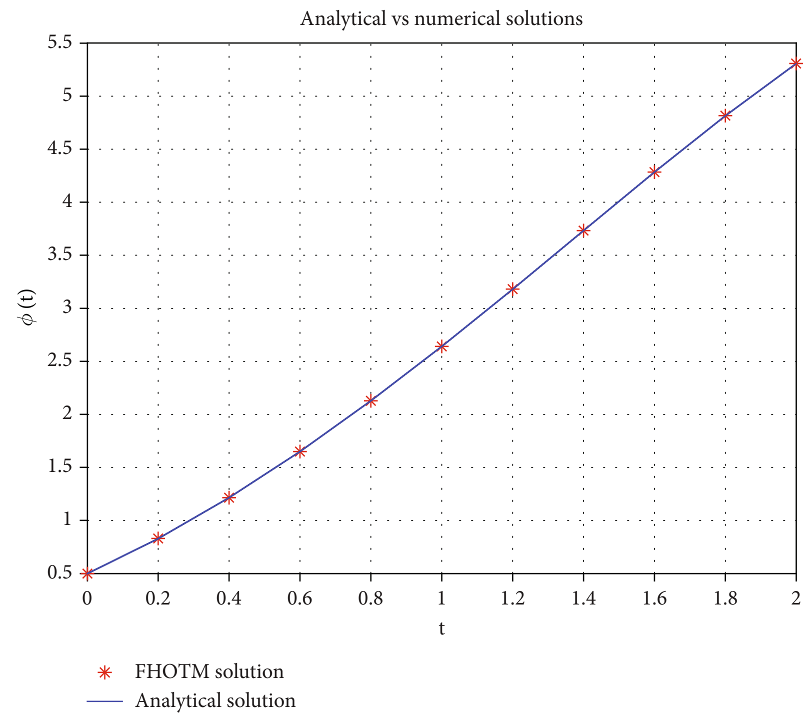

Figure 6: A graphical comparison of Example 2: an approximate solution of order 3β for $\beta = 1$ and the analytical solution.

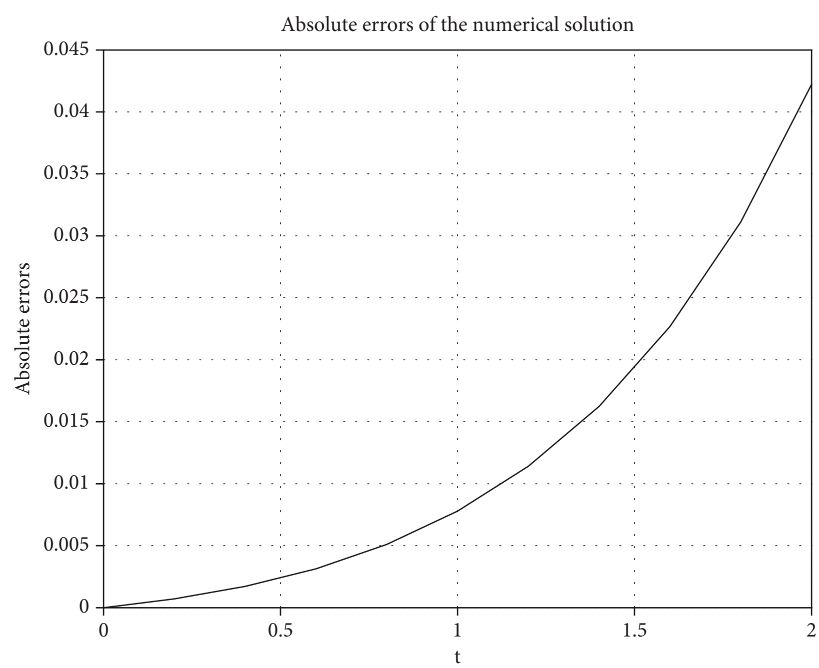

Figure 7: A graphical representation of Example 2 for the absolute errors of the numerical solutions of order 2β for $\beta = 1$.

$$
\begin{aligned}
&\varkappa_0 = 0.5 \\
&\varkappa_{i+1} = \varkappa_i + (0.2)^{\beta}\left[\frac{1}{\Gamma(\beta+1)}\left(\varkappa_i - 0.04i^2 + 1\right) + \frac{(0.2)^{\beta}}{\Gamma(2\beta+1)}\right. \\
&\quad \cdot \left(\varkappa_i - 0.04i^2 + 1 - \frac{2}{\Gamma(3-\beta)}(0.2i)^{2-\beta}\right) + \frac{(0.2)^{2\beta}}{\Gamma(3\beta+1)} \\
&\quad \left.\cdot \left(\varkappa_i - 0.04i^2 + 1 - \frac{2}{\Gamma(3-\beta)}(0.2i)^{2-\beta} - \frac{2}{\Gamma(3-2\beta)}(0.2i)^{2-2\beta}\right)\right].
\end{aligned}
\tag{25}
$$

Similarly, by utilizing Equations (24) and (25), we illustrate two approximated solutions for problem (21). Figures 4 and 5 show a numerical solution for $\beta = 1$ and for different values of β, which was achieved by applying the FHOTM of order 2β. Next, using the FHOTM with an order of 3β, the second approximation for the problem mentioned in (21) is obtained. Figure 6 illustrates this for $\beta = 1$.

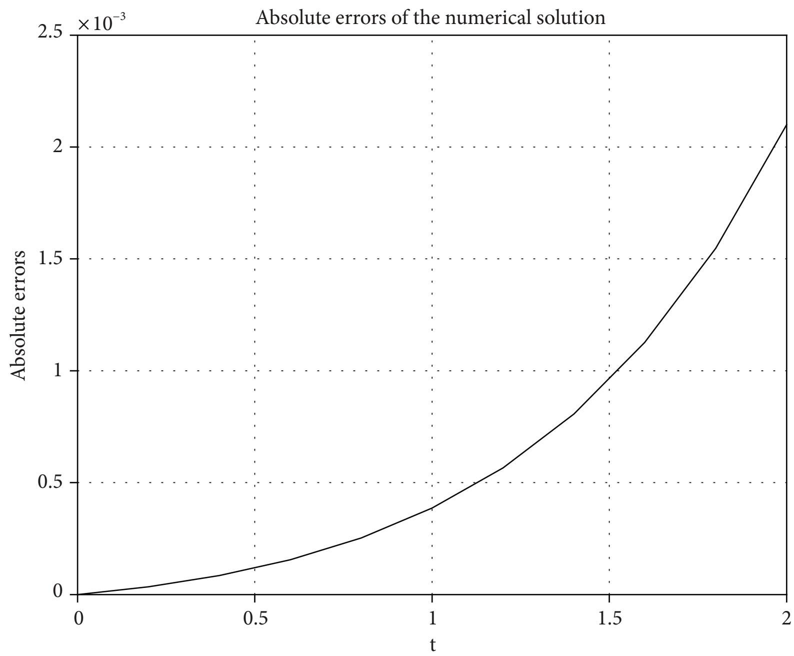

FIGURE 8: A graphical representation of Example 2 for the absolute errors of the numerical solutions of order 3β for $\beta = 1$.

TABLE 1: A comparison between the absolute errors of the numerical solutions for 2β and 3β order.

t	Absolute error for 2β	Absolute error for 3β
0	0	0
0.1	1×10^{-3}	1×10^{-4}
0.2	2×10^{-3}	1.4×10^{-4}
0.3	3×10^{-3}	1.8×10^{-4}
0.4	4×10^{-3}	2.2×10^{-4}
0.5	4.5×10^{-3}	2.6×10^{-4}
0.6	4.7×10^{-3}	3×10^{-4}
0.7	4.9×10^{-3}	3.4×10^{-4}
0.8	5×10^{-3}	3.8×10^{-4}
0.9	6×10^{-3}	4.2×10^{-4}
1	8×10^{-3}	4.6×10^{-4}

The absolute values of the errors between the exact solution for the case of $\beta = 1$ and the approximate solutions computed by FHOTM with the orders of 2β and 3β are displayed in Figures 7 and 8 and Table 1, respectively.

The accuracy of the suggested method is readily apparent in regard to previous numerical results, and it gets better as the method's order increases.

5. Conclusion

Applications such as the ones listed above have shown that the proposed method produces a good approximation of the FIVP solution, particularly when contrasted with the precise solution. This insight shows that there are numerous scenarios in which the formula can be applied. In particular, the FHOTM may have the following characteristics:

- The beneficial result of the FHOTM encourages the application of the formula in various real-world situations.
- The FHOTM indicates that the approach is practical and has the potential to progress a number of fields where fractional calculus is crucial.
- It has been concluded that the FHOTM can generate more degrees of freedom by considering various fractional order values. This means that we may take any value of fractional order into account instead of adhering to only one value that represents only the integer-order case.

Data Availability Statement

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

Conceptualization: Iqbal M. Batiha. Methodology: Iqbal H. Jebril and Amira Abdelnebi. Software: Zoubir Dahmani and Shawkat Alkhazaleh. Validation: Nidal Anakira. Formal analysis: Iqbal M. Batiha and Nidal Anakira. Investigation: Iqbal H. Jebril. Data curation: Amira Abdelnebi and Shawkat Alkhazaleh. Supervision: Zoubir Dahmani and Nidal Anakira. All authors have read and agreed to the published version of the manuscript.

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