Hindawi
Computational and Mathematical Methods
Volume 2024, Article ID 7326616, 19 pages
https://doi.org/10.1155/2024/7326616

Research Article

Redefined Quintic B-Spline Collocation Method to Solve the Time-Fractional Whitham-Broer-Kaup Equations

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Received 20 December 2023; Revised 6 February 2024; Accepted 6 March 2024; Published 5 April 2024

Academic Editor: Qichun Zhang

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This article proposes a collocation approach based on a redefined quintic B-spline basis for solving the time-fractional Whitham-Broer-Kaup equations. The presented method involves discretizing the time-fractional derivatives using an L_1-approximation scheme and then approximating the spatial derivatives using the redefined quintic B-spline basis. The von Neumann technique has been used to demonstrate that the proposed method is unconditionally stable. The error estimates are discussed and show that the proposed method is third-order convergent. The results demonstrate the potential of the proposed method as a reliable tool for solving fractional differential equations.

1. Introduction

Fractional calculus has grown significantly in relevance in recent years. The fractional derivatives and integrals have been used in numerous applications in the fields of science and engineering, including but not limited to fluid mechanics, chemical physics, electricity, control theory, epidemic diseases, biomedicine, signal processing, and issues with heat conduction and diffusion [1–5]. There are several definitions of fractional-order derivatives, each with a variety of uses [6–10].

The Whitham-Broer-Kaup (WBK) equations are a set of coupled nonlinear partial differential equations that describe the propagation of shallow water waves in a channel. Its fractional counterpart describes shallow water in a porous medium, which can absorb wave energy and prevent tsunamis. Several analytical and numerical methods have been developed to solve the WBK equations [11–14]. Wang and Chen [15] applied an analytic iterative technique called the residual power series method to solve time-fractional WBK equations. Wang et al. [14] proposed the generalized exponential rational function method to elucidate the basic solution properties of the WBK equation. Wang et al. [16] used the generalized projective Riccati equation method to solve the classical WBK equations. Yasmin [17] used the Yang decomposition method for fractional-order nonlinear WBK equations. Sadat and Kassem [18] used Lie point symmetries for the fractional Riemann-Liouville system to reduce fractional WBK equations to nonlinear fractional ordinary differential equations using the prolongation theorem. Wang and Li [19] provided a streamlined homogeneous balance technique to investigate the shallow water small-amplitude WBK model equations. Nonlaopon et al. [20] used the Laplace homotopy perturbation transform technique to solve the fractional-order WBK equations. Cao et al. [13] used the conformal fractional derivative to transform the nonlinear space-time fraction WBK equation into an ordinary differential equation and then used the complete polynomial discriminant system to find the exact solutions. Ali et al. [21] applied the Laplace Adomian decomposition technique to obtain an approximate solution of the nonlinear coupled system of WBK equations of time-space fractional order. Shah et al. [22] used the q-homotopy analysis transform method and the natural decomposition method to solve time-fractional WBK equations. Our paper focuses on the homogeneous and nonhomogeneous fractional WBK equations that take the following form [23]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial v(x,t)}{\partial x} + q\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (1)$$

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial v(x,t)}{\partial x} + v(x,t)\frac{\partial u(x,t)}{\partial x} + p\frac{\partial^3 u(x,t)}{\partial x^3} - q\frac{\partial^2 v(x,t)}{\partial x^2} = g(x,t), x \in [a,b], t \in [0,T], \quad (2)$$

subject to the initial conditions

$$u(x,0) = \eta_1(x), \quad v(x,0) = \eta_2(x), \quad x \in [a,b], \quad (3)$$

and boundary conditions

$$\begin{array}{ll} u(a,t) = \psi_1(t), & u(b,t) = \psi_2(t), \\ u_x(a,t) = \psi_3(t), & u_x(b,t) = \psi_4(t), \\ v(a,t) = \phi_1(t), & v(b,t) = \phi_2(t), \\ v_x(a,t) = \phi_3(t), & v_x(b,t) = \phi_4(t), \end{array} \quad t \in [0,t], \quad (4)$$

where $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$, $\psi_4(t)$, $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, and $\phi_4(t)$ are supposed to be smooth functions with continuous first-order derivatives and $u(x,t)$ represents the horizontal velocity, while $v(x,t)$ denotes the height that deviates from the equilibrium position. The constants p and q are real numbers, which are expressed as different diffusion powers, and d^α/dt^α is the Caputo derivative operator, where $0 < \alpha \leq 1$. When $\alpha = 1$, the resulting equations are the usual WBK equations. Importantly, setting $p = 1$ and $q = 0$ yields the fractional-order modified Boussinesq (MB) equation, while setting $p = 0$ and $q = 1/2$ produces the fractional-order approximate long wave (ALW) equation.

The collocation method is widely used to obtain solutions for partial differential equations [24–27]. Depending on the situation, it can be useful to find the solution of fractional partial differential equations (FPDEs) at various locations within the given problem domain. In such cases, spline solutions can provide information on spline interpolation between mesh points. The nonpolynomial, cubic, quadratic, trigonometric, and quintic B-spline methods are used to solve many fractional-order partial differential equations [28–32].

In the usual collocation method, the basis functions are required to vanish on the boundary where the Dirichlet-type boundary conditions are specified. However, in the set of quintic B-splines $\{Q_{-2}, Q_{-1}, Q_0, \cdots, Q_N, Q_{N+1}, Q_{N+2}\}$, the basis functions $Q_{-2}, Q_{-1}, Q_0, \cdots, Q_N, Q_{N+1}, Q_{N+2}$ do not vanish at one of the boundary points. Therefore, it is necessary to redefine the basis functions into a new set of basis functions that vanish on the boundary where the Dirichlet-type boundary conditions are specified. The primary goal of this work is to propose an efficient computational approach based on a redefined quintic B-spline (RQBS) algorithm for obtaining the numerical solution of time-fractional WBK equations. RQBS functions are essentially a generalization of typical quintic B-spline functions that include a free parameter that gives the ability to adjust the solution curve. We used the L_1-approximation formula to discretize the Caputo time-fractional derivative, whereas RQBS functions are used to discretize the spatial derivatives. This approach is developed for numerical solutions of fractional-order WBK equations. Moreover, this scheme is equally effective for homogeneous and nonhomogeneous FPDEs. The redefined quintic B-spline collocation discretization for the problem considered leads to a system with the pentadiagonal matrix.

This paper's brief outline is as follows. In Section 2, we provide some basic definitions and lemmas. In Section 3, we explain the quintic B-spline collocation scheme and its redefinition. Then, we describe the method and apply it to the coupled time-fractional WBK equation. Section 4 discusses the von Neumann technique for ensuring the stability of the method. Error analysis is discussed in Section 5. In Section 6, numerical examples are presented to demonstrate the applicability and accuracy of the proposed method. Finally, we finished this paper with the conclusion.

2. Basic Concepts

In this section, we will introduce some fundamental definitions of the fractional derivative of order α where $\alpha > 0$. Numerous definitions of the fractional derivative can be found in the literature, but the Riemann-Liouville and Caputo fractional derivatives are the most widely utilized ones.

Definition 1. The Riemann-Liouville fractional derivative of order $\alpha \geq 0, n-1 < \alpha \leq n, n \in \mathbb{N}$ of a function $f \in C[a,b]$, is defined by

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau. \quad (5)$$

The fractional derivative used in this study is in the Caputo meaning, which is defined as follows.

Definition 2 (see [6]). The Caputo fractional derivative of order $\alpha \geq 0, n-1 < \alpha \leq n, n \in \mathbb{N}$ of a function $f \in C[a,b]$, is defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\partial^n u(x,s)}{\partial t^n}(t-s)^{n-\alpha-1}ds. \quad (6)$$

3. Derivation Method

3.1. Temporal Discretization. Let $t_n = n\tau$ denote the integration time $t_n > 0$; the time-fractional derivative is approximated by the L_1-approximation [30, 33–35], which is valid for $0 \leq \alpha < 1$. Explicitly, the time Caputo derivative of order α is replaced by the L_1-approximation at t_n which is given in the following lemma.

Lemma 3 (see [36]). *Suppose $0 < \alpha < 1$ and $g(t) \in C^2[0, t_n]$, it holds that*

$$\left|\frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}\frac{g'(t)}{(t_n-t)^\alpha}dt - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[g(t_n) - \delta^\alpha_{n-1}g(t_0) - \sum_{k=1}^{n-1}(\delta^\alpha_{n-k-1} - \delta^\alpha_{n-k})g(t_k)\right]\right| \leq \frac{1}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})\right]\max_{0\leq t\leq t_n}|g''(t)|\tau^{2-\alpha}, \tag{7}$$

where $\delta^\alpha_k = (k+1)^{1-\alpha} - k^{1-\alpha}, q \geq 0$.

Lemma 4 (see [36]). *Let $0 < \alpha < 1$ and $\delta^\alpha_k = (k+1)^{1-\alpha} - k^{1-\alpha}$, $k = 0, 1, \cdots$; then, $1 = \delta^\alpha_0 > \delta^\alpha_1 > \cdots > \delta^\alpha_k \longrightarrow 0$, as $k \longrightarrow \infty$.*

Following Lemma 3 and some algebraic simplifications, we can approximate the time Caputo derivative at t_{n+1} as follows:

$$\frac{\partial^\alpha U^{n+1}(x)}{\partial t^\alpha} = \frac{\partial^\alpha U(x, t_{n+1})}{\partial t^\alpha} = \frac{(\tau)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\delta^\alpha_k\left(U^{n-k+1}(x) - U^{n-k}(x)\right) + \mathcal{O}\left(\tau^{2-\alpha}\right), \tag{8}$$

$$\frac{\partial^\alpha V^{n+1}(x)}{\partial t^\alpha} = \frac{\partial^\alpha V(x, t_{n+1})}{\partial t^\alpha} = \frac{(\tau)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\delta^\alpha_k \cdot \left(V^{n-k+1}(x) - V^{n-k}(x)\right) + \mathcal{O}\left(\tau^{2-\alpha}\right). \tag{9}$$

3.2. Quintic B-Spline Method. This section introduces the quintic B-spline collocation method. We begin by dividing the domain $[a, b]$ into N subinterval $[x_j, x_{j+1}]$ where $a = x_0 < x_1 < x_2 < \cdots < x_N = b$ with uniform step $h = x_{j+1} - x_j$ for $j = 0, 1, \cdots, N$. The quintic B-spline functions $Q_j(x)$ for $j = -2, -1, \cdots, M+1, M+2$ are described by the following relationships [32, 37–39]:

$$Q_j(x) = \frac{1}{120h^5}\begin{cases} (x - x_{j-3})^5, & x \in [x_{j-3}, x_{j-2}), \\ (x - x_{j-3})^5 - 6(x - x_{j-2})^5, & x \in [x_{j-2}, x_{j-1}), \\ (x - x_{j-3})^5 - 6(x - x_{j-2})^5 + 15(x - x_{j-1})^5, & x \in [x_{j-1}, x_j), \\ (x_{j+3} - x)^5 - 6(x_{j+2} - x)^5 + 15(x_{j+1} - x)^5, & x \in [x_j, x_{j+1}), \\ (x_{j+3} - x)^5 - 6(x_{j+2} - x)^5, & x \in [x_{j+1}, x_{j+2}), \\ (x_{j+3} - x)^5, & x \in [x_{j+2}, x_{j+3}), \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

The values of $Q_j(x)$ and its first three derivatives are given in Table 1.

Let $U(x, t)$ and $V(x, t)$ be the quintic B-spline approximations of the exact solutions $u(x, t)$ and $v(x, t)$, respectively, of the system considered in eqs. (1) and (2). Since the set of quintic B-splines forms a basis over the domain $a \leq x \leq b$, the approximate solutions $U(x, t)$ and $V(x, t)$ can be written as

$$u(x, t) = \sum_{j=-2}^{N+2}\gamma_j(t)Q_j(x), \tag{11}$$

$$v(x, t) = \sum_{j=-2}^{N+2}\rho_j(t)Q_j(x), \tag{12}$$

where $\gamma_j(t)$ and $\rho_j(t)$ are the time-dependent unknown quantities to be computed and $Q_j(x)$ are the quintic B-spline basis function as shown in Table 1.

Let $U^n_j = U(x_j, t_n)$ and $V^n_j = V(x_j, t_n)$ be the approximate solutions of $u(x, t)$ and $v(x, t)$, respectively; then, U^n_j, V^n_j, and their first three-order derivatives are determined at the nth time level and the nodal points x_j in terms of γ^n_j and ρ^n_j as

$$\begin{aligned} 120\, U^n_j &= \gamma^n_{j-2} + 26\gamma^n_{j-1} + 66\gamma^n_j + 26\gamma^n_{j+1} + \gamma^n_{j+2}, & 120\, V^n_j &= \rho^n_{j-2} + 26\rho^n_{j-1} + 66\rho^n_j + 26\rho^n_{j+1} + \rho^n_{j+2}, \\ 24h\,(U_x)^n_j &= -\gamma^n_{j-2} - 10\gamma^n_{j-1} + 10\gamma^n_{j+1} + \gamma^n_{j+2}, & 24h\,(V_x)^n_j &= -\rho^n_{j-2} - 10\rho^n_{j-1} + 10\rho^n_{j+1} + \rho^n_{j+2}, \\ 6h^2\,(U_{xx})^n_j &= \gamma^n_{j-2} + 2\gamma^n_{j-1} - 6\gamma^n_j + 2\gamma^n_{j+1} + \gamma^n_{j+2}, & 6h^2\,(V_{xx})^n_j &= \rho^n_{j-2} + 2\rho^n_{j-1} - 6\rho^n_j + 2\rho^n_{j+1} + \rho^n_{j+2}, \\ 2h^3\,(U_{xxx})^n_j &= -\gamma^n_{j-2} + 2\gamma^n_{j-1} - 2\gamma^n_{j+1} + \gamma^n_{j+2}, & 2h^3\,(V_{xxx})^n_j &= -\rho^n_{j-2} + 2\rho^n_{j-1} - 2\rho^n_{j+1} + \rho^n_{j+2}. \end{aligned} \tag{13}$$

3.3. Redefined Quintic B-Spline Method. To obtain an approximate solution to systems (1) and (2), we have redefined the quintic B-spline basis functions into a new set of basis functions that vanish on the boundary points since $Q_{-2}(x)$, $Q_{-1}(x)$, $Q_0(x)$, $Q_1(x)$, $Q_2(x)$, $Q_{N-2}(x)$, $Q_{N-1}(x)$, $Q_N(x)$, $Q_{N+1}(x)$, and $Q_{N+2}(x)$ are nonzero at one of the boundary points. The basis functions are redefined as follows. Allowing the approximate solutions $u(x, t)$ and $v(x, t)$ given by Eqs. (11) and (12) to satisfy the boundary conditions (Eq. (4)) and eliminating $\gamma_{-2}(t)$, $\gamma_{-1}(t)$,

Table 1: Quintic B-splines and their corresponding derivatives.

x	x_{j-3}	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}	x_{j+3}
Q_j	0	$\frac{1}{120}$	$\frac{13}{60}$	$\frac{22}{40}$	$\frac{13}{60}$	$\frac{1}{120}$	0
Q'_j	0	$\frac{1}{24h}$	$\frac{5}{12h}$	0	$-\frac{5}{12h}$	$-\frac{1}{24h}$	0
Q''_j	0	$\frac{1}{6h^2}$	$\frac{1}{3h^2}$	$-\frac{1}{h^2}$	$\frac{1}{3h^2}$	$\frac{1}{6h^2}$	0
Q'''_j	0	$\frac{1}{2h^3}$	$-\frac{1}{h^3}$	0	$\frac{1}{h^3}$	$-\frac{1}{2h^3}$	0

$\gamma_{N+1}(t)$, $\gamma_{N+2}(t)$, $\rho_{-2}(t)$, $\rho_{-1}(t)$, $\rho_{N+1}(t)$, and $\rho_{N+2}(t)$ from the resultant equations, we obtain the approximate solutions for $u(x,t)$ and $v(x,t)$ as

$$U(x,t) = W_1(x,t) + \sum_{j=0}^{N} \gamma_j(t)\tilde{Q}_j(x), \tag{14}$$

$$V(x,t) = W_2(x,t) + \sum_{j=0}^{N} \rho_j(t)\tilde{Q}_j(x), \tag{15}$$

where the weight functions $W_1(x,t)$ and $W_2(x,t)$ are given by

$$\begin{aligned} W_1(x,t) = {} & \frac{\left(Q'_{-1}(x_0)\psi_1(t) + Q_{-1}(x_0)\psi_3(t)\right)Q_{-2}(x) - \left(Q'_{-2}(x_0)\psi_1(t) + Q_{-2}(x_0)\psi_3(t)\right)Q_{-1}(x)}{Q_{-2}(x_0)Q'_{-1}(x_0) - Q'_{-2}(x_0)Q_{-1}(x_0)} \\ & + \frac{\left(Q'_{N+2}(x_N)\psi_2(t) + Q_{N+2}(x_N)\psi_4(t)\right)Q_{N+1}(x) - \left(Q'_{N+1}(x_N)\psi_2(t) + Q_{N+1}(x_N)\psi_4(t)\right)Q_{N+2}(x)}{Q'_{N+2}(x_N)Q_{N+1}(x_N) - Q'_{N+1}(x_N)Q_{N+2}(x_N)}, \end{aligned} \tag{16}$$

$$\begin{aligned} W_2(x,t) = {} & \frac{\left(Q'_{-1}(x_0)\phi_1(t) + Q_{-1}(x_0)\phi_3(t)\right)Q_{-2}(x) - \left(Q'_{-2}(x_0)\phi_1(t) + Q_{-2}(x_0)\phi_3(t)\right)Q_{-1}(x)}{Q_{-2}(x_0)Q'_{-1}(x_0) - Q'_{-2}(x_0)Q_{-1}(x_0)} \\ & + \frac{\left(Q'_{N+2}(x_N)\phi_2(t) + Q_{N+2}(x_N)\phi_4(t)\right)Q_{N+1}(x) - \left(Q'_{N+1}(x_N)\phi_2(t) + Q_{N+1}(x_N)\phi_4(t)\right)Q_{N+2}(x)}{Q'_{N+2}(x_N)Q_{N+1}(x_N) - Q'_{N+1}(x_N)Q_{N+2}(x_N)}, \end{aligned} \tag{17}$$

and the basis functions $\tilde{Q}_j(x)$ as

$$\tilde{Q}_j(x) = \begin{cases} Q_j(x) - \dfrac{Q'_{-1}(x_0)Q_j(x_0) - Q_{-1}(x_0)Q'_j(x_0)}{Q_{-2}(x_0)Q'_{-1}(x_0) - Q'_{-2}(x_0)Q_{-1}(x_0)}Q_{-2}(x) + \dfrac{Q'_{-2}(x_0)Q_j(x_0) - Q_{-2}(x_0)Q'_j(x_0)}{Q_{-2}(x_0)Q'_{-1}(x_0) - Q'_{-2}(x_0)Q_{-1}(x_0)}Q_{-1}(x), & j = 0, 1, 2, \\ Q_j(x), & 3 \le j \le N-3, \\ Q_j(x) - \dfrac{Q'_{N+2}(x_N)Q_j(x_N) - Q_{N+2}(x_N)Q'_j(x_N)}{Q'_{N+2}(x_N)Q_{N+1}(x_N) - Q'_{N+1}(x_N)Q_{N+2}(x_N)}Q_{N+1}(x) + \dfrac{Q'_{N+1}(x_N)\,Q_j(x_N) - Q_{N+1}(x_N)\,Q'_j(x_N)}{Q'_{N+2}(x_N)\,Q_{N+1}(x_N) - Q'_{N+1}(x_N)\,Q_{N+2}(x_N)}Q_{N+2}(x), & j = N-2, N-1, N. \end{cases} \tag{18}$$

Substituting the values of Q_j from Table 1 into Eqs. (16)–(18), we get

$$\begin{aligned}W_1(x,t)=&-3(25\,\psi_1(t)+13h\,\psi_3(t))Q_{-2}(x)\\&+\frac{3}{2}(5\,\psi_1(t)+h\,\psi_3(t))Q_{-1}(x)\\&+\frac{3}{2}(5\,\psi_2(t)-h\,\psi_4(t))Q_{N+1}(x)\\&+3(13h\,\psi_4(t)-25\,\psi_2(t))Q_{N+2}(x),\end{aligned}$$

$$\begin{aligned}W_2(x,t)=&-3(25\,\phi_1(t)+13h\,\phi_3(t))Q_{-2}(x)\\&+\frac{3}{2}(5\phi_1(t)+h\,\phi_3(t))Q_{-1}(x)\\&+\frac{3}{2}(5\,\phi_2(t)-h\,\phi_4(t))Q_{N+1}(x)\\&+3(13h\,\phi_4(t)-25\,\phi_2(t))Q_{N+2}(x),\end{aligned}\quad(19)$$

and $\tilde{Q}_j(x)$ as

$$\tilde{Q}_j(x)=\begin{cases}Q_j(x)+3\left(13\,h\,Q_j'(x_0)+25\,Q_j(x_0)\right)Q_{-2}(x)-\frac{3}{2}\left(h\,Q_j'(x_0)+5\,Q_j(x_0)\right)Q_{-1}(x), & j=0,1,2,\\ Q_j(x), & 3\le j\le N-3,\\ Q_j(x)-\frac{3}{2}\left(5\,Q_j(x_N)-h\,Q_j'(x_N)\right)Q_{N+1}(x)+3\left(25\,Q_j(x_N)-13h\,Q_j'(x_N)\right)Q_{N+2}(x), & j=N-2,N-1,N.\end{cases}\quad(20)$$

3.4. Description of Numerical Method. In order to apply the suggested method using the redefined set of quintic B-splines basis functions $\tilde{Q}_j(x)$ to systems (1)–(4), we write the system by approximate solutions $U_i^{n+1}=W_1(x_i,t_{n+1})+\sum_{j=0}^N\gamma_j(t_{n+1})\tilde{Q}_j(x_i)=(W_1)_i^{n+1}+\sum_{j=0}^N\gamma_j^{n+1}\tilde{Q}_j(x_i)$ and $V_i^{n+1}=W_2(x_i,t_{n+1})+\sum_{j=0}^N\rho_j(t_{n+1})\tilde{Q}_j(x_i)=(W_2)_i^{n+1}+\sum_{j=0}^N\rho_j^{n+1}\tilde{Q}_j(x_i)$ at $(n+1)$ time level and nodal points x_i, $i=1,2,\cdots,N$ as follows:

$$\begin{aligned}&\frac{\partial^\alpha U_i^{n+1}}{\partial t^\alpha}+(UU_x)_i^{n+1}+(V_x)_i^{n+1}+q(U_{xx})_i^{n+1}=f_i^{n+1},\\&\frac{\partial^\alpha V_i^{n+1}}{\partial t^\alpha}+(UV_x)_i^{n+1}+(VU_x)_i^{n+1}-q(V_{xx})_i^{n+1}+p(U_{xxx})_i^{n+1}=g_i^{n+1},\end{aligned}\quad(21)$$

where $f_i^{n+1}=f_1(x_i,t_{n+1})$ and $g_i^{n+1}=f_2(x_i,t_{n+1})$.

The Caputo fractional derivatives are discretized using the L_1-approximation as described in Eqs. (8) and (9), and the nonlinear terms $(UU_x)_i^{n+1}$, $(UV_x)_i^{n+1}$, and $(VU_x)_i^{n+1}$ are linearized using the linearization form given by Rubin and Graves [40], $(UU_x)_i^{n+1}=U_i^{n+1}(U_x)_i^n+U_i^n(U_x)_i^{n+1}-U_i^n(U_x)_i^n$; we get

$$\begin{aligned}&r\sum_{k=0}^n\delta_k^\alpha\left(U_i^{n-k+1}-U_i^{n-k}\right)+U_i^{n+1}(U_x)_i^n+U_i^n(U_x)_i^{n+1}\\&\quad-U_i^n(U_x)_i^n+(V_x)_i^{n+1}+q(U_{xx})_i^{n+1}=f_i^{n+1},\end{aligned}\quad(22)$$

$$\begin{aligned}&r\sum_{k=0}^n\delta_k^\alpha\left(V_i^{n-k+1}-V_i^{n-k}\right)+U_i^{n+1}(V_x)_i^n+U_i^n(V_x)_i^{n+1}\\&\quad-U_i^n(V_x)_i^n+V_i^{n+1}(U_x)_i^n+V_i^n(U_x)_i^{n+1}-V_i^n(U_x)_i^n\\&\quad-q(V_{xx})_i^{n+1}+p(U_{xxx})_i^{n+1}=g_i^{n+1},\end{aligned}\quad(23)$$

where $r=(\tau)^{-\alpha}/(\Gamma(2-\alpha))$. After simplifying Eqs. (22) and (23), we get

$$\begin{aligned}&\left(r+(U_x)_i^n\right)U_i^{n+1}+U_i^n(U_x)_i^{n+1}+(V_x)_i^{n+1}+q(U_{xx})_i^{n+1}\\&\quad=\left(r+(U_x)_i^n\right)U_i^n-r\sum_{k=1}^n\delta_k^\alpha\left(U_i^{n-k+1}-U_i^{n-k}\right)+f_i^{n+1},\end{aligned}\quad(24)$$

$$\begin{aligned}&\left(r+(U_x)_i^n\right)V_i^{n+1}+(V_x)_i^nU_i^{n+1}+V_i^n(U_x)_i^{n+1}\\&\quad+U_i^n(V_x)_i^{n+1}-q(V_{xx})_i^{n+1}+p(U_{xxx})_i^{n+1}\\&\quad=\left(r+(U_x)_i^n\right)V_i^n+U_i^n(V_x)_i^n\\&\quad-r\sum_{k=1}^n\delta_k^\alpha\left(V_i^{n-k+1}-V_i^{n-k}\right)+g_i^{n+1}.\end{aligned}\quad(25)$$

Using Eqs. (14) and (15) in Eqs. (24) and (25), we obtain

$$\begin{aligned}&(r+(U_x)_i^n)\left((W_1)_i^{n+1}+\sum_{j=0}^N\gamma_j^{n+1}\tilde{Q}_j(x_i)\right)+U_i^n\left((W_{1x})_i^{n+1}+\sum_{j=0}^N\gamma_j^{n+1}\tilde{Q}_j'(x_i)\right)\\&\quad+\left((W_{2x})_i^{n+1}+\sum_{j=0}^N\rho_j^{n+1}\tilde{Q}_j'(x_i)\right)+q\left((W_{1xx})_i^{n+1}+\sum_{j=0}^N\gamma_j^{n+1}\tilde{Q}_j''(x_i)\right)\\&\quad=(r+(U_x)_i^n)U_i^n-r\sum_{k=1}^n\delta_k^\alpha\left(U_i^{n-k+1}-U_i^{n-k}\right)+f_i^{n+1},\\&(r+(U_x)_i^n)\left((W_2)_i^{n+1}+\sum_{j=0}^N\rho_j^{n+1}\tilde{Q}_j(x_i)\right)+(V_x)_i^n\left((W_1)_i^{n+1}+\sum_{j=0}^N\gamma_j^{n+1}\tilde{Q}_j(x_i)\right)\\&\quad+V_i^n\left((W_{1x})_i^{n+1}+\sum_{j=0}^N\gamma_j^{n+1}\tilde{Q}_j'(x_i)\right)+U_i^n\left((W_{2x})_i^{n+1}+\sum_{j=0}^N\rho_j^{n+1}\tilde{Q}_j'(x_i)\right)\\&\quad-q\left((W_{2xx})_i^{n+1}+\sum_{j=0}^N\rho_j^{n+1}\tilde{Q}_j''(x_i)\right)+p\left((W_{1xxx})_i^{n+1}+\sum_{j=0}^N\gamma_j^{n+1}\tilde{Q}_j'''(x_i)\right)\\&\quad=(r+(U_x)_i^n)V_i^n+U_i^n(V_x)_i^n-r\sum_{k=1}^n\delta_k^\alpha\left(V_i^{n-k+1}-V_i^{n-k}\right)+g_i^{n+1}.\end{aligned}\quad(26)$$

This leads to that

$$\sum_{j=0}^{N}\left(\left(r+(U_x)_i^n\right)\tilde{Q}_j(x_i)+U_i^n\tilde{Q}_j'(x_i)+q\tilde{Q}_j''(x_i)\right)\gamma_j^{n+1}+\sum_{j=0}^{N}\tilde{Q}_j'(x_i)\rho_j^{n+1} = \left(r+(U_x)_i^n\right)U_i^n-r\sum_{k=1}^{n}\delta_k^\alpha\left(U_i^{n-k+1}-U_i^{n-k}\right)-\widetilde{W_1}_i^{n+1}+f_i^{n+1}, \quad (27)$$

$$\sum_{j=0}^{N}\left(\left(r+(U_x)_i^n\right)\tilde{Q}_j(x_i)+U_i^n\tilde{Q}_j'(x_i)-q\tilde{Q}_j''(x_i)\right)\rho_j^{n+1} + \sum_{j=0}^{N}\left((V_x)_i^n\tilde{Q}_j(x_i)+V_i^n\tilde{Q}_j'(x_i)+p\tilde{Q}_j'''(x_i)\right)\gamma_j^{n+1} = \left(r+(U_x)_i^n\right)V_i^n+U_i^n(V_x)_i^n-r\sum_{k=1}^{n}\delta_k^\alpha\left(V_i^{n-k+1}-V_i^{n-k}\right) - \widetilde{W_2}_i^{n+1}+g_i^{n+1}, \quad (28)$$

where $(\widetilde{W}_1)_i^{n+1}$ and $(\widetilde{W}_2)_i^{n+1}$ are the resulting terms from the weight functions

$$\begin{aligned}\widetilde{W_1}_i^{n+1} &= \left(r+(U_x)_i^n\right)(W_1)_i^{n+1}+U_i^n(W_{1x})_i^{n+1} + (W_{2x})_i^{n+1}+q(W_{1xx})_i^{n+1},\\ \widetilde{W_2}_i^{n+1} &= \left(r+(U_x)_i^n\right)(W_2)_i^{n+1}+U_i^n(W_{2x})_i^{n+1} - q(W_{2xx})_i^{n+1}+(V_x)_i^n(W_1)_i^{n+1} + V_i^n(W_{1x})_i^{n+1}+p(W_{1xxx})_i^{n+1}.\end{aligned} \quad (29)$$

Using Eq. (20) and Table 1 for the coefficients $\tilde{Q}_j(x_i)$, $\tilde{Q}_j''(x_i)$, $\tilde{Q}_j'(x_i)$, and $\tilde{Q}_j'''(x_i)$, we may rewrite systems (27) and (28) as follows:

$$\left.\begin{aligned}&\frac{9q}{2}\gamma_0^{n+1}+5q\gamma_1^{n+1}+\frac{q}{2}\gamma_2^{n+1}=h^2F_0^{n+1},\\ &-\frac{99p}{4}\gamma_0^{n+1}-\frac{39p}{2}\gamma_1^{n+1}-\frac{3p}{4}\gamma_2^{n+1}-\frac{9qh}{2}\rho_0^{n+1}-5qh\rho_1^{n+1}-\frac{qh}{2}\rho_2^{n+1}=h^3G_0^{n+1},\end{aligned}\right\}, i=0, \quad (30)$$

$$\left.\begin{aligned}&a_1^1\gamma_0^{n+1}+a_2^1\gamma_1^{n+1}+a_3^1\gamma_2^{n+1}+a_4^1\gamma_3^{n+1}-\frac{47h}{192}\rho_0^{n+1}+\frac{3h}{32}\rho_1^{n+1}+\frac{27h}{64}\rho_2^{n+1}+\frac{h}{24}\rho_3^{n+1}=h^2F_1^{n+1},\\ &c_1^1\gamma_0^{n+1}+c_2^1\gamma_1^{n+1}+c_3^1\gamma_2^{n+1}+c_4^1\gamma_3^{n+1}+d_1^1\rho_0^{n+1}+d_2^1\rho_1^{n+1}+d_3^1\rho_2^{n+1}d_4^1\rho_3^{n+1}=h^3G_1^{n+1},\end{aligned}\right\}, i=1, \quad (31)$$

$$\left.\begin{aligned}&a_5^i\gamma_{i-2}^{n+1}+a_6^i\gamma_{i-1}^{n+1}+a_7^i\gamma_i^{n+1}+a_8^i\gamma_{i+1}^{n+1}+a_9^i\gamma_{i+2}^{n+1}-\frac{h}{24}\rho_{i-2}^{n+1}-\frac{5h}{12}\rho_{i-1}^{n+1}+\frac{5h}{12}\rho_{i+1}^{n+1}+\frac{h}{24}\rho_{i+2}^{n+1}=h^2F_i^{n+1},\\ &c_5^i\gamma_{i-2}^{n+1}+c_6^i\gamma_{i-1}^{n+1}+c_7^i\gamma_i^{n+1}+c_8^i\gamma_{i+1}^{n+1}+c_9^i\gamma_{i+2}^{n+1}+d_5^i\rho_{i-2}^{n+1}+d_6^i\rho_{i-1}^{n+1}+d_7^i\rho_i^{n+1}+d_8^i\rho_{i+1}^{n+1}+d_9^i\rho_{i+2}^{n+1}=h^3G_i^{n+1},\end{aligned}\right\}, i=2,\cdots,N-2, \quad (32)$$

$$\left.\begin{aligned}&a_{10}^{N-1}\gamma_{N-3}^{n+1}+a_{11}^{N-1}\gamma_{N-2}^{n+1}+a_{12}^{N-1}\gamma_{N-1}^{n+1}+a_{13}^{N-1}\gamma_N^{n+1}-\frac{h}{24}\rho_{N-3}^{n+1}-\frac{27h}{64}\rho_{N-2}^{n+1}-\frac{3h}{32}\rho_{N-1}^{n+1}+\frac{47h}{192}\rho_N^{n+1}=h^2F_{N-1}^{n+1},\\ &c_{10}^{N-1}\gamma_{N-3}^{n+1}+c_{11}^{N-1}\gamma_{N-2}^{n+1}+c_{12}^{N-1}\gamma_{N-1}^{n+1}+c_{13}^{N-1}\gamma_N^{n+1}+d_{10}^{N-1}\rho_{N-3}^{n+1}+d_{11}^{N-1}\rho_{N-2}^{n+1}+d_{12}^{N-1}\rho_{N-1}^{n+1}+d_{13}^{N-1}\rho_N^{n+1}=h^3G_{N-1}^{n+1},\end{aligned}\right\}, i=N-1, \quad (33)$$

$$\left.\begin{aligned}&\frac{q}{2}\gamma_{N-2}^{n+1}+5q\gamma_{N-1}^{n+1}+\frac{9q}{2}\gamma_N^{n+1}=h^2F_N^{n+1},\\ &\frac{3p}{4}\gamma_{N-2}^{n+1}+\frac{39p}{2}\gamma_{N-1}^{n+1}+\frac{99p}{4}\gamma_N^{n+1}-\frac{qh}{2}\rho_{N-2}^{n+1}-5qh\rho_{N-1}^{n+1}-\frac{9qh}{2}\rho_N^{n+1}=h^3G_N^{n+1},\end{aligned}\right\}, i=N, \quad (34)$$

where

$$F_i^{n+1}=\left(r+(U_x)_i^n\right)U_i^n-r\sum_{k=1}^{n}\delta_k^\alpha\left(U_i^{n-k+1}-U_i^{n-k}\right)-\widetilde{W_1}_i^{n+1}+f_i^{n+1},$$

$$G_i^{n+1}=\left(r+(U_x)_i^n\right)V_i^n+U_i^n(V_x)_i^n-r\sum_{k=1}^{n}\delta_k^\alpha\left(V_i^{n-k+1}-V_i^{n-k}\right) - \widetilde{W_2}_i^{n+1}+g_i^{n+1}, i=0,1,\cdots,N, \quad (35)$$

and the following coefficients

$$a_1^1 = \frac{35h^2}{192}\left((U_x)_1^n + r\right) - \frac{47U_1^n h}{192} - \frac{17q}{48},$$
$$a_2^1 = \frac{17h^2}{32}\left((U_x)_1^n + r\right) + \frac{3U_1^n h}{32} - \frac{11q}{8},$$
$$a_3^1 = \frac{69h^2}{320}\left((U_x)_1^n + r\right) + \frac{27U_1^n h}{64} + \frac{5q}{16},$$
$$a_4^1 = \frac{h^2}{120}\left((U_x)_1^n + r\right) + \frac{U_1^n h}{24} + \frac{q}{6},$$
$$c_1^1 = \frac{35h^3}{192}(V_x)_1^n - \frac{47h^2 V_1^n}{192} + \frac{49p}{16},$$
$$c_2^1 = \frac{17h^3}{32}(V_x)_1^n + \frac{3V_1^n h^2}{32} + \frac{9p}{8},$$
$$c_3^1 = \frac{69h^3}{320}(V_x)_1^n + \frac{27V_1^n h^2}{64} - \frac{15p}{16},$$
$$c_4^1 = \frac{h^3}{120}(V_x)_1^n + \frac{V_1^n h^2}{24} + \frac{p}{2},$$
$$d_1^1 = \frac{35h^2}{192}\left((U_x)_1^n + r\right) - \frac{47U_1^n h}{192} + \frac{17q}{48},$$
$$d_2^1 = \frac{17h^2}{32}\left((U_x)_1^n + r\right) + \frac{3U_1^n h}{32} + \frac{11q}{8},$$
$$d_3^1 = \frac{69h^2}{320}\left((U_x)_1^n + r\right) + \frac{27U_1^n h}{64} - \frac{5q}{16},$$
$$d_4^1 = \frac{h^2}{120}\left((U_x)_1^n + r\right) + \frac{U_1^n h}{24} - \frac{q}{6},$$
$$\left.\begin{array}{lll}
a_5^i = \frac{h^2}{120}\left((U_x)_i^n + r\right) - \frac{U_i^n h}{24} + \frac{q}{6}, & a_6^i = \frac{13h^2}{60}\left((U_x)_i^n + r\right) - \frac{5U_i^n h}{12} + \frac{q}{3}, & a_7^i = \frac{11h^2}{20}\left((U_x)_i^n + r\right) - q, \\
a_8^i = \frac{13h^2}{60}\left((U_x)_i^n + r\right) + \frac{5U_i^n h}{12} + \frac{q}{3}, & a_9^i = \frac{h^2}{120}\left((U_x)_i^n + r\right) + \frac{U_i^n h}{24} + \frac{q}{6}, & c_5^i = \frac{h^3}{120}(V_x)_i^n - \frac{V_i^n h^2}{24} - \frac{p}{2}, \\
c_6^i = \frac{13h^3}{60}(V_x)_i^n - \frac{5V_i^n h^2}{12} + p, & c_7^i = \frac{11h^3}{20}(V_x)_i^n, & c_8^i = \frac{13h^3}{60}(V_x)_i^n + \frac{5V_i^n h^2}{12} - p, \\
c_9^i = \frac{h^3}{120}(V_x)_i^n + \frac{V_i^n h^2}{24} + \frac{p}{2}, & d_5^i = \frac{h^2}{120}\left((U_x)_i^n + r\right) - \frac{U_i^n h}{24} - \frac{q}{6}, & d_6^i = \frac{13h^2}{60}\left((U_x)_i^n + r\right) - \frac{5U_i^n h}{12} - \frac{q}{3}, \\
d_7^i = \frac{11h^2}{20}\left((U_x)_i^n + r\right) + q, & d_8^i = \frac{13h^2}{60}\left((U_x)_i^n + r\right) + \frac{5U_i^n h}{12} - \frac{q}{3}, & d_9^i = \frac{h^2}{120}\left((U_x)_i^n + r\right) + \frac{U_i^n h}{24} - \frac{q}{6},
\end{array}\right\} i = 2, \cdots, N-2, \quad (36)$$
$$a_{10}^{N-1} = \frac{h^2}{120}\left((U_x)_{N-1}^n + r\right) - \frac{U_{N-1}^n h}{24} + \frac{q}{6},$$
$$a_{11}^{N-1} = \frac{69h^2}{320}\left((U_x)_{N-1}^n + r\right) - \frac{27U_{N-1}^n h}{64} + \frac{5q}{16},$$
$$a_{12}^{N-1} = \frac{17h^2}{32}\left((U_x)_{N-1}^n + r\right) - \frac{3U_{N-1}^n h}{32} - \frac{11q}{8},$$
$$a_{13}^{N-1} = \frac{35h^2}{192}\left((U_x)_{N-1}^n + r\right) + \frac{47U_{N-1}^n h}{192} - \frac{17q}{48},$$
$$c_{10}^{N-1} = \frac{h^3}{120}(V_x)_{N-1}^n - \frac{V_{N-1}^n h^2}{24} - \frac{p}{2},$$
$$c_{11}^{N-1} = \frac{69h^3}{320}(V_x)_{N-1}^n - \frac{27V_{N-1}^n h^2}{64} + \frac{15p}{16},$$
$$c_{12}^{N-1} = \frac{17h^3}{32}(V_x)_{N-1}^n - \frac{3V_{N-1}^n h^2}{32} - \frac{9p}{8},$$
$$c_{13}^{N-1} = \frac{35h^3}{192}(V_x)_{N-1}^n + \frac{47V_{N-1}^n h^2}{192} - \frac{49p}{16},$$
$$d_{10}^{N-1} = \frac{h^2}{120}\left((U_x)_{N-1}^n + r\right) - \frac{U_{N-1}^n h}{24} - \frac{q}{6},$$
$$d_{11}^{N-1} = \frac{69h^2}{320}\left((U_x)_{N-1}^n + r\right) - \frac{27U_{N-1}^n h}{64} - \frac{5q}{16},$$
$$d_{12}^{N-1} = \frac{17h^2}{32}\left((U_x)_{N-1}^n + r\right) - \frac{3U_{N-1}^n h}{32} + \frac{11q}{8},$$
$$d_{13}^{N-1} = \frac{35h^2}{192}\left((U_x)_{N-1}^n + r\right) + \frac{47U_{N-1}^n h}{192} + \frac{17q}{48}.$$

System (30) to (34) can be written in blocks of five-diagonal matrices as follows:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c} \gamma^{n+1} \\ \hline \rho^{n+1} \end{array}\right]=\left[\begin{array}{c} h^2F^{n+1} \\ \hline h^3G^{n+1} \end{array}\right], \tag{37}$$

where

$$\gamma^{n+1}=\begin{bmatrix}\gamma_0^{n+1}\\ \gamma_1^{n+1}\\ \vdots\\ \gamma_N^{n+1}\end{bmatrix},\quad \rho^{n+1}=\begin{bmatrix}\rho_0^{n+1}\\ \rho_1^{n+1}\\ \vdots\\ \rho_N^{n+1}\end{bmatrix},\quad F^{n+1}=\begin{bmatrix}F_0^{n+1}\\ F_1^{n+1}\\ \vdots\\ F_N^{n+1}\end{bmatrix},\quad G^{n+1}=\begin{bmatrix}G_0^{n+1}\\ G_1^{n+1}\\ \vdots\\ G_N^{n+1}\end{bmatrix}, \tag{38}$$

and A, B, C, and D are $(N+1)\times(N+1)$ matrices where

$$A=\begin{bmatrix}\frac{9q}{2} & 5q & \frac{q}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ a_1^1 & a_2^1 & a_3^1 & a_4^1 & 0 & 0 & 0 & \cdots & 0 & 0\\ a_5^2 & a_6^2 & a_7^2 & a_8^2 & a_9^2 & 0 & 0 & \cdots & 0 & 0\\ 0 & a_5^3 & a_6^3 & a_7^3 & a_8^3 & a_9^3 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & a_5^{N-2} & a_6^{N-2} & a_7^{N-2} & a_8^{N-2} & a_9^{N-2}\\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{10}^{N-1} & a_{11}^{N-1} & a_{12}^{N-1} & a_{13}^{N-1}\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{q}{2} & 5q & \frac{9q}{2}\end{bmatrix},$$

$$B=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ -\frac{47h}{192} & \frac{3h}{32} & \frac{27h}{64} & \frac{h}{24} & 0 & 0 & \cdots & 0 & 0\\ -\frac{h}{24} & -\frac{5h}{12} & 0 & \frac{5h}{12} & \frac{h}{24} & 0 & \cdots & 0 & 0\\ 0 & -\frac{h}{24} & -\frac{5h}{12} & 0 & \frac{5h}{12} & \frac{h}{24} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -\frac{h}{24} & -\frac{5h}{12} & 0 & \frac{5h}{12} & \frac{h}{24}\\ 0 & 0 & 0 & \cdots & 0 & -\frac{h}{24} & -\frac{27h}{64} & -\frac{3h}{32} & \frac{47h}{192}\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \tag{39}$$

$$C=\begin{bmatrix}-\frac{297}{4} & -\frac{117}{2} & -\frac{9}{4} & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ c_1^1 & c_2^1 & c_3^1 & c_4^1 & 0 & 0 & 0 & \cdots & 0 & 0\\ c_5^2 & c_6^2 & c_7^2 & c_8^2 & c_9^2 & 0 & 0 & \cdots & 0 & 0\\ 0 & c_5^3 & c_6^3 & c_7^3 & c_8^3 & c_9^3 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & c_5^{N-2} & c_6^{N-2} & c_7^{N-2} & c_8^{N-2} & c_9^{N-2}\\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{10}^{N-1} & c_{11}^{N-1} & c_{12}^{N-1} & c_{13}^{N-1}\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{9}{4} & \frac{117}{2} & \frac{297}{4}\end{bmatrix},$$

$$D=\begin{bmatrix}-\frac{9q}{2} & -5q & -\frac{q}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ d_1^1 & d_2^1 & d_3^1 & d_4^1 & 0 & 0 & 0 & \cdots & 0 & 0\\ d_5^2 & d_6^2 & d_7^2 & d_8^2 & d_9^2 & 0 & 0 & \cdots & 0 & 0\\ 0 & d_5^3 & d_6^3 & d_7^3 & d_8^3 & d_9^3 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & d_5^{N-2} & d_6^{N-2} & d_7^{N-2} & d_8^{N-2} & d_9^{N-2}\\ 0 & 0 & 0 & 0 & \cdots & 0 & d_{10}^{N-1} & d_{11}^{N-1} & d_{12}^{N-1} & d_{13}^{N-1}\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{q}{2} & -5q & -\frac{9q}{2}\end{bmatrix}. \tag{40}$$

The initial vectors γ_i^0 and ρ_i^0 may be calculated from Eq. (13) and the initial conditions (Eq. (4)) as follows:

$$120\eta_1(x_j)=\gamma_{j-2}^0+26\gamma_{j-1}^0+66\gamma_j^0+26\gamma_{j+1}^0+\gamma_{j+2}^0,\quad 120\eta_2(x_j)=\rho_{j-2}^0+26\rho_{j-1}^0+66\rho_j^0+26\rho_{j+1}^0+\rho_{j+2}^0, \tag{41}$$

$$24h\eta_1'(x_j)=-\gamma_{j-2}^0-10\gamma_{j-1}^0+10\gamma_{j+1}^0+\gamma_{j+2}^0,\quad 24h\eta_2'(x_j)=-\rho_{j-2}^0-10\rho_{j-1}^0+10\rho_{j+1}^0+\rho_{j+2}^0, \tag{42}$$

$$6h^2\eta_1''(x_j)=\gamma_{j-2}^0+2\gamma_{j-1}^0-6\gamma_j^0+2\gamma_{j+1}^0+\gamma_{j+2}^0,\quad 6h^2\eta_2''(x_j)=\rho_{j-2}^0+2\rho_{j-1}^0-6\rho_j^0+2\rho_{j+1}^0+\rho_{j+2}^0. \tag{43}$$

Equation (41) creates two systems, each consisting of $N+1$ equations and $N+5$ unknown variables. So, using Eqs. (42) and (43) at $j=0$ and $j=N$ to eliminate each γ_{-2}^0, γ_{-1}^0, γ_{N+1}^0, γ_{N+2}^0, ρ_{-2}^0, ρ_{-1}^0, ρ_{N+1}^0, and ρ_{N+2}^0 from Eq. (41), we get

$$\begin{cases} 54\gamma_0^0+60\gamma_1^0+6\gamma_2^0=12\left(h^2\eta_1''(x_0)+6h\eta_1'(x_0)\right)+120\eta_1(x_0), & i=0,\\ 101\gamma_0^0+270\gamma_1^0+105\gamma_2^0+4\gamma_3^0=3\left(h^2\eta_1''(x_1)+4h\eta_1'(x_1)\right)+480\eta_1(x_1), & i=1,\\ \gamma_{i-2}^0+26\gamma_{i-1}^0+66\gamma_i^0+26\gamma_{i+1}^0+\gamma_{i+2}^0=120\eta_1(x_2), & i=2,\cdots,N-2,\\ 4\gamma_{N-3}^0+105\gamma_{N-2}^0+270\gamma_{N-1}^0+101\gamma_N^0=480\eta_1(x_{N-1})-3\left(4h\eta_1'(x_{N-1})-h^2\eta_1''(x_{N-1})\right), & i=N-1,\\ 6\gamma_{N-2}^0+60\gamma_{N-1}^0+54\gamma_N^0=120\eta_1(x_N)-12\left(6h\eta_{1'}(x_N)-h^2\eta_1''(x_N)\right), & i=N, \end{cases} \tag{44}$$

$$\begin{cases} 54\rho_0^0+60\rho_1^0+6\rho_2^0=12\left(h^2\eta_2''(x_0)+6h\eta_2'(x_0)\right)+120\eta_2(x_0), & i=0,\\ 101\rho_0^0+270\rho_1^0+105\rho_2^0+4\rho_3^0=3\left(h^2\eta_2''(x_1)+4h\eta_2'(x_1)\right)+480\eta_2(x_1), & i=1,\\ \rho_{i-2}^0+26\rho_{i-1}^0+66\rho_i^0+26\rho_{i+1}^0+\rho_{i+2}^0=120\eta_2(x_2), & i=2,\cdots,N-2,\\ 4\rho_{N-3}^0+105\rho_{N-2}^0+270\rho_{N-1}^0+101\rho_N^0=480\eta_2(x_{N-1})-3\left(4h\eta_2'(x_{N-1})-h^2\eta_2''(x_{N-1})\right), & i=N-1,\\ 6\rho_{N-2}^0+60\rho_{N-1}^0+54\rho_N^0=120\eta_2(x_N)-12\left(6h\eta_2'(x_N)-h^2\eta_2''(x_N)\right), & i=N. \end{cases} \tag{45}$$

Systems (44) and (45) can be written in matrix form as follows:

$$\begin{bmatrix} 54 & 60 & 6 & 0 & 0 & 0 & 0 & \cdots & 0\\ 101 & 270 & 105 & 4 & 0 & 0 & 0 & \cdots & 0\\ 1 & 26 & 66 & 26 & 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 26 & 66 & 26 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & & \ddots & \ddots & & \ddots & & \vdots\\ 0 & 0 & \cdots & 0 & 1 & 26 & 66 & 26 & 1\\ 0 & 0 & \cdots & 0 & 0 & 4 & 105 & 270 & 101\\ 0 & 0 & \cdots & 0 & 0 & 0 & 6 & 60 & 54 \end{bmatrix} \begin{bmatrix} \gamma_0^0\\ \gamma_1^0\\ \gamma_2^0\\ \gamma_3^0\\ \vdots\\ \gamma_{N-2}^0\\ \gamma_{N-1}^0\\ \gamma_N^0 \end{bmatrix} = \begin{bmatrix} 12\left(h^2\eta_1''(x_0)+6h\eta_1'(x_0)\right)+120\eta_1(x_0)\\ 3\left(h^2\eta_1''(x_1)+4h\eta_1'(x_1)\right)+480\eta_1(x_1)\\ 120\eta_1(x_2)\\ 120\eta_1(x_3)\\ \vdots\\ 120\eta_1(x_{N-2})\\ 480\eta_1(x_{N-1})-3\left(4h\eta_1'(x_{N-1})-h^2\eta_1''(x_{N-1})\right)\\ 120\eta_1(x_N)-12\left(6h\eta_1'(x_N)-h^2\eta_1''(x_N)\right) \end{bmatrix},$$
$$\begin{bmatrix} 54 & 60 & 6 & 0 & 0 & 0 & 0 & \cdots & 0\\ 101 & 270 & 105 & 4 & 0 & 0 & 0 & \cdots & 0\\ 1 & 26 & 66 & 26 & 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 26 & 66 & 26 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & & \ddots & \ddots & & \ddots & & \vdots\\ 0 & 0 & \cdots & 0 & 1 & 26 & 66 & 26 & 1\\ 0 & 0 & \cdots & 0 & 0 & 4 & 105 & 270 & 101\\ 0 & 0 & \cdots & 0 & 0 & 0 & 6 & 60 & 54 \end{bmatrix} \begin{bmatrix} \rho_0^0\\ \rho_1^0\\ \rho_2^0\\ \rho_3^0\\ \vdots\\ \rho_{N-2}^0\\ \rho_{N-1}^0\\ \rho_N^0 \end{bmatrix} = \begin{bmatrix} -12\left(h^2\eta_2''(x_0)+6h\eta_2'(x_0)\right)+120\eta_2(x_0)\\ 3\left(h^2\eta_2''(x_1)+4h\eta_2'(x_1)\right)+480\eta_2(x_1)\\ 120\eta_2(x_2)\\ 120\eta_2(x_3)\\ \vdots\\ 120\eta_2(x_{N-2})\\ 480\eta_2(x_{N-1})-3\left(4h\eta_2'(x_{N-1})-h^2\eta_2''(x_{N-1})\right)\\ 120\eta_2(x_N)-12\left(6h\eta_2'(x_N)-h^2\eta_2''(x_N)\right) \end{bmatrix}. \tag{46}$$

4. Stability Analysis

In this section, the stability of the suggested approach has been investigated through the implementation of the von Neumann technique. In order to execute this technique, the nonlinear terms $u(x,t)((\partial u(x,t))/\partial x)$, $u(x,t)((\partial v(x,t))/\partial x)$, and $v(x,t)((\partial u(x,t))/\partial x)$ in Eqs. (1) and (2) have been linearized by regarding $u(x,t)$ and $v(x,t)$ as local constants μ_1 and μ_2, respectively.

By implementing the L_1-approximation method as shown in Eqs. (8) and (9) and subsequently linearizing the nonlinear terms using the linearization form given by Rubin and Graves [40], the substitution of the approximate solutions for u, v, and their respective derivatives at the knots in the modified equation results in a difference equation with the variables γ_i and ρ_i given as

$$\begin{aligned} &a_5\gamma_{i-2}^{n+1}+a_6\gamma_{i-1}^{n+1}+a_7\gamma_i^{n+1}+a_8\gamma_{i+1}^{n+1}+a_9\gamma_{i+2}^{n+1}-\frac{1}{24h}\rho_{i-2}^{n+1}-\frac{5}{12h}\rho_{i-1}^{n+1}+\frac{5}{12h}\rho_{i+1}^{n+1}\\ &\quad+\frac{1}{24h}\rho_{i+2}^{n+1}=2r\left(\frac{1}{120}\gamma_{i-2}^{n}+\frac{13}{60}\gamma_{i-1}^{n}+\frac{11}{20}\gamma_i^{n}+\frac{13}{60}\gamma_{i+1}^{n}+\frac{1}{120}\gamma_{i+2}^{n}\right)\\ &\quad-2r\sum_{k=1}^{n}\delta_k^{\alpha}\left(\frac{1}{120}\left(\gamma_{i-2}^{n-k+1}-\gamma_{i-2}^{n-k}\right)+\frac{13}{60}\left(\gamma_{i-1}^{n-k+1}-\gamma_{i-1}^{n-k}\right)+\frac{11}{20}\left(\gamma_{i}^{n-k+1}-\gamma_{i}^{n-k}\right)\right.\\ &\quad\left.+\frac{13}{60}\left(\gamma_{i+1}^{n-k+1}-\gamma_{i+1}^{n-k}\right)+\frac{1}{120}\left(\gamma_{i+2}^{n-k+1}-\gamma_{i+2}^{n-k}\right)\right), \end{aligned} \tag{47}$$

$$\begin{aligned} &c_5\gamma_{i-2}^{n+1}+c_6\gamma_{i-1}^{n+1}-c_6\gamma_{i+1}^{n+1}-c_5\gamma_{i+2}^{n+1}+d_5\rho_{i-2}^{n+1}+d_6\rho_{i-1}^{n+1}\\ &\quad+d_7\rho_i^{n+1}+d_8\rho_{i+1}^{n+1}+d_9\rho_{i+2}^{n+1}\\ &\quad=2r\left(\frac{1}{120}\rho_{i-2}^{n}+\frac{13}{60}\rho_{i-1}^{n}+\frac{11}{20}\rho_i^{n}+\frac{13}{60}\rho_{i+1}^{n}+\frac{1}{120}\rho_{i+2}^{n}\right)\\ &\quad-2r\sum_{k=1}^{n}\delta_k^{\alpha}\left(\frac{1}{120}\left(\rho_{i-2}^{n-k+1}-\rho_{i-2}^{n-k}\right)\right.\\ &\quad+\frac{13}{60}\left(\rho_{i-1}^{n-k+1}-\rho_{i-1}^{n-k}\right)+\frac{11}{20}\left(\rho_{i}^{n-k+1}-\rho_{i}^{n-k}\right)\\ &\quad\left.+\frac{13}{60}\left(\rho_{i+1}^{n-k+1}-\rho_{i+1}^{n-k}\right)+\frac{1}{120}\left(\rho_{i+2}^{n-k+1}-\rho_{i+2}^{n-k}\right)\right), \end{aligned} \tag{48}$$

where

$$\begin{aligned} a_5&=\frac{r}{60}-\frac{\mu_1}{24h}+\frac{q}{6h^2},\\ a_6&=\frac{13r}{30}-\frac{5\mu_1}{12h}+\frac{q}{3h^2},\\ a_7&=\frac{11r}{10}-\frac{q}{h^2},\\ a_8&=\frac{13r}{30}+\frac{5\mu_1}{12h}+\frac{q}{3h^2},\\ a_9&=\frac{r}{60}+\frac{\mu_1}{24h}+\frac{q}{6h^2},\\ c_5&=-\frac{\mu_2}{24h}-\frac{p}{2h^3},\\ c_6&=-\frac{5\mu_2}{12h}+\frac{p}{h^3},\\ d_5&=\frac{2r}{120}-\frac{\mu_1}{24h}-\frac{q}{6h^2},\\ d_6&=\frac{13r}{30}-\frac{5\mu_1}{12h}-\frac{q}{3h^2},\\ d_7&=\frac{11r}{10}+\frac{q}{h^2},\\ d_8&=\frac{13r}{30}+\frac{5\mu_1}{12h}-\frac{q}{3h^2},\\ d_9&=\frac{r}{60}+\frac{\mu_1}{24h}-\frac{q}{6h^2}. \end{aligned} \tag{49}$$

Now, we consider the solutions in terms of Fourier series $\gamma_j^n=A\xi^n e^{Ij\theta h}$ and $\rho_j^n=B\xi^n e^{Ij\theta h}$ at a given point x_j, where A and B are the harmonic amplitude, θ and h are the mode number and element size, respectively, and $I=\sqrt{-1}$. Substituting these solutions in Eq. (47) and simplifying the terms, we get

$$\begin{aligned} &\xi^{n+1}\left[A\left(a_5e^{-2I\theta h}+a_6e^{-I\theta h}+a_7+a_8e^{I\theta h}+a_9e^{2I\theta h}\right)\right.\\ &\quad\left.+B\left(\frac{-1}{24h}e^{-2I\theta h}-\frac{5}{12h}e^{-I\theta h}+\frac{5}{12h}e^{I\theta h}+\frac{1}{24h}e^{2I\theta h}\right)\right]\\ &\quad=2Ar\left(\frac{1}{120}e^{-2I\theta h}+\frac{13}{60}e^{-I\theta h}+\frac{11}{20}+\frac{13}{60}e^{I\theta h}+\frac{1}{120}e^{2I\theta h}\right)\\ &\quad\times\left(\xi^n-\sum_{k=1}^{n}\delta_k^{\alpha}\left(\xi^{n-k+1}-\xi^{n-k}\right)\right)\\ &\quad=2Ar\left(\frac{1}{120}e^{-2I\theta h}+\frac{13}{60}e^{-I\theta h}+\frac{11}{20}+\frac{13}{60}e^{I\theta h}+\frac{1}{120}e^{2I\theta h}\right)\\ &\quad\times\left(\delta_n^{\alpha}\xi^0+\sum_{k=0}^{n-1}(\delta_k^{\alpha}-\delta_{k+1}^{\alpha})\xi^{n-k}\right). \end{aligned} \tag{50}$$

After performing algebraic simplifications on the given terms, we obtain

$$\xi^{n+1}=\frac{\Theta}{\Theta+\Phi-I\psi}\left(\delta_n^{\alpha}\xi^0+\sum_{k=0}^{n-1}(\delta_k^{\alpha}-\delta_{k+1}^{\alpha})\xi^{n-k}\right), \tag{51}$$

where $\psi=((B+A\mu_1)/6h)(5+\cos(\theta h))\sin(\theta h)$, $\Theta=A(r/30)(33+26\cos(\theta h)+\cos(2\theta h))$, and $\Phi=A(q/3h^2)(3-2\cos(\theta h)-\cos(2\theta h))$.

When we take the absolute value on both sides of Eq. (51), we obtain

$$\left|\xi^{n+1}\right|\le\sqrt{\frac{\Theta^2}{(\Theta+\Phi)^2+\psi^2}}\left(\delta_n^{\alpha}\left|\xi^0\right|+\sum_{k=0}^{n-1}(\delta_k^{\alpha}-\delta_{k+1}^{\alpha})\left|\xi^{n-k}\right|\right), \tag{52}$$

where $(\delta_k^{\alpha}-\delta_{k+1}^{\alpha})>0$ using Lemma 4. The necessary and sufficient condition for $\sqrt{\Theta^2/((\Theta+\Phi)^2+\psi^2)}\le 1$ is that $\Theta\Phi\ge 0$. Since q and r are positive, it follows that $\Theta\ge$

0 and $\Phi \geq 0$. Hence, $\Theta\Phi \geq 0$ and

$$\left|\xi^{n+1}\right| \leq \delta_n^\alpha \left|\xi^0\right| + \sum_{k=0}^{n-1} (\delta_k^\alpha - \delta_{k+1}^\alpha) \left|\xi^{n-k}\right|. \tag{53}$$

Using Eq. (53), we get $|\xi^{n+1}| \leq |\xi^0|$ for all $n \geq 0$.

Similarly, by substitution of the solution in terms of Fourier series into Eq. (48) and simplification, we get

$$\xi^{n+1} = \frac{\Theta}{\Theta + \Phi + I\psi} \left(\delta_n^\alpha \xi^0 + \sum_{k=0}^{n-1} (\delta_k^\alpha - \delta_{k+1}^\alpha) \xi^{n-k} \right), \tag{54}$$

where $\psi = (1/6h^3) \sin(\theta h)[h^2(5 + \cos(\theta h))(B\mu_1 + A\mu_2) + 12Ap(-1 + \cos(\theta h))]$, $\Theta = B(r/30)(33 + 26\cos(\theta h) + \cos(2\theta h))$, and $\Phi = B(q/3h^2)(3 - 2\cos(\theta h) - \cos(2\theta h))$, and then, $|\xi^{n+1}| \leq |\xi^0|$ for all $n \geq 0$. As a result, the schemes are unconditionally stable.

5. Error Analysis

Theorem 5 (see [41, 42]). *Assume that the exact solutions $u(x,t)$, $v(x,t) \in C^6[a,b]$, and $\mathscr{P} = \{a = x_0, x_1, \cdots, x_N = b\}$ are an equidistant partition, each of length h, over the interval $[a,b]$ such that $x_i = ih$, $i = 1, \cdots, N$. Let $\tilde{U}(x,t)$ and $\tilde{V}(x,t)$ be the unique spline approximations to the given problem at the spatial grid points $x_i \in \mathscr{P}$, $i = 0, \cdots, N$; then, for all $t \geq 0$, there exist κ_i, λ_i, independent of h, such that*

$$\begin{aligned} \left\|D^i\left(u - \tilde{U}\right)\right\| &\leq \kappa_i h^{6-i}, \\ \left\|D^i\left(v - \tilde{V}\right)\right\| &\leq \lambda_i h^{6-i}, \\ i &= 0, 1, 2, 3. \end{aligned} \tag{55}$$

Theorem 6. *Let U and V be the numerical approximations obtained by the redefined quintic B-spline method to the analytical exact solutions u and v, respectively, for Eqs. (1)–(4). If $f, g \in C^2[a,b]$, then for sufficiently small h and τ, we have*

$$|U - u| \leq \mathcal{O}\left(h^3 + \tau^{2-\alpha}\right), |V - v| \leq \mathcal{O}\left(h^3 + \tau^{2-\alpha}\right). \tag{56}$$

Proof. Let $\tilde{U} = \sum_{j=0}^N \varepsilon_j(t)\tilde{Q}_j(x)$, $\tilde{V} = \sum_{j=0}^N \zeta_j(t)\tilde{Q}_j(x)$ be the calculated spline for the approximate solutions $U(x,t)$, $V(x,t)$ and the exact solution $u(x,t), v(x,t)$, respectively. Let $Lu(x_i,t) = LU(x_i,t) = F(x_i,t), Lv(x_i,t) = LV(x_i,t) = G(x_i,t), i = 0, \cdots, N$, be the collocating conditions. Then, $L\tilde{U}(x_i,t) = F(x_i,t), L\tilde{V}(x_i,t) = G(x_i,t), i = 0, \cdots, N$.

Using the difference system (32), the nth time step of $L(\tilde{U}(x_i,t) - U(x_i,t)), L(\tilde{V}(x_i,t) - V(x_i,t))$ can be written as

$$\left.\begin{aligned} &\frac{9q}{2h^2}\varrho_0^{n+1} + \frac{5q}{h^2}\varrho_1^{n+1} + \frac{q}{2h^2}\varrho_2^{n+1} = \mathscr{F}_0^{n+1}, \\ &-\frac{99p}{4h^3}\varrho_0^{n+1} - \frac{39p}{2h^3}\varrho_1^{n+1} - \frac{3p}{4h^3}\varrho_2^{n+1} - \frac{9q}{2h^2}\sigma_0^{n+1} - \frac{5q}{h^2}\sigma_1^{n+1} - \frac{q}{2h^2}\sigma_2^{n+1} = \mathscr{G}_0^{n+1}, \end{aligned}\right\}, i = 0, \tag{57}$$

$$\left.\begin{aligned} &a_1^1\varrho_0^{n+1} + a_2^1\varrho_1^{n+1} + a_3^1\varrho_2^{n+1} + a_4^1\varrho_3^{n+1} - \frac{47}{192h}\sigma_0^{n+1} + \frac{3}{32h}\sigma_1^{n+1} + \frac{27}{64h}\sigma_2^{n+1} + \frac{1}{24h}\sigma_3^{n+1} = \mathscr{F}_1^{n+1}, \\ &c_1^1\varrho_0^{n+1} + c_2^1\varrho_1^{n+1} + c_3^1\varrho_2^{n+1} + c_4^1\varrho_3^{n+1} + d_1^1\sigma_0^{n+1} + d_2^1\sigma_1^{n+1} + d_3^1\sigma_2^{n+1}d_4^1\sigma_3^{n+1} = \mathscr{G}_1^{n+1}, \end{aligned}\right\}, i = 1, \tag{58}$$

$$\left.\begin{aligned} &a_5^i\varrho_{i-2}^{n+1} + a_6^i\varrho_{i-1}^{n+1} + a_7^i\varrho_i^{n+1} + a_8^i\varrho_{i+1}^{n+1} + a_9^i\varrho_{i+2}^{n+1} - \frac{1}{24h}\sigma_{i-2}^{n+1} - \frac{5}{12h}\sigma_{i-1}^{n+1} + \frac{5}{12h}\sigma_{i+1}^{n+1} + \frac{1}{24h}\sigma_{i+2}^{n+1} = \mathscr{F}_i^{n+1}, \\ &c_5^i\varrho_{i-2}^{n+1} + c_6^i\varrho_{i-1}^{n+1} + c_7^i\varrho_i^{n+1} + c_8^i\varrho_{i+1}^{n+1} + c_9^i\varrho_{i+2}^{n+1} + d_5^i\sigma_{i-2}^{n+1} + d_6^i\sigma_{i-1}^{n+1} + d_7^i\sigma_i^{n+1} + d_8^i\sigma_{i+1}^{n+1} + d_9^i\sigma_{i+2}^{n+1} = \mathscr{G}_i^{n+1}, \end{aligned}\right\}, i = 2, \cdots, N-2, \tag{59}$$

$$\left.\begin{aligned} &a_{10}^{N-1}\varrho_{N-3}^{n+1} + a_{11}^{N-1}\varrho_{N-2}^{n+1} + a_{12}^{N-1}\varrho_{N-1}^{n+1} + a_{13}^{N-1}\varrho_N^{n+1} - \frac{1}{24h}\sigma_{N-3}^{n+1} - \frac{27}{64h}\sigma_{N-2}^{n+1} - \frac{3}{32h}\sigma_{N-1}^{n+1} + \frac{47}{192h}\sigma_N^{n+1} = \mathscr{F}_{N-1}^{n+1}, \\ &c_{10}^{N-1}\varrho_{N-3}^{n+1} + c_{11}^{N-1}\varrho_{N-2}^{n+1} + c_{12}^{N-1}\varrho_{N-1}^{n+1} + c_{13}^{N-1}\rho_N^{n+1} + d_{10}^{N-1}\sigma_{N-3}^{n+1} + d_{11}^{N-1}\sigma_{N-2}^{n+1} + d_{12}^{N-1}\sigma_{N-1}^{n+1} + d_{13}^{N-1}\sigma_N^{n+1} = \mathscr{G}_{N-1}^{n+1}, \end{aligned}\right\}, i = N-1, \tag{60}$$

$$\left.\begin{aligned} &\frac{q}{2h^2}\varrho_{N-2}^{n+1} + \frac{5q}{h^2}\varrho_{N-1}^{n+1} + \frac{9q}{2h^2}\varrho_N^{n+1} = \mathscr{F}_N^{n+1}, \\ &\frac{3p}{4h^3}\varrho_{N-2}^{n+1} + \frac{39p}{2h^3}\varrho_{N-1}^{n+1} + \frac{99p}{4h^3}\varrho_N^{n+1} - \frac{q}{2h^2}\sigma_{N-2}^{n+1} - \frac{5q}{h^2}\sigma_{N-1}^{n+1} - \frac{9q}{2h^2}\sigma_N^{n+1} = \mathscr{G}_N^{n+1}, \end{aligned}\right\}, i = N, \tag{61}$$

where $\varrho_i^n = \gamma_i^n - \varepsilon_i^n, \sigma_i^n = \rho_i^n - \zeta_i^n, i = 0, 1, \cdots, N$, and $\mathscr{F}_i^n = h^2[F_i^n - \tilde{F}_i^n], \mathscr{G}_i^n = h^3[G_i^n - \tilde{G}_i^n], i = 0, 1, \cdots, N$.

It is evident from Eq. (55) that

$$|\mathscr{F}_i^n| = h^2\left|F_i^n - \tilde{F}_i^n\right| \le \kappa h^5, |\mathscr{G}_i^n| = h^3\left|G_i^n - \tilde{G}_i^n\right| \le \lambda h^6. \quad (62)$$

We define $|\mathscr{F}^n| = \max\{|\mathscr{F}_i^n|, 0 \le i \le N\}$ and $|\mathscr{G}^n| = \max\{|\mathscr{G}_i^n|, 0 \le i \le N\}$.

Now, we can write Eqs. (57) to (61) in the matrix form as

$$QE = Z, \quad (63)$$

where $Q = \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]$, $E = [\varrho^{n+1}, \sigma^{n+1}]^T$, $Z = [\mathscr{F}^{n+1}, \mathscr{G}^{n+1}]^T$, and

$$\varrho^{n+1} = \begin{bmatrix} \varrho_0^{n+1} \\ \varrho_1^{n+1} \\ \vdots \\ \varrho_N^{n+1} \end{bmatrix}, \quad \sigma^{n+1} = \begin{bmatrix} \sigma_0^{n+1} \\ \sigma_1^{n+1} \\ \vdots \\ \sigma_N^{n+1} \end{bmatrix}, \quad \mathscr{F}^{n+1} = \begin{bmatrix} \mathscr{F}_0^{n+1} \\ \mathscr{F}_1^{n+1} \\ \vdots \\ \mathscr{F}_N^{n+1} \end{bmatrix}, \quad \mathscr{G}^{n+1} = \begin{bmatrix} \mathscr{G}_0^{n+1} \\ \mathscr{G}_1^{n+1} \\ \vdots \\ \mathscr{G}_N^{n+1} \end{bmatrix}. \quad (64)$$

The submatrices A, B, C, and D are defined by Eqs. (39) and (40).

By defining

$$U_x{}^n = \begin{bmatrix} r + U_{x0}^n \\ \vdots \\ r + U_{xN}^n \end{bmatrix}, \quad U^n = \begin{bmatrix} U_0^n \\ \vdots \\ U_N^n \end{bmatrix}, \quad V^n = \begin{bmatrix} V_0^n \\ \vdots \\ V_N^n \end{bmatrix}, \quad V_x{}^n = \begin{bmatrix} V_{x0}^n \\ \vdots \\ V_{xN}^n \end{bmatrix}, \quad (65)$$

we can rewrite A, C, and D as follows:

$$\begin{aligned} A &= h^2 \operatorname{Diag}(U_x^n) A_0 + h \operatorname{Diag}(U^n) B - A_1, \\ C &= h^3 \operatorname{Diag}(V_x^n) A_0 + h^2 \operatorname{Diag}(V^n) B - A_2, \\ D &= h^2 \operatorname{Diag}(U_x^n) A_0 + h \operatorname{Diag}(U^n) B + A_1, \end{aligned} \quad (66)$$

where A_0, A_1, and A_2 are $(N+1) \times (N+1)$ matrices as follows:

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \\ \frac{35}{192} & \frac{17}{32} & \frac{69}{320} & \frac{1}{120} & 0 & 0 & \cdots & 0 & 0 & \\ \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & 0 & \cdots & 0 & 0 & \\ 0 & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & \cdots & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{120} & \frac{69}{320} & \frac{17}{32} & \frac{35}{192} & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \end{bmatrix},$$

$$A_1 = q\begin{bmatrix} -\frac{9}{2} & -5 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{17}{48} & \frac{11}{18} & \frac{-5}{16} & \frac{-1}{6} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{-1}{6} & \frac{-1}{3} & 1 & \frac{-1}{3} & \frac{-1}{6} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{-1}{6} & \frac{-1}{3} & 1 & \frac{-1}{3} & \frac{-1}{6} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-1}{6} & \frac{-1}{3} & 1 & \frac{-1}{3} & \frac{-1}{6} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{6} & \frac{-5}{16} & \frac{11}{8} & \frac{17}{48} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{-1}{2} & -5 & -\frac{9}{2} \end{bmatrix},$$

Table 2: Maximum absolute error for Example 1 at $t = 5$, $p = 3$, $q = 1$, $\alpha = 0.5$, $M = 15$, $N = 10$, and $x \in [0, 1]$.

x_i	$k = 5, l = \frac{1}{3}$		$k = 10, l = \frac{1}{6}$	
	E_{u_1}	E_{v_1}	E_{u_1}	E_{v_1}
0.1	9.57704×10^{-8}	8.872206×10^{-6}	1.443626×10^{-9}	1.343622×10^{-7}
0.2	1.477581×10^{-7}	4.894196×10^{-6}	2.258622×10^{-9}	7.30574×10^{-8}
0.3	2.226532×10^{-7}	4.081005×10^{-6}	3.428509×10^{-9}	5.952567×10^{-8}
0.4	2.37724×10^{-7}	1.936386×10^{-6}	3.693375×10^{-9}	2.62933×10^{-8}
0.5	2.592446×10^{-7}	3.772594×10^{-7}	4.059209×10^{-9}	1.473031×10^{-9}
0.6	2.328105×10^{-7}	1.250917×10^{-6}	3.674178×10^{-9}	2.361684×10^{-8}
0.7	2.128223×10^{-7}	3.336017×10^{-6}	3.390102×10^{-9}	5.66162×10^{-8}
0.8	1.384045×10^{-7}	4.285224×10^{-6}	2.222081×10^{-9}	7.067857×10^{-8}
0.9	8.624864×10^{-8}	8.096193×10^{-6}	1.406435×10^{-9}	1.313294×10^{-7}
Time	7.78125 sec		8.875 sec	

Table 3: Maximum absolute error for Example 1 at $t = 1$, $p = 3$, $q = 1$, $\alpha = 0.1$, $M = N = 10$, and $x \in [0, 1]$.

x_i	$k = 5, l = 0.5$		$k = 30, l = 1$	
	E_{u_2}	E_{v_2}	E_{u_2}	E_{v_2}
0.1	7.404186×10^{-8}	6.474259×10^{-6}	2.676951×10^{-9}	2.426883×10^{-7}
0.2	1.228618×10^{-7}	3.460206×10^{-6}	4.328953×10^{-9}	1.305957×10^{-7}
0.3	1.928312×10^{-7}	2.650239×10^{-6}	6.67696×10^{-9}	1.035607×10^{-7}
0.4	2.171545×10^{-7}	9.071887×10^{-7}	7.353315×10^{-9}	4.147997×10^{-8}
0.5	2.464134×10^{-7}	6.065784×10^{-7}	8.190674×10^{-9}	7.802135×10^{-9}
0.6	2.303387×10^{-7}	2.011775×10^{-6}	7.520783×10^{-9}	5.568671×10^{-8}
0.7	2.186918×10^{-7}	4.14558×10^{-6}	7.005586×10^{-9}	1.226692×10^{-7}
0.8	1.465566×10^{-7}	4.957014×10^{-6}	4.630311×10^{-9}	1.496303×10^{-7}
0.9	9.483132×10^{-8}	9.101146×10^{-6}	2.942323×10^{-9}	2.758954×10^{-7}
Time	4.78125 sec		5.546875 sec	

$$A_2 = p\begin{bmatrix} \frac{99}{4} & \frac{39}{2} & \frac{3}{4} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{-49}{16} & \frac{-9}{8} & \frac{15}{16} & \frac{-1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 1 & \frac{-1}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 1 & \frac{-1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & -1 & 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & \frac{-15}{6} & \frac{9}{8} & \frac{49}{16} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-3}{4} & \frac{-39}{2} & \frac{-99}{4} \end{bmatrix}. \tag{67}$$

The pentadiagonal matrices A and D are invertible and hold the following condition:

$$\left\|B D^{-1}\right\|_\infty \left\|C A^{-1}\right\|_\infty < 1. \tag{68}$$

According to [43], matrix Q is invertible. Moreover,

$$\left\|Q^{-1}\right\|_\infty \le \frac{\max\left\{\left\|A^{-1}\right\|_\infty, \left\|D^{-1}\right\|_\infty\right\}\left(1 + \left\|B D^{-1}\right\|_\infty\right)\left(1 + \left\|C A^{-1}\right\|_\infty\right)}{1 - \left\|B D^{-1}\right\|_\infty \left\|C A^{-1}\right\|_\infty}. \tag{69}$$

From Eq. (63) and norm inequalities, we have

$$\|E\|_\infty \le \left\|Q^{-1}\right\|_\infty \|Z\|_\infty. \tag{70}$$

From the classifications of the matrices A, C, and D defined in Eq. (66) and the truncation error of time-fractional discretization shown in Eqs. (8) and (9) and the fact that $\|Z\|_\infty \le \mathcal{O}(h^5)$, we have

$$\|E\|_\infty \le \mathcal{O}\left(h^3 + \tau^{2-\alpha}\right). \tag{71}$$

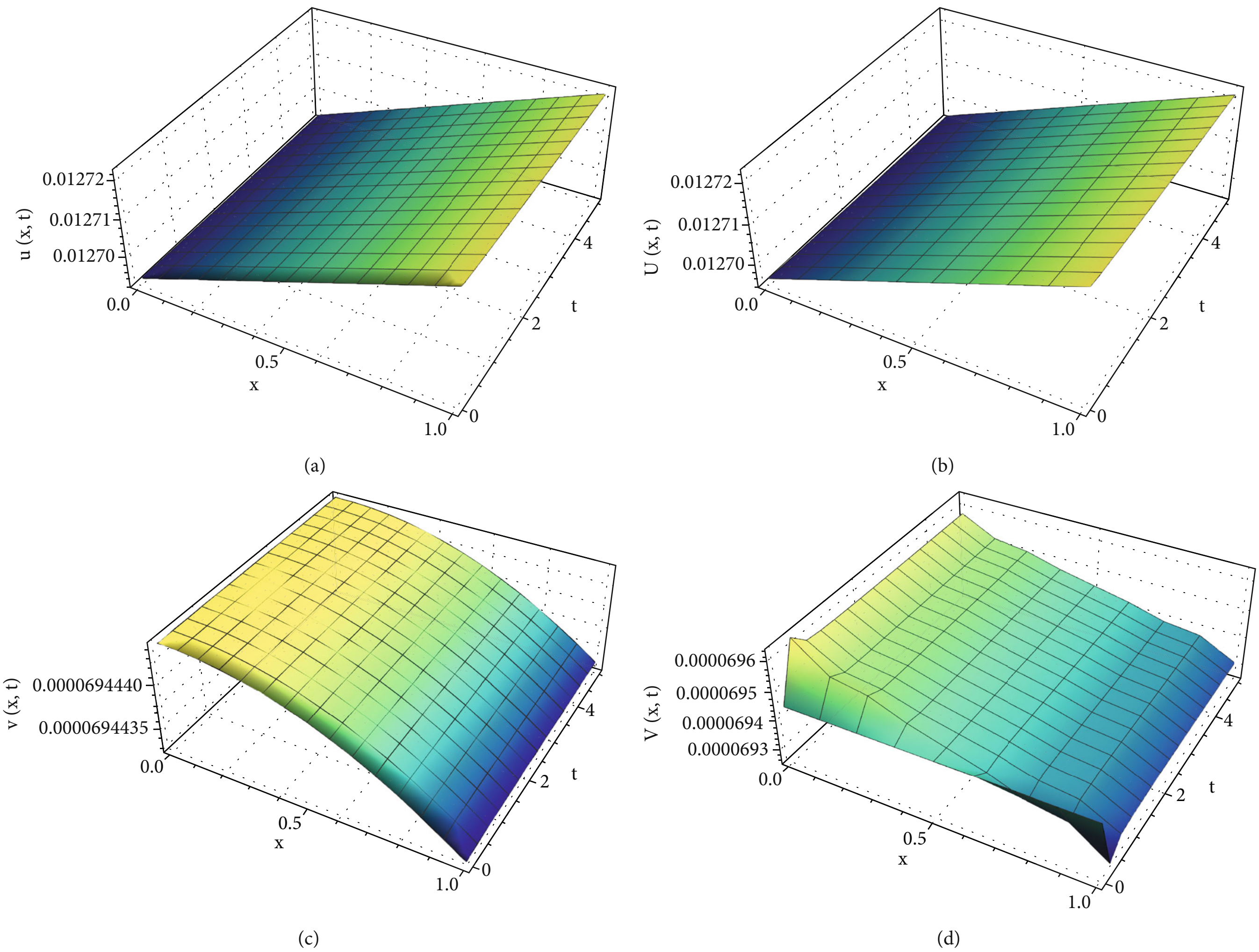

FIGURE 1: Comparison between (a, c) the exact solutions and (b, d) the approximate solutions of $u(x, t)$ and $v(x, t)$ for Example 1 at $t = 5$, $p = 3$, $q = 1$, $\alpha = 0.5$, $M = 15$, $N = 10$, $k = 10$, $l = 1/6$, and $x \in [0, 1]$.

Based on the previous analysis, we deduce that $\|E\| \longrightarrow 0$ as $h \longrightarrow 0$. Moreover, the convergence rate of the proposed method is of third order. □

6. Numerical Results

This section provides some illustrated cases to demonstrate the applicability and efficiency of the proposed technique. All the computations associated with the experiments discussed above were performed in Wolfram Mathematica 13.2 on a PC with Windows 11 64-bit OS + processor Intel Core i7~2.4 GHz.

In order to calculate the maximum absolute error E_u and E_v, we use the following formula:

$$\begin{aligned} E_u &= \max_{\substack{0\le i\le N\\ 0\le n}} \{|u(x_i, t_n) - U(x_i, t_n)|\}, \\ E_v &= \max_{\substack{0\le i\le N\\ 0\le n}} \{|v(x_i, t_n) - V(x_i, t_n)|\}. \end{aligned} \tag{72}$$

Accordingly [44], the convergence order (CO) of the proposed approach is given by

$$\mathrm{CO} = \frac{\ln(e_1) - \ln(e_2)}{\ln(h_1/h_2)} = \frac{\ln(e_1) - \ln(e_2)}{\ln(N_2/N_1)}, \tag{73}$$

where e_1 and e_2 are errors that correspond to grids with mesh size h_1 and h_2, respectively, and $h_1 = (b - a)/N1$ and $h_2 = (b - a)/N2$.

Example 1. Considering the system of the Whitham-Broer-Kaup equations (Eqs. (1)–(4)) with $f(x, t) = 0$ and $g(x, t) = 0$, we obtain the suggested equation in [13] taking into account that α is only fractional for the temporal variable and $\alpha = 1$ for the spatial variable follows

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial v(x,t)}{\partial x} + q\frac{\partial^2 u(x,t)}{\partial x^2} = 0,$$

Table 4: Maximum absolute error for Example 1 at $t = 1$, $p = 0$, $q = 1/2$, $\alpha = 0.5$, $M = N = 10$, and $x \in [0, 1]$.

x_i	$k = 5, l = \frac{1}{6}$		$k = 20, l = \frac{1}{3}$	
	E_{u_2}	E_{v_2}	E_{u_2}	E_{v_2}
0.1	4.577737×10^{-7}	3.888425×10^{-8}	5.82965×10^{-8}	2.44551×10^{-9}
0.2	1.03076×10^{-6}	8.846013×10^{-8}	1.319732×10^{-7}	5.556138×10^{-9}
0.3	1.372999×10^{-6}	1.18648×10^{-7}	1.764323×10^{-7}	7.444041×10^{-9}
0.4	1.591104×10^{-6}	1.379636×10^{-7}	2.047178×10^{-7}	8.647319×10^{-9}
0.5	1.664434×10^{-6}	1.443402×10^{-7}	2.139754×10^{-7}	9.038127×10^{-9}
0.6	1.602753×10^{-6}	1.385319×10^{-7}	2.054294×10^{-7}	8.666063×10^{-9}
0.7	1.392487×10^{-6}	1.196261×10^{-7}	1.776245×10^{-7}	7.47629×10^{-9}
0.8	1.051622×10^{-6}	8.958753×10^{-8}	1.33253×10^{-7}	5.593265×10^{-9}
0.9	4.665763×10^{-7}	3.958754×10^{-8}	5.883528×10^{-8}	2.468551×10^{-9}
Time	4.859375 sec		5.421875 sec	

Table 5: Maximum absolute error for Example 1 at $t = 1$, $p = 3$, $q = 1$, $\alpha = 0.1$, and $x \in [0, 1]$.

(x_i, t_j)	ADM [46]		VIM [47]		Present method	
	E_u	E_v	E_u	E_v	E_u	E_v
(0.1. 0.1)	1.04892×10^{-4}	6.41419×10^{-3}	1.23033×10^{-4}	1.10430×10^{-4}	1.8389×10^{-9}	1.483×10^{-7}
(0.1, 0.3)	9.64474×10^{-5}	5.99783×10^{-3}	3.69597×10^{-4}	3.31865×10^{-4}	1.488×10^{-9}	1.363×10^{-7}
(0.1, 0.5)	8.88312×10^{-5}	5.61507×10^{-3}	6.16873×10^{-4}	5.54071×10^{-4}	1.4388×10^{-9}	1.34×10^{-7}
(0.2, 0.1)	4.25408×10^{-4}	1.33181×10^{-2}	1.19869×10^{-4}	1.07016×10^{-4}	3.3846×10^{-9}	7.966×10^{-8}
(0.2, 0.3)	3.91098×10^{-4}	1.24441×10^{-2}	3.60098×10^{-4}	3.21601×10^{-4}	2.3682×10^{-9}	7.416×10^{-8}
(0.2, 0.5)	3.60161×10^{-4}	1.16416×10^{-2}	6.01006×10^{-4}	5.36927×10^{-4}	2.2466×10^{-9}	7.2915×10^{-8}
(0.3, 0.1)	9.71922×10^{-4}	2.07641×10^{-2}	1.16789×10^{-4}	1.03737×10^{-4}	5.346×10^{-9}	6.413×10^{-8}
(0.3, 0.3)	8.93309×10^{-4}	1.93852×10^{-2}	3.50866×10^{-4}	3.11737×10^{-4}	3.615×10^{-9}	6.039×10^{-8}
(0.3, 0.5)	8.22452×10^{-4}	1.81209×10^{-2}	5.85610×10^{-4}	5.20447×10^{-4}	3.409×10^{-9}	5.942×10^{-8}
(0.4, 0.1)	1.75596×10^{-3}	2.88100×10^{-2}	1.13829×10^{-4}	1.00579×10^{-4}	6.1408×10^{-9}	2.7807×10^{-8}
(0.4, 0.3)	1.61430×10^{-3}	2.68724×10^{-2}	3.41948×10^{-4}	3.02245×10^{-4}	3.9297×10^{-9}	2.658×10^{-8}
(0.4, 0.5)	1.48578×10^{-3}	2.50985×10^{-2}	5.70710×10^{-4}	5.04593×10^{-4}	3.67×10^{-9}	2.623×10^{-8}
(0.5, 0.1)	2.79519×10^{-3}	3.75193×10^{-2}	1.10936×10^{-4}	9.75385×10^{-5}	6.7274×10^{-9}	6.521×10^{-10}
(0.5, 0.3)	2.56714×10^{-3}	3.49617×10^{-2}	3.33274×10^{-4}	2.93107×10^{-4}	4.3194×10^{-9}	1.228×10^{-9}
(0.5, 0.5)	2.36184×10^{-3}	3.26239×10^{-2}	5.56235×10^{-4}	4.89335×10^{-4}	4.0343×10^{-9}	1.4237×10^{-9}

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial v(x,t)}{\partial x} + v(x,t)\frac{\partial u(x,t)}{\partial x} + p\frac{\partial^3 u(x,t)}{\partial x^3} - q\frac{\partial^2 v(x,t)}{\partial x^2} = 0. \tag{74}$$

According to [13], the exact solutions for this system are

$$u_{1,2}(x,t) = \pm\frac{l}{k}\left(\tanh\left(\frac{l}{2k^2\sqrt{p+q^2}}\left(kx - \frac{l\,t^\alpha}{\Gamma(\alpha+1)}\right)+\xi_0\right)\pm 1\right),$$
$$v_{1,2}(x,t) = \frac{l^2\sqrt{p+q^2}\mp q}{2k^2\sqrt{p+q^2}}\,\sec\mathrm{h}^2\left(\frac{l}{2k^2\sqrt{p+q^2}}\left(kx - \frac{l\,t^\alpha}{\Gamma(\alpha+1)}\right)+\xi_0\right), \tag{75}$$

where k, l, and ξ_0 are arbitrary constants. The initial and boundary conditions (Eqs. (3) and (4)) can be obtained from the exact solution.

In this example, we calculate the maximum absolute error between the approximate solution obtained by the proposed method and two exact solutions u_1, v_1 and u_2, v_2 computed by Aminikhah et al. [45] by setting $p = 3$, $q = 1$, and different values of $k, l, \alpha = 0.1$ as shown in Tables 2 and 3. Table 2 shows the maximum absolute error between the first exact solutions u_1, v_1 and the solutions obtained by the proposed method where $t = 5$, $\alpha = 0.5$, $M = 15$, $N = 10$, and $x \in [0, 1]$. Moreover, the exact and approximate solutions at the same previous values in addition to $k = 10, l = 1/6$ are represented for both $u(x, t)$ and $v(x, t)$ of Example 1 in

Table 6: Maximum absolute error for Example 2 at $t = 1$, $p = 3$, $q = 1$, $\alpha = 0.1$, and $x \in [0, 1]$.

x_i	$N = 20, M = 10$		$N = 50, M = 50$		Convergence order	
	E_u	E_v	E_u	E_v	CO_u	CO_v
0.1	3.426221×10^{-6}	1.424237×10^{-4}	1.701902×10^{-7}	7.56968×10^{-6}	3.28	3.20
0.2	8.796811×10^{-6}	1.193187×10^{-4}	4.08002×10^{-7}	5.937313×10^{-6}	3.35	3.27
0.3	1.477359×10^{-5}	6.35068×10^{-5}	6.719145×10^{-7}	3.354831×10^{-6}	3.37	3.21
0.4	1.96627×10^{-5}	9.069374×10^{-5}	8.858668×10^{-7}	4.511937×10^{-6}	3.38	3.27
0.5	2.154264×10^{-5}	1.128726×10^{-4}	9.663949×10^{-7}	4.615728×10^{-6}	3.39	3.49
0.6	1.918838×10^{-5}	1.786663×10^{-4}	8.610035×10^{-7}	7.656247×10^{-6}	3.39	3.44
0.7	1.301923×10^{-5}	1.829759×10^{-4}	5.885956×10^{-7}	8.017675×10^{-6}	3.38	3.41
0.8	5.513356×10^{-6}	1.102128×10^{-4}	2.565339×10^{-7}	4.970789×10^{-6}	3.35	3.38
0.9	4.292588×10^{-7}	8.694919×10^{-6}	2.576748×10^{-8}	4.045807×10^{-7}	3.07	3.35
Time	19.6875 sec		409.547 sec			

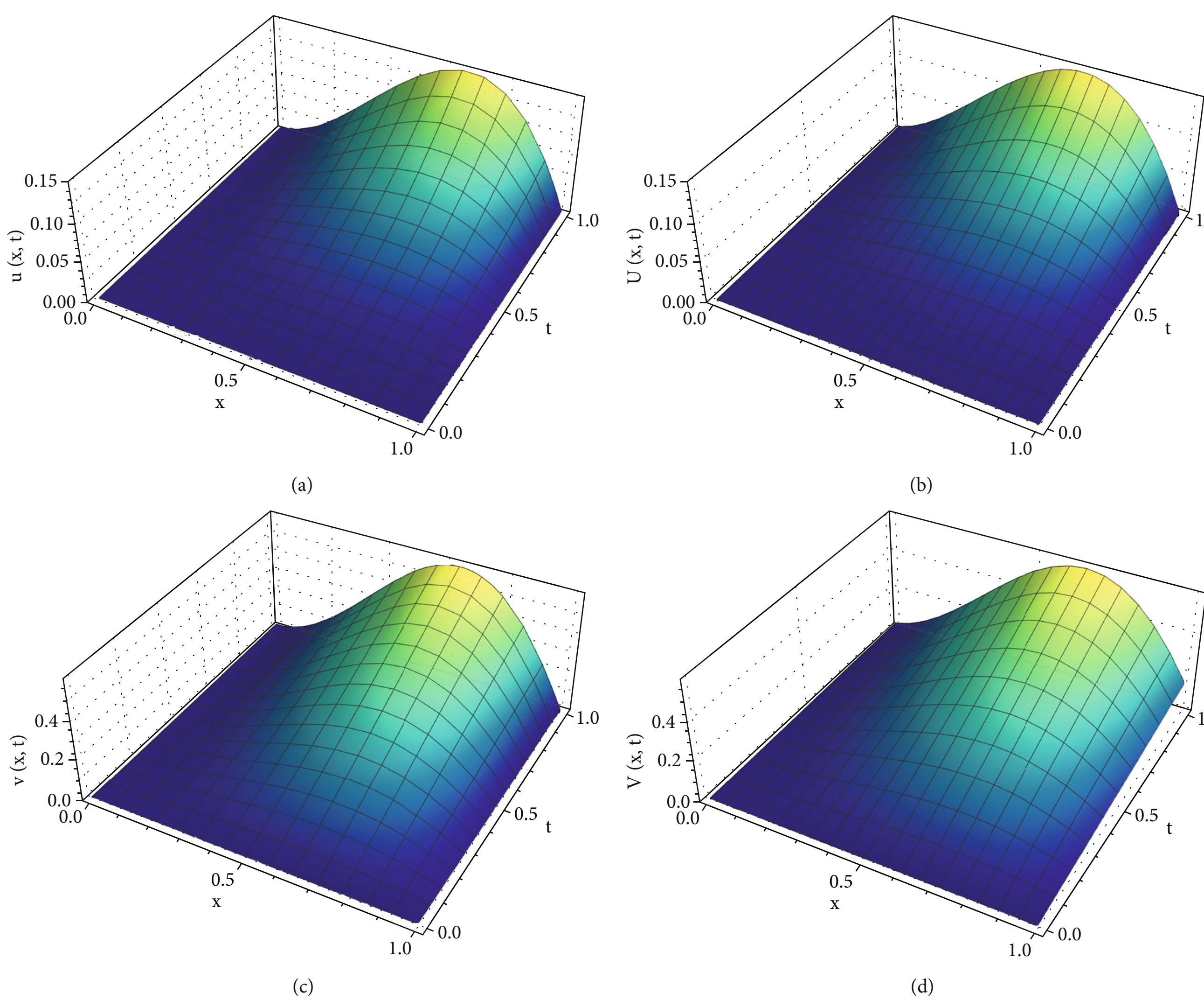

Figure 2: Comparison between (a, c) the exact solutions and (b, d) the approximate solutions of $u(x, t)$ and $v(x, t)$ for Example 2 at $t = 1$, $p = 3$, $q = 1$, $\alpha = 0.1$, $M = 10$, $N = 20$, and $x \in [0, 1]$.

Figure 1. The maximum absolute error between the second exact solutions u_2, v_2 and the approximate solutions for two sets of parameters $t = 1, p = 3, q = 1, \alpha = 0.1, M = N = 10$ and $t = 1, p = 0, q = 1/2, \alpha = 0.5, M = N = 10$ are tabulated in Tables 3 and 4, respectively. Table 5 compares the accuracy of the proposed method and the other popular existing methods for Example 1 at $\alpha = 1$.

Example 2. Consider systems (1)–(4) with $p = 3$, $q = 1$, and

$$f(x,t) = 2qt^2(1-3x) + \frac{2x^2(1-x)t^{2-\alpha}}{\Gamma(3-\alpha)} + t^4x^3\left(3x^2-5x+2\right) + t(\sin(\pi x) + \pi x\cos(\pi x)),$$

$$g(x,t) = t\left(-6pt + \pi\cos(\pi x)\left(-2q - t^2(x-1)x^3\right) + x\sin(\pi x)\left(\pi^2 q + t^2 x(3-4x)\right) + \frac{xt^{-\alpha}\sin(\pi x)}{\Gamma(2-\alpha)}\right). \quad (76)$$

The exact solution to this problem is as follows:

$$u(x,t) = t^2x^2(1-x), \quad v(x,t) = tx\sin(\pi x). \quad (77)$$

The initial and boundary conditions (Eqs. (3) and (4)) can be obtained from the exact solution. In this example, we applied the proposed method for solving the nonhomogeneous systems (1)–(4) for two different sets $N = 20$, $M = 10$ and $N = M = 50$. The calculated solutions using the suggested approach are compared to the exact solutions at $t = 1$, $p = 3$, $q = 1$, $\alpha = 0.1$, and $x \in [0,1]$, and the maximum absolute errors and convergence rates are tabulated in Table 6. Figure 2 shows the exact and approximate solutions of $u(x,t)$ and $v(x,t)$ of Example 2 at $t = 1, p = 3, q = 1, \alpha = 0.1$, $M = 10$, $N = 20$, and $x \in [0,1]$.

7. Conclusion

In this study, the time-fractional WBK equations were successfully solved using the redefined quintic B-spline collocation method. To achieve this, the L_1-approximation technique in time and the redefined quintic B-spline collocation scheme in space have been combined. We conducted a von Neumann stability analysis, which confirmed that the method used for solving the time-fractional WBK equations is unconditionally stable. The order of convergence is shown to be $\mathcal{O}(h^3 + \tau^{2-\alpha})$. To evaluate the accuracy of our approach, we compared our solutions to the exact solutions obtained by Aminikhah et al. [45]. The comparison revealed that our method is highly effective in solving the given equations, as it produced results that closely matched the exact solutions.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare no potential conflict of interest.

Authors' Contributions

All authors contributed equally. All authors read and approved the final manuscript.

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