Research Article

Stability and Hopf Bifurcation Analysis of a Vector-Borne Disease Model with Two Delays and Reinfection

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Received 30 November 2020; Revised 9 May 2021; Accepted 20 May 2021; Published 8 June 2021

Academic Editor: Alireza Jolfaei

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In this paper, a vector-borne disease model with two delays and reinfection is established and considered. First of all, the existence of the equilibrium of the system, under different cases of two delays, is discussed through analyzing the corresponding characteristic equation of the linear system. Some conditions that the system undergoes Hopf bifurcation at the endemic equilibrium are obtained. Furthermore, by employing the normal form method and the center manifold theorem for delay differential equations, some explicit formulas used to describe the properties of bifurcating periodic solutions are derived. Finally, the numerical examples and simulations are presented to verify our theoretical conclusions. Meanwhile, the influences of the degree of partial protection for recovered people acquired by a primary infection on the endemic equilibrium and the critical values of the two delays are analyzed.

1. Introduction

Vector-borne diseases (VBDs) are one of the complex infectious diseases that endanger human beings. Vectors are living biological agents, such as ticks, mosquitoes, and fleas, with the ability to transmit parasites, bacteria, or viruses between people or from animals to people. It is reported that VBDs cause more than 1 billion infections and 1 million deaths worldwide every year [1]. Malaria is the most prevalent parasitic vector-borne disease caused by plasmodium parasitizing the human body [2–5]. Plasmodium enters stem cells through the blood to parasitize and reproduce. It then invades red blood cells to reproduce after maturity, causing red blood cells to burst in batches and attack. The source of malaria infection is malaria patients and those with Plasmodium. The natural transmission medium of malaria is female mosquitoes of the genus *Anopheles*, which is transmitted by biting the human body, and a few by blood transfusion and vertical transmission, with *Anopheles gambiae* being a major carrier of the disease [2, 5, 6]. Different types of malaria have different incubation

periods, some of which are about 7–12 days, and some of which are more than 6 months. The morbidity and mortality of malaria are high [7]. According to the latest WHO reports, in 2018, there were estimated 228 million malaria cases worldwide, of which the death toll was about 405,000. It has led to great global economic and social losses, especially in the tropical and subtropical regions on five continents [3].

The population is generally susceptible to malaria. Although there is a certain degree of immunity after several infections, its acquisition process is so slow that it may take years or decades to develop, and it gradually weakens over time [5, 7, 8]. The remaining plasmodium may escape from the immune function due to the antigenic variation and reproduce again (recrudescence); even if the parasite has been eliminated by human immunity or drugs after the initial onset of malaria, the possibility of relapse is not excluded with time. That is, reinfection likely occurs [6, 7]. It is shown that time delay is of great significance in many biological modelling, and its change may affect the dynamic behavior of the system [4–6, 9–19], such as stability, uniqueness, and

oscillation of solution. So, it is instructive to consider a mathematical model with time delays to research the influence of immunity on disease control of malaria transmission.

Considering the incubation period of virus transmission in the vector population, some vector-borne epidemic models with a delay were considered [4–6]. In [5], Xu and Zhou proposed a delayed vector-borne epidemic model and reinfection, investigated its existence and stability of equilibrium, and analyzed its dynamical behavior. It suggested that there are two effective preventive measures to reduce infections: one is to minimize vector to human contacts and the other is to use insecticides to control vector. According to [4, 5], a vector-borne disease model with delay-saturated infection rate and cure rate was given by [6]. The existence and local stability of the epidemic equilibrium were discussed, and the length of the delay of the system preserving stability was estimated. This paper develops an improved vector-borne disease model with two delays and reinfection to consider the time required for the malaria virus to spread to the host population and vector population.

The remainder of this paper is organized as follows. In Section 2, an improved vector-borne disease model is formulated. In Section 3, the stability of the equilibrium and the existence of local Hopf bifurcation are discussed. In Section 4, some explicit formulas determining properties bifurcating periodic solutions are obtained by employing the normal form method and the center manifold theorem for the delay differential equations developed by Hassard et al. [20]. Some numerical examples and simulations are performed in Section 5 to demonstrate the main theoretical results, and the conclusions of this paper are summarized in Section 6.

2. Model Description

Generally, in the mathematical model of a vector-transmitted disease, the host population size at time t , denoted by $N_1(t)$, is divided into three subclasses: susceptible, infected, and recovered, with numbers denoted by $S(t)$, $I(t)$, and $R(t)$, respectively. The vector population size at time t , given by $N_2(t)$, is partitioned into two subclasses: susceptible vectors $M(t)$ and infectious vectors $V(t)$. Based on the models [4–6], we consider an improved vector-borne disease model with two delays and reinfection as follows:

$$\begin{cases} \frac{dS}{dt} = \Lambda_1 - b\beta_1 S(t - \tau_1) V(t - \tau_1) - \mu_1 S(t), \\ \frac{dI}{dt} = b\beta_1 S(t - \tau_1) V(t - \tau_1) + \sigma b\beta_1 R(t) V(t) - (\mu_1 + \gamma) I(t), \\ \frac{dR}{dt} = \gamma I(t) - \sigma b\beta_1 R(t) V(t) - \mu_1 R(t), \\ \frac{dM}{dt} = \Lambda_2 - b\beta_2 M(t - \tau_2) I(t - \tau_2) - \mu_2 M(t), \\ \frac{dV}{dt} = b\beta_2 M(t - \tau_2) I(t - \tau_2) - \mu_2 V(t), \end{cases} \quad (1)$$

where Λ_1 and Λ_2 are the recruitment rates of the hosts and

FIGURE 1: The phase graph of system (2) with $\tau_1 = \tau_2 = 0$.

vectors, respectively. b is the average number of bites per mosquito per day. β_1 and β_2 represent the infection rates from vector to human and human to vector, respectively. μ_1 and μ_2 are the natural death rates of the hosts and vectors, respectively. σ ($0 \leq \sigma \leq 1$) is the degree of partial protection of individuals that recovered from primary infection. γ is the recovery rate per capita of the infective host population. τ_1 and τ_2 are two delays, in which τ_1 represents the incubation period of the host population, and τ_2 denotes the incubation period of the vector population. The term $b\beta_1 S(t - \tau_1) V(t - \tau_1)$ represents the incidence number of the susceptible host infections caused by the infective vector at time $t - \tau_1$. $b\beta_2 M(t - \tau_2) I(t - \tau_2)$ denotes the number of infections of the susceptible vectors that bite the infected host at time $t - \tau_2$ and become infective at time t .

According to [21], system (1) is equivalent to the following model:

$$\begin{cases} \frac{dS}{dt} = \Lambda_1 - b\beta_1 S(t - \tau_1) V(t - \tau_1) - \mu_1 S(t), \\ \frac{dI}{dt} = b\beta_1 S(t - \tau_1) V(t - \tau_1) + \sigma b\beta_1 \left(\frac{\Lambda_1}{\mu_1} - S(t) - I(t) \right) V(t) - (\mu_1 + \gamma) I(t), \\ \frac{dV}{dt} = b\beta_2 \left(\frac{\Lambda_2}{\mu_2} - V(t - \tau_2) \right) I(t - \tau_2) - \mu_2 V(t). \end{cases} \quad (2)$$

Notice that $N_i'(t) = \Lambda_i - \mu_i N_i(t)$, $i = 1, 2$, and its solution is $N_i(t) = N_i(0)e^{-\mu_i t} + (\Lambda_i/\mu_i)(1 - e^{-\mu_i t})$.

Obviously, $\lim_{t \rightarrow \infty} N_i(t) = \Lambda_i/\mu_i$, ($i = 1, 2$). So, for $t \rightarrow \infty$ and any $\varepsilon > 0$, $0 \leq S + I \leq N_1(t) < (\Lambda_1/\mu_1) + \varepsilon$ holds by limit theorem, which follows that $0 \leq S + I \leq \Lambda_1/\mu_1$. Similarly, $0 \leq V \leq \Lambda_2/\mu_2$ also holds. Thus, all the solutions of system (2) enter into the region:

$$\Omega = \left\{ (S, I, V) \in \mathbb{R}_+^3 \mid 0 \leq S + I \leq \frac{\Lambda_1}{\mu_1}, \quad 0 \leq V \leq \frac{\Lambda_2}{\mu_2}, \quad S \geq 0, I \geq 0 \right\}. \quad (3)$$

For the existence of the equilibrium of system (2), according to [5], one can have the following result.

Lemma 1. For system (2), $E^0 = (\Lambda_1/\mu_1, 0, 0)$ is a disease-free equilibrium if $R_0 \leq 1$, and $E^* = (S^*, I^*, V^*)$ is an endemic equilibrium if $R_0 > 1$, where $R_0 = (b^2 \Lambda_1 \Lambda_2 \beta_1 \beta_2) / (\mu_1 \mu_2^2 (\mu_1 + \gamma))$ is the basic reproductive number, $S^* = (\Lambda_1 \mu_2 (b \beta_2 I^* + \mu_2)) / (b^2 \Lambda_2 \beta_1 \beta_2 I^* + \mu_1 \mu_2 (b \beta_2 I^* + \mu_2))$, $V^* = (b \Lambda_2 \beta_2 I^*) / (\mu_2 (b \beta_2 I^* + \mu_2))$, and I^* satisfy the following equation:

$$P_2(I^*)^2 + P_1 I^* + P_0 = 0, \quad (4)$$

where

$$P_2 = \mu_1 b^2 \beta_2^2 (b \sigma \beta_1 \Lambda_2 + \mu_1 \mu_2) (b \Lambda_2 \beta_1 + \mu_1 \mu_2) + \gamma b^2 \beta_2^2 \mu_1 \mu_2 (b \Lambda_2 \beta_1 + \mu_1 \mu_2),$$

$$P_1 = b \beta_2 \mu_1 \mu_2^2 (\mu_1 + \gamma) (b \Lambda_2 \beta_1 + \mu_1 \mu_2) + b \beta_2 \mu_1^2 \mu_2^2 (b \sigma \beta_1 \Lambda_2 + \mu_1 \mu_2) + b \gamma \beta_2 \mu_1^2 \mu_2^3 - b^3 \Lambda_1 \Lambda_2 \beta_1 \beta_2^2 (b \sigma \beta_1 \Lambda_2 + \mu_1 \mu_2),$$

$$P_0 = \mu_1^2 \mu_2^4 (\mu_1 + \gamma) (1 - R_0). \quad (5)$$

The disease-free equilibrium E^0 denotes no infection, and the endemic equilibrium E^* represents that the disease will exist and persist. The basic reproductive number R_0 describes the expected number of secondary infections, which is mainly sensitive to parameters b and β_1 but not affected by parameter σ [5]. The process of deriving R_0 using the next-generation method [22–24] is presented in Appendix A.

3. Stability of Endemic Equilibrium and Hopf Bifurcation

The Jacobian matrix of the linear system (2) at $E^* = (S^*, I^*, V^*)$ is

$$J = \begin{pmatrix} -\mu_1 - b \beta_1 V^* e^{-\lambda \tau_1} & 0 & -b \beta_1 S^* e^{-\lambda \tau_1} \\ -(\sigma - e^{-\lambda \tau_1}) b \beta_1 V^* & -\mu_1 - \gamma - b \sigma \beta_1 V^* & b \beta_1 S^* e^{-\lambda \tau_1} + b \sigma \beta_1 \left(\frac{\Lambda_1}{\mu_1} - I^* - S^* \right) \\ 0 & b \beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^* \right) e^{-\lambda \tau_2} & -\mu_2 - b \beta_2 I^* e^{-\lambda \tau_2} \end{pmatrix}, \quad (6)$$

and the characteristic equation $\det(\lambda I - J) = 0$ is

$$\lambda^3 + m_2 \lambda^2 + m_1 \lambda + m_0 + (n_2 \lambda^2 + n_1 \lambda + n_0) e^{-\lambda \tau_1} + (q_2 \lambda^2 + q_1 \lambda + q_0) e^{-\lambda \tau_2} + (r_1 \lambda + r_0) e^{-\lambda(\tau_1 + \tau_2)} = 0, \quad (7)$$

where

$$m_2 = \mu_2 + 2\mu_1 + \gamma + b \sigma \beta_1 V^*,$$

$$m_1 = (\mu_1 + \gamma + b \sigma \beta_1 V^*)(\mu_1 + \mu_2) + \mu_1 \mu_2,$$

$$m_0 = \mu_1 \mu_2 (\mu_1 + \gamma + b \sigma \beta_1 V^*),$$

$$n_2 = b \beta_1 V^*,$$

$$n_1 = b \beta_1 V^* (\mu_2 + \mu_1 + \gamma + b \sigma \beta_1 V^*),$$

$$n_0 = b \beta_1 V^* (\mu_1 + \gamma + b \sigma \beta_1 V^*) \mu_2,$$

$$q_2 = b \beta_2 I^*,$$

$$q_1 = b \beta_2 I^* (2\mu_1 + \gamma + b \sigma \beta_1 V^*) - b \sigma \beta_1 \left(\frac{\Lambda_1}{\mu_1} - I^* - S^* \right) b \beta_2 \cdot \left(\frac{\Lambda_2}{\mu_2} - V^* \right),$$

$$q_0 = b \beta_2 I^* \mu_1 (\mu_1 + \gamma + b \sigma \beta_1 V^*) - b \sigma \beta_1 \left(\frac{\Lambda_1}{\mu_1} - I^* - S^* \right) b \beta_2 \cdot \left(\frac{\Lambda_2}{\mu_2} - V^* \right) \mu_1,$$

$$r_1 = b^2 \beta_1 \beta_2 V^* I^* - b \beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^* \right) b \beta_1 S^*,$$

$$r_0 = b^2 \beta_1 \beta_2 V^* I^* (\mu_1 + \gamma + b \sigma \beta_1 V^*) - b^3 \beta_1 \beta_2 S^* \sigma V^* \left(\frac{\Lambda_2}{\mu_2} - V^* \right) - \left[b^3 \beta_1^2 \beta_2 \sigma V^* \left(\frac{\Lambda_1}{\mu_1} - I^* - S^* \right) \left(\frac{\Lambda_2}{\mu_2} - V^* \right) + \mu_1 b^2 \beta_1 \beta_2 S^* \left(\frac{\Lambda_2}{\mu_2} - V^* \right) \right]. \quad (8)$$

Case (1). $\tau_1 = \tau_2 = 0$.

FIGURE 2: The endemic equilibrium E^* is asymptotically stable with $\tau_1 = 23 < \tau_{10}$.

Characteristic equation (7) becomes

$$\lambda^3 + A_{12}\lambda^2 + A_{11}\lambda + A_{10} = 0, \quad (9)$$

where $A_{12} = m_2 + n_2 + q_2$, $A_{11} = m_1 + n_1 + q_1 + r_1$, and $A_{10} = m_0 + n_0 + q_0 + r_0$.

When the following condition (H1) holds, all roots of equation (9) have negative real parts.

$$(H1): A_{12} > 0, A_{11} > 0, A_{10} > 0, A_{12}A_{11} - A_{10} > 0. \quad (10)$$

Hence, according to the Routh-Hurwitz criterion, the following conclusion can be drawn.

Theorem 2. *If (H1) holds, the endemic equilibrium E^* is locally asymptotically stable when $\tau_1 = \tau_2 = 0$.*

Case (2). $\tau_1 > 0, \tau_2 = 0$.

The characteristic equation (7) becomes

$$\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20} + (B_{22}\lambda^2 + B_{21}\lambda + B_{20})e^{-\lambda\tau_1} = 0, \quad (11)$$

where $A_{22} = m_2 + q_2$, $A_{21} = m_1 + q_1$, $A_{20} = m_0 + q_0$, $B_{22} = n_2$, $B_{21} = n_1 + r_1$, and $B_{20} = n_0 + r_0$.

Suppose $\lambda = i\omega_1$ ($\omega_1 > 0$) is a root of equation (11), replacing it into equation (11) and separating the real and imaginary parts, then, we can obtain

$$\begin{cases} A_{22}\omega_1^2 - A_{20} = (B_{20} - B_{22}\omega_1^2) \cos \omega_1\tau_1 + B_{21}\omega_1 \sin \omega_1\tau_1, \\ \omega_1^3 - A_{21}\omega_1 = -(B_{20} - B_{22}\omega_1^2) \sin \omega_1\tau_1 + B_{21}\omega_1 \cos \omega_1\tau_1. \end{cases} \quad (12)$$

It follows that

$$\omega_1^6 + (A_{22}^2 - 2A_{21} - B_{22}^2)\omega_1^4 + (A_{21}^2 - B_{21}^2 - 2A_{20}A_{22} + 2B_{20}B_{22})\omega_1^2 + A_{20}^2 - B_{20}^2 = 0. \quad (13)$$

FIGURE 3: The endemic equilibrium E^* is unstable with $\tau_1 = 24 > \tau_{10}$.

Let $z = \omega_1^2$, $p_2 = A_{22}^2 - 2A_{21} - B_{22}^2$, $p_1 = A_{21}^2 - B_{21}^2 - 2A_{20}A_{22} + 2B_{20}B_{22}$, and $p_0 = A_{20}^2 - B_{20}^2$, then, equation (13) is equivalent to

$$f(z) = z^3 + p_2 z^2 + p_1 z + p_0 = 0. \quad (14)$$

It is easy to get that the two roots of $f'(z) = 3z^2 + 2p_2 z + p_1 = 0$ are $z_1 = (1/3)(-p_2 + \sqrt{p_2^2 - 3p_1})$ and $z_2 = (1/3)(-p_2 - \sqrt{p_2^2 - 3p_1})$.

According to [25, 26], we can get the conditions that equation (14) exists positive roots.

Lemma 3. For equation (14),

- (i) If $p_0 < 0$, then equation (14) has at least one positive root
- (ii) If $p_0 \geq 0$ and $\Delta_1 = p_2^2 - 3p_1 \leq 0$, then equation (14) has no positive roots
- (iii) If $p_0 \geq 0$ and $\Delta_1 = p_2^2 - 3p_1 > 0$, then equation (14) has positive roots if and only if $z_1 = (1/3)(-p_2 + \sqrt{\Delta_1}) > 0$ and $f(z_1) \leq 0$

By Lemma 3, it is easy to see that the stability of E^* will not change when τ changes if (ii) is set up. However, the stability may change when τ crosses through some critical values if equation (14) has a positive root.

Assume that z_1 , z_2 , and z_3 are three positive roots of equation (14), then, there are three positive numbers $\omega_{11} = \sqrt{z_1}$, $\omega_{12} = \sqrt{z_2}$, and $\omega_{13} = \sqrt{z_3}$. It follows from equation (12) that

$$\cos \omega_1 \tau_1 = \frac{(A_{22}\omega_1^2 - A_{20})(B_{20} - B_{22}\omega_1^2) + B_{21}\omega_1(\omega_1^3 - A_{21}\omega_1)}{(B_{20} - B_{22}\omega_1^2)^2 + B_{21}^2\omega_1^2}. \quad (15)$$

Denote

$$\tau_{1k}^{(j)} = \frac{1}{\omega_{1k}} \arccos \left\{ \frac{(A_{22}\omega_1^2 - A_{20})(B_{20} - B_{22}\omega_1^2) + B_{21}\omega_1(\omega_1^3 - A_{21}\omega_1)}{(B_{20} - B_{22}\omega_1^2)^2 + B_{21}^2\omega_1^2} \right\} + \frac{2j\pi}{\omega_{1k}}, \quad (16)$$

where $j = 0, 1, 2, \dots, k = 1, 2, 3$, then, $\pm i\omega_{1k}$ is a pair of purely imaginary roots of equation (11) for $\tau_1 = \tau_{1k}^{(j)}$. Let $\tau_{10} =$

FIGURE 4: The bifurcation diagrams with τ_1 as parameter and $\tau_2 = 0$. A Hopf bifurcation exists at $\tau_1 = \tau_{10}$.

$\min_{k \in \{1,2,3\}} \{\tau_{1k}^{(0)}\}$, ω_{10} be the ω_{1k} that corresponds to the minimum $\tau_{1k}^{(0)}$, and $\lambda(\tau_1) = \alpha(\tau_1) + i\omega(\tau_1)$ be the root of equation (11) near $\tau_1 = \tau_{1k}^{(j)}$ satisfying $\alpha(\tau_{1k}^{(j)}) = 0$ and $\omega(\tau_{1k}^{(j)}) = \omega_{1k}$.

Furthermore, we consider the transversality condition. By differentiating both sides of equation (11) with respect to τ_1 , we get

$$\left[\frac{d\lambda}{d\tau_1} \right]^{-1} = \frac{3\lambda^2 + 2A_{22}\lambda + A_{21}}{-\lambda(\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20})} + \frac{2B_{22}\lambda + B_{21}}{\lambda(B_{22}\lambda^2 + B_{21}\lambda + B_{20})} - \frac{\tau_1}{\lambda}. \quad (17)$$

It follows that

$$\begin{aligned} \operatorname{Re} \left\{ \left[\frac{d\lambda}{d\tau_1} \right]^{-1} \right\}_{\tau_1=\tau_{10}} &= \operatorname{Re} \left\{ \frac{(3\lambda^2 + 2A_{22}\lambda + A_{21})e^{\lambda\tau_1} + 2B_{22}\lambda + B_{21}}{\lambda(B_{22}\lambda^2 + B_{21}\lambda + B_{20})} - \frac{\tau_1}{\lambda} \right\}_{\tau_1=\tau_{10}} \\ &= \frac{[3\omega_1^4 + (2A_{22}^2 - 4A_{21} - 2B_{22}^2)\omega_1^2 + A_{21}^2 - 2A_{20}A_{22} - B_{21}^2 + 2B_{20}B_{22}]\omega_1^2}{B_{21}^2\omega_1^4 + (B_{20} - B_{22}\omega_1^2)^2\omega_1^2} \\ &= \frac{\omega_1^2 f'(\omega_1^2)}{B_{21}^2\omega_1^4 + (B_{20} - B_{22}\omega_1^2)^2\omega_1^2}. \end{aligned} \quad (18)$$

Thus,

$$\operatorname{sign} \left\{ \frac{d(\operatorname{Re} \lambda)}{d\tau_1} \right\}_{\tau_1=\tau_{10}} = \operatorname{sign} \left\{ \operatorname{Re} \left\{ \left[\frac{d\lambda}{d\tau_1} \right]^{-1} \right\}_{\tau_1=\tau_{10}} \right\} = \operatorname{sign} \{ f'(\omega_1^2) \}. \quad (19)$$

The transversality condition is satisfied when $f'(\omega_1^2) \neq 0$.

According to [27], and applying Lemma 3 and the above transversal condition to system (2), the following theorem is obtained.

Theorem 4. For system (2), when $\tau_2 = 0$,

- (i) if $p_0 \geq 0$ and $\Delta_1 = p_2^2 - 3p_1 \leq 0$ hold, then, the endemic equilibrium E^* is locally asymptotically stable for all $\tau_1 > 0$
- (ii) if $p_0 < 0$ or $p_0 \geq 0$, $\Delta_1 = p_2^2 - 3p_1 > 0$, $z_1 > 0$, and $f(z_1) \leq 0$, then, the endemic equilibrium E^* is locally asymptotically stable for $\tau_1 \in (0, \tau_{10})$
- (iii) if the conditions of (ii) and $f'(z_1) \neq 0$ hold, then, it undergoes a Hopf bifurcation at E^* when $\tau_1 = \tau_{10}$

Case (3). $\tau_1 > 0, \tau_2 > 0$.

Fixing τ_1 in its stable interval and regarding τ_2 as a parameter, we analyze the roots of characteristic equation (7). Let $\lambda = i\omega$ ($\omega > 0$) be a characteristic root of equation (7). By performing some calculations as those in case (2), we can obtain

$$\begin{aligned} &(\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0)e^{\lambda\tau_2} + (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau_1}e^{\lambda\tau_2} \\ &+ (q_2\lambda^2 + q_1\lambda + q_0) + (r_1\lambda + r_0)e^{-\lambda\tau_1} = 0, \end{aligned} \quad (20)$$

FIGURE 5: The endemic equilibrium E^* is asymptotically stable with $\tau_1 = 23$, $\tau_2 = 21 < \tau_2^*$.

$$\begin{cases} A_{31} \cos \omega \tau_2 + A_{32} \sin \omega \tau_2 = A_{34}, \\ A_{31} \sin \omega \tau_2 - A_{32} \cos \omega \tau_2 = A_{35}, \end{cases} \quad (21)$$

$$\omega^6 + k_5 \omega^5 + k_4 \omega^4 + k_3 \omega^3 + k_2 \omega^2 + k_1 \omega + k_0 = 0, \quad (22)$$

where

$$A_{31} = (n_0 - n_2 \omega^2) \cos \omega \tau_1 + n_1 \omega \sin \omega \tau_1 + m_0 - m_2 \omega^2,$$

$$A_{32} = (n_0 - n_2 \omega^2) \sin \omega \tau_1 - n_1 \omega \cos \omega \tau_1 + \omega^3 - m_1 \omega,$$

$$A_{34} = -r_1 \omega \sin \omega \tau_1 - r_0 \cos \omega \tau_1 + q_2 \omega^2 - q_0,$$

$$A_{35} = -r_1 \omega \cos \omega \tau_1 + r_0 \sin \omega \tau_1 - q_1 \omega,$$

$$k_0 = n_0^2 + m_0^2 - r_0^2 - q_0^2 + 2n_0 m_0 \cos \omega \tau_1 - 2r_0 q_0 \cos \omega \tau_1,$$

$$k_1 = 2(n_1 m_0 - n_0 m_1 - r_1 q_0 + r_0 q_1) \sin \omega \tau_1,$$

$$\begin{aligned} k_2 = & n_1^2 + m_1^2 - 2n_0 n_2 - 2n_0 m_2 \cos \omega \tau_1 - 2n_2 m_0 \cos \omega \tau_1 \\ & - 2m_0 m_2 + 2n_1 m_1 \cos \omega \tau_1 - r_1^2 + 2r_0 q_2 \cos \omega \tau_1 + 2q_0 q_2 \\ & - 2r_1 q_1 \cos \omega \tau_1 - q_1^2, \end{aligned}$$

$$k_3 = 2(n_0 + m_1 n_2 - n_1 m_2 + r_1 q_2) \sin \omega \tau_1,$$

$$k_4 = n_2^2 + m_2^2 + 2n_2 m_2 \cos \omega \tau_1 - 2n_1 \cos \omega \tau_1 - 2m_1 - q_2^2,$$

$$k_5 = -2n_2 \sin \omega \tau_1. \quad (23)$$

Furthermore, we suppose that

(H2): equation (22) has finite positive roots $\omega_2^{(i)}$, ($i = 1, 2, \dots, k$).

Then, for every fixed $\omega_2^{(i)}$, there exists a sequence of critical values $\tau_{2i}^{(j)}$ such that equation (22) holds, where

$$\tau_{2i}^{(j)} = \frac{1}{\omega_2^{(i)}} \arccos \left\{ \frac{A_{31} A_{34} - A_{32} A_{35}}{A_{31}^2 + A_{32}^2} \right\} + \frac{2j\pi}{\omega_2^{(i)}}, \quad (i = 1, 2, \dots, k; j \in \mathbb{N}_0). \quad (24)$$

FIGURE 6: The endemic equilibrium E^* is unstable with $\tau_1 = 23$ and $\tau_2 = 22 > \tau_2^*$.

Let $\tau_2^* = \min \{\tau_{2i}^0 | i = 1, 2, \dots, k\}$, when $\tau_2 = \tau_2^*$, then equation (7) has a pair of purely imaginary roots $\pm i\omega_2^*$.

Next, we also check the transversality condition. By taking the differentiation of (20) with respect to τ_2 and further calculating, we can obtain

$$\left[\frac{d\lambda}{d\tau_2} \right]^{-1} = -\frac{f_{31}(\lambda)}{f_{32}(\lambda)} + \frac{\tau_2}{\lambda}, \quad (25)$$

where

$$\begin{aligned} f_{31}(\lambda) = & (3\lambda^2 + 2m_2\lambda + m_1)e^{\lambda\tau_2} + (2n_2\lambda + n_1)e^{\lambda(\tau_2-\tau_1)} \\ & - (n_2\lambda^2 + n_1\lambda + n_0)\tau_1 e^{\lambda(\tau_2-\tau_1)} + (2q_2\lambda + q_1) \\ & + r_1 e^{-\lambda\tau_1} - (r_1\lambda + r_0)\tau_1 e^{-\lambda\tau_1}, \end{aligned}$$

$$f_{32}(\lambda) = (\lambda^4 + m_2\lambda^3 + m_1\lambda^2 + m_0\lambda)e^{\lambda\tau_2} + (n_2\lambda^3 + n_1\lambda^2 + n_0\lambda)e^{\lambda(\tau_2-\tau_1)}. \quad (26)$$

Thus, we can get

$$\operatorname{Re} \left\{ \left[\frac{d\lambda}{d\tau_2} \right]^{-1} \right\}_{\tau_2=\tau_2^*} = -\frac{P_{31}Q_{31} + P_{32}Q_{32}}{Q_{31}^2 + Q_{32}^2}, \quad (27)$$

where

$$\begin{aligned} Q_{31} = & \left[(n_0\omega_2^* - n_2(\omega_2^*)^3) \sin \omega_2^*\tau_1 - n_1(\omega_2^*)^2 \cos \omega_2^*\tau_1 + (\omega_2^*)^4 \right. \\ & \left. - m_1(\omega_2^*)^2 \cos \omega_2^*\tau_2^* - \left[(n_0\omega_2^* - n_2(\omega_2^*)^3) \cos \omega_2^*\tau_1 \right. \right. \\ & \left. \left. + n_1(\omega_2^*)^2 \sin \omega_2^*\tau_1 + m_0\omega_2^* - m_2(\omega_2^*)^3 \right] \sin \omega_2^*\tau_2^* \right], \end{aligned}$$

$$\begin{aligned} Q_{32} = & \left[(n_0\omega_2^* - n_2(\omega_2^*)^3) \sin \omega_2^*\tau_1 - n_1(\omega_2^*)^2 \cos \omega_2^*\tau_1 + (\omega_2^*)^4 \right. \\ & \left. - m_1(\omega_2^*)^2 \right] \sin \omega_2^*\tau_2^* + \left[(n_0\omega_2^* - n_2(\omega_2^*)^3) \cos \omega_2^*\tau_1 \right. \\ & \left. + n_1(\omega_2^*)^2 \sin \omega_2^*\tau_1 - m_2(\omega_2^*)^3 + m_0\omega_2^* \right] \cos \omega_2^*\tau_2^*, \end{aligned}$$

$$\begin{aligned} P_{31} = & \left[(\tau_1 n_2 (\omega_2^*)^2 - \tau_1 n_0 + n_1) \cos \omega_2^*\tau_1 \right. \\ & \left. + (2n_2 - \tau_1 n_1) \omega_2^* \sin \omega_2^*\tau_1 - 3(\omega_2^*)^2 + m_1 \right] \cos \omega_2^*\tau_2^* \\ & + \left[(\tau_1 n_2 (\omega_2^*)^2 - \tau_1 n_0 + n_1) \sin \omega_2^*\tau_1 \right. \\ & \left. - (2n_2 - \tau_1 n_1) \omega_2^* \cos \omega_2^*\tau_1 - 2m_2 \omega_2^* \right] \sin \omega_2^*\tau_2^* \\ & + (r_1 - r_0 \tau_1) \cos \omega_2^*\tau_1 - \tau_1 r_1 \omega_2^* \sin \omega_2^*\tau_1 + q_1, \end{aligned}$$

$$\begin{aligned}
P_{32} = & \left[\left(\tau_1 n_2 (\omega_2^*)^2 - \tau_1 n_0 + n_1 \right) \cos \omega_2^* \tau_1 \right. \\
& + \left. (2n_2 - \tau_1 n_1) \omega_2^* \sin \omega_2^* \tau_1 - 3(\omega_2^*)^2 + m_1 \right] \sin \omega_2^* \tau_2^* \\
& - \left[\left(\tau_1 n_2 (\omega_2^*)^2 - \tau_1 n_0 + n_1 \right) \sin \omega_2^* \tau_1 \right. \\
& - \left. (2n_2 - \tau_1 n_1) \omega_2^* \cos \omega_2^* \tau_1 - 2m_2 \omega_2^* \right] \cos \omega_2^* \tau_2^* \\
& - (r_1 - r_0 \tau_1) \sin \omega_2^* \tau_1 - \tau_1 r_1 \omega_2^* \cos \omega_2^* \tau_1 + 2q_2 \omega_2^*.
\end{aligned} \tag{28}$$

Therefore, when

(H3): $P_{31}Q_{31} + P_{32}Q_{32} \neq 0$ holds, then $\text{Re} \{ [d\lambda/d\tau_2]_{\tau_2=\tau_2^*}^{-1} \} \neq 0$, i.e., the transversality condition is satisfied.

According to the above discussions and based on [27], we can get the following result.

Theorem 5. Let $\tau_1 \in (0, \tau_{10})$, if (H2) and (H3) hold, then, the endemic equilibrium E^* is locally asymptotically stable for $\tau_2 \in [0, \tau_2^*)$ and is unstable for $\tau_2 \in (\tau_2^*, +\infty)$. System (2) undergoes Hopf bifurcation at the endemic equilibrium E^* for $\tau_2 = \tau_2^*$.

4. Properties of Hopf Bifurcation

From the analysis in the last section, we can see that for some critical values of delays, system (2) can occur in a series of periodic solutions at the equilibrium. On the basis of Theorem 5, we will employ the normal form method and the center manifold theorem introduced by Hassard et al. [20] to provide the properties of bifurcating periodic solutions in this section. Without loss of generality, we assume that $\tau_1^* < \tau_2^*$, $\tau_1^* \in (0, \tau_{10})$, $\tau_2 = \tau_2^* + \mu$, $u_1(t) = S(\tau t) - S^*$, $u_2(t) = I(\tau t) - I^*$, and $u_3(t) = V(\tau t) - V^*$. Then, equation (2) can be rewritten into the following functional differential equation in the Banach Space $C = C([-1, 0], \mathbb{R}^3)$.

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{29}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in \mathbb{R}^3$, $u_t(\theta) = u(t + \theta)$, $L_\mu : C \rightarrow \mathbb{R}^3$, and $F(\mu, \cdot) : \mathbb{R} \times C \rightarrow \mathbb{R}^3$ are defined as follows:

$$\begin{aligned}
L_\mu(\phi) = & (\tau_2^* + \mu) \left[A' \phi(0) + B' \phi\left(-\frac{\tau_1^*}{\tau_2^*}\right) + C' \phi(-1) \right], \\
F(\mu, \phi) = & (\tau_2^* + \mu) (F_1, F_2, F_3)^T,
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
\phi(\theta) = & (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C, \\
A' = & \begin{pmatrix} -\mu_1 & 0 & 0 \\ -\sigma b \beta_1 V^* & -(\mu_1 + \gamma + b \sigma \beta_1 V^*) & b \sigma \beta_1 \left(\frac{\Lambda_1}{\mu_1} - I^* - S^* \right) \\ 0 & 0 & -\mu_2 \end{pmatrix}, \\
B' = & \begin{pmatrix} -b \beta_1 V^* & 0 & -b \beta_1 S^* \\ b \beta_1 V^* & 0 & b \beta_1 S^* \\ 0 & 0 & 0 \end{pmatrix}, \\
C' = & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b \beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^* \right) & -b \beta_2 I^* \end{pmatrix}, \\
F_1 = & -b \beta_1 \phi_1 \left(-\frac{\tau_1^*}{\tau_2^*} \right) \phi_3 \left(-\frac{\tau_1^*}{\tau_2^*} \right), \\
F_2 = & b \beta_1 \phi_1 \left(-\frac{\tau_1^*}{\tau_2^*} \right) \phi_3 \left(-\frac{\tau_1^*}{\tau_2^*} \right) - b \sigma \beta_1 \phi_1(0) \phi_3(0) - b \sigma \beta_1 \phi_2(0) \phi_3(0), \\
F_3 = & -b \beta_2 \phi_2(-1) \phi_3(-1).
\end{aligned} \tag{31}$$

Based on the Riesz representation theorem, we know that there exists a function of bounded variation components $\eta(\theta, \mu)$, $\theta \in [-1, 0]$, such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu) \phi(\theta), \phi(\theta) \in C([-1, 0], \mathbb{R}^3). \tag{32}$$

In fact, we can take

$$\eta(\theta, \mu) = \begin{cases} (\tau_2^* + \mu) (A' + B' + C'), & \theta = 0, \\ (\tau_2^* + \mu) (B' + C'), & \theta \in \left[-\frac{\tau_1^*}{\tau_2^*}, 0 \right], \\ (\tau_2^* + \mu) C', & \theta \in \left(-1, -\frac{\tau_1^*}{\tau_2^*} \right), \\ 0, & \theta = -1. \end{cases} \tag{33}$$

Define $A(\mu)$ and $R(\mu)$ by

$$A(\mu) \phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(s, \mu) \phi(s), & \theta = 0, \end{cases} \tag{34}$$

$$R(\mu) \phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0. \end{cases} \tag{35}$$

Then, system (29) can be further represented as

$$\dot{u}(t) = A(\mu) u_t + R(\mu) u_t, \tag{36}$$

FIGURE 7: The bifurcation diagrams with τ_2 as parameter and $\tau_1 = 23$. A Hopf bifurcation exists at $\tau_2 = \tau_2^*$.

where $u_t(\theta) = u(t + \theta) = (u_1(t + \theta), u_2(t + \theta), u_3(t + \theta))$, $\theta \in [-1, 0]$.

For $\varphi \in C([0, 1], (\mathbb{R}^3)^*)$, we define

$$A^* \varphi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(t, 0) \varphi(-t), & s = 0, \end{cases} \quad (37)$$

and the bilinear inner product

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0) \phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{\varphi}(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi, \quad (38)$$

where $\eta(\theta) = \eta(\theta, 0)$. Let $A = A(0)$, then A and A^* are a pair of adjoint operators. From the discussions in Section 3, we know that $\pm i\omega_2^* \tau_2^*$ are a pair of eigenvalues of $A(0)$, it follows that they are also a pair of eigenvalues of A^* . Next, we calculate the eigenvectors of $A(0)$ and A^* with respect to $i\omega_2^* \tau_2^*$ and $-i\omega_2^* \tau_2^*$, respectively.

Suppose that $q(\theta)$ is an eigenfunction of $A(0)$ corresponding to $i\omega_2^* \tau_2^*$. Then, by the definition of A , we can take

$$q(\theta) = (1, q_2, q_3)^T e^{i\omega_2^* \tau_2^* \theta}, \quad (39)$$

where

$$q_2 = \frac{(i\omega_2^* + \mu_2 + b\beta_2 I^* e^{-i\omega_2^* \tau_2^*})(i\omega_2^* + \mu_1 + b\beta_1 V^* e^{-i\omega_2^* \tau_1^*})}{-b\beta_2((\Lambda_2/\mu_2) - V^*)e^{-i\omega_2^* \tau_2^*}},$$

$$q_3 = \frac{i\omega_2^* + \mu_1 + b\beta_1 V^* e^{-i\omega_2^* \tau_1^*}}{-b\beta_1 S^* e^{-i\omega_2^* \tau_1^*}}. \quad (40)$$

Similarly, $q^*(s) = D(1, q_2^*, q_3^*)^T e^{i\omega_2^* \tau_2^* s}$, $s \in [0, 1]$ is the eigenfunction of $A^*(0)$ corresponding to $-i\omega_2^* \tau_2^*$, where

$$q_2^* = -\frac{-i\omega_2^* + \mu_1 + b\beta_1 V^* e^{i\omega_2^* \tau_1^*}}{(\sigma - e^{i\omega_2^* \tau_1^*})b\beta_1 V^*},$$

$$q_3^* = \frac{(-i\omega_2^* + \mu_1 + \gamma + b\sigma\beta_1 V^*)(-i\omega_2^* + \mu_1 + b\beta_1 V^* e^{i\omega_2^* \tau_1^*})}{b\beta_2((\Lambda_2/\mu_2) - V^*)e^{i\omega_2^* \tau_2^*}(\sigma - e^{i\omega_2^* \tau_1^*})b\beta_1 V^*}. \quad (41)$$

From equation (38), we can get

$$\begin{aligned}
 \langle q^*(s), q(\theta) \rangle &= \bar{q}^*(0)q(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{q}^*(\xi - \theta) d\eta(\theta) q(\xi) d\xi \\
 &= \bar{D}(1, \bar{q}_2^*, \bar{q}_3^*)(1, q_2, q_3)^T - \bar{D} \int_{-1}^0 \int_{\xi=0}^{\theta} \\
 &\quad \cdot (1, \bar{q}_2^*, \bar{q}_3^*) e^{i\omega_2^* \tau_2^* (\theta - \xi)} d\eta(\theta) (1, q_2, q_3)^T e^{i\omega_2^* \tau_2^* \xi} d\xi \\
 &= \bar{D} \left[1 + q_2 \bar{q}_2^* + q_3 \bar{q}_3^* - (1, \bar{q}_2^*, \bar{q}_3^*) \int_{-1}^0 \theta e^{i\omega_2^* \tau_2^* \theta} d\eta(\theta) (1, q_2, q_3)^T \right] \\
 &= \bar{D} \left[1 + q_2 \bar{q}_2^* + q_3 \bar{q}_3^* + \tau_1^* b \beta_1 e^{-i\omega_2^* \tau_1^*} (-V^* + \bar{q}_2^* - S^* q_3 + \bar{q}_2^* q_3) \right. \\
 &\quad \left. + \tau_2^* b \beta_2 e^{-i\omega_2^* \tau_2^*} \left(\left(\frac{\Lambda_2}{\mu_2} - V^* \right) q_2 - I^* q_3 \right) \bar{q}_3^* \right].
 \end{aligned} \tag{42}$$

Thus, one can choose D as

$$\begin{aligned}
 D &= \left[1 + \bar{q}_2 q_2^* + \bar{q}_3 q_3^* + \tau_1^* b \beta_1 e^{i\omega_2^* \tau_1^*} (-V^* + \bar{q}_2^* - S^* q_3 + \bar{q}_2^* q_3) \right. \\
 &\quad \left. + \tau_2^* b \beta_2 e^{i\omega_2^* \tau_2^*} \left(\left(\frac{\Lambda_2}{\mu_2} - V^* \right) q_2 - I^* q_3 \right) \bar{q}_3^* \right]^{-1},
 \end{aligned} \tag{43}$$

which satisfies $\langle q^*(s), q(\theta) \rangle = 1$.

Next, using the methods given in [20], we can calculate some explicit expressions as follows that are used to determine the qualities of bifurcating periodic solution.

$$\begin{aligned}
 g_{20} &= 2\bar{D}\tau_2^* \left[-b\beta_1 q_3 e^{-2i\omega_2^* \tau_1^*} + \bar{q}_2^* \left(b\beta_1 q_3 e^{-2i\omega_2^* \tau_1^*} - \sigma b\beta_1 q_3 - \sigma b\beta_1 q_2 q_3 \right) \right. \\
 &\quad \left. - \bar{q}_3^* b\beta_2 q_2 q_3 e^{-2i\omega_2^* \tau_2^*} \right],
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 g_{11} &= \bar{D}\tau_2^* [-b\beta_1 (\bar{q}_3 + q_3) + \bar{q}_2^* ((1 - \sigma)b\beta_1 (\bar{q}_3 + q_3) \\
 &\quad - b\sigma\beta_1 (q_2 \bar{q}_3 + \bar{q}_2 q_3)) - \bar{q}_3^* b\beta_2 (q_2 \bar{q}_3 + \bar{q}_2 q_3)],
 \end{aligned} \tag{45}$$

$$\begin{aligned}
 g_{02} &= 2\bar{D}\tau_2^* \left[-b\beta_1 \bar{q}_3 e^{2i\omega_2^* \tau_1^*} + \bar{q}_2^* \left(b\beta_1 \bar{q}_3 e^{2i\omega_2^* \tau_1^*} - \sigma b\beta_1 \bar{q}_3 - \sigma b\beta_1 \bar{q}_2 \bar{q}_3 \right) \right. \\
 &\quad \left. - \bar{q}_3^* b\beta_2 \bar{q}_2 \bar{q}_3 e^{2i\omega_2^* \tau_2^*} \right],
 \end{aligned} \tag{46}$$

$$\begin{aligned}
 g_{21} &= \bar{D}\tau_2^* \left\{ -b\beta_1 \left[2W_{11}^{(3)} \left(-\frac{\tau_1^*}{\tau_2^*} \right) e^{-i\omega_2^* \tau_1^*} + W_{20}^{(3)} \left(-\frac{\tau_1^*}{\tau_2^*} \right) e^{i\omega_2^* \tau_1^*} \right. \right. \\
 &\quad \left. \left. + W_{20}^{(1)} \left(-\frac{\tau_1^*}{\tau_2^*} \right) \bar{q}_3 e^{i\omega_2^* \tau_1^*} + 2W_{11}^{(1)} \left(-\frac{\tau_1^*}{\tau_2^*} \right) q_3 e^{-i\omega_2^* \tau_1^*} \right] \right. \\
 &\quad \left. + \bar{q}_2^* \left[b\beta_1 \left(2W_{11}^{(3)} \left(-\frac{\tau_1^*}{\tau_2^*} \right) e^{-i\omega_2^* \tau_1^*} + W_{20}^{(3)} \left(-\frac{\tau_1^*}{\tau_2^*} \right) e^{i\omega_2^* \tau_1^*} \right. \right. \right. \\
 &\quad \left. \left. + W_{20}^{(1)} \left(-\frac{\tau_1^*}{\tau_2^*} \right) \bar{q}_3 e^{i\omega_2^* \tau_1^*} + 2W_{11}^{(1)} \left(-\frac{\tau_1^*}{\tau_2^*} \right) q_3 e^{-i\omega_2^* \tau_1^*} \right) \right. \right. \\
 &\quad \left. - \sigma b\beta_1 \left(2W_{11}^{(3)}(0) + W_{20}^{(3)}(0) + W_{20}^{(1)}(0) \bar{q}_3 + 2W_{11}^{(1)}(0) q_3 \right) \right. \\
 &\quad \left. - b\sigma\beta_1 \left(2W_{11}^{(3)}(0) q_2 + W_{20}^{(3)}(0) \bar{q}_2 + W_{20}^{(2)}(0) \bar{q}_3 2W_{11}^{(2)}(0) q_3 \right) \right] \\
 &\quad \left. - \bar{q}_3^* b\beta_2 \left[2W_{11}^{(3)}(-1) q_2 e^{-i\omega_2^* \tau_2^*} + W_{20}^{(3)}(-1) \bar{q}_2 e^{i\omega_2^* \tau_2^*} \right. \right. \\
 &\quad \left. \left. + W_{20}^{(2)}(-1) \bar{q}_3 e^{i\omega_2^* \tau_2^*} + 2W_{11}^{(2)}(-1) q_3 e^{-i\omega_2^* \tau_2^*} \right] \right\},
 \end{aligned} \tag{47}$$

where

$$\begin{aligned}
 W_{20}(\theta) &= \frac{i g_{20}}{\omega_2^* \tau_2^*} q(0) e^{i\omega_2^* \tau_2^* \theta} + \frac{i \bar{g}_{02}}{3\omega_2^* \tau_2^*} \bar{q}(0) e^{-i\omega_2^* \tau_2^* \theta} + E_1 e^{2i\omega_2^* \tau_2^* \theta}, \\
 W_{11}(\theta) &= -\frac{i g_{11}}{\omega_2^* \tau_2^*} q(0) e^{i\omega_2^* \tau_2^* \theta} + \frac{i \bar{g}_{11}}{\omega_2^* \tau_2^*} \bar{q}(0) e^{-i\omega_2^* \tau_2^* \theta} + E_2, \\
 E_1 &= 2 \left(\frac{\Delta_{11}}{\Delta_1}, \frac{\Delta_{12}}{\Delta_1}, \frac{\Delta_{13}}{\Delta_1} \right), \\
 E_2 &= \left(\frac{\Delta_{21}}{\Delta_2}, \frac{\Delta_{22}}{\Delta_2}, \frac{\Delta_{23}}{\Delta_2} \right), \\
 \Delta_1 &= \begin{vmatrix} 2i\omega_2^* + \mu_1 + b\beta_1 V^* e^{-2i\omega_2^* \tau_1^*} & 0 & b\beta_1 S^* e^{-2i\omega_2^* \tau_1^*} \\ \left(\sigma - e^{-2i\omega_2^* \tau_1^*} \right) b\beta_1 V^* & 2i\omega_2^* + \mu_1 + \gamma + b\sigma\beta_1 V^* & -b\beta_1 S^* e^{-2i\omega_2^* \tau_1^*} - b\sigma\beta_1 \left(\frac{\Lambda_1}{\mu_1} - I^* - S^* \right) \\ 0 & -b\beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^* \right) e^{-2i\omega_2^* \tau_2^*} & 2i\omega_2^* + \mu_2 + b\beta_2 I^* e^{-2i\omega_2^* \tau_2^*} \end{vmatrix}, \\
 \Delta_2 &= \begin{vmatrix} \mu_1 + b\beta_1 V^* & 0 & b\beta_1 S^* \\ -(1 - \sigma)b\beta_1 V^* & \mu_1 + \lambda + b\sigma\beta_1 V^* & -\frac{(\mu_1 + \gamma)I^*}{V^*} \\ 0 & -b\beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^* \right) & \mu_2 + b\beta_2 I^* \end{vmatrix}.
 \end{aligned} \tag{48}$$

FIGURE 8: The bifurcation periodic solutions of $I(t)$ with $\tau_1 = 23$ and $\tau_2 = 22, 28, 38$ (from the top to the first of the bottom). The bifurcation periodic solution is unstable with $\tau_1 = 23$ and $\tau_2 = 38$.

Δ_{11} , Δ_{12} , and Δ_{13} are the third-order determinants obtained by substituting the first, second, and third columns of Δ_1 by vector $(P_1, P_2, P_3)^T$, respectively. Δ_{21} , Δ_{22} , and Δ_{23} are also third-order determinants obtained by replacing the first, second, and third columns of Δ_2 by vector $(Q_1, Q_2, Q_3)^T$, respectively, where

$$\begin{aligned}
 P_1 &= -b\beta_1 q_3 e^{2i\omega_2^* \tau_1^*}, \\
 P_2 &= b\beta_1 q_3 e^{-2i\omega_2^* \tau_1^*} - \sigma b\beta_1 q_3 - \sigma b\beta_1 q_2 q_3, \\
 P_3 &= -b\beta_2 q_2 q_3 e^{-2i\omega_2^* \tau_2^*}, \\
 Q_1 &= -b\beta_1 \operatorname{Re} \{q_3\}, \\
 Q_2 &= (1 - \sigma)b\beta_1 \operatorname{Re} \{q_3\} - b\sigma\beta_1 \operatorname{Re} \{q_2 \bar{q}_3\}, \\
 Q_3 &= -b\beta_2 \operatorname{Re} \{q_2 \bar{q}_3\}.
 \end{aligned} \tag{49}$$

Thus, the following quantities can be calculated:

$$\begin{cases}
 c_1(0) = \frac{i}{2\omega_2^* \tau_2^*} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{1}{2}g_{21}, \\
 \mu_2 = -\frac{\operatorname{Re} \{c_1(0)\}}{\operatorname{Re} \{\lambda'(\tau_2^*)\}}, \\
 \beta_2 = 2 \operatorname{Re} \{c_1(0)\}, \\
 T_2 = -\frac{1}{\omega_2^* \tau_2^*} \left(\operatorname{Im} \{c_1(0)\} + \mu_2 \operatorname{Im} \{\lambda'(\tau_2^*)\} \right).
 \end{cases} \tag{50}$$

Therefore, based on [20], we can obtain the main results in the present section as follows.

Theorem 6. For system (2), the periodic solution is supercritical (resp., subcritical) if $\mu_2 > 0$ (resp., $\mu_2 < 0$). The bifurcating periodic solutions are orbitally asymptotically stable with an

FIGURE 9: The system appears irregular periodic behaviors when $\tau_1 = 31$ and $\tau_2 = 31$.

asymptotical phase (resp., unstable) if $\beta_2 < 0$ (resp., $\beta_2 > 0$). The period of the bifurcating periodic solutions increases (resp., decreases) if $T_2 > 0$ (resp. $T_2 < 0$).

5. Numerical Examples and Simulations

We select the parameters to simulate the above theoretical results as follows:

$$\begin{aligned}
 \Lambda_1 &= 9, b = 0.3, \\
 \beta_1 &= 0.002, \mu_1 = 0.01, \\
 \sigma &= 0.8, \gamma = 0.6, \\
 \Lambda_2 &= 9, \\
 \beta_2 &= 0.005, \\
 \mu_2 &= 0.05.
 \end{aligned} \tag{51}$$

By calculations, we can obtain that the endemic equilibrium of system (2) is $E^* = (S^*, I^*, V^*) = (102.5734, 85.6437, 129.5701)$. Under the above parameters, the condition (H1) is satisfied. According to Theorem 2, we know that

TABLE 1: The relationships among the partial immunity σ , the endemic equilibrium E^* , and the critical values τ_{10} and τ_2^* .

σ	1	0.8	0.6	0.4	0.2
S^*	97.4478	102.5734	111.3948	129.1222	170.4044
I^*	107.0566	85.6437	63.4246	41.2030	21.8956
V^*	137.2619	129.5701	117.9896	99.5023	71.3613
τ_{10}	21.5895	23.3291	26.6112	34.3647	56.7849
τ_2^*	17.8120	21.4739	25.5925	30.6267	44.0800

the endemic equilibrium E^* of system (2) is asymptotically stable when $\tau_1 = \tau_2 = 0$ (see Figure 1). When $\tau_1 > 0$, $\tau_2 = 0$, we get $p_0 = -4.7847 \times 10^{-5} < 0$, $\tau_{10} = 23.3291$, $z_1 = 0.0053$, and $f'(z_1) = 0.017 \neq 0$; from Theorem 4, the endemic equilibrium E^* is locally asymptotically stable for $\tau_1 \in (0, \tau_{10})$ and is unstable for $\tau_1 > \tau_{10}$, and a Hopf bifurcation occurs at $\tau_1 = \tau_{10}$ (see Figures 2–4). When $\tau_2 > 0$, fixing $\tau_1 = 23$ that belongs to its stable interval $(0, \tau_{10})$, then, by calculation, we obtain that a positive root of equation (22) is $\omega = 0.0933$ and $\tau_2^* = 21.4739$, which follow from Theorem 5 that the endemic equilibrium E^* is locally asymptotically stable for $\tau_2 \in (0, \tau_2^*)$

and is unstable for $\tau_2 \in (\tau_2^*, +\infty)$, a Hopf bifurcation exits at $\tau_2 = \tau_2^*$ (see Figures 5–7).

The numerical results show that $c_1(0) = 1.0266 \times 10^{-5} - 4.7626 \times 10^{-6}i$, $\mu_2 = -0.0024 < 0$, $\beta_2 = 2.0531 \times 10^{-5} > 0$, and $T_2 = 2.3115 \times 10^{-6} > 0$, which imply that, for system (2), its periodic solution is subcritical at τ_2^* and unstable, and the period of the bifurcating periodic solutions increases (see Figure 8). Also, when the two delays are greater than their critical values, the system will produce irregular periodic behaviors, which is depicted in Figure 9.

Besides, the influences of the parameter σ on the endemic equilibrium E^* and the critical values τ_{10} and τ_2^* of the two delays τ_1 and τ_2 are shown in Table 1. When σ decreases, I^* and V^* also decrease, but S^* , τ_{10} , and τ_2^* increase, which suggests that the degree of partial protection for recovered people acquired by a primary infection can change the densities of distributions of susceptible and infective people and can also affect on the stability of positive equilibrium. It is noteworthy that it can not change the direction of Hopf bifurcation and the stability of bifurcating periodic solutions.

6. Conclusions

In this paper, we considered a vector-borne disease model with two delays and reinfection based on the models proposed by [4–6]. Some theoretical analyses of this model were implemented. From the characteristic equations, firstly, we discussed the conditions of the endemic equilibrium E^* , which is locally asymptotically stable in the model without delays by the Routh-Hurwitz criterion. Then, by setting $\tau_2 = 0$, regarding τ_1 as a bifurcating parameter, and setting τ_1 in its stable interval, regarding τ_2 as a parameter, respectively, we employed the Hopf bifurcation theorem to analyze the conditions for the stability of equilibrium and existence of Hopf bifurcation. Furthermore, when τ_1 is in its stable interval, for parameter τ_2 , the directions and stability of Hopf bifurcation were investigated, and some explicit formulas were derived by using the normal form theory and center manifold theorem. In the end, our numerical examples and simulations verified the correctness of theoretical conclusions. As the facts show, the disease will be persistent under the set of parameters, and the number of the host population and vector population will fluctuate. Also, when both delays exceed their critical values, the stability of the system and the properties and behaviors of the bifurcating periodic solutions may lead to change. The oscillatory behavior and phenomenon that an increased partial immunity may give rise to instability will prompt medical workers engaged in infectious diseases to remain vigilant against the virus even if they have noticed that the number of infected populations decreases. The qualitative analysis and numerical simulation used in the research of dynamic behavior of system can provide theoretical basis for revealing the development process, transmission pattern, and change trend of diseases and give important reference to the analysis, prediction, and control of human infectious diseases.

Appendix

A. The Derivation Process of R_0 Based on the Next-Generation Method

R_0 is the dominant eigenvalue of the next generation matrix $G = FV^{-1}$ whose entries are interpreted as the number of secondary infections produced by infected vectors and hosts in the process of their infection. Here, the notation we use follows [22–24]. For system (2), we calculate

$$\begin{aligned} F &= \begin{pmatrix} 0 & \frac{b\beta_1\Lambda_1}{\mu_1} \\ \frac{b\beta_2\Lambda_2}{\mu_2} & 0 \end{pmatrix}, \\ V &= \begin{pmatrix} \mu_1 + \gamma & 0 \\ 0 & \mu_2 \end{pmatrix}, \\ G &= \begin{pmatrix} 0 & \frac{b\beta_1\Lambda_1}{\mu_1\mu_2} \\ \frac{b\beta_2\Lambda_2}{(\mu_1 + \gamma)\mu_2} & 0 \end{pmatrix}. \end{aligned} \quad (\text{A.1})$$

Thus, $R_0 = (b^2\Lambda_1\Lambda_2\beta_1\beta_2)/(\mu_1\mu_2(\mu_1 + \gamma))$. Note that we use R_0^2 as in [5], though it is called R_0 . Either way, its threshold behavior $R_0 > 1$ remains unchanged.

B. The Derivation Process of Coefficients g_{20} , g_{11} , g_{02} , and g_{21} of Equation (47)

Using the algorithms and marks presented in [20], when $\mu = 0$, the projection system of system (2) on the center manifold C_0 can be calculated. For the solution u_t of equation (29), we define $z(t) = \langle q^*, u_t \rangle$, which follows from equations (35)–(39) that

$$\dot{z}(t) = \langle q^*, \dot{u}_t \rangle = \langle A^*(0)q^*, u_t \rangle + \bar{q}^*(0)F(0, u_t) \triangleq i\omega_2^*\tau_2^*z + g(z, \bar{z}), \quad (\text{B.1})$$

where $g(z, \bar{z}) = \bar{q}^*(0)F(0, u_t) = g_{20}(\theta)(z^2/2) + g_{11}(\theta)z\bar{z} + g_{02}(\theta)(\bar{z}^2/2) + g_{21}(\theta)(z^2\bar{z}/2) \dots$.

Let

$$W(t, \theta) = u_t(\theta) - 2 \operatorname{Re} \{z(t)q(\theta)\}, \quad (\text{B.2})$$

then, on the center manifold C_0 , we can have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \quad (\text{B.3})$$

where

$$W(z, \bar{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \dots \quad (\text{B.4})$$

z and $\bar{z}$ are local coordinates on C_0 in the directions of q^*

and $\bar{q}^*$. From equations (39) and (B.2), we can see

$$\begin{aligned} u_t(\theta) &= W(t, \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta) = (1, q_2, q_3)^T e^{i\omega_2^* \tau_2^* \theta} \\ &\quad + (1, \bar{q}_2, \bar{q}_3)^T e^{-i\omega_2^* \tau_2^* \theta} + W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z\bar{z} \\ &\quad + W_{02}(\theta) \frac{\bar{z}^2}{2} + W_{30}(\theta) \frac{z^3}{6} + \dots \end{aligned} \quad (\text{B.5})$$

Therefore, we can obtain

$$\begin{aligned} u_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \dots, \\ u_{2t}(0) &= zq_2 + \bar{z}\bar{q}_2 + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots, \\ u_{3t}(0) &= zq_3 + \bar{z}\bar{q}_3 + W_{20}^{(3)}(0) \frac{z^2}{2} + W_{11}^{(3)}(0) z\bar{z} + W_{02}^{(3)}(0) \frac{\bar{z}^2}{2} + \dots, \\ u_{1t}(-1) &= ze^{-i\omega_2^* \tau_2^*} + \bar{z}e^{i\omega_2^* \tau_2^*} + W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} \\ &\quad + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + \dots, \\ u_{2t}(-1) &= zq_2 e^{-i\omega_2^* \tau_2^*} + \bar{z}\bar{q}_2 e^{i\omega_2^* \tau_2^*} + W_{20}^{(2)}(-1) \frac{z^2}{2} + W_{11}^{(2)}(-1) z\bar{z} \\ &\quad + W_{02}^{(2)}(-1) \frac{\bar{z}^2}{2} + \dots, \end{aligned}$$

$$\begin{aligned} u_{3t}(-1) &= zq_3 e^{-i\omega_2^* \tau_2^*} + \bar{z}\bar{q}_3 e^{i\omega_2^* \tau_2^*} + W_{20}^{(3)}(-1) \frac{z^2}{2} + W_{11}^{(3)}(-1) z\bar{z} \\ &\quad + W_{02}^{(3)}(-1) \frac{\bar{z}^2}{2} + \dots, \end{aligned}$$

$$\begin{aligned} u_{1t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) &= ze^{-i\omega_2^* \tau_1^*} + \bar{z}e^{i\omega_2^* \tau_1^*} + W_{20}^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) \frac{z^2}{2} \\ &\quad + W_{11}^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) z\bar{z} + W_{02}^{(1)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) \frac{\bar{z}^2}{2} + \dots, \end{aligned}$$

$$\begin{aligned} u_{2t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) &= zq_2 e^{-i\omega_2^* \tau_1^*} + \bar{z}\bar{q}_2 e^{i\omega_2^* \tau_1^*} + W_{20}^{(2)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) \frac{z^2}{2} \\ &\quad + W_{11}^{(2)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) z\bar{z} + W_{02}^{(2)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) \frac{\bar{z}^2}{2} + \dots, \end{aligned}$$

$$\begin{aligned} u_{3t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) &= zq_3 e^{-i\omega_2^* \tau_1^*} + \bar{z}\bar{q}_3 e^{i\omega_2^* \tau_1^*} + W_{20}^{(3)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) \frac{z^2}{2} \\ &\quad + W_{11}^{(3)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) z\bar{z} + W_{02}^{(3)}\left(-\frac{\tau_1^*}{\tau_2^*}\right) \frac{\bar{z}^2}{2} + \dots \end{aligned} \quad (\text{B.6})$$

Furthermore, we can get $g(z, \bar{z})$ as follows:

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0)F(0, u_t) = \bar{D}(1, \bar{q}_2^*, \bar{q}_3^*)F(0, u_t) = \bar{D}(1, \bar{q}_2^*, \bar{q}_3^*)\tau_2^* \\ &\quad \begin{pmatrix} -b\beta_1 u_{1t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) u_{3t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) \\ b\beta_1 u_{1t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) u_{3t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) - \sigma b\beta_1 u_{1t}(0) u_{3t}(0) - b\sigma\beta_1 u_{2t}(0) u_{3t}(0) \\ -b\beta_2 u_{2t}(-1) u_{3t}(-1) \end{pmatrix} \\ &= \bar{D}\tau_2^* \left\{ -b\beta_1 u_{1t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) u_{3t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) + \bar{q}_2^* \left[b\beta_1 u_{1t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) u_{3t}\left(-\frac{\tau_1^*}{\tau_2^*}\right) - \sigma b\beta_1 u_{1t}(0) u_{3t}(0) - b\sigma\beta_1 u_{2t}(0) u_{3t}(0) \right] \right. \\ &\quad \left. - \bar{q}_3^* b\beta_2 u_{2t}(-1) u_{3t}(-1) \right\}. \end{aligned} \quad (\text{B.7})$$

Substituting $u_{1t}(0)$, $u_{2t}(0)$, $u_{3t}(0)$, $u_{2t}(-1)$, $u_{3t}(-1)$, $u_{1t}(-\tau_1^*/\tau_2^*)$, and $u_{3t}(-\tau_1^*/\tau_2^*)$ into the above expression of $g(z, \bar{z})$ and comparing the coefficients yield the coefficients g_{20} , g_{11} , g_{02} , and g_{21} of equation (47).

Obviously, g_{20} , g_{11} , and g_{02} can be determined directly by substituting system parameters, but $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21} need to be calculated through some methods [25, 26].

By equations (36) and (B.2), we can get

$$\begin{aligned} \dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} &= \begin{cases} AW - 2 \operatorname{Re} \{ \bar{q}^*(0)F(0, u_t)q(\theta) \}, \theta \in [-1, 0), \\ AW - 2 \operatorname{Re} \{ \bar{q}^*(0)F(0, u_t)q(0) \} + F(0, u_t), \theta = 0, \end{cases} \\ &\triangleq AW + H(z, \bar{z}, \theta), \end{aligned} \quad (\text{B.8})$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \quad (\text{B.9})$$

Thus, we have

$$AW(t, \theta) - \dot{W} = -H(z, \bar{z}, \theta) = -H_{20}(\theta) \frac{z^2}{2} - H_{11}(\theta) z\bar{z} - H_{02}(\theta) \frac{\bar{z}^2}{2} - \dots \quad (\text{B.10})$$

From equation (B.4), we can get

$$\begin{aligned} AW(t, \theta) &= AW_{20}(\theta) \frac{z^2}{2} + AW_{11}(\theta) z\bar{z} + AW_{02}(\theta) \frac{\bar{z}^2}{2} + AW_{30}(\theta) \frac{z^3}{6} + \dots, \\ \dot{W} &= W_{20}(\theta) z\bar{z} + W_{11}(\theta) (\dot{z}\bar{z} + z\dot{\bar{z}}) + \dots = 2i\omega_2^* \tau_2^* W_{20}(\theta) \frac{z^2}{2} + \dots \end{aligned} \quad (\text{B.11})$$

Therefore, we can obtain

$$(A - 2i\omega_2^* \tau_2^*) W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta). \quad (\text{B.12})$$

For $\theta \in [-1, 0)$, we have

$$H(z, \bar{z}, \theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \quad (\text{B.13})$$

By comparing coefficients of (B.9) with (B.13), one can get

$$H_{20}(\theta) = -(g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta)), \quad (\text{B.14})$$

$$H_{11}(\theta) = -(g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta)). \quad (\text{B.15})$$

From equations (B.12) and (B.14) and the definition of A , we can obtain

$$\dot{W}_{20}(\theta) = 2i\omega_2^* \tau_2^* W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta). \quad (\text{B.16})$$

Since $q(\theta) = q(0)e^{i\omega_2^* \tau_2^* \theta} = (1, q_2, q_3)^T e^{i\omega_2^* \tau_2^* \theta}$, then according to (B.16), we can get

$$W_{20}(\theta) = \frac{i g_{20}}{\omega_2^* \tau_2^*} q(0) e^{i\omega_2^* \tau_2^* \theta} + \frac{i \bar{g}_{02}}{3\omega_2^* \tau_2^*} \bar{q}(0) e^{-i\omega_2^* \tau_2^* \theta} + E_1 e^{2i\omega_2^* \tau_2^* \theta}, \quad (\text{B.17})$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}) \in \mathbb{R}^3$.

Similarly,

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta), \quad (\text{B.18})$$

$$W_{11}(\theta) = -\frac{i g_{11}}{\omega_2^* \tau_2^*} q(0) e^{i\omega_2^* \tau_2^* \theta} + \frac{i \bar{g}_{11}}{\omega_2^* \tau_2^*} \bar{q}(0) e^{-i\omega_2^* \tau_2^* \theta} + E_2, \quad (\text{B.19})$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}) \in \mathbb{R}^3$.

Next, we seek the appropriate constant vectors E_1 and E_2 in expressions (B.17) and (B.19). From equation (B.12) and the definition of A , we can have

$$\int_{-1}^0 d\eta(\theta) W_{20}(\theta) = 2i\omega_2^* \tau_2^* W_{20}(0) - H_{20}(0), \quad (\text{B.20})$$

$$\int_{-1}^0 d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \quad (\text{B.21})$$

where $\eta(\theta) = \eta(0, \theta)$. Thus,

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_2^*(P_1, P_2, P_3)^T, \quad (\text{B.22})$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_2^*(Q_1, Q_2, Q_3)^T. \quad (\text{B.23})$$

where $P_1 = -b\beta_1 q_3 e^{2i\omega_2^* \tau_1^*}$, $P_2 = b\beta_1 q_3 e^{-2i\omega_2^* \tau_1^*} - \sigma b\beta_1 q_3 - \sigma b\beta_1 q_2 q_3$, and $P_3 = -b\beta_2 q_2 q_3 e^{-2i\omega_2^* \tau_2^*}$ and $Q_1 = -b\beta_1 \text{Re}\{q_3\}$, $Q_2 = (1 - \sigma)b\beta_1 \text{Re}\{q_3\} - b\sigma\beta_1 \text{Re}\{q_2 \bar{q}_3\}$, and $Q_3 = -b\beta_2 \text{Re}\{q_2 \bar{q}_3\}$.

Note that

$$(i\omega_2^* \tau_2^* I - \int_{-1}^0 e^{i\omega_2^* \tau_2^* \theta} d\eta(\theta)) q(0) = 0,$$

$$(-i\omega_2^* \tau_2^* I - \int_{-1}^0 e^{-i\omega_2^* \tau_2^* \theta} d\eta(\theta)) \bar{q}(0) = 0.$$

Substituting expressions (39) and (B.22) into equation (B.20), we can have

$$\left(2i\omega_2^* \tau_2^* I - \int_{-1}^0 e^{2i\omega_2^* \tau_2^* \theta} d\eta(\theta) \right) E_1 = 2\tau_2^*(P_1, P_2, P_3)^T, \quad (\text{B.24})$$

which leads to $K_1 E_1 = 2(P_1, P_2, P_3)^T$, where

$$K_1 = \begin{pmatrix} 2i\omega_2^* + \mu_1 + b\beta_1 V^* e^{-2i\omega_2^* \tau_1^*} & 0 & b\beta_1 S^* e^{-2i\omega_2^* \tau_1^*} \\ \left(\sigma - e^{-2i\omega_2^* \tau_1^*} \right) b\beta_1 V^* & 2i\omega_2^* + \mu_1 + \gamma + b\sigma\beta_1 V^* & -b\beta_1 S^* e^{-2i\omega_2^* \tau_1^*} - b\sigma\beta_1 \left(\frac{\Lambda_1}{\mu_1} - I^* - S^* \right) \\ 0 & -b\beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^* \right) e^{-2i\omega_2^* \tau_2^*} & 2i\omega_2^* + \mu_2 + b\beta_2 I^* e^{-2i\omega_2^* \tau_2^*} \end{pmatrix}. \quad (\text{B.25})$$

It follows that

$$\begin{aligned} E_1^{(1)} &= \frac{2\Delta_{11}}{\Delta_1}, \\ E_1^{(2)} &= \frac{2\Delta_{12}}{\Delta_1}, \\ E_1^{(3)} &= \frac{2\Delta_{13}}{\Delta_1}, \end{aligned} \quad (\text{B.26})$$

where $\Delta_1 = |K_1|$,

$$\begin{aligned} \Delta_{11} &= \begin{vmatrix} P_1 & 0 & b\beta_1 S^* e^{-2i\omega_2^* \tau_1^*} \\ P_2 & 2i\omega_2^* + \mu_1 + \gamma + b\sigma\beta_1 V^* & -b\beta_1 S^* e^{-2i\omega_2^* \tau_1^*} - b\sigma\beta_1 \left(\frac{\Lambda_1}{\mu_1} - I^* - S^*\right) \\ P_3 & -b\beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^*\right) e^{-2i\omega_2^* \tau_2^*} & 2i\omega_2^* + \mu_2 + b\beta_2 I^* e^{-2i\omega_2^* \tau_2^*} \end{vmatrix}, \\ \Delta_{12} &= \begin{vmatrix} 2i\omega_2^* + \mu_1 + b\beta_1 V^* e^{-2i\omega_2^* \tau_1^*} & P_1 & b\beta_1 S^* e^{-2i\omega_2^* \tau_1^*} \\ \left(\sigma - e^{-2i\omega_2^* \tau_1^*}\right) b\beta_1 V^* & P_2 & -b\beta_1 S^* e^{-2i\omega_2^* \tau_1^*} - b\sigma\beta_1 \left(\frac{\Lambda_1}{\mu_1} - I^* - S^*\right) \\ 0 & P_3 & 2i\omega_2^* + \mu_2 + b\beta_2 I^* e^{-2i\omega_2^* \tau_2^*} \end{vmatrix}, \\ \Delta_{13} &= \begin{vmatrix} 2i\omega_2^* + \mu_1 + b\beta_1 V^* e^{-2i\omega_2^* \tau_1^*} & 0 & P_1 \\ \left(\sigma - e^{-2i\omega_2^* \tau_1^*}\right) b\beta_1 V^* & 2i\omega_2^* + \mu_1 + \gamma + b\sigma\beta_1 V^* & P_2 \\ 0 & -b\beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^*\right) e^{-2i\omega_2^* \tau_2^*} & P_3 \end{vmatrix}. \end{aligned} \quad (\text{B.27})$$

Similarly, substituting expressions (B.19) and (B.23) into equation (B.21), we can have

$$\int_{-1}^0 d\eta(\theta) E_2 = -\tau_2^* (Q_1, Q_2, Q_3)^T, \quad (\text{B.28})$$

which leads to $K_2 E_2 = (Q_1, Q_2, Q_3)^T$, where

$$K_2 = \begin{pmatrix} \mu_1 + b\beta_1 V^* & 0 & b\beta_1 S^* \\ -(1-\sigma)b\beta_1 V^* & \mu_1 + \gamma + b\sigma\beta_1 V^* & -\frac{(\mu_1 + \gamma)I^*}{V^*} \\ 0 & -b\beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^*\right) & \mu_2 + b\beta_2 I^* \end{pmatrix}. \quad (\text{B.29})$$

It follows that

$$\begin{aligned} E_2^{(1)} &= \frac{\Delta_{21}}{\Delta_2}, \\ E_2^{(2)} &= \frac{\Delta_{22}}{\Delta_2}, \\ E_2^{(3)} &= \frac{\Delta_{23}}{\Delta_2}, \end{aligned} \quad (\text{B.30})$$

where $\Delta_2 = |K_2|$,

$$\begin{aligned} \Delta_{21} &= \begin{vmatrix} Q_1 & 0 & b\beta_1 S^* \\ Q_2 & \mu_1 + \gamma + b\sigma\beta_1 V^* & -\frac{(\mu_1 + \gamma)I^*}{V^*} \\ Q_3 & -b\beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^*\right) & \mu_2 + b\beta_2 I^* \end{vmatrix}, \\ \Delta_{22} &= \begin{vmatrix} \mu_1 + b\beta_1 V^* & Q_1 & b\beta_1 S^* \\ -(1-\sigma)b\beta_1 V^* & Q_2 & -\frac{(\mu_1 + \gamma)I^*}{V^*} \\ 0 & Q_3 & \mu_2 + b\beta_2 I^* \end{vmatrix}, \\ \Delta_{23} &= \begin{vmatrix} \mu_1 + b\beta_1 V^* & 0 & Q_1 \\ -(1-\sigma)b\beta_1 V^* & \mu_1 + \gamma + b\sigma\beta_1 V^* & Q_2 \\ 0 & -b\beta_2 \left(\frac{\Lambda_2}{\mu_2} - V^*\right) & Q_3 \end{vmatrix}. \end{aligned} \quad (\text{B.31})$$

Data Availability

The data included in this paper is only to demonstrate the validity of the results.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

This work was supported in part by the Fundamental Research Funds for the Central Universities, China (SWU020005), in part by the Natural Science Foundation Project of Chongqing CSTC (cstc2018jcyjAX0810), in part by the Science and Technology Research Program of Chongqing Municipal Education Commission (KJQN201801605), and in part by CQUE (KY201708B and KY20200146).

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