Supplementary Material for "The Clebsch-Gordan Coefficients and Their Application to Magnetic Resonance"

The effect of the lowering operators on spins 1/2-4 are shown in the sections below.

Section S1. SPIN (RANK) 1/2

j	m	$j(j+1)$	$m(m-1)$	$j(j+1)-m(m-1)$	$\sqrt{j(j+1)-m(m-1)}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{4}$	1	1
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	0	0

Section S2. SPIN (RANK) 1

j	m	$j(j+1)$	$m(m-1)$	$j(j+1)-m(m-1)$	$\sqrt{j(j+1)-m(m-1)}$
1	1	2	0	2	$\sqrt{2}$
1	0	2	0	2	$\sqrt{2}$
1	-1	2	2	0	0

Section S3. SPIN (RANK) 3/2

j	m	$j(j+1)$	$m(m-1)$	$j(j+1)-m(m-1)$	$\sqrt{j(j+1)-m(m-1)}$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{15}{4}$	$\frac{3}{4}$	3	$\sqrt{3}$
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{15}{4}$	$-\frac{1}{4}$	4	2
$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{15}{4}$	$\frac{3}{4}$	3	$\sqrt{3}$
$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{15}{4}$	$\frac{15}{4}$	0	0

Section S4. SPIN (RANK) 2

j	m	$j(j+1)$	$m(m-1)$	$j(j+1)-m(m-1)$	$\sqrt{j(j+1)-m(m-1)}$
2	2	6	2	4	2
2	1	6	0	6	$\sqrt{6}$
2	0	6	0	6	$\sqrt{6}$
2	-1	6	2	4	2
2	-2	6	6	0	0

Section S5. SPIN (RANK) 5/2

j	m	$j(j+1)$	$m(m-1)$	$j(j+1)-m(m-1)$	$\sqrt{j(j+1)-m(m-1)}$
$\frac{5}{2}$	$\frac{5}{2}$	$\frac{35}{4}$	$\frac{15}{4}$	5	$\sqrt{5}$
$\frac{5}{2}$	$\frac{3}{2}$	$\frac{35}{4}$	$\frac{3}{4}$	8	$\sqrt{8}$
$\frac{5}{2}$	$\frac{1}{2}$	$\frac{35}{4}$	$-\frac{1}{4}$	9	3
$\frac{5}{2}$	$-\frac{1}{2}$	$\frac{35}{4}$	$\frac{3}{4}$	8	$\sqrt{8}$
$\frac{5}{2}$	$-\frac{3}{2}$	$\frac{35}{4}$	$\frac{15}{4}$	5	$\sqrt{5}$
$\frac{5}{2}$	$-\frac{5}{2}$	$\frac{35}{4}$	$\frac{35}{4}$	0	0

Section S6. SPIN (RANK) 3

j	m	$j(j+1)$	$m(m-1)$	$j(j+1)-m(m-1)$	$\sqrt{j(j+1)-m(m-1)}$
3	3	12	6	6	$\sqrt{6}$
3	2	12	2	10	$\sqrt{10}$
3	1	12	0	12	$\sqrt{12}$
3	0	12	0	12	$\sqrt{12}$
3	-1	12	2	10	$\sqrt{10}$
3	-2	12	6	6	$\sqrt{6}$
3	-3	12	12	0	0

Section S7. SPIN (RANK) 7/2

j	m	$j(j+1)$	$m(m-1)$	$j(j+1)-m(m-1)$	$\sqrt{j(j+1)-m(m-1)}$
$\frac{7}{2}$	$\frac{7}{2}$	$\frac{63}{4}$	$\frac{35}{4}$	7	$\sqrt{7}$
$\frac{7}{2}$	$\frac{5}{2}$	$\frac{63}{4}$	$\frac{15}{4}$	12	$\sqrt{12}$
$\frac{7}{2}$	$\frac{3}{2}$	$\frac{63}{4}$	$\frac{3}{4}$	15	$\sqrt{15}$
$\frac{7}{2}$	$\frac{1}{2}$	$\frac{63}{4}$	$-\frac{1}{4}$	16	4
$\frac{7}{2}$	$-\frac{1}{2}$	$\frac{63}{4}$	$\frac{3}{4}$	15	$\sqrt{15}$
$\frac{7}{2}$	$-\frac{3}{2}$	$\frac{63}{4}$	$\frac{15}{4}$	12	$\sqrt{12}$
$\frac{7}{2}$	$-\frac{5}{2}$	$\frac{63}{4}$	$\frac{35}{4}$	7	$\sqrt{7}$
$\frac{7}{2}$	$-\frac{7}{2}$	$\frac{63}{4}$	$\frac{63}{4}$	0	0

Section S8. SPIN (RANK) 4

j	m	$j(j+1)$	$m(m-1)$	$j(j+1)-m(m-1)$	$\sqrt{j(j+1)-m(m-1)}$
4	4	20	12	8	$\sqrt{8}$
4	3	20	6	14	$\sqrt{14}$
4	2	20	2	18	$\sqrt{18}$
4	1	20	0	20	$\sqrt{20}$
4	0	20	0	20	$\sqrt{20}$
4	-1	20	2	18	$\sqrt{18}$
4	-2	20	6	14	$\sqrt{14}$
4	-3	20	12	8	$\sqrt{8}$
4	-4	20	20	0	0

Section S9. JUSTIFICATION OF $\hat{j}_{-_{m,m'}}(j) = \langle j,m|\hat{j}_{-}(j)|j,m'\rangle$ AS THE m,m' MATRIX ELEMENT OF THE OPERATOR $\hat{j}_{-}(j)$

In Equation 12b of the main text, we state that the matrix elements of the lowering operator can be evaluated with the equation $\hat{j}_{-_{m,m'}}(j) = \langle j,m|\hat{j}_{-}(j)|j,m'\rangle$. Here we provide a justification for this in a quantum mechanical context. In the primary example worked through in the main text, we concerned ourselves with the total spin of a system of two spin-1 particles. The set of $|j,m\rangle$ states form an orthonormal basis. In the example of a spin-1 particle, this orthonormal basis can be written, without loss of generality, in the manner shown in Equations S1a-S1c:

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{S1a}$$

$$|1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{S1b}$$

$$|1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{S1c}$$

The bras corresponding to these three kets are shown in Equations S2a-S2c:

$$\langle 1,1| = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{S2a}$$

$$\langle 1,0| = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \tag{S2b}$$

$$\langle 1,-1| = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \tag{S2c}$$

The lowering operator can be written as is shown in Equation S3 below:

$$\hat{j}_{-}(1) = \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \tag{S3}$$

Thus, by applying Equation 12b of the main text, we obtain for the 9 matrix elements of $\hat{j}_{-}(1)$

the results of Equations S4a-S4i:

$$\langle 1,1|\hat{j}_{-}(1)|1,1\rangle = \begin{pmatrix}1 & 0 & 0\end{pmatrix} \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} = \hat{j}_{-1,1}(1) \quad \text{(S4a)}$$

$$\langle 1,1|\hat{j}_{-}(1)|1,0\rangle = \begin{pmatrix}1 & 0 & 0\end{pmatrix} \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} = \hat{j}_{-1,0}(1) \quad \text{(S4b)}$$

$$\langle 1,1|\hat{j}_{-}(1)|1,-1\rangle = \begin{pmatrix}1 & 0 & 0\end{pmatrix} \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} = \hat{j}_{-1,-1}(1) \quad \text{(S4c)}$$

$$\langle 1,0|\hat{j}_{-}(1)|1,1\rangle = \begin{pmatrix}0 & 1 & 0\end{pmatrix} \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} = \hat{j}_{-0,1}(1) \quad \text{(S4d)}$$

$$\langle 1,0|\hat{j}_{-}(1)|1,0\rangle = \begin{pmatrix}0 & 1 & 0\end{pmatrix} \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} = \hat{j}_{-0,0}(1) \quad \text{(S4e)}$$

$$\langle 1,0|\hat{j}_{-}(1)|1,-1\rangle = \begin{pmatrix}0 & 1 & 0\end{pmatrix} \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} = \hat{j}_{-0,-1}(1) \quad \text{(S4f)}$$

$$\langle 1,-1|\hat{j}_{-}(1)|1,1\rangle = \begin{pmatrix}0 & 0 & 1\end{pmatrix} \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} = \hat{j}_{--1,1}(1) \quad \text{(S4g)}$$

$$\langle 1,-1|\hat{j}_{-}(1)|1,0\rangle = \begin{pmatrix}0 & 0 & 1\end{pmatrix} \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} = \hat{j}_{--1,0}(1) \quad \text{(S4h)}$$

$$\langle 1,-1|\hat{j}_{-}(1)|1,-1\rangle = \begin{pmatrix}0 & 0 & 1\end{pmatrix} \begin{pmatrix} \hat{j}_{-1,1}(1) & \hat{j}_{-1,0}(1) & \hat{j}_{-1,-1}(1) \\ \hat{j}_{-0,1}(1) & \hat{j}_{-0,0}(1) & \hat{j}_{-0,-1}(1) \\ \hat{j}_{--1,1}(1) & \hat{j}_{--1,0}(1) & \hat{j}_{--1,-1}(1) \end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} = \hat{j}_{--1,-1}(1) \quad \text{(S4i)}$$

For systems of different spins the same concepts presented here apply, demonstrating how Equation 12b of the main text selects the m, m' matrix element of the operator of interest.