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Research Article

Dynamics of Pair of Entangled Spin-1/2 Particles and Quantification of the Dynamics in terms of Correlations

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The dynamics of an identical pair of entangled spin-1/2 particles, both subjected to the same random magnetic field, are studied. The dynamics of the pure joint state of the pair are derived using stochastic calculus. An ensemble of such pure states is combined using the modified spin joint density matrix, and the joint relaxation time for the pair of spin-1/2 particles is obtained. The dynamics can be interpreted as a special kind of correlation involving the spatial components of the Bloch polarization vectors of the constituent entangled spin-1/2 particles.

1. Introduction

Entanglement is an important feature of quantum mechanics that is useful in the area of "quantum computing and information." Two-qubit maximally entangled states are of particular interest in the implementation of quantum communication protocols like quantum teleportation and superdense coding [1, 2]. They are also useful to generate a secured key distribution between sender and receiver for communicating information [3]. This is because entangled qubits are strongly correlated such that the behaviour of one of the qubits can decide the behaviour of the other, irrespective of the separation, provided the qubits are undisturbed. For example, consider an entangled state in which a pair of spin-1/2 particles' component of the angular momentum along a preferred direction (usually the z-axis) is zero, i.e., $(1/\sqrt{2})(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$. If one of them is measured along the z-axis and the component of the angular momentum is found to be $(1/2)\hbar$, then the result of the other will be forced to be $-(1/2)\hbar$. It is important to understand the properties of the entangled states and their behaviour in the environment in which they exist. This paper considers one such idea, in which the entangled state of a pair of spin-1/2 particles is considered and subjected to a random magnetic field. The random field arises due to various couplings and molecular motions [4]. The dynamics of a single spin-1/2 particle in its pure state under a random magnetic field has already been studied [5]. An ensemble of spins is combined using modified spin density to notice the fluctuations, and the relaxation times were obtained in the context of nuclear magnetic resonance (a phenomenon in which nuclei respond to the surrounding magnetic fields [5, 6]). Relaxation theory and spin entanglement find relevance in the context of nuclear magnetic resonance (NMR) (for example, [7, 8]). In the current article, we restrict our attention to the "extreme narrowed" case in which the autocorrelation time of the random magnetic fields driving relaxation is negligibly small and thereby the transverse/longitudinal relaxation times are equal (this same assumption applies to our considerations for both a single spin-1/2 particle and a pair of entangled spin-1/2 particles.) Additionally, here, we do not include interaction operators in the

Hamiltonian explicitly; cf. The ideas pertaining to long-lived spin states described in [7, 8]. The ideas in [5] are used and extended for an entangled state of a pair of spin-1/2 particles, and the dynamics are derived. The definition of modified spin density is also extended to a pair of entangled spins (modified spin joint density), and the joint relaxation time is obtained.

This paper is organized as follows: in Section 2, we recall important definitions related to spin-1/2 and pairs of spin-1/2 systems and the tensor representation of their density matrices to familiarize the reader with the notations. In Section 3, some important ideas in [5] are explained. Section 3 is based on the dynamics of a single spin-1/2 particle under a random magnetic field, in which the dynamics of the density matrix in the form of a stochastic differential equation (SDE) gives relaxation time and steady state. It demonstrates that in the extreme narrowed approximation, the autocorrelation frequency of the dynamics of the spin corresponds to the relaxation time of the spin, which is a familiar parameter in the context of standard approaches to NMR. Section 4 is an extended study on the dynamics of an entangled pair of spin-1/2 particles, motivated by the ideas in Section 3. The SDE pertaining to the entangled pair gives the steady state and the timescale associated for it to be reached by the joint state of the pair of spins. The correlation matrix concept is invoked to elucidate the entanglement notion and is motivated by some other entanglement-based measures in the literature. The SDE and the associated volatility are discussed for the correlation matrix components to illustrate that a maximally entangled state has stronger fluctuations than its unentangled counterpart. These ideas were developed in the present article to convey some extra intuition concerning the properties of entangled spins. Section 4 thus contains the central results of the paper, beginning with the idea of the correlation matrix, whose components are the correlations involving the Bloch polarization vector components of the constituent spins. We define the Hamiltonian for the pair of spins and derive the dynamics for a single pair of entangled spins. An ensemble of these pure states is combined using the modified spin joint density, and fluctuations in various components of the density matrix are noted. These dynamics are interpreted as correlations from the tensor representation of the density matrix. Using the idea of a correlation matrix, it is shown that entangled states have stronger scales of fluctuations than those of the unentangled states. We obtain the joint relaxation time for the pair of spins and conclude in Section 5 with a discussion of the results, including the steady state density matrix and persistence of entanglement when the constituent spins are subjected to the same random magnetic field and a timescale associated with the joint relaxation time.

2. Definitions and Prerequisites

2.1. Spin States and Their Representations. The spin-1/2 is the fundamental unit of spin system. The spin-j angular momentum observed along a preferred direction (usually z-axis) takes the eigenvalues $m = -j$ to j. The spin-1/2 eigenstates in $|j, m\rangle$ notation are

$$\left|j = \frac{1}{2}, m = \frac{1}{2}\right\rangle = |\uparrow\rangle, \quad \left|j = \frac{1}{2}, m = -\frac{1}{2}\right\rangle = |\downarrow\rangle. \tag{1}$$

In the presence of a strong and constant magnetic field applied along the z-axis (say $B_0\hat{z}$), the spin-1/2 particle tends to align *along* or *opposite to* the direction of the magnetic field, following the Boltzmann fraction [5, 6], where the minimum energy dominates (elaborated in Section 3). These states are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. Any other direction parameterized by θ, ϕ is a superposition of these two eigenstates. For example,

$$|\psi\rangle = e^{-i\phi/2}\cos\left(\frac{1}{2}\theta\right)|\uparrow\rangle + e^{i\phi/2}\sin\left(\frac{1}{2}\theta\right)|\downarrow\rangle. \tag{2}$$

This direction gives the polarization of the spin-1/2 particle. We say that the particle is polarized along $\vec{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

The spin-1/2 eigenstates can be represented using the vectors in $\mathbb{C}^2$ (two-dimensional complex vector space) as

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3}$$

so that a general quantum state of a particle polarized along $\vec{p}$ can be represented in $\mathbb{C}^2$ as

$$|\psi\rangle = \begin{pmatrix} e^{-i\phi/2}\cos\left(\frac{1}{2}\theta\right) \\ e^{i\phi/2}\sin\left(\frac{1}{2}\theta\right) \end{pmatrix}. \tag{4}$$

We denote this state as $|\nearrow\rangle$. Two spin states corresponding to oppositely polarized vectors can be considered as a basis for the spin system. The state orthogonal to $|\nearrow\rangle$ is denoted as $|\swarrow\rangle$ and is represented in $\mathbb{C}^2$ as

$$|\swarrow\rangle = \begin{pmatrix} -e^{-i\phi/2}\sin\left(\frac{1}{2}\theta\right) \\ e^{i\phi/2}\cos\left(\frac{1}{2}\theta\right) \end{pmatrix}, \tag{5}$$

whose polarization is $-\vec{p}$. Defining $|\nearrow\rangle$ and $|\swarrow\rangle$ should not confuse the readers. The intention was to convey that the two basis states can be chosen corresponding to any antipodal directions. The most general state parameterized by θ, ϕ facilitates the analysis of the dynamics with the initial state/direction given by θ, ϕ, thereby the trajectory is a random walk on the sphere that started from that initial direction.

Higher spin systems like spin-1, 3/2 can be constructed using the spin-1/2 system via tensor product $\otimes$. A spin-j state can be expressed in terms of tensor products of "$2j$" spin-1/2 states [9]. Physically, these states can correspond to a single spin-j particle or a combination of $2j$ spin-1/2 particles. The eigenstates of spin-1 system in terms of the constituent spin-1/2 states in $|j,m\rangle$ notation are

$$\begin{aligned}|1,1\rangle &= \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle = |\uparrow\rangle \otimes |\uparrow\rangle,\\ |1,-1\rangle &= \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle = |\downarrow\rangle \otimes |\downarrow\rangle,\\ |1,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle\right).\end{aligned} \tag{6}$$

In addition, we also have a spin-0 state

$$\begin{aligned}|0,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\right).\end{aligned} \tag{7}$$

The states like $|0,0\rangle, |1,0\rangle$, which cannot be expressed as product of single spin-1/2 states (as in case of $|1,1\rangle$) are called *entangled* states. Correspondingly, the measurement outcomes of the constituent spins are correlated. The states which are not entangled are *unentangled* states (such as $|1,1\rangle$). The spin-1 eigenstates span the three-dimensional*triplet* subspace of the spin-1 system and the spin-0 state spans the unidimensional *singlet* subspace. We can represent these four states in terms of the standard orthonormal vectors in $\mathbb{C}^4$ as

$$\begin{aligned}|1,1\rangle &= \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\\ |1,-1\rangle &= \begin{pmatrix}0\\0\\0\\1\end{pmatrix},\\ |1,0\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\\ |0,0\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}.,\end{aligned} \tag{8}$$

Using the basis vectors, we can also define the spin-1 states corresponding to an arbitrary direction (parameterized by θ, ϕ) along which the observable component of the angular momentum takes the eigenvalues $m = 0, \pm 1$. The three states of the triplet are defined as $|\nearrow\rangle \otimes |\nearrow\rangle$, $|\swarrow\rangle \otimes |\swarrow\rangle$ and $|s\rangle \triangleq (1/\sqrt{2})(|\nearrow\rangle \otimes |\swarrow\rangle + |\swarrow\rangle \otimes |\nearrow\rangle)$. In this article, we are interested in $|s\rangle$ to derive the dynamics of the entangled state of pair of spins.

2.2. The Spin Density Matrix and Tensor Representation. The quantum mechanical density matrix of a system containing the states $\{|\phi_m\rangle\}$ occurring with the probability $\{p_m\}$ is given by

$$\rho = \sum_m p_m \,|\,\phi_m\rangle\langle\phi_m\,|, \tag{9}$$

where $\sum_m p_m = 1$. If the decomposition above has only one state, then it is called a *pure* state. Otherwise, it is called a *mixed* state. Geometrically, spin-1/2 pure states correspond to points on the Bloch sphere, whereas mixed states correspond to points inside the sphere. This can be elaborated from the tensor representation, as we are about to demonstrate.

The density matrix of a spin-j system can be represented using some special tensors with "$2j$" indices [9]. The spin-1/2 density matrix of a pure state is given by the projection operator $|\nearrow\rangle\langle\nearrow|(=\rho)$.

$$\rho = \frac{1}{2}\begin{pmatrix}1+\cos\theta & e^{-i\phi}\sin\theta\\ e^{i\phi}\sin\theta & 1-\cos\theta\end{pmatrix}. \tag{10}$$

This can be expressed in terms of Pauli matrices and the identity matrix as (cf. [9])

$$\rho = \frac{1}{2}\sum_{i=0}^{3} X_i\sigma_i, \tag{11}$$

where $X_i = \mathrm{Tr}(\rho\sigma_i)$, $X_0 = 1$, $\sigma_0 = \mathbb{I}$, and the Pauli matrices are

$$\begin{aligned}\sigma_1 &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\\ \sigma_2 &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\\ \sigma_3 &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\end{aligned} \tag{12}$$

(X_1, X_2, X_3) is the Bloch polarization vector whose modulus is unity. Hence, pure states correspond to points on the unit sphere. Now, consider a mixture of two pure states ρ_1 and ρ_2 with probabilities $p, 1-p$. Let the Bloch unit vectors be (X_1, X_2, X_3) and (Y_1, Y_2, Y_3), respectively. The density matrix of the mixed state is

$$\rho = p\,\rho_1 + (1-p)\rho_2. \tag{13}$$

From the tensor representation of ρ, the polarization vector of the mixed state ρ is

$$\vec{P} = p(X_1, X_2, X_3) + (1-p)(Y_1, Y_2, Y_3). \tag{14}$$

The magnitude of $\vec{P}$ is less than unity because it lies on the line segment joining the points (X_1, X_2, X_3) and (Y_1, Y_2, Y_3). Hence, mixed states correspond to points inside the unit sphere (the appendix).

The density matrix for a pair of spin-1/2 particles can be expressed in terms of tensors with two indices. The density matrix of a state $|\nearrow\rangle_1 \otimes |\nearrow\rangle_2$ whose constituent spin-1/2 particles are polarized along different directions is

$$\begin{aligned}\rho &= |\nearrow\rangle_1\langle\nearrow|_1 \otimes |\nearrow\rangle_2\langle\nearrow|_2 \\ &= \left(\frac{1}{2}\sum_{i=0}^{3} X_i\sigma_i\right) \otimes \left(\frac{1}{2}\sum_{i=0}^{3} Y_i\sigma_i\right) \\ &= \frac{1}{4}\sum_{a,b=0}^{3} X_a Y_b (\sigma_a \otimes \sigma_b).\end{aligned} \tag{15}$$

$X_a Y_b = \mathrm{Tr}[\rho(\sigma_a \otimes \sigma_b)]$. The vector (X_1, X_2, X_3) is the polarization of the first spin and (Y_1, Y_2, Y_3) is the polarization of the second spin. The coefficients $X_a Y_b$ can be interpreted as correlations between the constituent qubits [10]. It should be noted that the indices 1, 2 in the above equation indicate that the constituents are polarized along different directions.

The density matrix of $|\nearrow\rangle \otimes |\nearrow\rangle$ can be expressed as

$$\begin{aligned}\rho &= |\nearrow\rangle\langle\nearrow| \otimes |\nearrow\rangle\langle\nearrow| \\ &= \left(\frac{1}{2}\sum_{i=0}^{3} X_i\sigma_i\right) \otimes \left(\frac{1}{2}\sum_{i=0}^{3} X_i\sigma_i\right) \\ &= \frac{1}{4}\sum_{a,b=0}^{3} X_{ab} (\sigma_a \otimes \sigma_b),\end{aligned} \tag{16}$$

where $X_{ab} = X_a X_b$. In this case, the constituents are polarized along same direction. From now on, we consider the states in which the constituents are polarized along same or opposite directions.

The density matrix of a pair of entangled spins can also be expressed in a similar way.

$$\begin{aligned}\rho &= |s\rangle\langle s| \\ &= \frac{1}{2}(|\nearrow\rangle\langle\nearrow| \otimes |\swarrow\rangle\langle\swarrow| + |\swarrow\rangle\langle\swarrow| \otimes |\nearrow\rangle\langle\nearrow| \\ &\quad + |\nearrow\rangle\langle\swarrow| \otimes |\swarrow\rangle\langle\nearrow| + |\swarrow\rangle\langle\nearrow| \otimes |\nearrow\rangle\langle\swarrow|).\end{aligned} \tag{17}$$

The various terms arising above can be expressed as

$$\begin{aligned}|\nearrow\rangle\langle\nearrow| &= \frac{1}{2}(\mathbb{I} + X_1\sigma_1 + X_2\sigma_2 + X_3\sigma_3), \\ |\swarrow\rangle\langle\swarrow| &= \frac{1}{2}(\mathbb{I} - X_1\sigma_1 - X_2\sigma_2 - X_3\sigma_3), \\ |\nearrow\rangle\langle\swarrow| &= \frac{1}{2}\left[(\cos\theta\cos\phi - i\sin\phi)\sigma_1 + (\cos\theta\sin\phi + i\cos\phi)\sigma_2 - (\sin\theta)\sigma_3\right], \\ |\swarrow\rangle\langle\nearrow| &= \frac{1}{2}\left[(\cos\theta\cos\phi + i\sin\phi)\sigma_1 + (\cos\theta\sin\phi - i\cos\phi)\sigma_2 - (\sin\theta)\sigma_3\right].\end{aligned} \tag{18}$$

Writing in terms of $(\sigma_a \otimes \sigma_b)$, we get

$$\rho = \frac{1}{4}\sum_{a,b=0}^{3} X_{ab}(\sigma_a \otimes \sigma_b), \tag{19}$$

with $X_{ab} (= X_{ba})$ defined as

$$\begin{aligned}X_{00} &= 1, \\ X_{a0} &= 1; a = 1, 2, 3, \\ X_{aa} &= 1 - 2X_a^2; a = 1, 2, 3, \\ X_{ab} &= -2X_a X_b; a \neq b; a, b = 1, 2, 3.\end{aligned} \tag{20}$$

In this way, we could proceed for higher spins and also extend for mixed states. We restrict ourselves to the pair of entangled spin-1/2 particles. It is interesting to see that the coefficients in the entangled state are different from those in unentangled states. Later, we also prove a well-known fact that the entangled states have stronger correlations than the unentangled ones by defining the correlation matrix.

In the presence of a random magnetic field, the polarization of the spin-1/2 pure state is affected. It changes randomly as the magnetic field changes. The polarization is a vector random process whose dynamics were derived in [5]. In the case of an entangled pair of spin-1/2 particles, the correlation matrix is a random process whose dynamics can be derived from the ideas in [5]. For more discussion on spin states, see [11, 12].

3. Dynamics of a Spin-1/2 Particle

In this section, we briefly explain the idea of [5], which considers a single spin-1/2 particle in its pure state. The dynamics in the presence of random magnetic field modelled as Gaussian white noise process can be derived using the stochastic calculus. An ensemble of spins is combined in a special way using the *modified spin density* as opposed to the conventionally used *ensemble density matrix*, which conceals the information about the fluctuations. The ensemble density matrix and associated relaxation times can be obtained from the modified spin density via ensemble averaging (law of large numbers), thereby making contact with conventional nuclear magnetic resonance (NMR).

Here, we extract and present some important ideas from [5] and use them to derive the dynamics of the entangled state. The random magnetic field experienced by the spins may be the result, *inter alia*, of molecular motions and dipolar interactions; this constitutes the environment [4]. Such microfield dynamics (here treated classically, i.e., first quantized) leads to relaxation in the case of (an ensemble of) unentangled spin-1/2 particles, wherein, in the present context, the standard relaxation time is obtained from the autocorrelation properties of the spin noise process. In this way, the amplitude of the magnetic field fluctuations determines the corresponding relaxation time. In a similar way, by extending these considerations for a single spin-1/2 particle to a pair of entangled spin-1/2 particles, the random environment is principally due to these kinds of dipole-dipole interactions. The total magnetic field experienced by a spin-1/2 particle is the sum of main field $B_0\hat{z}$ and the random field $\vec{B}_1 \propto k^{1/2}(\Gamma_t^{(x)}, \Gamma_t^{(y)}, \Gamma_t^{(z)})$. The vector random process is zero mean Gaussian whose autocorrelation function is $\langle \Gamma_t^{(a)} \Gamma_{t'}^{(b)} \rangle = \delta_{ab}\delta(t-t')$. The Hamiltonian associated with the main field B_0 is

$$\mathbf{H_0} = -\vec{\mu} \cdot \vec{B}_0 = -\frac{\gamma}{2}\vec{\sigma} \cdot B_0\hat{z} = \frac{\omega^0}{2}\sigma_3. \tag{21}$$

The spin-1/2 particle precesses about z-axis under constant field. The random Hamiltonian $\mathbf{H_1}(t)$ is

$$\begin{aligned}\mathbf{H_1}(t) &= \frac{k^{1/2}}{2}\left(\Gamma_t^{(x)}\sigma_1 + \Gamma_t^{(y)}\sigma_2 + \Gamma_t^{(z)}\sigma_3\right) \\ &= \frac{k^{1/2}}{2}\begin{pmatrix} \Gamma_t^{(z)} & \Gamma_t^{(x)} - i\,\Gamma_t^{(y)} \\ \Gamma_t^{(x)} + i\,\Gamma_t^{(y)} & -\Gamma_t^{(z)} \end{pmatrix}.\end{aligned} \tag{22}$$

The Hamiltonian above, which is fundamentally here an operator-valued stochastic process, is *represented* (in appropriate basis) as a random matrix process. In terms of Wiener differentials, we can write [13]

$$\mathbf{H_1}(t)\mathrm{d}t = \frac{k^{1/2}}{2}\begin{pmatrix} \mathrm{dW}_t^{(z)} & \mathrm{dW}_t^{(x)} - i\,\mathrm{dW}_t^{(y)} \\ \mathrm{dW}_t^{(x)} + i\,\mathrm{dW}_t^{(y)} & -\mathrm{dW}_t^{(z)} \end{pmatrix}. \tag{23}$$

Following (23), we emphasize that the Hamiltonian (to be precise) is not expressed in terms of Wiener differentials. The Hamiltonian is modelled using a Gaussian white noise process Γ_t (as in a Langevin approach [14]); thereby the Hamiltonian $\mathbf{H_1}(t)$ does not follow the Wiener process. It is actually the operator $\mathbf{H_1}(t)\mathrm{d}t$ that corresponds to Wiener differentials (via $\mathbf{H_1}(t) \sim \Gamma_t$, $\mathbf{H_1}(t)\mathrm{d}t \sim \mathrm{dW}_t$). In other words, the Wiener process (which corresponds directly to the spin phase) is obtained from the Gaussian white noise process via the time integral; thus $\Gamma_t = \mathrm{dW}_t/\mathrm{d}t$, in the sense of Langevin. In such an *extreme narrowed* treatment the autocorrelation time of the random magnetic field is negligibly small (corresponding mathematically to the Dirac delta-function autocorrelation for the white noise process); in a more general description, beyond extreme narrowing, and of relevance to slow molecular motions for instance, the magnetic field noise could instead be modelled by a stochastic process with finite (i.e., positive) autocorrelation time.

The total Hamiltonian affecting the spin-1/2 is $\mathbf{H_0} + \mathbf{H_1}(t)$. The dynamics can be derived assuming the spin is initially polarized along an arbitrary direction with $\mathbf{H_0} = \mathbf{0}$ (switching off the main field). The solution to the equation of motion of the density matrix integrated up to second order gives the dynamics of the spin.

$$i\frac{\mathrm{d}\rho}{\mathrm{d}t} = [\mathbf{H}, \rho]. \tag{24}$$

The equation of motion integrated up to second-order gives [5, 15]

$$\mathrm{d}\rho(t) = -i\,\mathrm{d}t[\mathbf{H_1}(t), \rho(0)] - \mathrm{d}t\int_0^t \mathrm{d}\tau[\mathbf{H_1}(t), [\mathbf{H_1}(t-\tau), \rho(0)]]. \tag{25}$$

The Wiener terms appearing in the expression of $\mathbf{H_1}(t)\mathrm{d}t$ in (23) are such that $\mathrm{d}W_t^2 = \mathrm{d}t$ and all other higher powers $\mathrm{d}W_t^n = 0$ for $n > 2$. Therefore, the second-order solution is *exact* [4, 5]. Assuming the initial state $\rho(0)$ as the pure state projection operator (10).

$$\rho(0) = \frac{1}{2}\begin{pmatrix} 1 + \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1 - \cos\theta \end{pmatrix}. \tag{26}$$

The components of $\mathrm{d}\rho(t)$ on the left side of the equation of motion (25) (up to second-order following the property of Wiener differential) are

$$\begin{aligned} \mathrm{d}\left(\cos\theta_t\right) &= -\sin\theta_t \mathrm{d}\theta_t - \frac{1}{2}\cos\theta_t \mathrm{d}\theta_t^2, \\ \mathrm{d}\left(e^{i\phi_t}\sin\theta_t\right) &= e^{i\phi_t}\left(\cos\theta_t \mathrm{d}\theta_t - \frac{1}{2}\sin\theta_t \mathrm{d}\theta_t^2 + i\sin\theta_t d\phi_t - \frac{1}{2}\sin\theta_t d\phi_t^2\right). \end{aligned} \tag{27}$$

Comparing the stochastic differential equations (SDEs) on both sides of equation (25), we get

$$\mathrm{d}\theta_t = \frac{1}{2}k\cot\theta_t \mathrm{d}t + k^{1/2}\mathrm{d}W_t^{\theta}, \tag{28}$$

$$\mathrm{d}\phi_t = \frac{k^{1/2}}{\sin\theta_t}\mathrm{d}W_t^{\phi}, \tag{29}$$

where

$$\begin{aligned} \mathrm{d}W_t^{\theta} &= \cos\phi_t \mathrm{d}W_t^{(y)} - \sin\phi_t \mathrm{d}W_t^{(x)}, \\ \mathrm{d}W_t^{\phi} &= \sin\theta_t \mathrm{d}W_t^{(z)} - \cos\theta_t\left(\cos\phi_t \mathrm{d}W_t^{(x)} + \sin\phi_t \mathrm{d}W_t^{(y)}\right). \end{aligned} \tag{30}$$

The term with the double commutator in the evolution equation (25) contains the following integral which is evaluated using the properties of Wiener differentials (cf. [5]):

$$\int_0^t \mathrm{d}W_t^{(a)} \circ dW_{t-\tau}^{(b)} = -\frac{1}{2}\mathrm{d}t\,\delta_{ab}, \tag{31}$$

where $a, b = x, y, z$. The product "$\circ$" in the integral should be understood in the Stratonovich sense (cf. [5]). From the above SDEs equations (28) and (29), we can obtain the dynamics of the density matrix as

$$\mathrm{d}\rho_t = k\,\mathrm{d}t\left(\frac{1}{2}\mathbb{I} - \rho_t\right) + \frac{1}{2}k^{1/2}\begin{pmatrix} f & g^* \\ g & -f \end{pmatrix}, \tag{32}$$

where $f = -\sin\theta_t \mathrm{d}W_t^{\theta}$ and $g = e^{i\phi_t}(\cos\theta_t \mathrm{d}W_t^{\theta} + i\mathrm{d}W_t^{\phi})$.

The dynamics can also be derived using the concept of rotational diffusion on a unit sphere by considering Laplacian in spherical coordinates (cf. [5]). The joint probability distribution of the diffusion process is

$$p_{\theta\phi} = \frac{1}{4\pi}\sin\theta, \tag{33}$$

θ, ϕ are statistically independent. Therefore,

$$\begin{aligned} p_{\theta} &= \frac{1}{2}\sin\theta, 0 \le \theta \le \pi, \\ p_{\phi} &= \frac{1}{2\pi}, 0 \le \phi \le 2\pi. \end{aligned} \tag{34}$$

The dynamics of a spin-1/2 particle implies the dynamics of the spin-1/2 Bloch vector. Using the tensor representation, the dynamics of the components of the vector (X_1, X_2, X_3) are obtained as

$$\begin{aligned} \mathrm{d}X_1 &= -k\,\mathrm{d}t\,X_1 + k^{1/2}\left(\cos\theta_t\cos\phi_t \mathrm{d}W_t^{\theta} - \sin\phi_t \mathrm{d}W_t^{\phi}\right), \\ \mathrm{d}X_2 &= -k\,\mathrm{d}t\,X_2 + k^{1/2}\left(\cos\theta_t\sin\phi_t \mathrm{d}W_t^{\theta} + \cos\phi_t \mathrm{d}W_t^{\phi}\right), \\ \mathrm{d}X_3 &= -k\,\mathrm{d}t\,X_3 - k^{1/2}\sin\theta_t \mathrm{d}W_t^{\theta}. \end{aligned} \tag{35}$$

For an ensemble of spin-1/2 population, the *modified spin density* is defined [5] as

$$\Sigma_t = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\rho_t^{(j)}, \tag{36}$$

as opposed to the mean density matrix $\widehat{\Sigma}_t$

$$\widehat{\Sigma}_t = \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\rho_t^{(j)} = \mathbb{E}(\rho), \tag{37}$$

which can be expressed in terms of the modified spin density as

$$\widehat{\Sigma}_t = \lim_{N\longrightarrow\infty}\frac{1}{\sqrt{N}}\Sigma_t. \tag{38}$$

Since the density matrix can be associated with the Bloch vector, the advantage of modified spin density is that it gives information about the stochastic volatility/variances due to the random fields affecting the spin Bloch vector. Pertaining to each pure state in the ensemble, we define $\mathrm{d}\chi_t^{(j)} = \sin\theta_t^{(j)}\mathrm{d}W_t^{\theta^{(j)}}$ (see (32)), so that

$$\begin{aligned} \mathrm{d}\chi_t &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathrm{d}\chi_t^{(j)}, \\ \lim_{N\to\infty}\mathrm{d}\chi_t^2 &= \mathbb{E}\left(\sin^2\theta\right)\mathrm{d}t = \frac{2}{3}\mathrm{d}t, \\ \mathrm{d}\chi_t &= \sqrt{\frac{2}{3}}\mathrm{d}W_t. \end{aligned} \tag{39}$$

For some Wiener process W_t, such that $\mathrm{d}W_t^2 = \mathrm{d}t$.

Also consider $\mathrm{d}\zeta_t^{(j)} = e^{i\phi_t^{(j)}}(\cos\theta_t^{(j)}\mathrm{d}W_t^{\theta^{(j)}} + i\,\mathrm{d}W_t^{\phi^{(j)}})$, so that

$$
\begin{aligned}
\mathrm{d}\zeta_t &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\mathrm{d}\zeta_t^{(j)},\\
\lim_{N\to\infty}\mathrm{d}\zeta_t^2 &= \mathbb{E}\left[e^{2i\phi}\left(\cos^2\theta-1\right)\right]\mathrm{d}t\\
&= \mathbb{E}\left(e^{2i\phi}\right)\mathbb{E}\left[\left(\cos^2\theta-1\right)\right]\mathrm{d}t = 0,\\
\lim_{N\to\infty}\mathrm{d}\zeta_t\,\mathrm{d}\zeta_t^* &= \mathbb{E}\left[\left(\cos^2\theta+1\right)\right]\mathrm{d}t\\
&= \frac{4}{3}\mathrm{d}t,\\
\mathrm{d}\zeta_t &= \frac{2}{\sqrt{3}}\mathrm{d}\xi_t,
\end{aligned}
\tag{40}
$$

for some complex Wiener process ξ_t, such that $\mathrm{d}\xi_t^2 = 0, \mathrm{d}\xi_t \mathrm{d}\xi_t^* = \mathrm{d}t$. The coefficients 2/3 and 4/3 we obtained above correspond to the variances in the longitudinal and transverse spin components (i.e., $X_3 = \cos\theta$ and $X_1 + iX_2 = e^{i\phi}\sin\theta$), respectively. When the effect of random field is prevalent, the statistical information associated with it cannot be ignored. The SDE of the modified spin density is

$$
\mathrm{d}\Sigma_t = \left(\frac{N^{1/2}}{2}\mathbb{I}-\Sigma_t\right)k\,\mathrm{d}t + \frac{1}{2}k^{1/2}\begin{pmatrix} -\sqrt{\frac{2}{3}}\mathrm{d}W_t & \frac{2}{\sqrt{3}}\mathrm{d}\xi_t^* \\ \frac{2}{\sqrt{3}}\mathrm{d}\xi_t & \sqrt{\frac{2}{3}}\mathrm{d}W_t \end{pmatrix}. \tag{41}
$$

The mean density matrix can be obtained from (38) as

$$
\mathrm{d}\widehat{\Sigma}_t = \left(\frac{1}{2}\mathbb{I}-\widehat{\Sigma}_t\right)k\,\mathrm{d}t, \tag{42}
$$

$$
\widehat{\Sigma}_t = \frac{1}{2}\mathbb{I} + \left(\widehat{\Sigma}_0 - \frac{1}{2}\mathbb{I}\right)e^{-kt}. \tag{43}
$$

The modified spin density matrix Σ_t can be expressed in tensors to reveal the variances in each of the spatial components of the spin Bloch vector. $\widehat{\Sigma}_t$ can also be expressed using tensors to see how the mean Bloch vector components $\overline{X}_1, \overline{X}_2, \overline{X}_3$ decay exponentially to zero (geometrically the centre of the sphere). From the expression of the ensemble density matrix (43), we see that the steady state as $t \longrightarrow \infty$ is the maximally mixed state $(1/2)\mathbb{I}$, which is geometrically the centre of the Bloch sphere. In the presence of the main field, the steady state density matrix is given by the Boltzmann fraction [5]. The spin longitudinal and transverse relaxation times are each equal to k^{-1}.

3.1. The Boltzmann Density Matrix. In the presence of main field $B_0\widehat{z}$, the density matrix ρ_0 under steady state assumes Boltzmann distribution [5, 15].

$$
\rho_0 = \frac{\exp\left(-(\hbar\mathbf{H_0}/k_BT)\right)}{\mathrm{Tr}\{\exp\left(-\hbar\mathbf{H_0}/k_BT\right)\}}, \tag{44}
$$

where k_B is the Boltzmann constant, T is the temperature, and $\mathbf{H_0}$ is the Hamiltonian associated with the main field $B_0\widehat{z}$ (see equation (21)).

$$
\begin{aligned}
\mathbf{H_0} &= \frac{1}{2}\omega^0\sigma_3\\
&= \frac{1}{2}\omega^0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}
\tag{45}
$$

Therefore, the matrix exponential in the expression of ρ_0 is

$$
\exp\left(-\frac{\hbar\mathbf{H_0}}{k_BT}\right) = \begin{pmatrix} \exp\left(-\frac{\hbar\omega^0}{2k_BT}\right) & 0 \\ 0 & \exp\left(\frac{\hbar\omega^0}{2k_BT}\right) \end{pmatrix}. \tag{46}
$$

For practical purposes, $|\hbar\omega^0/2k_BT| \ll 1$ (cf. [6]). So, we can approximate the exponential terms in the above matrix expression as

$$
\exp\left(\pm\frac{\hbar\omega^0}{2k_BT}\right) \cong 1 \pm \frac{\hbar\omega^0}{2k_BT}. \tag{47}
$$

The density matrix ρ_0 can be written as

$$
\begin{aligned}
\rho_0 &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\hbar\omega^0}{4k_BT}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\\
\rho_0 &= \frac{1}{2}\mathbb{I} - \frac{\hbar\omega^0}{4k_BT}\sigma_3.
\end{aligned}
\tag{48}
$$

Since $\omega^0 < 0$, the quantity $-(\hbar\omega^0/4k_BT) > 0$. We denote it as p. So, ρ_0 becomes

$$
\rho_0 = \begin{pmatrix} \frac{1}{2}+p & 0 \\ 0 & \frac{1}{2}-p \end{pmatrix}, \tag{49}
$$

which can be expressed as

$$
\rho_0 = \left(\frac{1}{2}+p\right)|\uparrow\rangle\langle\uparrow| + \left(\frac{1}{2}-p\right)|\downarrow\rangle\langle\downarrow|. \tag{50}
$$

This means that spins in the lower energy state $|\uparrow\rangle$ slightly outnumber the spins in higher energy state $|\downarrow\rangle$ (as p is a small positive number). The dynamics were obtained assuming $B_0 = 0$. The steady state density matrix in this case is $\rho_0 = (1/2)\mathbb{I}$, which complies with equation (43) as $t \longrightarrow \infty$.

4. The Dynamics of Entangled State of Pair of Spin-1/2 Particles

The entangled pair of spin-1/2 particles is studied in the same way as we did for the case of spin-1/2. We begin with a single pair of entangled spin-1/2 particles in their pure state and combine using the modified spin joint density to capture the fluctuations; the dynamics are then interpreted in terms of correlations. This section is organized as follows: we define the correlation matrix and prove that the entangled states have stronger correlations than the unentangled states based on the positive semi-definiteness of the correlation matrix. We consider the entangled state $|s\rangle$ and derive the dynamics using the equation of motion. We interpret the dynamics of the components of the density matrix $|s\rangle\langle s|$ using tensor representation.

4.1. The Correlation Matrix. The correlation matrix we define here quantifies quantum correlations between the constituent spins in the language of classical probability. Recall the density matrix of a pure state consisting of a pair of spins polarized in different directions, $|\nearrow\rangle_1 \otimes |\nearrow\rangle_2$ is

$$\begin{aligned}\rho &= |\nearrow\rangle_1\langle\nearrow|_1 \otimes |\nearrow\rangle_2\langle\nearrow|_2 \\ &= \left(\frac{1}{2}\sum_{i=0}^{3} X_i\sigma_i\right) \otimes \left(\frac{1}{2}\sum_{i=0}^{3} Y_i\sigma_i\right) \\ &= \frac{1}{4}\left(\sum_{a,b=0}^{3} X_a Y_b \left(\sigma_a\right) \otimes \sigma_b\right).\end{aligned} \tag{51}$$

$X_a Y_b = \mathrm{Tr}[\rho(\sigma_a \otimes \sigma_b)]$. As explained in Section 2, (X_1, X_2, X_3) is the polarization of the first spin and (Y_1, Y_2, Y_3) is the polarization of the second spin. We define a tensor $\widehat{C}_{ij}$ as

$$\widehat{C}_{ij} = \frac{1}{2}\left(\sigma_i \otimes \sigma_j + \sigma_j \otimes \sigma_i\right). \tag{52}$$

The components of the correlation matrix are defined as

$$C_{ij} = \mathrm{Tr}\left(\rho \widehat{C}_{ij}\right). \tag{53}$$

In case of two spins with different polarization vectors,

$$\begin{aligned}C_{ii} &= X_i Y_i, \\ C_{ij} &= \frac{1}{2}\left(X_i Y_j + X_j Y_i\right).\end{aligned} \tag{54}$$

Similarly, the correlation matrices can be defined for states like $|\nearrow\rangle \otimes |\nearrow\rangle$, $|\nearrow\rangle \otimes |\swarrow\rangle$, $|s\rangle \triangleq (1/\sqrt{2})(|\nearrow\rangle \otimes |\swarrow\rangle + |\swarrow\rangle \otimes |\nearrow\rangle)$. The state $|s\rangle$ is called maximally entangled state as we demonstrate using the correlation matrix. A general entangled state can be represented as

$$|\chi\rangle = e^{-i\beta/2}\cos\left(\frac{1}{2}\alpha\right)|\nearrow\rangle \otimes |\swarrow\rangle + e^{i\beta/2}\sin\left(\frac{1}{2}\alpha\right)|\swarrow\rangle \otimes |\nearrow\rangle. \tag{55}$$

For $\alpha = 0$, $\beta = 0$ and $\alpha = \pi$, $\beta = 0$, we obtain the unentangled states $|\nearrow\rangle \otimes |\swarrow\rangle$ and $|\swarrow\rangle \otimes |\nearrow\rangle$ as special cases, respectively. Now the Bloch vectors of $|\nearrow\rangle$ and $|\swarrow\rangle$ are $\overrightarrow{X}$ and $-\overrightarrow{X}$, respectively. Consider the correlation matrix defined from the tensor representation of density matrix $|\chi\rangle\langle\chi|$. The components of the matrix $\mathbf{C}$ are

$$\begin{aligned}C_{ii} &= -X_i^2 + \sin\alpha\cos\beta\left(1 - X_i^2\right), \\ C_{ij} &= -X_i X_j - X_i X_j \sin\alpha\cos\beta.\end{aligned} \tag{56}$$

If we represent $\overrightarrow{X}$ by the column vector $\mathbf{X}$, the correlation matrix $\mathbf{C}$ is

$$\mathbf{C} = -\mathbf{X}\mathbf{X}^{\mathbf{T}} + \sin\alpha\cos\beta\left(\mathbb{I} - \mathbf{X}\mathbf{X}^{\mathbf{T}}\right). \tag{57}$$

For $\alpha = 0$, π and $\beta = 0$, the correlation matrix corresponds to the unentangled states $|\nearrow\rangle \otimes |\swarrow\rangle$ and $|\swarrow\rangle \otimes |\nearrow\rangle$. The correlation coefficient

$$\max_{\alpha,\beta}\ \sin\alpha\cos\beta = 1, \tag{58}$$

is attained for $\alpha = (\pi/2)$, $\beta = 0$ and $\alpha = -(\pi/2)$, $\beta = \pi$. The quantum states corresponding to these pairs are $|s\rangle$ and $-i|s\rangle$. Since the overall phase does not distinguish quantum states as both correspond to the same point in their state space (the complex projective space $\mathbb{CP}^n$ for an $n+1$ dimensional quantum system). Both the pairs of (α, β) correspond to $|s\rangle$. The correlation matrix of the maximally entangled state $\mathbf{C}^{(\mathbf{s})}$ is

$$\mathbf{C}^{(\mathbf{s})} = \left(\mathbb{I} - 2\,\mathbf{X}\mathbf{X}^{\mathbf{T}}\right). \tag{59}$$

The correlation matrix of the general entangled state:

$$\mathbf{C} = -\mathbf{X}\mathbf{X}^{\mathbf{T}} + \sin\alpha\cos\beta\left(\mathbb{I} - \mathbf{X}\mathbf{X}^{\mathbf{T}}\right). \tag{60}$$

Also,

$$\mathbf{C}^{(\mathbf{s})} - \mathbf{C} = (1 - \sin\alpha\cos\beta)\left(\mathbb{I} - \mathbf{X}\mathbf{X}^{\mathbf{T}}\right), \tag{61}$$

$(1 - \sin\alpha\cos\beta) \geq 0$ and the matrix $(\mathbb{I} - \mathbf{X}\mathbf{X}^{\mathbf{T}})$ has the positive semi-definite property as its eigenvalues are $0, 1, 1$. Therefore, the matrix $\mathbf{C}^{(\mathbf{s})} - \mathbf{C} \succcurlyeq \mathbf{0}$. So, we can write

$$\mathbf{C} \preccurlyeq \mathbf{C}^{(\mathbf{s})} \forall \alpha, \beta. \tag{62}$$

Thus, $\mathbf{C}^{(\mathbf{s})}$ is the maximal correlation matrix corresponding to the maximum value of $\sin\alpha\cos\beta$.Consistently, $|s\rangle$ is referred to as *maximally entangled state* with strong correlations between the constituent spins.

4.2. Dynamics of Entangled Spin-1/2 Pair under Random Magnetic Field. In this section, we derive the dynamics of a pair of entangled spin-1/2 particles experiencing the same random magnetic field. The joint random Hamiltonian affecting the entangled pair is

$$\mathbf{H}^{(\mathbf{tot})} = \mathbf{H}^{(\mathbf{1})} \otimes \mathbb{I} + \mathbb{I} \otimes \mathbf{H}^{(\mathbf{2})}. \tag{63}$$

Since both the spins are under the same field $\mathbf{H}^{(\mathbf{1})} = \mathbf{H}^{(\mathbf{2})} = \mathbf{H}$,

$$\mathbf{H} = \frac{k^{1/2}}{2}\left(\Gamma_t^{(x)}\sigma_1 + \Gamma_t^{(y)}\sigma_2 + \Gamma_t^{(z)}\sigma_3\right) = \frac{k^{1/2}}{2}\begin{pmatrix} \Gamma_t^{(z)} & \Gamma_t^{(x)} - i\,\Gamma_t^{(y)} \\ \Gamma_t^{(x)} + i\,\Gamma_t^{(y)} & -\Gamma_t^{(z)} \end{pmatrix}. \quad (64)$$

In terms of Wiener differentials, we can write [13] (according to Langevin)

$$\mathbf{H}\,dt = \frac{k^{1/2}}{2}\begin{pmatrix} \mathrm{dW}_t^{(z)} & \mathrm{dW}_t^{(x)} - i\,\mathrm{dW}_t^{(y)} \\ \mathrm{dW}_t^{(x)} + i\,\mathrm{dW}_t^{(y)} & -\mathrm{dW}_t^{(z)} \end{pmatrix}. \quad (65)$$

The total joint Hamiltonian can be expressed as

$$\mathbf{H}^{(\mathbf{tot})}\mathrm{d}t = \frac{k^{1/2}}{2}\begin{pmatrix} 2\,\mathrm{d}W_t^{(z)} & \mathrm{d}W_t^{(w)}* & \mathrm{d}W_t^{(w)}* & 0 \\ \mathrm{d}W_t^{(w)} & 0 & 0 & \mathrm{d}W_t^{(w)}* \\ \mathrm{d}W_t^{(w)} & 0 & 0 & \mathrm{d}W_t^{(w)}* \\ 0 & \mathrm{d}W_t^{(w)} & \mathrm{d}W_t^{(w)} & -2\,\mathrm{d}W_t^{(z)} \end{pmatrix}, \quad (66)$$

where $\mathrm{d}W_t^{(w)} \triangleq \mathrm{dW}_t^{(x)} + i\,\mathrm{dW}_t^{(y)}$. Let the initial joint density matrix of the entangled pair be $\rho(0) = |s\rangle\langle s|$. The equation of motion of the joint density matrix integrated up to second-order [5, 15] is then

$$\mathrm{d}\rho(t) = -i\,\mathrm{d}t\left[\mathbf{H}^{(\mathbf{tot})}(t), \rho(0)\right] - \mathrm{d}t\int_0^t \mathrm{d}\tau\left[\mathbf{H}^{(\mathbf{tot})}(t), \left[\mathbf{H}^{(\mathbf{tot})}(t-\tau), \rho(0)\right]\right]. \quad (67)$$

Comparing the expressions on both sides of the equation, we get the SDEs of θ, ϕ as

$$\mathrm{d}\theta_t = \frac{1}{2}k\cot\theta_t \mathrm{d}t + k^{1/2}\mathrm{d}W_t^{\theta}, \quad (68)$$

$$\mathrm{d}\phi_t = \frac{k^{1/2}}{\sin\theta_t}\mathrm{d}W_t^{\phi}, \quad (69)$$

where

$$\mathrm{d}W_t^{\theta} = \cos\phi_t \mathrm{d}W_t^{(y)} - \sin\phi_t \mathrm{d}W_t^{(x)}, \quad (70)$$

$$\mathrm{d}W_t^{\phi} = \sin\theta_t \mathrm{d}W_t^{(z)} - \cos\theta_t\left(\cos\phi_t \mathrm{d}W_t^{(x)} + \sin\phi_t \mathrm{d}W_t^{(y)}\right). \quad (71)$$

The term with the double commutator in the equation of motion (67) contains the following integral [5]:

$$\int_0^t \mathrm{d}W_t^{(a)} \circ \mathrm{d}W_{t-\tau}^{(b)} = -\frac{1}{2}\mathrm{d}t\,\delta_{ab}, \quad (72)$$

where $a, b = x, y, z$ as in case of spin-1/2. The product "∘" in the integral should be understood in the Stratonovich sense (cf. [5]). The dynamics of the density matrix can be obtained as

$$\mathrm{d}\rho_t = 3k(\rho_M - \rho_t)\mathrm{d}t + k^{1/2}\begin{pmatrix} g_1 & -g_2^* & -g_2^* & g_3^* \\ -g_2 & -g_1 & -g_1 & g_2^* \\ -g_2 & -g_1 & -g_1 & g_2^* \\ g_3 & g_2 & g_2 & g_1 \end{pmatrix}, \quad (73)$$

where the fluctuating terms in the matrix can be read off as

$$g_1 = \sin\theta_t\cos\theta_t \mathrm{d}W_t^{\theta},$$
$$g_2 = \frac{1}{2}e^{i\phi_t}\left(\cos 2\theta_t \mathrm{d}W_t^{\theta} + i\cos\theta_t \mathrm{d}W_t^{\phi}\right), \quad (74)$$
$$g_3 = -e^{i2\phi_t}\left(\sin\theta_t\cos\theta_t \mathrm{d}W_t^{\theta} + i\sin\theta_t \mathrm{d}W_t^{\phi}\right).$$

The density matrix ρ_M in (73) is given by

$$\rho_M = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (75)$$

which can be expressed as

$$\rho_M = \frac{1}{3}|1,1\rangle\langle 1,1| + \frac{1}{3}|1,0\rangle\langle 1,0| + \frac{1}{3}|1,-1\rangle\langle 1,-1|. \quad (76)$$

By the simulation of noise associated with the random magnetic field, ρ_M can be computed via ensemble averaging of SDE (73). For instance, let Q_t be a stochastic process being characterized by the SDE.

$$\mathrm{d}Q_t = b_t\mathrm{d}t + S_t\mathrm{d}W_t, \quad (77)$$

where b_t is the drift and S_t is the volatility. Following [16], we can obtain b_t and S_t from the process Q_t that is being observed. $b_t = \mathbb{E}(\mathrm{d}Q_t)/\mathrm{d}t$ where $\mathbb{E}(.)$ is the expectation (mean/average) operator. This means that we need an ensemble of sample paths Q_t to compute the drift b_t. In other words, generate a large number of $Q(t)$ and $Q(t+\Delta t)$ and compute the ensemble mean of the difference $Q(t+\Delta t) - Q(t)$ so that via the law of large numbers the drift can be computed as $b_t = \lim_{N\longrightarrow\infty} (1/N\Delta t)\sum_{i=1}^{N}(Q_i(t+\Delta t) - Q_i(t))$. Using this idea, the steady state density matrix ρ_M can be verified from the SDE (73). The steady state density matrix denoted by $\rho_{M,\text{num}}$ computed by simulating noise pertaining to random magnetic field looks as shown below:

$$\rho_{M,\text{num}} = \begin{pmatrix} a & \delta_1 & \delta_1 & \delta_2 \\ \delta_1^* & b & b & -\delta_1 \\ \delta_1^* & b & b & -\delta_1 \\ \delta_2^* & -\delta_1^* & -\delta_1^* & a \end{pmatrix}, \tag{78}$$

where $a \approx (1/3)$, $b \approx (1/6)$ and $|\delta_1|, |\delta_2| \approx 0$ (cf. ρ_M in (75)).

Using the modified spin density, we can combine an ensemble of pairs to reveal the fluctuations that are the origin of joint relaxation. Later, we give the interpretation for various components of the matrix in the SDE (73). The modified spin pair density Σ_t is defined [5] as

$$\Sigma_t = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\rho_t^{(j)}. \tag{79}$$

Now we calculate the following terms to write the SDE of Σ_t:

$$\begin{aligned} G_1 &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N} g_1^{(j)}, \\ G_1 &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N} \sin\theta_t^{(j)}\cos\theta_t^{(j)} dW_t^{\theta^{(j)}} = \frac{2}{15}\mathrm{d}t, \\ \lim_{N\to\infty}\left[G_1^2\right] &= \mathbb{E}\left(\sin^2\theta\cos^2\theta\right)\mathrm{d}t, \\ G_1 &= \sqrt{\frac{2}{15}}\mathrm{d}W_t, \end{aligned} \tag{80}$$

for some Wiener process W_t, such that $\mathrm{d}W_t^2 = \mathrm{d}t$. This is possible from the *stability* property of the Gaussian distribution. The sum of Gaussian distributed random variables is Gaussian distributed [5, 14]. From the independence of the random variables $g_1^{(j)}$, the squared sum, $[G_1^2]$, corresponds to the sample mean of the independent and identically distributed (iid) random variables $\sin^2\theta_t^{(j)}\cos^2\theta_t^{(j)}\mathrm{d}t$ via the law of large numbers, the sample mean tends to the expected value as $N \longrightarrow \infty$. Similarly,

$$G_2 = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} g_2^{(j)}. \tag{81}$$

Since θ, ϕ are statistically independent,

$$\begin{aligned} \lim_{N\longrightarrow\infty}\left[G_2^2\right] &= \frac{1}{4}\mathbb{E}\left(e^{i2\phi}\right)\mathbb{E}\left(\cos 2\theta\,\mathrm{d}W_t^{\theta} + i\cos\theta\,\mathrm{d}W_t^{\phi}\right)^2\mathrm{d}t = 0, \\ \lim_{N\longrightarrow\infty} G_2^* G_2 &= \frac{1}{4}\mathbb{E}\left(\cos^2 2\theta + \cos^2\theta\right)\mathrm{d}t = \frac{1}{5}\mathrm{d}t, \\ G_2 &= \sqrt{\frac{1}{5}}d\xi_t, \end{aligned} \tag{82}$$

for some complex Wiener process ξ_t, such that $\mathrm{d}\xi_t\mathrm{d}\xi_t^* = \mathrm{d}t$ and $\mathrm{d}\xi_t^2 = 0$.

$$G_3 = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} g_3^{(j)}. \tag{83}$$

Since θ, ϕ are statistically independent,

$$\begin{aligned} \lim_{N\longrightarrow\infty}\left[G_3^2\right] &= \mathbb{E}\left(e^{i4\phi}\right)\mathbb{E}\left(\sin\theta\cos\theta\,\mathrm{d}W_t^{\theta} + i\sin\theta\,\mathrm{d}W_t^{\phi}\right)^2\mathrm{d}t = 0, \\ \lim_{N\longrightarrow\infty} G_3^* G_3 &= \mathbb{E}\left(\sin^2\theta\cos^2\theta + \sin^2\theta\right)\mathrm{d}t = \frac{4}{5}\mathrm{d}t, \\ G_3 &= \sqrt{\frac{4}{5}}\mathrm{d}\zeta_t. \end{aligned} \tag{84}$$

Thus, we can write for some complex Wiener process ζ_t, such that $\mathrm{d}\zeta_t^2 = 0$ and $\mathrm{d}\zeta_t\mathrm{d}\zeta_t^* = \mathrm{d}t$. It is to be noticed that $G_i's$ and $g_i's$ are stochastic differential forms. All the expected values are calculated based on the SDEs obtained from θ, ϕ whose joint probability density is

$$p_{\theta\phi} = \frac{1}{4\pi}\sin\theta;\ 0 \le \theta \le \pi, 0 \le \phi \le 2\pi. \tag{85}$$

Now we can write the SDE of the modified spin pair density.

$$\mathrm{d}\Sigma_t = 3k\,\mathrm{d}t\left(N^{1/2}\rho_M - \Sigma_t\right) + k^{1/2}\begin{pmatrix} G_1 & -G_2^* & -G_2^* & G_3^* \\ -G_2 & -G_1 & -G_1 & G_2^* \\ -G_2 & -G_1 & -G_1 & G_2^* \\ G_3 & G_2 & G_2 & G_1 \end{pmatrix}. \tag{86}$$

In the above SDE, $N^{1/2}$ in the numerator shows that the components of Σ_t become large as $N \longrightarrow \infty$. Physically, it means that the overall signal obtained from an ensemble of particles also becomes large as N becomes large. Following [5], we can now define the mean density matrix $\widehat{\Sigma}_t = \mathbb{E}(\rho)$. The SDE (73) of ρ_t is mean reverting. The entangled pair of spins asymptotically attain the state ρ_M of (75) at the rate $3k$. This can be understood better by calculating $\mathbb{E}(\rho)$ via the law of large numbers,

$$\mathbb{E}(\rho) = \widehat{\Sigma}_t = \lim_{N\longrightarrow\infty}\frac{1}{\sqrt{N}}\Sigma_t. \tag{87}$$

The (deterministic) differential equation in $\widehat{\Sigma}_t$ akin to conventional NMR can be obtained from the SDE (86) of Σ_t as

$$\mathrm{d}\widehat{\Sigma}_t = \left(\rho_M - \widehat{\Sigma}_t\right)3k\,\mathrm{d}t. \tag{88}$$

Solving, we get

$$\widehat{\Sigma}_t = \rho_M + \left(\widehat{\Sigma}_0 - \rho_M\right)e^{-3kt}, \tag{89}$$

for some initial mean density matrix $\widehat{\Sigma}_0$. The joint relaxation time constant is $(3k)^{-1}$.

4.3. Interpretation of Spin Dynamics in terms of Correlations. The tensor representations of the density matrix yield various correlations involving the spatial components of the Bloch vectors of the constituent spins. The dynamics of the density matrix determine the dynamics of these correlation components. As we already mentioned, in the case of spin-1/2, the Bloch vector is a random process, whereas in the case of an entangled pair of spin-1/2 particles, it is the correlation matrix that is the random process of primary relevance. The SDEs can be derived from the correlation components of a single pair of entangled spins. Let $\mathbf{C}^{(\mathbf{s})}$ denote the correlation matrix of the maximally entangled state $|s\rangle$. The SDEs of various components of the (symmetric) matrix $\mathbf{C}^{(\mathbf{s})}$ (see equation (59)) are

$$\mathrm{d}C_{11}^{(s)} = k\mathrm{d}t\left(1 - 3C_{11}^{(s)}\right) - 4k^{1/2}\left(\sin\theta_t\cos\theta_t\cos^2\phi_t\mathrm{d}W_t^{\theta} - \sin\theta_t\sin\phi_t\cos\phi_t\mathrm{d}W_t^{\phi}\right), \tag{90}$$

$$\mathrm{d}C_{22}^{(s)} = k\mathrm{d}t\left(1 - 3C_{22}^{(s)}\right) - 4k^{1/2}\left(\sin\theta_t\cos\theta_t\sin^2\phi_t\mathrm{d}W_t^{\theta} + \sin\theta_t\sin\phi_t\cos\phi_t\mathrm{d}W_t^{\phi}\right), \tag{91}$$

$$\mathrm{d}C_{33}^{(s)} = k\mathrm{d}t\left(1 - 3C_{33}^{(s)}\right) + 4k^{1/2}\sin\theta_t\cos\theta_t\mathrm{d}W_t^{\theta}, \tag{92}$$

$$\mathrm{d}C_{12}^{(s)} = -3k\mathrm{d}t\,C_{12}^{(s)} - 2k^{1/2}\left(\sin\theta_t\cos\theta_t\sin 2\phi_t\mathrm{d}W_t^{\theta} + \sin\theta_t\cos 2\phi_t\mathrm{d}W_t^{\phi}\right), \tag{93}$$

$$\mathrm{d}C_{23}^{(s)} = -3k\mathrm{d}t\,C_{23}^{(s)} - 2k^{1/2}\left(\cos 2\theta_t\sin\phi_t\mathrm{d}W_t^{\theta} + \cos\theta_t\cos\phi_t\mathrm{d}W_t^{\phi}\right), \tag{94}$$

$$\mathrm{d}C_{13}^{(s)} = -3k\mathrm{d}t\,C_{13}^{(s)} - 2k^{1/2}\left(\cos 2\theta_t\cos\phi_t\mathrm{d}W_t^{\theta} - \cos\theta_t\sin\phi_t\mathrm{d}W_t^{\phi}\right). \tag{95}$$

Figures 1–6 show the sample paths for the six correlation components of the correlation matrix $\mathbf{C}^{(\mathbf{s})}$ pertaining to $\alpha = (\pi/2)$, $\beta = 0$ (maximally entangled state). These are obtained numerically from the solution of the SDEs (90)–(95). These SDEs can also be derived from the SDEs of X_1, X_2, X_3 that we obtained in case of a single spin-1/2. We can also derive the SDEs for the correlation components of a general entangled state. Let $\mathbf{C}$ denote the correlation matrix of a general entangled state:

$$|\chi\rangle = e^{-i\beta/2}\cos\left(\frac{1}{2}\alpha\right)|\nearrow\rangle\otimes|\swarrow\rangle + e^{i\beta/2}\sin\left(\frac{1}{2}\alpha\right)|\swarrow\rangle\otimes|\nearrow\rangle. \tag{96}$$

These plots are obtained via the SDEs of θ_t, ϕ_t, assuming the Wiener fluctuating terms of the random magnetic field components are zero mean Gaussian $\mathrm{d}W_t \sim N\sqrt{\mathrm{d}t}$ where N is the normal random variable (zero mean and variance unity).

The dynamics of the general correlation matrix components can be derived from the definition, using the SDEs of X_1, X_2, X_3, as

$$\mathrm{d}C_{11} = k\,\mathrm{d}t\,(2\sin\alpha\cos\beta - 1 - 3C_{11}) - 2k^{1/2}\,(1 + \sin\alpha\cos\beta)\left(\sin\theta_t\cos\theta_t\cos^2\phi_t\mathrm{d}W_t^{\theta} - \sin\theta_t\sin\phi_t\cos\phi_t dW_t^{\phi}\right), \tag{97}$$

$$\mathrm{d}C_{22} = k\,\mathrm{d}t\,(2\sin\alpha\cos\beta - 1 - 3C_{22}) - 2k^{1/2}\,(1 + \sin\alpha\cos\beta)\left(\sin\theta_t\cos\theta_t\sin^2\phi_t\mathrm{d}W_t^{\theta} + \sin\theta_t\sin\phi_t\cos\phi_t\mathrm{d}W_t^{\phi}\right), \tag{98}$$

$$\mathrm{d}C_{33} = k\,\mathrm{d}t\,(2\sin\alpha\cos\beta - 1 - 3C_{33}) + 2k^{1/2}\,(1 + \sin\alpha\cos\beta)\left(\sin\theta_t\cos\theta_t\mathrm{d}W_t^{\theta}\right), \tag{99}$$

$$\mathrm{d}C_{12} = -3k\,\mathrm{d}t\,C_{12} - k^{1/2}\,(1 + \sin\alpha\cos\beta)\left(\sin\theta_t\cos\theta_t\sin 2\phi_t\mathrm{d}W_t^{\theta} + \sin\theta_t\cos 2\phi_t\mathrm{d}W_t^{\phi}\right), \tag{100}$$

$$\mathrm{d}C_{23} = -3k\,\mathrm{d}t\,C_{23} - k^{1/2}\,(1 + \sin\alpha\cos\beta)\left(\cos 2\theta_t\sin\phi_t\mathrm{d}W_t^{\theta} + \cos\theta_t\cos\phi_t\mathrm{d}W_t^{\phi}\right), \tag{101}$$

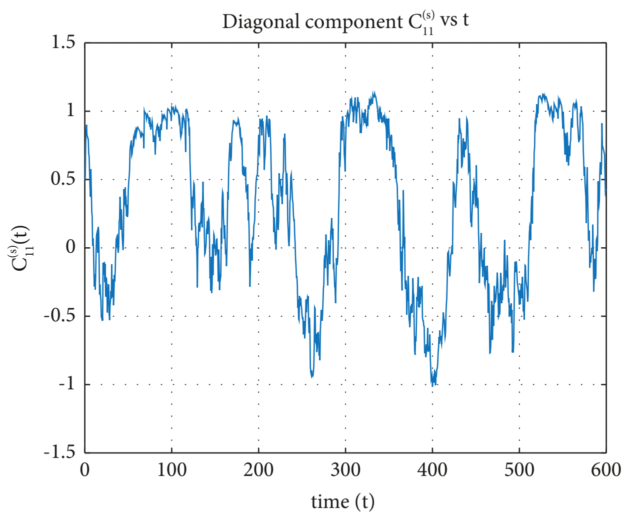

Figure 1: The diagonal correlation component sample path for $k = 0.01$, $dt = 0.05$.

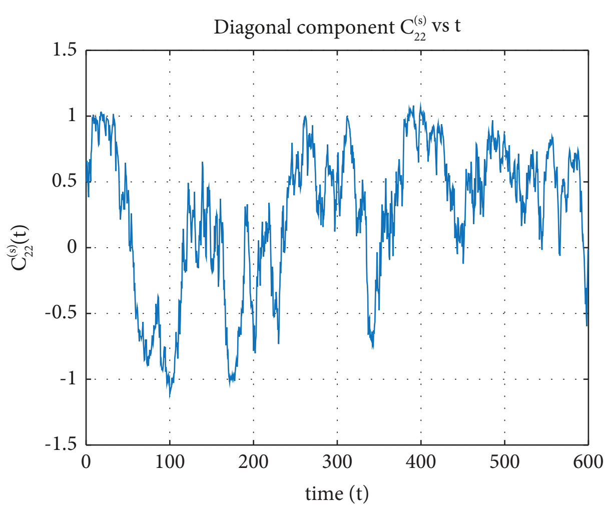

Figure 2: The diagonal correlation component sample path for $k = 0.01$, $dt = 0.5$.

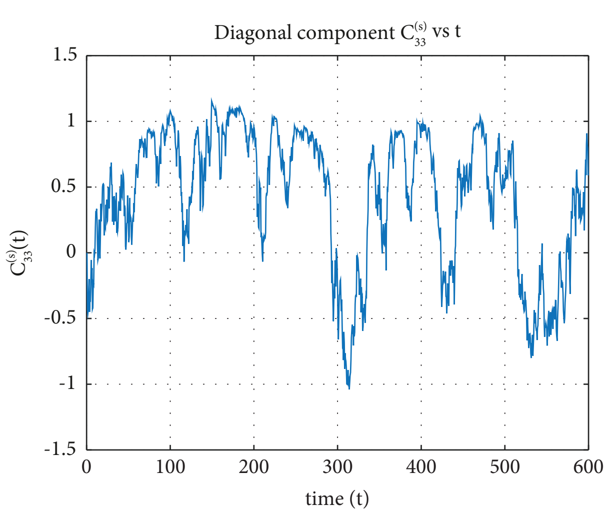

Figure 3: The diagonal correlation component sample path for $k = 0.01$, $dt = 0.5$.

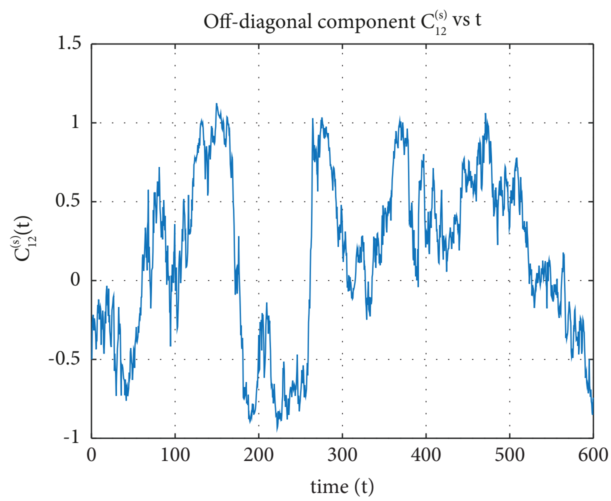

Figure 4: The off-diagonal correlation component sample path for $k = 0.01$, $dt = 0.5$.

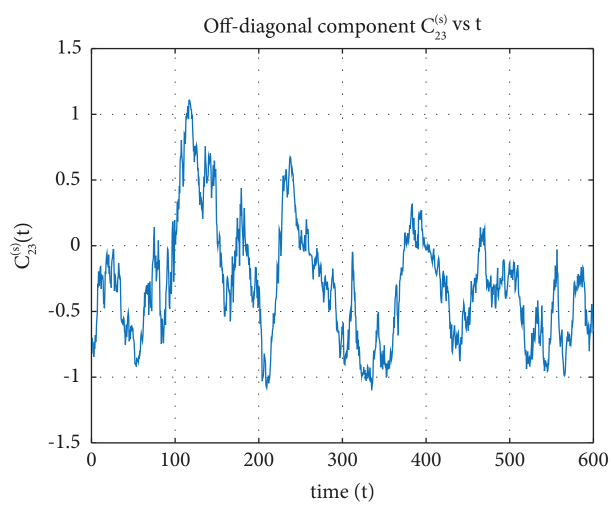

Figure 5: The off-diagonal correlation component sample path for $k = 0.01$, $dt = 0.5$.

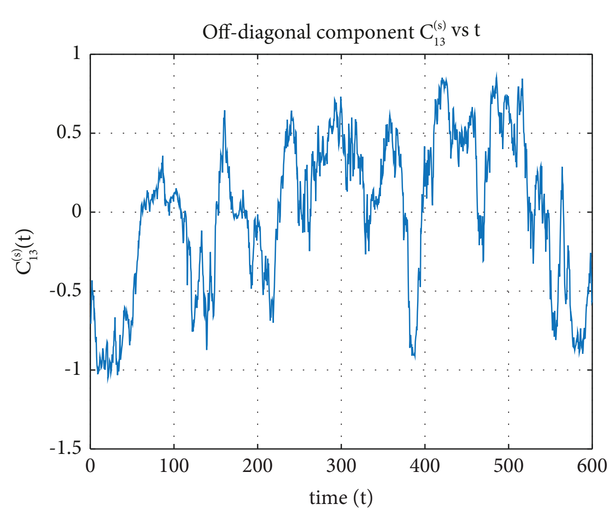

Figure 6: The off-diagonal correlation component sample path for $k = 0.01$, $dt = 0.5$.

$$\mathrm{d}C_{13} = -3k\,\mathrm{d}t\,C_{13} - k^{1/2}\,(1 + \sin\alpha\cos\beta)\Big(\cos 2\theta_t \cos\phi_t \mathrm{d}W_t^{\theta} - \cos\theta_t \sin\phi_t \mathrm{d}W_t^{\phi}\Big). \tag{102}$$

The stochastic volatility $S(R_t, t)$ in a general SDE,

$$\mathrm{d}R_t = b(R_t, t)\mathrm{d}t + S(R_t, t)\mathrm{d}W_t, \tag{103}$$

is a measure of how the magnitude of fluctuations in a stochastic process R_t vary randomly. In the above SDE, the drift term $b(R_t, t)$ is in general a function of R_t, t, in which case, the SDE corresponds to an *Itô* process R_t. Else, R_t is a diffusion.

Finally, we consider the mean squared volatility in the SDEs of the correlation components (97)–(102) in case of a general entangled state $|\chi\rangle$ in order to interpret how the fluctuations occur in a maximally entangled state, in contrast to an unentangled state. We therefore calculate $\mathbb{E}\,(dC_{ab}^2)/\mathrm{d}t$ in each case [$\mathbb{E}\,(.)$can be calculate d using $p_{\theta\phi}$]. The matrix is

$$\left[\frac{\mathbb{E}\big(dC_{ab}^2\big)}{\mathrm{d}t}\right] = k(1 + \sin\alpha\cos\beta)^2 \begin{pmatrix} \frac{8}{15} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{8}{15} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{8}{15} \end{pmatrix}, \tag{104}$$

with $\max_{\alpha,\beta} \sin\alpha\cos\beta = 1$. So, the infinitesimal fluctuation matrix corresponding to the maximally entangled state is

$$\left[\frac{\mathbb{E}\Big(dC^{(s)2}_{ab}\Big)}{\mathrm{d}t}\right] = k \begin{pmatrix} \frac{32}{15} & \frac{8}{5} & \frac{8}{5} \\ \frac{8}{5} & \frac{32}{15} & \frac{8}{5} \\ \frac{8}{5} & \frac{8}{5} & \frac{32}{15} \end{pmatrix}, \tag{105}$$

from which it is easy to see that

$$\frac{\mathbb{E}\Big(dC^{(s)2}_{ab}\Big)}{\mathrm{d}t} \geq \frac{\mathbb{E}\big(dC_{ab}^2\big)}{\mathrm{d}t}. \tag{106}$$

These results can be numerically verified by considering an ensemble of sample paths and calculating the mean squared volatility. To illustrate, consider equation (103) to generate a large number of sample paths to obtain their squared differences $(R_i(t+\Delta t) - R_i(t))^2$. Now, find the sample mean of the squared differences, which tends to the expected value of the squared stochastic volatility via the law of large numbers. In other words, $\mathbb{E}\,(S_t^2)$ is equal to $\lim_{N\longrightarrow\infty} (1/N\,\Delta t)\sum_{i=1}^{N} (R_i(t+\Delta t) - R_i(t))^2$, which is obtained numerically to verify the above results for an ensemble of sample paths of the correlation components to obtain the matrix elements of (105). To elaborate, fluctuating random magnetic field components are generated following the Wiener property $\mathrm{d}W_t = N\sqrt{\mathrm{d}t}$, where N is the standard normal random variable has a zero mean and unit variance.

Following equations (68)–(71), the quantities $\theta(t), \theta(t+\mathrm{d}t)$ and $\phi(t), \phi(t+\mathrm{d}t)$ can be obtained, which could be used to generate the components $C_{ab}^{(s)}(t), C_{ab}^{(s)}(t+\mathrm{d}t)$ via the SDEs (90)–(95). By considering an ensemble of sample paths for the correlation components, the expected value of the squared stochastic volatility can be calculated using the law of large numbers, as mentioned above. As an illustration, we shall simulate as demonstrated above to calculate $[\mathbb{E}\,(\mathrm{d}C^{(s)2}_{ab})/k\,\mathrm{d}t]$ numerically to obtain the matrix $[\mathbb{E}\,(dC^{(s)2}_{ab})/k\,\mathrm{d}t]_{\mathrm{num}}$ components in equation (105) (notice k in the denominator to avoid the dependence of k for convenience). The (expected) squared volatility matrix evaluated numerically for two runs is given below (for comparing with equation (105)):

$$\left[\frac{\mathbb{E}\Big(\mathrm{d}C^{(s)2}_{ab}\Big)}{k\,\mathrm{d}t}\right]_{\mathrm{num}} = \begin{pmatrix} 2.0784 & 1.5626 & 1.5343 \\ 1.5626 & 2.0784 & 1.5343 \\ 1.5343 & 1.5343 & 2.0630 \end{pmatrix},$$
$$\left[\frac{\mathbb{E}\Big(\mathrm{d}C^{(s)2}_{ab}\Big)}{k\,\mathrm{d}t}\right]_{\mathrm{num}} = \begin{pmatrix} 2.1956 & 1.6523 & 1.6599 \\ 1.6523 & 2.1956 & 1.6599 \\ 1.6599 & 1.6599 & 2.1732 \end{pmatrix}. \tag{107}$$

The correlation matrix structure we have described is therefore an effective measure to quantify the entanglement between constituent spins. Indeed, there are other measures to quantify entanglement, such as *concurrence*, which leads to the idea of *entanglement of formation* (cf. [17] for the definitions of concurrence and entanglement of formation). In this case, we assumed that the initial state is maximally entangled and the constituents are subjected to the same random magnetic field, meaning the maximally entangled state evolves into another maximally entangled pure state. As a result of the definition [17], the concurrence (and thus the entanglement of formation) *remains constant* for the maximally entangled state. In other words, the plot of concurrence versus time is a constant function. The correlation matrix we defined in this article for a maximally entangled state evolves as a stochastic matrix process as the random magnetic field fluctuates and thus is an Ito process in its own right via Ito's formula, the trajectories of which are plotted (Figures 1–6). The compact matrix SDE (of the maximally entangled state) for the process $\mathbf{C}_t^{(\mathbf{s})}$ can be obtained from the component SDEs as

$$d\mathbf{C}_t^{(\mathbf{s})} = k\mathrm{d}t\left(\mathbb{1} - 3\,\mathbf{C}_t^{(\mathbf{s})}\right) + k^{1/2}\mathbf{F}_t \tag{108}$$

where $\mathbf{F}_t$ is a matrix containing the fluctuation terms appearing from (90)–(95). It is necessary to mention the general Ito formula for any kind of entanglement quantifier that could be defined depending on the context. Let $G_t = g(X_t)$ be a function/functional (an entanglement quantifier in this context), dependent on a stochastic process X_t, where

X_t is characterized by the SDE $\mathrm{d}X_t = b_t\mathrm{d}t + S_t\mathrm{d}W_t$ analogous to the dependence of $\mathbf{C}_t^{(\mathbf{s})}$ on (θ_t, ϕ_t). Then, SDE of G_t is obtained as

$$\begin{aligned}\mathrm{d}G_t &= \frac{\partial g}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\mathrm{d}X_t^2 \\ &= \left(b_t\frac{\partial g}{\partial x} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}S_t^2\right)\mathrm{d}t + S_t\frac{\partial g}{\partial x}\mathrm{d}W_t.\end{aligned} \tag{109}$$

Similarly, we can proceed with the dependence of two or more stochastic processes (cf. [13, 14]).

5. Conclusion

This article could be taken as a pedagogical one. The spin-1/2 density matrix of a pure state follows the mean-reverting dynamics, from which the timescale and steady state can be obtained, as understood previously in the context of NMR [5]. As higher spin states are obtained from sums of tensor products, the dynamics of the density matrix of a pure state of an unentangled pair of spin-1/2 particles is obtained from the total Hamiltonian acting on the tensor product $|\nearrow\rangle \otimes |\nearrow\rangle$. Thereby, the dynamics of an unentangled spin pair can be obtained by considering the tensor product of individual spin-1/2 SDEs that are obtained in [5]. On the other hand, the entangled spin pair density matrix is not obtained with mere tensor products. Accordingly, in the current article, the dynamics of a pure state of an entangled pair of spins are derived based on the single spin-1/2 dynamics of [5], and it is found that the entangled spins reach a joint steady state within a certain timescale. Further, it could be considered that a new computational approach is followed, viz., the introduction of the correlation matrix (that resembles the correlation matrix of classical probability theory), and that was motivated by various measures of entanglement in the literature and its ensuing extremization, which is the rationale for the term and agrees with the name "maximally entangled state." This is further illuminating to the concept of entanglement and so included. Further, the idea of volatility in a general SDE is explained and analysed where it arises in the SDEs of the correlation components to demonstrate that, from this point of view, maximally entangled state states have a stronger scale of fluctuations compared to their unentangled counterparts. In the context of NMR, it is useful to see how a pair of entangled state of spin-1/2 particles evolve when subjected to the same random magnetic field, which is caused by various disturbances and molecular motion [4, 15], and how long it takes to relax to the steady state. The relaxation time gives the rate at which an entangled state evolves into the steady-state density matrix ρ_M. This further helps to study the dynamics of the maximally entangled state and the concept of disentanglement time when the constituent spins are subjected to independent random magnetic fields (see [18] for more discussion).

Appendix

The Spin-1/2 Density Matrix and the Bloch Sphere

The spin-1/2 density matrix for a pure state $\rho = |\nearrow\rangle\langle\nearrow|$ corresponding to an arbitrary direction of polarization, parameterized by θ, ϕ, is given by

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix}, \tag{A.1}$$

which can be expressed in terms of Pauli matrices and the identity matrix as

$$\rho = \frac{1}{2}\sum_{i=0}^{3} X_i\sigma_i, \tag{A.2}$$

where $X_i = \mathrm{Tr}(\rho\sigma_i)$, $X_0 = 1$, $\sigma_0 = \mathbb{I}$, and the Pauli matrices are

$$\begin{aligned}\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned} \tag{A.3}$$

(X_1, X_2, X_3) is the polarization vector. Pure states correspond to points on the Bloch sphere. Let ρ_1 and ρ_2 be two pure states with probabilities p and $1-p$, respectively, in a mixture. Let $\overrightarrow{X} \triangleq (X_1, X_2, X_3)$ and $\overrightarrow{Y} \triangleq (Y_1, Y_2, Y_3)$ be the polarization vectors associated with ρ_1 and ρ_2. Then, the density matrix ρ is given by

$$\rho = p\rho_1 + (1-p)\rho_2. \tag{A.4}$$

Using the tensor representation, the polarization vector $\overrightarrow{P}$ associated with ρ is given by

$$\overrightarrow{P} = p\overrightarrow{X} + (1-p)\overrightarrow{Y}, \tag{A.5}$$

which is a point inside the Bloch sphere as it lies on the line segment joining $\overrightarrow{X}$ and $\overrightarrow{Y}$. Geometrically, $\overrightarrow{P}$ divides the join of $\overrightarrow{X}$ and $\overrightarrow{Y}$ in the ratio $(1-p)\colon p$. If we denote the distance of $\overrightarrow{P}$ from $\overrightarrow{X}$, $\overrightarrow{Y}$ as d_{PX} and d_{PY}, respectively, we can write [5]

$$d_{PX}\colon d_{PY} = (1-p)\colon p. \tag{A.6}$$

As $p \longrightarrow 1$, $d_{PX} \longrightarrow 0$ meaning $\overrightarrow{P}$ approaches $\overrightarrow{X}$ which corresponds to the pure state ρ_1. This is evident from the density matrix expression of ρ. A mixed state ρ can be visualised as a point P (say) inside the Bloch sphere. There are infinitely many line segments passing through P that intersect the Bloch sphere at two points corresponding to pure states ρ_1 and ρ_2. Correspondingly, ρ can represent infinitely many mixtures. For example, consider the following density matrix as shown below:

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}. \tag{A.7}$$

This can be shown as a mixture of pure states $|\uparrow\rangle$ and $|\downarrow\rangle$ as

$$\rho = \frac{3}{4}|\uparrow\rangle\langle\uparrow| + \frac{1}{4}|\downarrow\rangle\langle\downarrow|. \tag{A.8}$$

Geometrically, ρ is represented by the point $(0, 0, 1/2)$ inside the sphere. Let $|\alpha\rangle\langle\alpha|$, $|\beta\rangle\langle\beta|$ be two pure states representing points $(\pm(\sqrt{3}/2), 0, (1/2))$.

$$< mfencedopen == \frac{1}{2}\begin{pmatrix} \frac{3}{2} & -\left(\frac{\sqrt{3}}{2}\right) \\ -\left(\frac{\sqrt{3}}{2}\right) & \frac{1}{2} \end{pmatrix}. \tag{A.9}$$

The point $(0, 0, 1/2)$ representing ρ is the midpoint of $(\pm(\sqrt{3}/2), 0, (1/2))$. Therefore, ρ can be shown as an equal mixture of $|\alpha\rangle, |\beta\rangle$.

$$\rho = \frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|\beta\rangle\langle\beta|. \tag{A.10}$$

However, ρ can be uniquely decomposed into orthogonal states that correspond to antipodal points on the join of P and the centre of the sphere. In the above example, the decomposition of ρ in terms of orthogonal states $|\uparrow\rangle$ and $|\downarrow\rangle$ is unique. Let $|\pm x\rangle$ be two pure states represented by the points $(\pm 1, 0, 0)$. Then

$$\begin{aligned} |+x\rangle\langle+x| &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ |-x\rangle\langle-x| &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned} \tag{A.11}$$

Then, ρ can be shown to be a mixture of $|\uparrow\rangle$ and $|\downarrow\rangle$, $|\pm x\rangle$ as

$$\rho = \frac{2}{3}|\uparrow\rangle\langle\uparrow| + \frac{1}{6}|\downarrow\rangle\langle\downarrow| + \frac{1}{12}|+x\rangle\langle+x| + \frac{1}{12}|-x\rangle\langle-x|. \tag{A.12}$$

This is to show that the number of constituent states in a mixture can be greater than the dimension of the system, which is 2 in the case of spin-1/2. (For more discussion about the properties of density matrix, see [5, 19, 20]).

Data Availability

This research is theoretical and there is no applicability for enclosing the data. Only simulated (random) data are included, and the methodology for producing this is detailed in the text.

Disclosure

The funder of the research is not involved in the manuscript writing, editing, approval, and decision to publish.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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