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Research Article

The Karush-Kuhn-Tucker Optimality Conditions for the Fuzzy Optimization Problems in the Quotient Space of Fuzzy Numbers

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We propose the solution concepts for the fuzzy optimization problems in the quotient space of fuzzy numbers. The Karush-Kuhn-Tucker (KKT) optimality conditions are elicited naturally by introducing the Lagrange function multipliers. The effectiveness is illustrated by examples.

1. Introduction

The fuzzy set theory was introduced initially in 1965 by Zadeh [1]. After that, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. The fuzziness occurring in the optimization problems is categorized as the fuzzy optimization problems. Bellman and Zadeh [2] inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. After this motivation and inspiration, there come out a lot of works dealing with the fuzzy optimization problems.

Zimmermann and Rödder initially applied fuzzy sets theory to the linear programing problems and linear multiobjective programing problems by using the aspiration level approach [3–6]. Durea and Tammer [7] derived the Lagrange multiplier rules for fuzzy optimization problems using the concept of abstract subdifferential. Bazine et al. [8] developed some fuzzy optimality conditions for fractional multiobjective optimization problems. In 2013, the solution approach for the lower level fuzzy optimization problem and the fuzzy bilevel optimization problem was investigated by Budnitzki [9]. Panigrahi et al. [10] extended and generalized these concepts to fuzzy mappings of several variables using the approach due to Buckley and Feuring [11] for fuzzy differentiation and derived the KKT conditions for the constrained fuzzy minimization problems. Wu [12, 13] presented the KKT conditions for the optimization problems with convex constraints and fuzzy-valued objective functions on the class of all fuzzy numbers by considering the concepts of Hausdorff metric and Hukuhara difference. Chalco-Cano et al. [14] discussed the KKT optimality conditions for a class of fuzzy optimization problems using strongly generalized differentiable fuzzy-valued functions, which is a concept of differentiability for fuzzy mappings more general than the Hukuhara differentiability.

These above results of fuzzy optimization are based on well-known and widely used algebraic structures of fuzzy numbers and the differentiability of fuzzy mappings was based on the concept of Hukuhara difference. However these operations can have some disadvantages for both theory and practical application. In [15], Qiu et al. intuitively showed a method of finding the inverse operation in the quotient space of fuzzy numbers based on the Mareš equivalence relation [16, 17], which have the desired group properties for the addition operation [18–20] midpoint function. As an application of the main results, it is shown that if we identify every fuzzy number with the corresponding equivalence class, there would be more differentiable fuzzy functions than what is found in the literature. In [21] Qiu et al. further investigated the differentiability properties of such functions in the quotient space of fuzzy numbers. In this paper, the KKT optimality conditions for the constrained fuzzy optimization problems in the quotient space of fuzzy numbers are derived.

2. Preliminaries

We start this section by recalling some pertinent concepts and key lemmas from the function of bounded variation, fuzzy numbers, and fuzzy number equivalence classes which will be used later.

Definition 1 (see [22]). Let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be of bounded variation if there exists a $C > 0$ such that

$$\sum_{i=1}^{n} \left| f(x_{i-1}) - f(x_{i-1}) \right| \le C \tag{1}$$

for every partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ on $[a, b]$. The set of all functions of bounded variation on $[a, b]$ is denoted by $\mathrm{BV}[a, b]$.

Definition 2 (see [22]). Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. The total variation of f on $[a, b]$, denoted by $V_a^b(f)$, is defined by

$$V_a^b(f) = \sup_p \sum_{i=1}^{n} \left| f(x_{i-1}) - f(x_i) \right|, \tag{2}$$

where p represents all partitions of $[a, b]$.

Lemma 3 (see [22]). *Let $f, g \in \mathrm{BV}[a, b]$, and then we have the following:*

(1) $cf + dg \in \mathrm{BV}[a, b]$ *and*

$$V_a^b(cf + dg) \le |c| V_a^b(f) + |d| V_a^b(g) \tag{3}$$

for any contents $c, d \in \mathbb{R}$.

(2) $f \cdot g \in \mathrm{BV}[a, b]$ *and*

$$\begin{aligned} V_a^b(f \cdot g) \le V_a^b(f) \sup_{x \in [a,b]} |g(x)| \\ + V_a^b(g) \sup_{x \in [a,b]} |f(x)|. \end{aligned} \tag{4}$$

Lemma 4 (see [22]). *Every monotonic function $f : [a, b] \to \mathbb{R}$ is of bounded variation and*

$$V_a^b(f) = |f(a) - f(b)|. \tag{5}$$

Any mapping $\tilde{x} : \mathbb{R} \to [0, 1]$ will be called a fuzzy set $\tilde{x}$ on $\mathbb{R}$. Its α-level set of $\tilde{x}$ is $[\tilde{x}]^\alpha = \{x \in \mathbb{R} : \tilde{x}(x) \ge \alpha\}$ for each $\alpha \in (0, 1]$. Specifically, for $\alpha = 0$, the set $[\tilde{x}]^0$ is defined by $[\tilde{x}]^0 = \mathrm{cl}\{x \in \mathbb{R} : \tilde{x}(x) > 0\}$, where $\mathrm{cl}A$ denotes the closure of a crisp set A. A fuzzy set $\tilde{x}$ is said to be a fuzzy number if it is normal, fuzzy convex, and upper semicontinuous and the set $[\tilde{x}]^0$ is compact.

Let F be the set of all fuzzy numbers on $\mathbb{R}$. Then for an $\tilde{x} \in F$ it is well known that the α-level set $[\tilde{x}]^\alpha = [\tilde{x}_L(\alpha), \tilde{x}_R(\alpha)]$ is a nonempty bounded closed interval in $\mathbb{R}$ for all $\alpha \in [0, 1]$, where $\tilde{x}_L(\alpha)$ denotes the left-hand end point of $[\tilde{x}]^\alpha$ and $\tilde{x}_R(\alpha)$ denotes the right one. For any $\tilde{x}, \tilde{y} \in F$ and $\lambda \in \mathbb{R}$, owing to Zadeh's extension principle [23], the addition and scalar multiplication can be, respectively, defined for any $x \in \mathbb{R}$ by

$$\begin{gathered} (\tilde{x} + \tilde{y})(x) = \sup_{x_1, x_2 : x_1 + x_2 = x} \min\{\tilde{x}(x_1), \tilde{x}(x_2)\}, \\ \lambda \tilde{x}(x) = \begin{cases} \tilde{x}\left(\dfrac{x}{\lambda}\right), & \lambda \ne 0, \\ 0, & \lambda = 0. \end{cases} \end{gathered} \tag{6}$$

We say that a fuzzy number $\tilde{s} \in F$ is symmetric if $\tilde{s} = -\tilde{s}$ [16]. We denote the set of all symmetric fuzzy numbers by φ.

Definition 5 (see [15]). Let $\tilde{x} \in F$, and we define a function $\tilde{x}_M : [0, 1] \to \mathbb{R}$ by assigning the midpoint of each α-level set to $\tilde{x}_M(\alpha)$ for all $\alpha \in [0, 1]$; that is,

$$\tilde{x}_M(\alpha) = \frac{\tilde{x}_L(\alpha) + \tilde{x}_R(\alpha)}{2}. \tag{7}$$

Then the function $\tilde{x}_M : [0, 1] \to \mathbb{R}$ will be called the midpoint function of the fuzzy number $\tilde{x}$.

Lemma 6 (see [15]). *For any $\tilde{x} \in F$, the midpoint function $\tilde{x}_M$ is continuous from the right at* 0 *and continuous from the left on* $[0, 1]$*. Furthermore, it is a function of bounded variation on* $[0, 1]$.

Definition 7 (see [24]). Let $\tilde{x}, \tilde{y} \in F$, and we say that $\tilde{x}$ is equivalent to $\tilde{y}$, if there exist two symmetric fuzzy numbers $\tilde{s}_1, \tilde{s}_2 \in \varphi$ such that $\tilde{x} + \tilde{s}_1 = \tilde{y} + \tilde{s}_2$ and then we denote this by $\tilde{x} \sim \tilde{y}$.

It is easy to verify that the equivalence relation defined above is reflexive, symmetric, and transitive [16]. Let $\langle \tilde{x} \rangle$ denote the fuzzy number equivalence class containing the element $\tilde{x}$ and denote the set of all fuzzy number equivalence classes by F/φ.

Definition 8 (see [17]). Let $\tilde{x} \in F$ and let $\hat{x}$ be a fuzzy number such that $\tilde{x} = \hat{x} + \tilde{s}$ for some $\tilde{s} \in \varphi$, and if $\hat{x} = \tilde{y} + \tilde{s}_1$ for some $\tilde{y} \in F$ and $\tilde{s}_1 \in \varphi$, then $\tilde{s}_1 = \tilde{0}$. Then the fuzzy number $\hat{x}$ will be called the Mareš core of the fuzzy number $\tilde{x}$.

Definition 9 (see [21]). Let $\langle \tilde{x} \rangle \in F/\varphi$, and we define the midpoint function $M_{\langle \tilde{x} \rangle} : [0, 1] \to \mathbb{R}$ by

$$M_{\langle \tilde{x} \rangle}(\alpha) = \hat{x}_M(\alpha) \tag{8}$$

for all $\alpha \in [0, 1]$, where $\hat{x}$ is the Mareš core of $\tilde{x}$.

Definition 10 (see [21]). Let $\langle \tilde{x} \rangle, \langle \tilde{y} \rangle \in F/\varphi$, and we define the sum of this two fuzzy number equivalence classes as a fuzzy equivalence class $\langle \tilde{z} \rangle \in F/\varphi$, which satisfies the condition

$$M_{\langle \tilde{x} \rangle}(\alpha) + M_{\langle \tilde{y} \rangle}(\alpha) = M_{\langle \tilde{z} \rangle}(\alpha) \tag{9}$$

for all $\alpha \in [0,1]$ and we denote this by

$$\langle \tilde{x} \rangle + \langle \tilde{y} \rangle = \langle \tilde{x} + \tilde{y} \rangle = \langle \tilde{z} \rangle . \quad (10)$$

Remark 11. The addition operation defined by Definition 10 is a group operation over the set of fuzzy number equivalence classes F/φ up to the equivalence relation in Definition 7. For the details of the discussion, please see [25, 26].

Definition 12 (see [15]). Let $\langle \tilde{x} \rangle, \langle \tilde{y} \rangle \in F/\varphi$, and we say that $\langle \tilde{z} \rangle \in F/\varphi$ is the product of $\langle \tilde{x} \rangle$ and $\langle \tilde{y} \rangle$ if their midpoint functions satisfy

$$M_{\langle \tilde{x} \rangle}(\alpha) \cdot M_{\langle \tilde{y} \rangle}(\alpha) = M_{\langle \tilde{z} \rangle}(\alpha) \quad (11)$$

for all $\alpha \in [0,1]$ and we denote this by

$$\langle \tilde{x} \rangle \cdot \langle \tilde{y} \rangle = \langle \tilde{z} \rangle . \quad (12)$$

Definition 13 (see [21]). For any $\langle \tilde{x} \rangle \in F/\varphi$ and $\lambda \in \mathbb{R}$, we define $\lambda \cdot \langle \tilde{x} \rangle = \lambda \langle \tilde{x} \rangle$ by

$$\lambda \langle \tilde{x} \rangle = \langle \tilde{x} \rangle \lambda = \langle \lambda \tilde{x} \rangle . \quad (13)$$

It is obvious that $M_{\lambda \langle \tilde{x} \rangle}(\alpha) = M_{\langle \lambda \tilde{x} \rangle}(\alpha) = \lambda M_{\langle \tilde{x} \rangle}(\alpha)$ for all $\alpha \in [0,1]$.

Definition 14 (see [15]). Let $\langle \tilde{x} \rangle, \langle \tilde{y} \rangle \in F/\varphi$, and we define $d_{\sup} : F/\varphi \times F/\varphi \to \mathbb{R}^+ \cup \{0\}$ by

$$d_{\sup}(\langle \tilde{x} \rangle, \langle \tilde{y} \rangle) = \sup_{\alpha \in [0,1]} \left| M_{\langle \tilde{x} \rangle}(\alpha) - M_{\langle \tilde{y} \rangle}(\alpha) \right| . \quad (14)$$

It is easy to see that $(F/\varphi, d_{\sup})$ is a metric space [15].

3. The Karush-Kuhn-Tucker Optimality Conditions

In this paper, we always suppose that the range of fuzzy mappings is the set of all fuzzy number equivalence classes.

Definition 15 (see [21]). Let $F : T \to F/\varphi$ be a fuzzy mapping, where $T = [a,b] \subseteq \mathbb{R}$. Then F is said to be differentiable at $t \in T$ if there exists an $F'(t) \in F/\varphi$ such that

$$\lim_{h \to 0} d_{\sup}\left(\frac{F(t+h) - F(t)}{h}, F'(t) \right) = 0. \quad (15)$$

If $t = a$ (or b), then we consider only $h \to 0^+$ (or $h \to 0^-$).

Lemma 16 (see [21]). *$F : T \to F/\varphi$ is differentiable on T if and only if*

(1) *$M_{F(t)}(\alpha)$ is differentiable with respect to $t \in T$ for all $\alpha \in [0,1]$. That is, $(\partial/\partial t)M_{F(t)}(\alpha)$ exists and is of bounded variation with respect to $\alpha \in [0,1]$ for all $t \in T$;*

(2) *the mappings $\{M_{F(t)}(\alpha)\}_{\alpha \in [0,1]}$ are uniformly differentiable with the derivatives $(\partial/\partial t)M_{F(t)}(\alpha)$. That is, for each $t \in T$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\left| \frac{M_{F(t+h)}(\alpha) - M_{F(t)}(\alpha)}{h} - \frac{\partial}{\partial t} M_{F(t)}(\alpha) \right| < \varepsilon \quad (16)$$

for all $|h| \in (0,\delta)$ and $\alpha \in [0,1]$.

Definition 17 (see [27]). Let $\langle \tilde{a} \rangle = (\langle \tilde{a}_1 \rangle, \langle \tilde{a}_2 \rangle, \ldots, \langle \tilde{a}_n \rangle)^T \in (F/\varphi)^n$ and $t = (t_1, t_2, \ldots, t_n)^T \in \mathbb{R}^n$ be an n-dimensional fuzzy number equivalence class vector and n-dimensional real vector, respectively. We define their product as

$$\langle \tilde{a} \rangle^T t = \sum_{i=1}^{n} \langle \tilde{a}_i \rangle t_i = \langle \tilde{a}_1 \rangle t_1 + \langle \tilde{a}_2 \rangle t_2 + \cdots + \langle \tilde{a}_n \rangle t_n, \quad (17)$$

which is a fuzzy number equivalence class.

Definition 18 (see [27]). Let $F : \Omega \to F/\varphi$ be a fuzzy mapping, where Ω is an open subset in $\mathbb{R}^n$. We say that F has a partial derivative at $t = (t_1, t_2, \ldots, t_n)^T \in \Omega$ with respect to the ith variable t_i if there exists an $(\partial/\partial t_i)F(t) \in F/\varphi$ such that

$$\lim_{h \to o} d_{\sup}\left(\frac{F\left(t + he^i\right) - F(t)}{h}, \frac{\partial}{\partial t_i} F(t) \right) = 0, \quad (18)$$

where e^i stands for the unit vector that the ith component is 1 and the others are 0.

Definition 19 (see [27]). Let $F : \Omega \to F/\varphi$ be a fuzzy mapping, where Ω is an open subset in $\mathbb{R}^n$. We say that F is differentiable at $t = (t_1, t_2, \ldots, t_n)^T \in \Omega$ if F has continuous partial derivatives $(\partial/\partial t_i)F(t)$ with respect to ith variable t_i $(i = 1, 2, \ldots, n)$ and satisfies

$$F(t+h) = F(t) + \tilde{\nabla} F(t)^T h + o(\|h\|), \quad h = (h_1, h_2, \ldots, h_n)^T \in \mathbb{R}^n, \quad (19)$$

where $\tilde{\nabla} F(t) \in (F/\varphi)^n$ is an n-dimensional fuzzy number equivalence class vector defined by

$$\tilde{\nabla} F(t) = \left(\frac{\partial F(t)}{\partial t_1}, \frac{\partial F(t)}{\partial t_2}, \ldots, \frac{\partial F(t)}{\partial t_n} \right)^T, \quad (20)$$

and $\|h\|$ is the usual Euclid norm of h and $o : [0, +\infty) \to F/\varphi$ is a fuzzy mapping that satisfies

$$\lim_{t \to 0} d_{\sup}\left(\frac{o(t)}{t}, \langle \tilde{0} \rangle \right) = 0. \quad (21)$$

Then we call $\tilde{\nabla} F(t)$ the gradient of the fuzzy mappings F at t.

Definition 20 (see [27]). Let $\langle \tilde{x} \rangle, \langle \tilde{y} \rangle \in F/\varphi$.

(1) We say that $\langle \tilde{x} \rangle \preceq \langle \tilde{y} \rangle$ if $M_{\langle \tilde{x} \rangle}(\alpha) \le M_{\langle \tilde{y} \rangle}(\alpha)$ for all $\alpha \in [0,1]$.

(2) We say that $\langle \tilde{x} \rangle \prec \langle \tilde{y} \rangle$ if $\langle \tilde{x} \rangle \preceq \langle \tilde{y} \rangle$ and there exists at least one $\alpha_0 \in [0,1]$ such that $M_{\langle \tilde{x} \rangle}(\alpha_0) < M_{\langle \tilde{y} \rangle}(\alpha_0)$.

(3) If $\langle \tilde{x} \rangle \preceq \langle \tilde{y} \rangle$ and $\langle \tilde{y} \rangle \preceq \langle \tilde{x} \rangle$ then $\langle \tilde{x} \rangle = \langle \tilde{y} \rangle$.

Sometimes we may write $\langle \tilde{y} \rangle \succeq \langle \tilde{x} \rangle$ instead of $\langle \tilde{x} \rangle \preceq \langle \tilde{y} \rangle$ and write $\langle \tilde{y} \rangle \succ \langle \tilde{x} \rangle$ instead of $\langle \tilde{x} \rangle \prec \langle \tilde{y} \rangle$. Note that $\preceq$ is a partial order relation on F/φ.

Definition 21. Let $\langle \tilde{a} \rangle \in F/\varphi$, and we say that $\langle \tilde{a} \rangle$ is nonnegative if $\langle \tilde{a} \rangle \succeq \langle \tilde{0} \rangle$; that is, $M_{\langle \tilde{a} \rangle}(\alpha) \ge 0$ for all $\alpha \in [0,1]$.

Let $F : \mathbb{R}^n \to F/\varphi$ be a fuzzy mapping. Consider the following optimization problem:

$$\begin{aligned} \min \quad & F(t) = F(t_1, t_2, \ldots, t_n), \\ \text{subject to} \quad & t = (t_1, t_2, \ldots, t_n)^T \in \Omega \subseteq \mathbb{R}^n, \end{aligned} \tag{22}$$

where the feasible set Ω is assumed to be convex subset of $\mathbb{R}^n$. Since $\preceq$ is a partial order relation on F/φ, we may follow the similar solution concept (the nondominated solution) used in multiobjective programing problems to interpret the meaning of minimization in problem (22).

Definition 22. Let t^* be a feasible solution of problem (22); that is, $t^* \in \Omega$.

(1) We say that t^* is a local nondominated solution of problem (22) if there exists an $\varepsilon > 0$ and there does not exist any $t \in N_\varepsilon(t^*) \cap \Omega$ such that $F(t) \prec F(t^*)$, where $N_\varepsilon(t^*)$ is an ε-neighborhood around t^*.

(2) We say that t^* is a (global) nondominated solution of problem (22) if there exists no $t \in \Omega$ such that $F(t) \prec F(t^*)$.

Definition 23. Let $F : \Omega \to F/\varphi$ be a fuzzy mapping, where Ω is a nonempty convex subset in $\mathbb{R}^n$. F is said to be convex on Ω if, for any $s, t \in \Omega$ and $\lambda \in (0,1)$, we always have $F(\lambda s + (1-\lambda)t) \preceq \lambda F(s) + (1-\lambda)F(t)$. F is said to be concave if $-F$ is convex.

Theorem 24. *Let $F : \Omega \to F/\varphi$ be a fuzzy mapping, where Ω is a nonempty convex subset in $\mathbb{R}^n$. Then F is convex on Ω if and only if $M_{F(t)}(\alpha)$ is convex with respect to $t \in \Omega$ for all $\alpha \in [0,1]$.*

Proof. The result follows from Definitions 20 and 23 immediately.

Let $f, g_j : \mathbb{R}^n \to \mathbb{R}$ be real-valued functions. Consider the following optimization problem:

$$\begin{aligned} \min \quad & f(t) = f(t_1, t_2, \ldots, t_n), \\ \text{subject to} \quad & g_j(t) \le 0, \quad j = 1, 2, \ldots, m. \end{aligned} \tag{23}$$

Suppose that the constraint functions g_j are convex on $\mathbb{R}^n$ for all $j = 1, 2, \ldots, m$, and then the feasible set $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$ is a convex subset of $\mathbb{R}^n$. The well-known KKT optimality conditions for problem (23) are stated as below. □

Theorem 25 (see [28, 29]). *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$ be the convex feasible set and $t^* \in \Omega$ be a feasible solution of problem (23). Suppose that the objective function f and constraint functions g_j are convex on $\mathbb{R}^n$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, m$. If there exist nonnegative Lagrange multipliers $u_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$, such that*

(1) $\nabla f(t^*) + \sum_{j=1}^{m} u_j \nabla g_j(t^*) = 0$,

(2) $u_j g_j(t^*) = 0$ *for all* $j = 1, 2, \ldots, m$,

then t^* is nondominated solution of problem (23).

Let $F : \mathbb{R}^n \to F/\varphi$ be a fuzzy mapping and $g_j : \mathbb{R}^n \to \mathbb{R}$ be real-valued functions, $j = 1, 2, \ldots, m$. Now we consider the following optimization problem:

$$\begin{aligned} \min \quad & F(t) = F(t_1, t_2, \ldots, t_n), \\ \text{subject to} \quad & g_j(t) \le 0, \quad j = 1, 2, \ldots, m. \end{aligned} \tag{24}$$

If we suppose that the constraint functions g_j are convex on $\mathbb{R}^n$ for all $j = 1, 2, \ldots, m$, then we can see that problem (24) follows from problem (22) by taking the convex feasible set as $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$.

Now we are in a position to present the KKT optimality conditions for nondominated solutions of problem (24).

Theorem 26. *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \le 0,\ j = 1, 2, \ldots, m\}$ be the convex feasible set and $t^* \in \Omega$ be a feasible solution of problem (24). Suppose that the fuzzy-valued objective function F and real-valued constraint functions g_j are convex on $\mathbb{R}^n$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, m$. If there exist nonnegative real-valued Lagrange function multipliers u_j for $j = 1, 2, \ldots, m$ defined on $[0,1]$ such that*

(1) $M_{\tilde{\nabla} F(t^*)}(\alpha) + \sum_{j=1}^{m} u_j(\alpha) \nabla g_j(t^*) = 0$ *for all* $\alpha \in [0,1]$,

(2) $u_j(\alpha) g_j(t^*) = 0$ *for all* $\alpha \in [0,1]$ *and* $j = 1, 2, \ldots, m$,

then t^* is a nondominated solution of problem (24).

Proof. Suppose that conditions (1) and (2) are satisfied and t^* is not a nondominated solution of problem (24). Then there exists a $\bar{t} \in \Omega$ such that $F(\bar{t}) \prec F(t^*)$; that is, for some $\alpha^* \in [0,1]$ we have that $M_{F(\bar{t})}(\alpha^*) < M_{F(t^*)}(\alpha^*)$. We now define a real-valued function f by $f(t) = M_{F(t)}(\alpha^*)$. Then we have

$$f(\bar{t}) < f(t^*). \tag{25}$$

Since the fuzzy mapping F is convex on $\mathbb{R}^n$ and continuously differentiable at t^*, by Theorem 24 and Lemma 16 we see that f is also convex on $\mathbb{R}^n$ and continuously differentiable at t^*. Furthermore, we have $\nabla f(t) = \nabla M_{F(t)}(\alpha^*) = M_{\tilde{\nabla} F(t)}(\alpha^*)$. Since conditions (1) and (2) are satisfied, we can obtain the following two new conditions for any fixed $\alpha^* \in [0,1]$:

(1′) $\nabla f(t^*) + \sum_{j=1}^{m} u_{j\alpha^*} \cdot \nabla g_j(t^*) = 0$;

(2′) $u_{j\alpha^*} \cdot g_j(t^*) = 0$ for all $j = 1, 2, \ldots, m$,

where $u_{j\alpha^*} = u_j(\alpha^*) \ge 0$ for $j = 1, 2, \ldots, m$. Now we consider the following constrained optimization problem:

$$\begin{aligned} \min \quad & f(t) = f(t_1, t_2, \ldots, t_n), \\ \text{subject to} \quad & g_j(t) \le 0, \quad j = 1, 2, \ldots, m \end{aligned} \tag{26}$$

which has the same constraints of problem (24). By Theorem 25, conditions (1′) and (2′) are the KKT conditions of problem (26). Therefore, we have that t^* is an optimal solution of problem (26) with the real-valued objective function f; that is, $f(t^*) \le f(t)$ for all $t \in \Omega$, which contradicts inequality (25). Then we get that t^* is indeed a nondominated solution of problem (24). □

Theorem 27. *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \leq 0,\ j = 1, 2, \ldots, m\}$ be the convex feasible set and $t^* \in \Omega$ be a feasible solution of problem (24). Suppose that the fuzzy-valued objective function F and real-valued constraint functions g_j are convex on $\mathbb{R}^n$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, m$. If there exist nonnegative fuzzy number equivalent class Lagrange multipliers $\langle \tilde{x}_j \rangle \in F/\varphi$ for $j = 1, 2, \ldots, m$ such that*

(1) $\tilde{\nabla} F(t^*) + \sum_{j=1}^{m} \langle \tilde{x}_j \rangle \cdot \nabla g_j(t^*) = \langle \tilde{0} \rangle$,

(2) $\langle \tilde{x}_j \rangle \cdot g_j(t^*) = \langle \tilde{0} \rangle$ *for all* $j = 1, 2, \ldots, m$,

then t^* is a nondominated solution of problem (24).

Proof. Since conditions (1) and (2) are satisfied, taking the midpoint function of (1) and (2), we obtain the following new conditions:

(1′) $M_{\tilde{\nabla} F(t^*)}(\alpha) + \sum_{j=1}^{m} M_{\langle \tilde{x}_j \rangle}(\alpha) \cdot \nabla g_j(t^*) = 0$ for all $\alpha \in [0, 1]$.

(2′) $M_{\langle \tilde{x}_j \rangle}(\alpha) \cdot g_j(t^*) = 0$ for all $\alpha \in [0, 1]$ and $j = 1, 2, \ldots, m$.

Since the fuzzy number equivalence classes $\langle \tilde{x}_j \rangle$ are nonnegative for all $j = 1, 2, \ldots, m$, then we can get that $M_{\langle \tilde{x}_j \rangle}$ are nonnegative real-valued functions defined on $[0, 1]$ for all $j = 1, 2, \ldots, m$. So, (1′) and (2′) verify the KKT optimality conditions (1) and (2) of Theorem 26, respectively. Therefore, we get that t^* is a nondominated solution of problem (24). □

Lemma 28 (see [28]). *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \leq 0,\ j = 1, 2, \ldots, m\}$ be a feasible set and $t^* \in \Omega$. Assume that g_j are differentiable at t^* for all $j = 1, 2, \ldots, m$. Let $J = \{j : g_j(t) = 0\}$ be the index set for the active constraints. Then we have*

$$D \subseteq \left\{ d \in \mathbb{R}^n : \nabla g_j \left(t^*\right)^T d \leq 0\ \forall j \in J \right\}, \tag{27}$$

where D is the cone of feasible directions of Ω at t^* defined by

$$\begin{aligned} D = \{ & d \in \mathbb{R}^n : d \neq 0,\ \text{there exists a } \delta \\ & > 0 \text{ such that } t^* + \eta d \in \Omega\ \forall \eta \in (0, \delta) \}. \end{aligned} \tag{28}$$

Lemma 29 (see [28]). *Let A and C be two matrices. Exactly one of the following systems has a solution:*

System I: $Ax \leq 0,\ Ax \neq 0,\ Cx \leq 0$ for some $x \in \mathbb{R}^n$.

System II: $A^T \lambda + C^T u = 0$ for some $(\lambda, u),\ \lambda > 0,\ u \geq 0$.

Theorem 30. *Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \leq 0,\ j = 1, 2, \ldots, m\}$ be the convex feasible set and $t^* \in \Omega$ be a feasible solution of problem (24). Suppose that the fuzzy-valued objective function F is differentiable and strictly pseudoconvex on Ω, and the real-valued constraint functions g_j are convex on $\mathbb{R}^n$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, m$. If there exist a $\alpha^* \in [0, 1]$ and nonnegative Lagrange multipliers $u_j \in \mathbb{R}$ for $j = 1, 2, \ldots, m$ such that*

(1) $M_{\tilde{\nabla} F(t^*)}(\alpha^*) + \sum_{j=1}^{m} u_j \cdot \nabla g_j(t^*) = 0$,

(2) $u_j \cdot g_j(t^*) = 0$ *for all* $j = 1, 2, \ldots, m$,

then t^* is a strongly nondominated solution of problem (24).

Proof. Suppose that conditions (1) and (2) are satisfied and t^* is not a strongly nondominated solution of problem (24). Then there exists a $\bar{t} \in \Omega$ with $\bar{t} \neq t^*$ such that $F(\bar{t}) \preceq F(t^*)$. Since F is differentiable and strictly pseudoconvex on Ω, we have

$$\tilde{\nabla} F \left(t^*\right)^T \left(\bar{t} - t^*\right) \prec \left\langle \tilde{0} \right\rangle; \tag{29}$$

that is,

$$M_{\tilde{\nabla} F(t^*)} \left(\alpha^*\right)^T \left(\bar{t} - t^*\right) < 0. \tag{30}$$

Let $d = \bar{t} - t^*$. Since Ω is a convex set and $\bar{t}, t^* \in \Omega$, we have

$$t^* + \eta d = t^* + \eta \left(\bar{t} - t^*\right) = \eta \bar{t} + \left(1 - \eta\right) t^* \in \Omega \tag{31}$$

for any $\eta \in (0, 1)$. By Lemma 28 we get that $d \in D$, which means that

$$\nabla g_j \left(t^*\right)^T d \leq 0 \quad \forall i \in J, \tag{32}$$

where D is the cone of feasible directions of Ω at t^* and $J = \{j : g_j(t) = 0\}$ is the index set for the active constraints. Now let $A = M_{\tilde{\nabla} F(t^*)}(\alpha^*)^T$ and C be the matrix whose rows are $\nabla g_j(t^*)^T$ for $j \in J$. We consider the following two systems:

System I: $Ax \leq 0,\ Ax \neq 0,\ Cx \leq 0$ for some $x \in \mathbb{R}^n$.

System II: $A^T \lambda + C^T u = 0$ for some $(\lambda, u),\ \lambda > 0,\ u \geq 0$.

Then by (30) and (32) we get that System I has a solution $d = \bar{t} - t^*$. Further, by Lemma 29 System II has no solutions, which means that there exist no multipliers $0 < \lambda \in \mathbb{R}$ and $0 \leq u_j \in \mathbb{R}$ for $j \in J$ such that

$$\lambda M_{\tilde{\nabla} F(t^*)} \left(\alpha^*\right) + \sum_{j \in J} u_j \cdot \nabla g_j \left(t^*\right) = 0. \tag{33}$$

Since $\lambda > 0$, dividing (33) by λ and denoting $\eta_j = u_j / \lambda$ for $j \in J$, we have that

$$M_{\tilde{\nabla} F(t^*)} \left(\alpha^*\right) + \sum_{j \in J} \eta_j \cdot \nabla g_j \left(t^*\right) = 0. \tag{34}$$

Since J is the index set for the active constraints, we have $g_j(t^*) < 0$ for $j \notin J$. Further, if $u_j \cdot g_j(t^*) = 0$ for all $j = 1, 2, \ldots, m$, we can get that $\eta_j = 0$ for $j \notin J$; that is,

$$\sum_{j \in J} \eta_j \cdot \nabla g_j \left(t^*\right) = \sum_{j=1}^{m} \eta_j \cdot \nabla g_j \left(t^*\right). \tag{35}$$

From (34) and (35), there exist no multipliers $0 \leq \eta_j \in \mathbb{R}$ for $j = 1, 2, \ldots, m$ such that

(1′) $M_{\tilde{\nabla} F(t^*)}(\alpha^*) + \sum_{j=1}^{m} \eta_j \nabla g_j(t^*) = 0$,

(2′) $u_j \cdot g_j(t^*) = 0$ for all $j = 1, 2, \ldots, m$,

which contradicts conditions (1) and (2) for the existence of multipliers $0 \leq u_j \in \mathbb{R}$ for $j = 1, 2, \ldots, m$. Hence, we have that t^* is indeed a strongly nondominated solution of problem (24). □

Example 31. Define a fuzzy mapping $F : \mathbb{R}^3 \to F/\varphi$ by

$$\begin{aligned} F(t) &= \langle \tilde{a} \rangle^T t + \|t\|^2 = \sum_{i=1}^{3} \left(\langle \tilde{a}_i \rangle t_i + t_i^{\,2} \right) \\ &= \langle \tilde{a}_1 \rangle t_1 + \langle \tilde{a}_2 \rangle t_2 + \langle \tilde{a}_3 \rangle t_3 + t_1^{\,2} + t_2^{\,2} + t_3^{\,2} \end{aligned} \tag{36}$$

for all $t = (t_1, t_2, t_3)^T \in \mathbb{R}^3$, where $\langle \tilde{a} \rangle = (\langle \tilde{a}_1 \rangle, \langle \tilde{a}_2 \rangle, \langle \tilde{a}_3 \rangle)^T \in (F/\varphi)^3$ and we define $\langle \tilde{a}_i \rangle$ by the level sets of its Mareš core $[\hat{a}_1]^\alpha = [-6, -12\alpha + 6]$, $[\hat{a}_2]^\alpha = [-1, -2\alpha + 1]$, and $[\hat{a}_3]^\alpha = [-4, -8\alpha + 4]$ for all $\alpha \in [0, 1]$ and $i = 1, 2, 3$, respectively. Thus, we have

$$\begin{aligned} M_{F(t_1, t_2, t_3)}(\alpha) &= -6\alpha t_1 + t_1^2 - \alpha t_2 + t_2^2 - 4\alpha t_3 + t_3^2 \\ &= \alpha \left(-6t_1 - t_2 - 4t_3 \right) + t_1^2 + t_2^2 + t_3^2 \end{aligned} \tag{37}$$

for all $\alpha \in [0, 1]$ and $t = (t_1, t_2, t_3)^T \in \mathbb{R}^3$. It is obvious that $M_{F(t)}(\alpha)$ is continuous from the right at 0 and continuous from the left on $[0, 1]$ with respect to α. Now we consider the following optimization problem:

$$\begin{aligned} \min \quad & F(t) = F(t_1, t_2, t_3), \\ \text{subject to} \quad & g_1(t_1, t_2, t_3) = 4t_1 - t_2 + 2t_3 - 8 \leq 0, \\ & g_2(t_1, t_2, t_3) = 3t_1 + 2t_2 - t_3 - 1 \leq 0, \\ & g_j(t_1, t_2, t_3) = -t_{j-2} \leq 0, \\ & \qquad\qquad \text{for } j = 3, 4, 5, \\ & g_j(t_1, t_2, t_3) = t_{j-5} - 2 \leq 0, \\ & \qquad\qquad \text{for } j = 6, 7, 8. \end{aligned} \tag{38}$$

It is obvious that the constraint functions g_j are convex on $\mathbb{R}^3$ for all $j = 1, 2, \ldots, 8$, and then we know that the feasible set $\Omega = \{t \in \mathbb{R}^3 : g_j(t) \leq 0,\ j = 1, 2, \ldots, 8\}$ is convex. Since $M_{F(t)}(\alpha)$ is decreasing with respect to α for all $t \in \Omega$, we get that

$$\begin{aligned} V_0^1 \left(M_{F(t)} \right) &= \left| M_{F(t)}(1) - M_{F(t)}(0) \right| = \left| 6t_1 + t_2 + 4t_3 \right| \\ &\leq 6 \left| t_1 \right| + \left| t_2 \right| + 4 \left| t_3 \right| \leq 22. \end{aligned} \tag{39}$$

Thus, we find that $M_{F(t)}(\alpha)$ is of bounded variation with respect to α for all $t = (t_1, t_2, t_3)^T \in \Omega \subseteq \mathbb{R}^3$. It is easy to verify that F is differentiable and strictly pseudoconvex on Ω, and g_j are convex on $\mathbb{R}^3$ and continuously differentiable at t^* for all $j = 1, 2, \ldots, 8$. Then we obtain

$$\begin{aligned} M_{\tilde{\nabla} F(t_1, t_2, t_3)}(\alpha) &= \left(2t_1 - 6\alpha, 2t_2 - \alpha, 2t_3 - 4\alpha \right)^T, \\ \nabla g_1(t_1, t_2, t_3) &= (4, -1, 2)^T, \\ \nabla g_2(t_1, t_2, t_3) &= (3, 2, -1)^T, \\ \nabla g_3(t_1, t_2, t_3) &= (-1, 0, 0)^T, \\ \nabla g_4(t_1, t_2, t_3) &= (0, -1, 0)^T, \\ \nabla g_5(t_1, t_2, t_3) &= (0, 0, -1)^T, \\ \nabla g_6(t_1, t_2, t_3) &= (1, 0, 0)^T, \\ \nabla g_7(t_1, t_2, t_3) &= (0, 1, 0)^T, \\ \nabla g_8(t_1, t_2, t_3) &= (0, 0, 1)^T, \end{aligned} \tag{40}$$

for all $\alpha \in [0, 1]$ and $t = (t_1, t_2, t_3)^T \in \Omega$. Now we consider the point $t^* = (t_1^*, t_2^*, t_3^*)^T = (1, 0, 2)^T \in \Omega$. Since

$$\begin{aligned} g_3(t^*) &\neq 0, \\ g_5(t^*) &\neq 0, \\ g_6(t^*) &\neq 0, \\ g_7(t^*) &\neq 0, \end{aligned} \tag{41}$$

from condition (2) in Theorem 30, we get that

$$u_3 = u_5 = u_6 = u_7 = 0. \tag{42}$$

Now, applying condition (2) of Theorem 30 at the point t^*, we obtain

$$\begin{aligned} & M_{\tilde{\nabla} F(t^*)}(\alpha^*) + \sum_{j=1}^{8} u_j \cdot \nabla g_j(t^*) \\ &= \begin{bmatrix} 2 - 6\alpha^* + 4u_1 + 3u_2 \\ -\alpha^* - u_1 + 2u_2 - u_4 \\ 4 - 4\alpha^* + 2u_1 - u_2 + u_8 \end{bmatrix} = 0. \end{aligned} \tag{43}$$

After these algebraic calculations, we obtain that there exist a $\alpha^* = 1 \in [0, 1]$ and nonnegative Lagrange multipliers

$$\begin{aligned} u_1 &= \frac{1}{4}, \\ u_2 &= 1, \\ u_4 &= \frac{3}{4}, \\ u_8 &= \frac{1}{2}, \\ u_j &= 0, \quad j = 3, 5, 6, 7, \end{aligned} \tag{44}$$

which satisfied conditions (1) and (2) of Theorem 30. Hence, we get that $t^* = (t_1^*, t_2^*, t_3^*)^T = (1, 0, 2)^T \in \Omega$ is a strongly nondominated solution of problem (38).

4. Conclusions

In this present investigation, the KKT optimality conditions are elicited naturally by introducing the Lagrange function multipliers, and we also provided some examples to illustrate the main results. The research on the quotient space of fuzzy numbers can be traced back to the works of Mareš [16, 17]. Hong and Do [24] improved this result and proposed a more refined equivalence relation. This equivalence relation can be used to partition the set of fuzzy numbers into equivalence class having the desired group properties for the addition operation. Since the quotient space of fuzzy numbers is characterized by the midpoint functions, there are more differentiable fuzzy mappings. As a matter of fact, there are still many other types of the KKT optimality conditions that can be derived using the similar techniques discussed in this paper on the quotient space of fuzzy numbers. However, for the nondifferentiable fuzzy optimization problem, we can follow the approach proposed by Ruziyeva and Dempe [30] to derive the necessary and sufficient optimality conditions in the quotient space of fuzzy numbers. In addition, Fuzzy sets and fuzzy optimization problems have several appropriate applications to today's world. But there are no sufficient examples and applications of the topics discussed in this paper. Therefore, we will develop the contribution of this research to practical problems in future studies.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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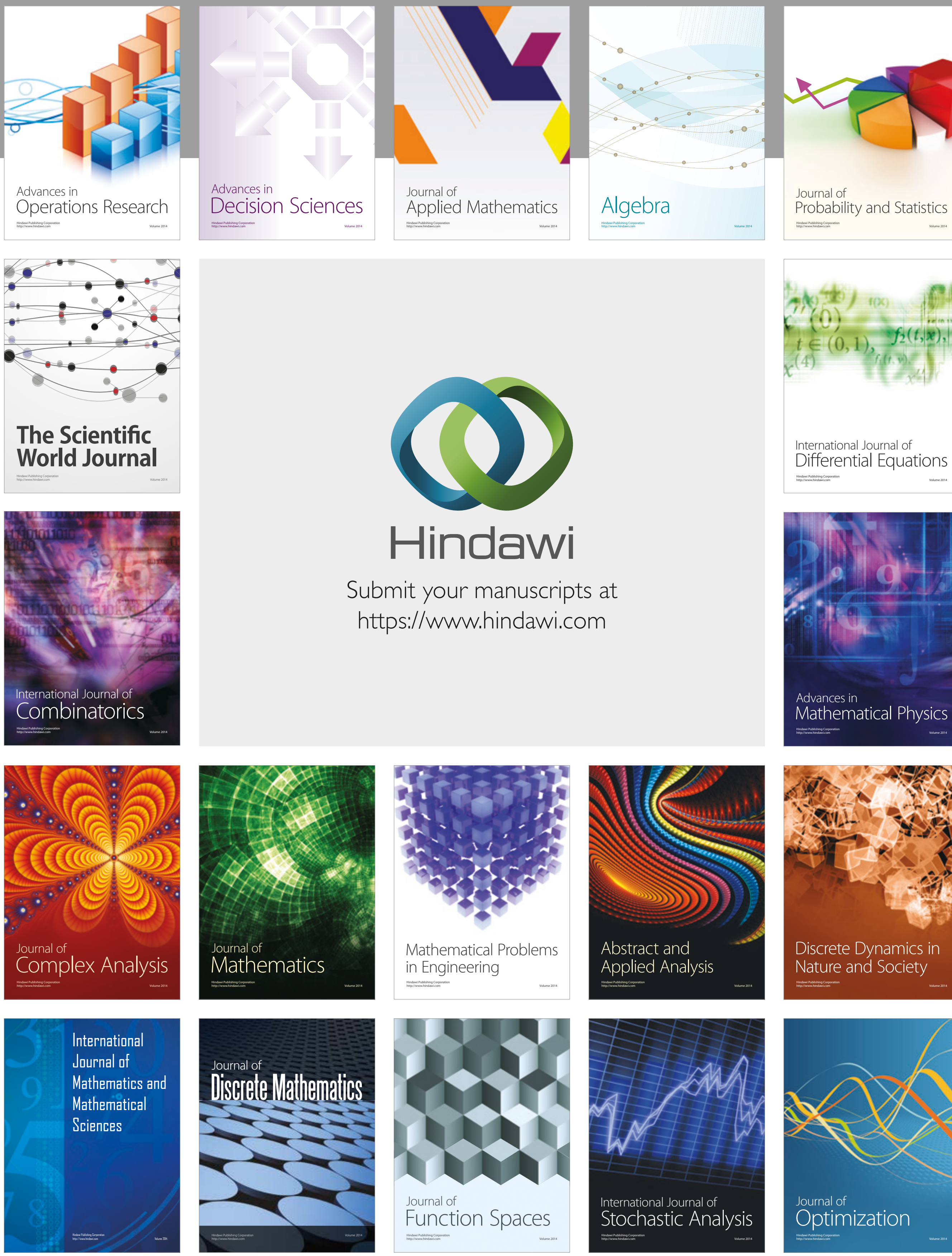
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