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Research Article

Multiattribute Group Decision Making Methods Based on Linguistic Intuitionistic Fuzzy Power Bonferroni Mean Operators

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This paper focuses on the multiattribute group decision making problems with linguistic intuitionistic fuzzy information. Firstly the concept of linguistic intuitionistic fuzzy numbers (LIFNs) is introduced, and then based on the LIFNs, some new aggregation operators based on Bonferroni mean and power operator are proposed, such as linguistic intuitionistic fuzzy power Bonferroni mean (LIFPBM) operator, linguistic intuitionistic fuzzy weighted power Bonferroni mean (LIFWPBM) operator, linguistic intuitionistic fuzzy geometric power Bonferroni mean (LIFGPBM) operator, and linguistic intuitionistic fuzzy weighted geometric power Bonferroni mean (LIFWGPBM) operator. Then, some properties are proved such as idempotency, permutation, and boundedness. Besides, some special situations of the operators are explored. After that, an approach based of the LIFWGPBM and LIFWGPBM operators is proposed. Finally an example is used to illustrate the validity of the developed method.

1. Introduction

Multiple attributes group decision making (MAGDM) plays an important role in the field of decision sciences. It aims to select the most satisfied one from the finite alternatives according to the evaluation information for different attributes given by decision makers [1–8]. Because of the complexity of the real decision problems, sometimes it is more suitable to express the evaluation information by fuzzy numbers rather than crisp numbers, for instance, interval numbers [9], intuitionistic fuzzy numbers [10], hesitant fuzzy numbers [11], and interval-valued hesitant uncertain linguistic variables [12]. Intuitionistic fuzzy set (IFS) [10] is composed of a membership degree and a nonmembership degree, respectively. The membership degree indicates the epistemic positiveness, while the nonmembership degree reveals the epistemic negativeness. Because of this advantage, IFSs have been widely applied to solve the fuzzy decision making problems. However, the membership degree and nonmembership degree in the form of crisp numbers are not always adequate to express the fuzzy and uncertain information in practice, especially for qualitative aspects; however it is easy to provide the evaluation information by the linguistic variables. So a possible solution is that membership degree and nonmembership degree are represented by linguistic variables, which is called the linguistic intuitionistic fuzzy numbers (LIFNs) firstly developed by Chen et al. [13]. Since then, Yager [14] developed the ordinal based intuitionistic fuzzy set.

In order to obtain the best choice more precisely, we need to not only consider the existing evaluation information, but also take the relationship between them into account. Yager [15] developed the power average (PA) operator and the power OWA (POWA) operator, which can avoid the effect of too large or too small data by the inputting different weights. Then, Xu and Yager [16] proposed some power geometric operators. Zhou and Chen [17] developed the generalized power average (GPA) operator and extended it to the linguistic information. Zhang [18] extended the PA operators to hesitant fuzzy numbers. Yu et al. [19] extended GPA to the interval numbers and intuitionistic fuzzy numbers. Afterwards Liu and Wang [20] extended GPA to the two-dimension linguistic variables.

The Bonferroni mean (BM) operator is another aggregation tool which can catch the interrelationship of individual input arguments. Since it was firstly defined by Bonferroni [21], BM has been applied to many fields and has attracted increasing attentions from researchers. Zhu et al. [22] further developed the geometric Bonferroni mean (GBM) operator. Xu and Yager [23] extended the BM operator to IFSs. Zhou and He [24] developed a normalized weighted Bonferroni mean (IFNWBM) operator for intuitionistic fuzzy numbers. Liu and Wang [25] introduced a normalized weighted Bonferroni mean (SVNNWBM) operator for single-valued neutrosophic numbers. Beliakov and James [26] extended the generalized BM to Atanassov orthopairs. Liu et al. [27] introduced an intuitionistic uncertain linguistic weighted Bonferroni OWA operator.

As mentioned above, because of the complexity of the decision making problems and environment, the linguistic intuitionistic fuzzy numbers (LIFNs) can easily express the fuzzy information by combined intuitionistic fuzzy numbers (IFNs) with linguistic information, and PA can relieve the effect of too large or too small data by the inputting different weights and BM can catch the interrelationship of individual input arguments. However, the study on the MAGDM problems with LIFNs is less; particularly the PA and BM operators cannot deal with the LIFNs. Obviously, it is necessary to extend the PA and BM operators to the LIFNs. Motivated by GIFPA [19] and IFNWBM [24], this paper is to develop the linguistic intuitionistic fuzzy power Bonferroni mean (LIFPBM) operator and the linguistic intuitionistic fuzzy geometric power Bonferroni mean (LIFGPBM) operator by combined PA with BM operators and extended it to the LIFNs, which can fully make use of the advantages of the PA operator, BM operator, and the LIFNs.

The rest of this paper is organized as follows: In Section 2, we briefly review some basic concepts, operational rules, and characters of linguistic information, IFNs, the PA, and BM operators. Section 3 firstly develops the linguistic intuitionistic fuzzy numbers and then combines the PA with the BM operator and extends them to linguistic intuitionistic fuzzy numbers. So linguistic intuitionistic fuzzy power Bonferroni mean (LIFPBM) operator, linguistic intuitionistic fuzzy weighted power Bonferroni mean (LIFWPBM) operator, linguistic intuitionistic fuzzy geometric power Bonferroni mean (LIFGPBM) operator, and linguistic intuitionistic fuzzy weighted geometric power Bonferroni mean (LIFWGPBM) operator are introduced later followed by properties and special cases of these operators. Section 4 puts forward a method for MAGDM problems based on the LIFWGPBM operator and gives the detail procedures. In Section 5, an illustrative example is given to verify the developed approach. In Section 6, this paper ends up with conclusion and future research scopes.

2. Preliminaries

2.1. Linguistic Terms Set

Definition 1 (see [28]). The linguistic terms set $S = (s_0, s_1, \ldots, s_{t-1})$ is made up of odd numbers of elements; that is, t should be odd positive integers. s_i denotes a possible value for a linguistic variable. In practice, t can take the values of 3, 5, 7, 9, and so on. For example, when $t = 7$, a set S can be given as follows:

$S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6)$ = {very low, low, slightly low, fair, slightly high, high, very high}.

For all linguistic terms sets, they should meet the conditions as follows:

(1) if $i > j$, then $s_i \succ s_j$ (i.e., s_i is superior to s_j);

(2) there exists negative operator $\operatorname{neg}(s_i) = s_j$; let $j = t - 1 - i$;

(3) if $s_i \geq s_j$ (which means s_i is not inferior to s_j), $\max(s_i, s_j) = s_i$;

(4) if $s_i \leq s_j$ (which means s_i is not inferior to s_j), $\min(s_i, s_j) = s_i$.

In order to relieve the loss of information in the calculation process of linguistic variables, the discrete linguistic set $S = (s_0, s_1, \ldots, s_{t-1})$ is extended to a continuous scale $\overline{S} = \{s_\alpha \mid \alpha \in R^+\}$. It still satisfies the relationship of strict monotonic increasing and has the operational laws as follows:

$$\beta \times s_i = s_{\beta \times i}, \quad \beta \geq 0, \tag{1}$$

$$s_i \oplus s_j = s_{i+j}, \tag{2}$$

$$s_i \otimes s_j = s_{i \times j}, \tag{3}$$

$$(s_i)^\beta = s_{i^\beta}, \quad \beta \geq 0, \tag{4}$$

2.2. Intuitionistic Fuzzy Numbers. Atanassov [10] proposed the intuitionistic fuzzy set (IFS) which is defined as follows.

Definition 2 (see [10]). Let X be a given fixed set; an IFS in X is an expression: $A = \{\langle x, \mu_{A(x)}, \nu_{A(x)} \rangle \mid x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ are the membership function and the nonmembership function of the element $\forall_x \in X$ to A, respectively, and $0 \leq \mu_{A(x)} + \nu_{A(x)} \leq 1$. In addition, $\pi_{A(x)} = 1 - \mu_{A(x)} - \nu_{A(x)}$ represents the indeterminacy or hesitation degree of x to A.

For convenience, for an IFS A and a given x, we denote an intuitionistic fuzzy number (IFN) by $\alpha = (\mu_\alpha, \nu_\alpha)$.

Some operational laws of IFNs are defined as follows.

Definition 3 (see [29]). Let $\alpha = (\mu_\alpha, \nu_\alpha)$ and $\beta = (\mu_\beta, \nu_\beta)$ be any two IFNs; then

$$\alpha \oplus \beta = \left(\mu_\alpha + \mu_\beta - \mu_\alpha \mu_\beta, \nu_\alpha \nu_\beta\right); \tag{5}$$

$$\alpha \otimes \beta = \left(\mu_\alpha \mu_\beta, \nu_\alpha + \nu_\beta - \nu_\alpha \nu_\beta\right); \tag{6}$$

$$\gamma \alpha = \left(1 - (1 - \mu_\alpha)^\gamma, \nu_\alpha^\gamma\right), \quad \gamma > 0; \tag{7}$$

$$\alpha^\gamma = \left(\mu_\alpha^\gamma, 1 - (1 - \nu_\alpha)^\gamma\right), \quad \gamma > 0. \tag{8}$$

2.3. The PA Operator. The PA operator, proposed by Yager [15], is defined as follows.

Definition 4 (see [15]). For all real numbers α_i ($i = 1, 2, \ldots, n$), a power aggregation (PA) operator of dimension n is a mapping PA: $R^n \to R$, such that

$$\mathrm{PA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{\sum_{i=1}^{n}(1 + T(\alpha_i)) \cdot \alpha_i}{\sum_{i=1}^{n}(1 + T(\alpha_i))}, \quad (9)$$

where $T(\alpha_i) = \sum_{j=1, j\neq i}^{n} \sup(\alpha_i, \alpha_j)$ and $\sup(\alpha_i, \alpha_j)$ is the degree to which α_j supports α_i. It satisfies some rules as below:

(1) $\sup(\alpha_i, \alpha_j) = \sup(\alpha_j, \alpha_i)$;

(2) $\sup(\alpha_i, \alpha_j) \in [0, 1]$;

(3) $\sup(\alpha_i, \alpha_j) \geq \sup(\alpha_m, \alpha_n)$, if $|\alpha_i - \alpha_j| \leq |\alpha_m - \alpha_n|$.

2.4. The BM Operator. The BM operator, proposed by Bonferroni [21], was defined as follows.

Definition 5 (see [21]). Let a_i ($i = 1, 2, \ldots, n$) be a set of nonnegative numbers, $p, q \geq 0$; a Bonferroni mean operator (BM) of dimension n is a mapping BM: $R^n \to R$, such that

$$\mathrm{BM}^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_i{}^p a_j{}^q\right)^{1/(p+q)}. \quad (10)$$

This operator ignores the situation that decision makers may give different weight to each input argument according to their interest. Thus, a weighted Bonferroni mean operator was introduced by Zhou and He [24].

Definition 6 (see [24]). Let a_i ($i = 1, 2, \ldots, n$) be a set of nonnegative numbers; $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of a_i, satisfying $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, $p, q \geq 0$. A weighted Bonferroni mean (WBM) operator of dimension n is a mapping WBM: $R^n \to R$, such that

$$\mathrm{WBM}^{p,q}(a_1, a_2, \ldots, a_n) = \left(\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{\omega_i \omega_j}{1 - \omega_i} a_i{}^p a_j{}^q\right)^{1/(p+q)}. \quad (11)$$

The properties of the WBM operator are shown as follows [24].

Theorem 7 (reducibility). *Let $\omega = (1/n, 1/n, \ldots, 1/n)^T$ be the weight vector of a_i ($i = 1, 2, \ldots, n$); then*

$$\begin{aligned} WBM^{p,q}(a_1, a_2, \ldots, a_n) &= \left(\frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_i{}^p a_j{}^q\right)^{1/(p+q)} \\ &= BM^{p,q}(a_1, a_2, \ldots, a_n). \end{aligned} \quad (12)$$

Theorem 8 (idempotency). *Let $a_j = a$ ($j = 1, 2, \ldots, n$); then $WBM^{p,q}(a_1, a_2, \ldots, a_n) = a$.*

Theorem 9 (permutation). *Let $(a_1, a_2, \ldots, a_n)$ be any permutation of $(a_1', a_2', \ldots, a_n')$; then*

$$WBM^{p,q}(a_1', a_2', \ldots, a_n') = WBM^{p,q}(a_1, a_2, \ldots, a_n). \quad (13)$$

Theorem 10 (monotonicity). *If $a_j \geq b_j$ ($j = 1, 2, \ldots, n$), then $WBM^{p,q}(a_1, a_2, \ldots, a_n) \geq WBM^{p,q}(b_1, b_2, \ldots, b_n)$.*

Theorem 11 (boundedness). *The $WBM^{p,q}$ operator lies between the max and min operators; that is,*

$$\begin{aligned} \min(a_1, a_2, \ldots, a_n) &\leq WBM^{p,q}(a_1, a_2, \ldots, a_n) \\ &\leq \max(a_1, a_2, \ldots, a_n). \end{aligned} \quad (14)$$

In the same way, the geometric BM (GBM) operator also has the characters of considering correlations of the input arguments.

Definition 12 (see [30]). Let a_i ($i = 1, 2, \ldots, n$) be a set of nonnegative numbers, and $p, q \geq 0$; a geometric BM operator of dimension n is a mapping GBM: $R^n \to R$, such that

$$\mathrm{GBM}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q} \prod_{i=1}^{n} \prod_{j=1, j\neq i}^{n} (pa_i + qa_j)^{1/n(n-1)}. \quad (15)$$

The GBM operator does not take the weights of aggregated arguments into account. Sun and Liu [31] further introduced the weighted GBM (WGBM) operator, since the extension process is similar to the WBM operator from the BM operator, so it is omitted here.

3. Some Power Bonferroni Mean Operators Based on Linguistic Intuitionistic Fuzzy Numbers

Now, the WBM and WGBM operators have not been applied to linguistic intuitionistic fuzzy numbers. In order to make up for this gap, in this section, we will extend the WBM and WGBM operators to deal with the linguistic intuitionistic fuzzy numbers and propose linguistic intuitionistic fuzzy power BM (LIFPBM) operator, linguistic intuitionistic fuzzy weighted power BM (LIFWPBM) operator, linguistic intuitionistic fuzzy geometric power BM (LIFGPBM) operator, and linguistic intuitionistic fuzzy weighted geometric power BM (LIFWGPBM) operator, which can be defined as follows.

3.1. Linguistic Intuitionistic Fuzzy Numbers (LIFNs). In practice, sometimes it is difficult to use crisp numbers to depict the membership function and nonmembership function of IFSs. So we introduce the concept of LIFNs where both membership and nonmembership are denoted by linguistic terms.

Definition 13 (see [13]). Let linguistic terms $s_\alpha, s_\beta \in S_{[0,t]}$ and $\gamma = (s_\alpha, s_\beta)$; s_α and s_β denote the membership and nonmembership, respectively. If $\alpha+\beta \le t$, then one can call γ the linguistic intuitionistic fuzzy number (LIFN) defined on $S_{[0,t]}$. For convenience, one uses $\Gamma[0,t]$ to express the set of all LIFNs.

In order to measure the deviation between any two LIFNs, we define the following distance formula.

Definition 14. Let $\gamma_1 = (s_{\alpha_1}, s_{\beta_1})$ and $\gamma_2 = (s_{\alpha_2}, s_{\beta_2})$ be any two LIFNs; then the distance d between γ_1 and γ_2 is expressed by

$$d(\gamma_1,\gamma_2) = \frac{|\alpha_1-\alpha_2| + |\beta_1-\beta_2|}{2t}. \tag{16}$$

Obviously, $d(\gamma_1,\gamma_2)$ satisfies

(1) $d(\gamma_1,\gamma_2) \ge 0$, $d(\gamma_2,\gamma_1) \ge 0$;

(2) $d(\gamma_1,\gamma_2) = d(\gamma_2,\gamma_1)$;

(3) suppose $\gamma_3 = (s_{\alpha_3}, s_{\beta_3})$ is any one LIFN; then $d(\gamma_1,\gamma_3) \le d(\gamma_1,\gamma_2) + d(\gamma_2,\gamma_3)$.

Definition 15 (see [13]). Let $(s_\alpha, s_\beta), (s_{\alpha 1}, s_{\beta 1}), (s_{\alpha 2}, s_{\beta 2}) \in \Gamma[0,t]$, $\lambda > 0$; then the operational laws for LIFNs are shown as follows:

$$\left(s_{\alpha_1}, s_{\beta_1}\right) \oplus \left(s_{\alpha_2}, s_{\beta_2}\right) = \left(s_{\alpha_1+\alpha_2-\alpha_1\alpha_2/t}, s_{\beta_1\beta_2/t}\right); \tag{17}$$

$$\left(s_{\alpha_1}, s_{\beta_1}\right) \otimes \left(s_{\alpha_2}, s_{\beta_2}\right) = \left(s_{\alpha_1\alpha_2/t}, s_{\beta_1+\beta_2-\beta_1\beta_2/t}\right); \tag{18}$$

$$\lambda\left(s_\alpha, s_\beta\right) = \left(s_{t-t(1-\alpha/t)^\lambda}, s_{t(\beta/t)^\lambda}\right); \tag{19}$$

$$\left(s_\alpha, s_\beta\right)^\lambda = \left(s_{t(\alpha/t)^\lambda}, s_{t-t(1-\beta/t)^\lambda}\right). \tag{20}$$

Theorem 16. *Let $(s_\alpha, s_\beta), (s_{\alpha_1}, s_{\beta_1}), (s_{\alpha_2}, s_{\beta_2}) \in \Gamma[0,t]$, $\lambda > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$; then*

$$\left(s_{\alpha_1}, s_{\beta_1}\right) \oplus \left(s_{\alpha_2}, s_{\beta_2}\right) = \left(s_{\alpha_2}, s_{\beta_2}\right) \oplus \left(s_{\alpha_1}, s_{\beta_1}\right); \tag{21}$$

$$\left(s_{\alpha_1}, s_{\beta_1}\right) \otimes \left(s_{\alpha_2}, s_{\beta_2}\right) = \left(s_{\alpha_2}, s_{\beta_2}\right) \otimes \left(s_{\alpha_1}, s_{\beta_1}\right); \tag{22}$$

$$\begin{aligned}&\left(s_\alpha, s_\beta\right) \oplus \left(\left(s_{\alpha_1}, s_{\beta_1}\right) \oplus \left(s_{\alpha_2}, s_{\beta_2}\right)\right)\\ &\quad = \left(\left(s_\alpha, s_\beta\right) \oplus \left(s_{\alpha_1}, s_{\beta_1}\right)\right) \oplus \left(s_{\alpha_2}, s_{\beta_2}\right);\end{aligned} \tag{23}$$

$$\begin{aligned}&\left(s_\alpha, s_\beta\right) \otimes \left(\left(s_{\alpha_1}, s_{\beta_1}\right) \otimes \left(s_{\alpha_2}, s_{\beta_2}\right)\right)\\ &\quad = \left(\left(s_\alpha, s_\beta\right) \otimes \left(s_{\alpha_1}, s_{\beta_1}\right)\right) \otimes \left(s_{\alpha_2}, s_{\beta_2}\right);\end{aligned} \tag{24}$$

$$\lambda\left(\left(s_{\alpha_1}, s_{\beta_1}\right) \oplus \left(s_{\alpha_2}, s_{\beta_2}\right)\right) = \lambda\left(s_{\alpha_1}, s_{\beta_1}\right) \oplus \lambda\left(s_{\alpha_2}, s_{\beta_2}\right); \tag{25}$$

$$\left(\left(s_{\alpha_1}, s_{\beta_1}\right) \otimes \left(s_{\alpha_2}, s_{\beta_2}\right)\right)^\lambda = \left(s_{\alpha_1}, s_{\beta_1}\right)^\lambda \otimes \left(s_{\alpha_2}, s_{\beta_2}\right)^\lambda; \tag{26}$$

$$\lambda_1\left(s_\alpha, s_\beta\right) \oplus \lambda_2\left(s_\alpha, s_\beta\right) = \left(\lambda_1+\lambda_2\right)\left(s_\alpha, s_\beta\right); \tag{27}$$

$$\left(s_\alpha, s_\beta\right)^{\lambda_1} \otimes \left(s_\alpha, s_\beta\right)^{\lambda_2} = \left(s_\alpha, s_\beta\right)^{(\lambda_1+\lambda_2)}; \tag{28}$$

$$\lambda_1\left(\lambda_2\left(s_\alpha, s_\beta\right)\right) = \lambda_1\lambda_2\left(s_\alpha, s_\beta\right); \tag{29}$$

$$\left(\left(s_\alpha, s_\beta\right)^{\lambda_2}\right)^{\lambda_1} = \left(s_\alpha, s_\beta\right)^{\lambda_1\lambda_2}. \tag{30}$$

The proof is easy, so it is omitted here.

Next, we will introduce the comparison method of two LIFNs.

Definition 17 (see [13]). Let $\gamma = (s_\alpha, s_\beta)$, $\gamma_1 = (s_{\alpha_1}, s_{\beta_1})$, $\gamma_2 = (s_{\alpha_2}, s_{\beta_2}) \in \Gamma[0,t]$, and

$$\begin{aligned} Ls(\gamma) &= \alpha - \beta,\\ Lh(\gamma) &= \alpha + \beta;\end{aligned} \tag{31}$$

then one calls $Ls(\gamma)$ the score function of γ and $Lh(\gamma)$ the accuracy function of γ.

If $Ls(\gamma_1) < Ls(\gamma_2)$, then γ_1 is smaller than γ_2, denoted by $\gamma_1 < \gamma_2$.

If $Ls(\gamma_1) > Ls(\gamma_2)$, then γ_1 is bigger than γ_2, denoted by $\gamma_1 > \gamma_2$.

If $Ls(\gamma_1) = Ls(\gamma_2)$, and if

(1) $Lh(\gamma_1) = Lh(\gamma_2)$, then γ_1 and γ_2 represent the same information, denoted by $\gamma_1 = \gamma_2$;

(2) $Lh(\gamma_1) < Lh(\gamma_2)$, then γ_1 is smaller than γ_2, denoted by $\gamma_1 < \gamma_2$;

(3) $Lh(\gamma_1) > Lh(\gamma_2)$, then γ_1 is bigger than γ_2, denoted by $\gamma_1 > \gamma_2$.

It is obvious that $(s_0, s_t) \le (s_\alpha, s_\beta) \le (s_t, s_0)$ for any $(s_\alpha, s_\beta) \in \Gamma[0,t]$ andif $\alpha_1 \le \alpha_2$ and $\beta_1 \ge \beta_2$, then $\gamma_1 \le \gamma_2$.

3.2. Some Linguistic Intuitionistic Fuzzy Numbers Power BM Operators

3.2.1. The Linguistic Intuitionistic Fuzzy Power BM Operator

Definition 18. Let $\gamma_i = (s_{\alpha_i}, s_{\alpha_j})$ $(i = 1, 2, \ldots, n)$ be a set of LIFNs, $p, q \geq 0$. If

$$\begin{aligned} &\mathrm{LIFPBM}^{p,q}(\gamma_1, \gamma_2, \ldots, \gamma_n) \\ &= \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\frac{n(1+T(\gamma_i))}{\sum_{t=1}^{n}(1+T(\gamma_t))} \gamma_i \right)^p \right.\right.\right. \\ &\left.\left.\left. \otimes \left(\frac{n(1+T(\gamma_j))}{\sum_{t=1}^{n}(1+T(\gamma_t))} \gamma_j \right)^q \right) \right) \right)^{1/(p+q)} \end{aligned} \tag{32}$$

then $\mathrm{LIFPBM}^{p,q}$ is called the linguistic intuitionistic fuzzy power BM operator, where $T(\gamma_i) = \sum_{j=1,i\neq j}^{n} \sup(\gamma_i, \gamma_j)$, and $\sup(\gamma_i, \gamma_j)$ is the support for γ_i from γ_j, which satisfies the following three properties:

(1) $\sup(\gamma_i, \gamma_j) \in [0, 1]$;

(2) $\sup(\gamma_i, \gamma_j) = \sup(\gamma_j, \gamma_i)$;

(3) if $d(\gamma_i, \gamma_j) \leq d(\gamma_l, \gamma_r)$,

and then $\sup(\gamma_i, \gamma_j) \geq \sup(\gamma_l, \gamma_r)$, where $d(a, b)$ is the distance between the LIFNs a and b.

Theorem 19. *Let $\gamma_i = (s_{\alpha_i}, s_{\beta_i})$ $(i = 1, 2, \ldots, n)$ be a set of LIFNs, $p, q \geq 0$; the aggregated value by Definition 18 is still a LIFN and can be denoted as*

$$\begin{aligned} &\mathrm{LIFPBM}^{p,q}(\gamma_1, \gamma_2, \ldots, \gamma_n) \\ &= \left[\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\frac{n(1+T(\gamma_i))}{\sum_{t}^{n}(1+T(\gamma_t))} \gamma_i \right)^p \right.\right.\right. \\ &\left.\left.\left. \otimes \left(\frac{n(1+T(\gamma_j))}{\sum_{t}^{n}(1+T(\gamma_t))} \gamma_j \right)^q \right) \right) \right]^{1/(p+q)} \\ &= \left(s_{t[1-\prod_{i,j=1, i\neq j}^{n}(1-(1-(1-\alpha_i/t)^{u_i})^p(1-(1-\alpha_i/t)^{u_j})^q)^{1/n(n-1)}]^{1/(p+q)}}, \right. \\ &\left. s_{t-t[1-(\prod_{i,j=1, i\neq j}^{n}(1-(1-(\beta_i/t)^{u_i})^p(1-(\beta_i/t)^{u_j})^q))^{1/n(n-1)}]^{1/(p+q)}} \right), \end{aligned} \tag{33}$$

where $T(\gamma_i) = \sum_{j=1,i\neq j}^{n} \sup(\gamma_i, \gamma_j)$ and $u_i = n(1 + T(\gamma_i))/\sum_{t=1}^{n}(1 + T(\gamma_t))$ $(i = 1, 2, \ldots, n)$.

Proof. By the operational laws of the LIFNs, we can easily deduce that

$$\begin{aligned} u_i\gamma_{\mathrm{i}} &= \left(s_{t-t(1-\alpha_i/t)^{u_i}}, s_{t(\beta_i/t)^{u_i}} \right), \\ (u_i\gamma_i)^p &= \left(s_{t(1-(1-\alpha_i/t)^{u_i})^p}, s_{t-t(1-(\beta_i/t)^{u_i})^p} \right). \end{aligned} \tag{34}$$

Similarly, we have

$$(u_j\gamma_j)^q = \left(s_{t(1-(1-\alpha_j/t)^{u_j})^p}, s_{t-t(1-(\beta_j/t)^{u_j})^p} \right); \tag{35}$$

then

$$\begin{aligned} &(u_i\gamma_i)^p \otimes (u_j\gamma_j)^q = \left(s_{t(1-(1-\alpha_i/t)^{u_i})^p(1-(1-\alpha_j/t)^{u_j})^q}, \right. \\ &\left. s_{t-t(1-(\beta_i/t)^{u_i})^p(1-(\beta_j/t)^{u_j})^q} \right), \\ &\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left((u_i\gamma_i)^p \otimes (u_j\gamma_j)^q \right) \\ &= \left(s_{t-t\prod_{i,j=1, i\neq j}^{n}(1-(1-\alpha_i/t)^{u_i})^p(1-(1-\alpha_j/t)^{u_j})^q}, \right. \\ &\left. s_{t\prod_{i,j=1}^{n}(1-(1-(\beta_i/t)^{u_i})^p(1-(\beta_i/t)^{u_i})^q)} \right). \end{aligned} \tag{36}$$

Further we have

$$\begin{aligned} &\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left((u_i\gamma_i)^p \otimes (u_j\gamma_j)^q \right) \right) \\ &= \left(s_{t-t(\prod_{i,j=1, i\neq j}^{n}(1-(1-\alpha_i/t)^{u_i})^p(1-(1-\alpha_j/t)^{u_j})^q)^{1/n(n-1)}}, \right. \\ &\left. s_{t(\prod_{i,j=1}^{n}(1-(1-(\beta_i/t)^{u_i})^p(1-(\beta_i/t)^{u_i})^q))^{1/n(n-1)}} \right). \end{aligned} \tag{37}$$

Finally, we have

$$\begin{aligned} &\left[\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\frac{n(1+T(\gamma_i))}{\sum_{t}^{n}(1+T(\gamma_t))} \gamma_i \right)^p \right.\right.\right. \\ &\left.\left.\left. \otimes \left(\frac{n(1+T(\gamma_j))}{\sum_{t}^{n}(1+T(\gamma_t))} \gamma_j \right)^q \right) \right) \right]^{1/(p+q)} \\ &= \left(s_{t[1-\prod_{i,j=1, i\neq j}^{n}(1-(1-(1-\alpha_i/t)^{u_i})^p(1-(1-\alpha_i/t)^{u_j})^q)^{1/n(n-1)}]^{1/(p+q)}}, \right. \\ &\left. s_{t-t[1-(\prod_{i,j=1, i\neq j}^{n}(1-(1-(\beta_i/t)^{u_i})^p(1-(\beta_i/t)^{u_j})^q))^{1/n(n-1)}]^{1/(p+q)}} \right) \end{aligned} \tag{38}$$

which ends the proof of Theorem 19. □

Theorem 20 (idempotency). *Let $\gamma_i = (s_{\alpha_i}, s_{\beta_i})$ $(i = 1, 2, \ldots, n)$ be a set of LIFNs, and $p, q \geq 0$; if for all i, $\gamma_i = \gamma = (s_\alpha, s_\beta)$, then $\mathrm{LIFPBM}^{p,q}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \gamma$.*

Proof. Since for all i, $\gamma_i = \gamma = (s_\alpha, s_\beta)$, then we have

$$u_i = \frac{n(1+T(\gamma))}{\sum_{t=1}^n (1+T(\gamma))} = 1$$

$$\begin{aligned}
&\mathrm{LIFPBM}^{p,q}(\gamma_1,\gamma_2,\ldots,\gamma_n) = \mathrm{LIFPBM}^{p,q}(\gamma,\gamma,\ldots,\gamma) \\
&= \left(s_{t[1-\prod_{i,j=1, i\neq j}^{n}(1-(1-(1-\alpha_i/t)^{u_i})^p(1-(1-\alpha_i/t)^{u_j})^q)^{1/n(n-1)}]^{1/(p+q)}}, \right. \\
&\left. s_{t-t[1-(\prod_{i,j=1, i\neq j}^{n}(1-(1-(\beta_i/t)^{u_i})^p(1-(\beta_i/t)^{u_j})^q))^{1/n(n-1)}]^{1/(p+q)}} \right) \\
&= \left(s_{t[1-\prod_{i,j=1, i\neq j}^{n}(1-(1-(1-\alpha/t)^1)^p(1-(1-\alpha/t)^1)^q)^{1/n(n-1)}]^{1/(p+q)}}, \right. \\
&\left. s_{t-t[1-(\prod_{i,j=1, i\neq j}^{n}(1-(1-(\beta/t)^1)^p(1-(\beta/t)^1)^q))^{1/n(n-1)}]^{1/(p+q)}} \right) \\
&= \left(s_{t[1-\prod_{i,j=1, i\neq j}^{n}(1-(\alpha/t)^{p+q})^{1/n(n-1)}]^{1/(p+q)}}, \right. \\
&\left. s_{t-t[1-(\prod_{i,j=1, i\neq j}^{n}(1-(1-\beta/t)^{p+q}))^{1/n(n-1)}]^{1/(p+q)}} \right) \\
&= \left(s_{t[1-(1-(\alpha/t)^{p+q})]^{1/(p+q)}}, s_{t-t[1-(1-(1-\beta/t)^{p+q})]^{1/(p+q)}} \right) \\
&= \left(s_{t(\alpha/t)}, s_{t-t(1-\beta/t)} \right) = (s_\alpha, s_\beta)
\end{aligned} \tag{39}$$

so this theorem is proved. □

Theorem 21 (permutation). *If $(\gamma_1', \gamma_2', \ldots, \gamma_n')$ is any permutation of $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, then one can get*

$$\begin{aligned} &LIFPBM^{p,q}(\gamma_1', \gamma_2', \ldots, \gamma_n') \\ &= LIFPBM^{p,q}(\gamma_1, \gamma_2, \ldots, \gamma_n). \end{aligned} \tag{40}$$

Proof. According to Theorem 19 and Definition 18, we can get

$$\begin{aligned}
&\left[\frac{1}{n(n-1)}\left(\bigoplus_{i,j=1, i\neq j}^{n}\left(\left(\frac{n(1+T(\gamma_i'))}{\sum_{t=1}^n(1+T(\gamma_t'))}\gamma_i'\right)^p \otimes \left(\frac{n(1+T(\gamma_j'))}{\sum_{t=1}^n(1+T(\gamma_t'))}\gamma_j'\right)^q\right)\right)\right]^{1/(p+q)} \\
&= \left[\frac{1}{n(n-1)}\left(\bigoplus_{i,j=1, i\neq j}^{n}\left(\left(\frac{n(1+T(\gamma_i))}{\sum_{t=1}^n(1+T(\gamma_t))}\gamma_i\right)^p \otimes \left(\frac{n(1+T(\gamma_j))}{\sum_{t=1}^n(1+T(\gamma_t))}\gamma_j\right)^q\right)\right)\right]^{1/(p+q)}
\end{aligned} \tag{41}$$

so $\mathrm{LIFPBM}^{p,q}(\gamma_1', \gamma_2', \ldots, \gamma_n') = \mathrm{LIFPBM}^{p,q}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ which ends the proof of this theorem. □

Theorem 22 (boundedness). *Let γ_i $(i = 1, 2, \ldots, n)$ be a collection of LIFNs, $\widetilde{\gamma} = \min(\gamma_1, \gamma_2, \ldots, \gamma_n) = (s_{\widetilde{\alpha}}, s_{\widetilde{\beta}})$, $\widehat{\gamma} = \max(\gamma_1, \gamma_2, \ldots, \gamma_n) = (s_{\widehat{\alpha}}, s_{\widehat{\beta}})$, and the LIFPBM^{p,q} operator lies at*

$$\widetilde{m} \leq LIFPBM^{p,q}(\gamma_1, \gamma_2, \ldots, \gamma_n) \leq \widehat{m}, \tag{42}$$

where

$$\begin{aligned}
\widetilde{m} &= \left(s_{t[1-\prod_{i,j=1, i\neq j}^{n}(1-(1-(1-\widetilde{\alpha}/t)^{u_i})^p(1-(1-\widetilde{\alpha}/t)^{u_j})^q)^{1/n(n-1)}]^{1/(p+q)}}, \right. \\
&\left. s_{t-t[1-(\prod_{i,j=1, i\neq j}^{n}(1-(1-(\widehat{\beta}/t)^{u_i})^p(1-(\widehat{\beta}/t)^{u_j})^q))^{1/n(n-1)}]^{1/(p+q)}} \right) \\
\widehat{m} &= \left(s_{t[1-\prod_{i,j=1, i\neq j}^{n}(1-(1-(1-\widehat{\alpha}/t)^{u_i})^p(1-(1-\widehat{\alpha}/t)^{u_j})^q)^{1/n(n-1)}]^{1/(p+q)}}, \right. \\
&\left. s_{t-t[1-(\prod_{i,j=1, i\neq j}^{n}(1-(1-(\widehat{\beta}/t)^{u_i})^p(1-(\widehat{\beta}/t)^{u_j})^q))^{1/n(n-1)}]^{1/(p+q)}} \right) \\
u_i &= \frac{n(1+T(\gamma_i))}{\sum_t^n (1+T(\gamma_t))}.
\end{aligned} \tag{43}$$

Proof. Since $1-(1-\widetilde{\alpha}/t)^{u_i} \leq 1-(1-\alpha_i/t)^{u_i}$,

$$\begin{aligned}
&1-\left(1-\left(1-\frac{\widetilde{\alpha}}{t}\right)^{u_i}\right)^p\left(1-\left(1-\frac{\widetilde{\alpha}}{t}\right)^{u_j}\right)^q \\
&\geq 1-\left(1-\left(1-\frac{\alpha_i}{t}\right)^{u_i}\right)^p\left(1-\left(1-\frac{\alpha_j}{t}\right)^{u_j}\right)^q
\end{aligned} \tag{44}$$

then

$$\begin{aligned}
&1-\prod_{i,j=1, i\neq j}^{n}\left(1-\left(1-\left(1-\frac{\widetilde{\alpha}}{t}\right)^{u_i}\right)^p\left(1-\left(1-\frac{\widetilde{\alpha}}{t}\right)^{u_j}\right)^q\right)^{1/n(n-1)} \\
&\leq 1 \\
&-\prod_{i,j=1, i\neq j}^{n}\left(1-\left(1-\left(1-\frac{\alpha_i}{t}\right)^{u_i}\right)^p\left(1-\left(1-\frac{\alpha_j}{t}\right)^{u_j}\right)^q\right)^{1/n(n-1)},
\end{aligned} \tag{45}$$

and finally we have

$$t\left[1-\prod_{i}^{n}\left(1-\left(1-\left(1-\frac{\widetilde{\alpha}}{t}\right)^{u_i}\right)^{p}\left(1-\left(1-\frac{\widetilde{\alpha}}{t}\right)^{u_j}\right)^{q}\right)^{1/n(n-1)}\right]^{1/(p+q)} \leq t\left[1-\prod_{i}^{n}\left(1-\left(1-\left(1-\frac{\alpha_i}{t}\right)^{u_i}\right)^{p}\left(1-\left(1-\frac{\alpha_j}{t}\right)^{u_j}\right)^{q}\right)^{1/n(n-1)}\right]^{1/(p+q)}. \tag{46}$$

Similarly, we have

$$t-t\left[1-\left(\prod_{i}^{n}\left(1-\left(1-\left(\frac{\widetilde{\beta}}{t}\right)^{u_i}\right)^{p}\left(1-\left(\frac{\widetilde{\beta}}{t}\right)^{u_j}\right)^{q}\right)\right)^{1/n(n-1)}\right]^{1/(p+q)} \leq t-t\left[1-\left(\prod_{i}^{n}\left(1-\left(1-\left(\frac{\beta_i}{t}\right)^{u_i}\right)^{p}\left(1-\left(\frac{\beta_j}{t}\right)^{u_j}\right)^{q}\right)\right)^{1/n(n-1)}\right]^{1/(p+q)}. \tag{47}$$

Further, we have

$$\widetilde{m} \leq \mathrm{LIFPBM}^{p,q}\left(\gamma_1,\gamma_2,\ldots,\gamma_n\right). \tag{48}$$

In the same way, we can get

$$\mathrm{LIFPBM}^{p,q}\left(\gamma_1,\gamma_2,\ldots,\gamma_n\right) \leq \widehat{m}. \tag{49}$$

So this theorem is proved. □

Definition 23. Let $\gamma_i = (s_{\alpha_i}, s_{\alpha_j})$ $(i = 1,2,\ldots,n)$ be a set of LIFNs, $p,q \geq 0$, and $\omega = (\omega_1,\omega_2,\ldots,\omega_n)$ is the associated weight vector of γ_i $(i = 1,2,\ldots,n)$. If

$$\mathrm{LIFWPBM}^{p,q}\left(\gamma_1,\gamma_2,\ldots,\gamma_n\right) = \left[\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\frac{\omega_i n\left(1+T\left(\gamma_i\right)\right)}{\sum_{t=1}^{n}\omega_t\left(1+T\left(\gamma_t\right)\right)}\gamma_i\right)^{p} \otimes\left(\frac{\omega_j n\left(1+T\left(\gamma_j\right)\right)}{\sum_{t=1}^{n}\omega_t\left(1+T\left(\gamma_t\right)\right)}\gamma_j\right)^{q}\right)\right)\right]^{1/(p+q)} \tag{50}$$

then $\mathrm{LIFWPBM}^{p,q}$ is called the linguistic intuitionistic fuzzy weighted power BM operator.

Theorem 24. *Let $\gamma_i = (s_{\alpha_i}, s_{\beta_i})$ $(i = 1,2,\ldots,n)$ be a set of LIFNs, $p,q \geq 0$, and LIFNs$^n \rightarrow$ LIFNs. So the aggregated value by Definition 23 is still a LIFN and can be denoted as*

$$\begin{aligned}
&\mathrm{LIFWPBM}^{p,q}\left(\gamma_1,\gamma_2,\ldots,\gamma_n\right) \\
&= \left[\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\frac{\omega_i n\left(1+T\left(\gamma_i\right)\right)}{\sum_{t=1}^{n}\omega_t\left(1+T\left(\gamma_t\right)\right)}\gamma_i\right)^{p} \otimes\left(\frac{\omega_j n\left(1+T\left(\gamma_j\right)\right)}{\sum_{t=1}^{n}\omega_t\left(1+T\left(\gamma_t\right)\right)}\gamma_j\right)^{q}\right)\right)\right]^{1/(p+q)} \\
&= \left(s_{t[1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-\eta_{\alpha i}^{u_i})^{p}(1-\eta_{\alpha j}^{u_j})^{q})^{1/n(n-1)}]^{1/(p+q)}},\ s_{t-t[1-(\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-\mu_{\beta i}^{u_i})^{p}(1-\mu_{\beta j}^{u_j})^{p}))^{1/n(n-1)}]^{1/(p+q)}}\right),
\end{aligned} \tag{51}$$

where $\eta_{\alpha i} = 1-\alpha_i/t$, $\eta_{\alpha j} = 1-\alpha_j/t$, $\mu_{\beta i} = \beta_i/t$, $\mu_{\beta j} = \beta_j/t$, and $T(\gamma_i) = \sum_{j=1,i\neq j}^{n}\sup(\gamma_i,\gamma_j)$,

$$u_i = \frac{\omega_i n\left(1+T\left(\gamma_i\right)\right)}{\sum_{t=1}^{n}\omega_t\left(1+T\left(\gamma_i\right)\right)} \quad (i = 1,2,\ldots,n). \tag{52}$$

The proof is similar to that of LIFPBM, so it is omitted here.

It is easy to prove that the LIFWPBM operator has the character of boundedness; it is omitted here.

In the following, we will discuss some special cases of the LIFWPBM operator about the parameters p and q.

(1) If $q = 0$, then

$$\mathrm{LIFWPBM}^{p,0}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \left(s_{t[1-\prod_{i,j=1, i\neq j}^{n}(1-(1-(1-\alpha_i/t)^{u_i})^p)^{1/n(n-1)}]^{1/p}}, s_{t-t[1-(\prod_{i,j=1, i\neq j}^{n}(1-(1-(\beta_i/t)^{u_i})^p))^{1/n(n-1)}]^{1/p}} \right). \quad (53)$$

(2) If $p = 1$ and $q = 0$, then

$$\mathrm{LIFWPBM}^{1,0}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \left(s_{t-t\prod_{i,j=1, i\neq j}^{n}(1-\alpha_i/t)^{u_i/n(n-1)}}, s_{t(\prod_{i,j=1, i\neq j}^{n}(\beta_i/t)^{u_i})^{1/n(n-1)}} \right). \quad (54)$$

(3) If $p = 1/2$ and $q = 1/2$, then

$$\mathrm{LIFWPBM}^{1/2,1/2}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \left(s_{t-t\prod_{i,j=1, i\neq j}^{n}(1-(1-(1-\alpha_i/t)^{u_i})^{1/2}(1-(1-\alpha_j/t)^{u_j})^{1/2})^{1/n(n-1)}}, s_{t(\prod_{i,j=1, i\neq j}^{n}(1-(1-(\beta_i/t)^{u_i})^{1/2}(1-(\beta_j/t)^{u_j})^{1/2}))^{1/n(n-1)}} \right). \quad (55)$$

(4) If $p = 1$ and $q = 1$, then

$$\mathrm{LIFWPBM}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \left(s_{t[1-\prod_{i,j=1, i\neq j}^{n}(1-(1-(\alpha_i/t))^{u_i}(1-(\alpha_j/t))^{u_j})^{1/n(n-1)}]^{1/2}}, s_{t-t[1-(\prod_{i,j=1, i\neq j}^{n}(1-(1-(\beta_i/t)^{u_i})(1-(\beta_j/t)^{u_j})))^{1/n(n-1)}]^{1/2}} \right). \quad (56)$$

3.2.2. The Linguistic Intuitionistic Fuzzy Geometric Power BM Operator

Definition 25. Let $\gamma_i = (s_{\alpha_i}, s_{\alpha_j})$ $(i = 1, 2, \ldots, n)$ be a set of LIFNs, and $p, q \geq 0$. If

$$\mathrm{LIFGPBM}^{p,q}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \frac{1}{p+q}\left[\bigotimes_{i,j=1, i\neq j}^{n} \left(p\gamma_i^{n(1+T(\gamma_i))/\sum_{t=1}^{n}(1+T(\gamma_t))} \oplus q\gamma_j^{n(1+T(\gamma_j))/\sum_{t=1}^{n}(1+T(\gamma_t))} \right)^{1/n(n-1)} \right] \quad (57)$$

then $\mathrm{LIFGPBM}^{p,q}$ is called the linguistic intuitionistic fuzzy geometric power BM (LIFGPBM) operator.

Theorem 26. *Let $\gamma_i = (s_{\alpha_i}, s_{\beta_i})$ $(i = 1, 2, \ldots, n)$ be a set of LIFNs, $p, q \geq 0$; then the aggregated result by Definition 25 is still a LIFN and can be denoted by*

$$\begin{aligned} &\mathit{LIFGPBM}^{p,q}(\gamma_1, \gamma_2, \ldots, \gamma_n) \\ &= \frac{1}{p+q}\left[\bigotimes_{i,j=1, i\neq j}^{n} \left(p\gamma_i^{n(1+T(\gamma_i))/\sum_{t=1}^{n}(1+T(\gamma_t))} \oplus q\gamma_j^{n(1+T(\gamma_j))/\sum_{t=1}^{n}(1+T(\gamma_t))} \right)^{1/n(n-1)} \right] \\ &= \left(s_{t-t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(\alpha_i/t)^{u_i})^p(1-(\alpha_j/t)^{u_j})^q]^{1/n(n-1)}]^{1/(p+q)}}, s_{t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(1-\beta_i/t)^{u_i})^p(1-(1-\beta_j/t)^{u_j})^q]^{1/n(n-1)}]^{1/(p+q)}} \right), \end{aligned} \quad (58)$$

where $T(\gamma_i) = \sum_{j=1, i\neq j}^{n} \sup(\gamma_i, \gamma_j)$ $u_i = n(1 + T(\gamma_i))/\sum_{t=1}^{n}(1 + T(\gamma_t))$ $(i = 1, 2, \ldots, n)$.

Proof. By the operational laws of the LIFNs, we have

$$\begin{aligned} p\gamma_i^{u_i} \oplus p\gamma_j^{u_j} &= \left(s_{t-t(1-(\alpha_i/t)^{u_i})^p(1-(\alpha_j/t)^{u_j})^q}, s_{t(1-(1-\beta_i/t)^{u_i})^p(1-(1-\beta_j/t)^{u_j})^q} \right), \\ \left(p\gamma_i^{u_i} \oplus p\gamma_j^{u_j} \right)^{1/n(n-1)} &= \left(s_{(t-t(1-(\alpha_i/t)^{u_i})^p(1-(\alpha_j/t)^{u_j})^q)^{1/n(n-1)}}, s_{t-t(1-(1-(1-\beta_i/t)^{u_i})^p(1-(1-\beta_j/t)^{u_j})^q)^{1/n(n-1)}} \right), \end{aligned} \quad (59)$$

then

$$\bigotimes_{i,j=1}^{n} \left(p\gamma_i^{u_i} \oplus p\gamma_j^{u_j} \right)^{1/n(n-1)} = \left(s_{t\prod_{i,j=1, i\neq j}^{n}(1-(1-(\alpha_i/t)^{u_i})^p(1-(\alpha_j/t)^{u_j})^q)^{1/n(n-1)}}, s_{t(1-\prod_{i,j=1, i\neq j}^{n}(1-(1-\beta_i/t)^{u_i})^p(1-(1-\beta_j/t)^{u_j})^q)^{1/n(n-1)}} \right) \quad (60)$$

and as a result

$$\frac{1}{p+q}\left[\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\gamma_i^{n(1+T(\gamma_i))/\sum_{t=1}^{n}(1+T(\gamma_t))}\right.\right.$$
$$\left.\left.\oplus q\gamma_j^{n(1+T(\gamma_j))/\sum_{t=1}^{n}(1+T(\gamma_t))}\right)^{1/n(n-1)}\right]$$
$$=\left(s_{t-t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(\alpha_i/t)^{u_i})^p(1-(\alpha_j/t)^{u_j})^q]^{1/n(n-1)}]^{1/(p+q)}},\right.$$
$$\left.s_{t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(1-\beta_i/t)^{u_i})^p(1-(1-\beta_j/t)^{u_j})^q]^{1/n(n-1)}]^{1/(p+q)}}\right) \tag{61}$$

so the theorem is proved. □

Theorem 27 (idempotency). *Let $\gamma_i = (s_{\alpha_i}, s_{\alpha_j})$ $(i = 1,2,\ldots,n)$ be a set of LIFNs, and $p, q \geq 0$; if for all i, $\gamma_i = \gamma = (s_\alpha, s_\beta)$, then $LIFGPBM^{p,q}(\gamma_1,\gamma_2,\ldots,\gamma_n) = \gamma$.*

Proof. Since for all i, $\gamma_i = \gamma = (s_\alpha, s_\beta)$, then we have

$$u_i = \frac{n(1+T(\gamma))}{\sum_{t=1}^{n}(1+T(\gamma))} = 1. \tag{62}$$

So

$$\mathrm{LIFGPBM}^{p,q}(\gamma,\gamma,\ldots,\gamma)$$
$$=\left(s_{t-t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(\alpha/t)^1)^p(1-(\alpha/t)^1)^q]^{1/n(n-1)}]^{1/(p+q)}},\right.$$
$$\left.s_{t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(1-\beta/t)^1)^p(1-(1-\beta/t)^1)^q]^{1/n(n-1)}]^{1/(p+q)}}\right)$$
$$=\left(s_{t-t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-\alpha/t)^{p+q}]^{1/n(n-1)}]^{1/(p+q)}},\right.$$
$$\left.s_{t[1-\prod_{i,j=1, i\neq j}^{n}[1-(\beta/t)^{p+q}]^{1/n(n-1)}]^{1/(p+q)}}\right) \tag{63}$$
$$=\left(s_{t-t[(1-\alpha/t)^{p+q}]^{1/(p+q)}},\ s_{t[(\beta/t)^{p+q}]^{1/(p+q)}}\right)$$
$$=\left(s_{t-t(1-\alpha/t)},\ s_{t(\beta/t)}\right) = \left(s_\alpha,\ s_\beta\right) = \gamma$$

so this theorem is proved. □

Theorem 28 (permutation). *If $(\gamma_1', \gamma_2', \ldots, \gamma_n')$ is any permutation of $(\gamma_1,\gamma_2,\ldots,\gamma_n)$, $\omega = (\omega_1, \omega_2', \ldots, \omega_n')$ is the weight vector of γ_i' $(i = 1,2,\ldots,n)$; then one can get*

$$LIFGPBM^{p,q}(\gamma_1',\gamma_2',\ldots,\gamma_n') = LIFGPBM^{p,q}(\gamma_1,\gamma_2,\ldots,\gamma_n). \tag{64}$$

Proof. According to Theorem 26 and Definition 25, we can get

$$\frac{1}{p+q}\left[\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\gamma_i'^{\omega_i' n(1+T(\gamma_i'))/\sum_{t=1}^{n}\omega_t'(1+T(\gamma_t'))}\right.\right.$$
$$\left.\left.\oplus q\gamma_j'^{\omega_j' n(1+T(\gamma_j'))/\sum_{t=1}^{n}\omega_t'(1+T(\gamma_t'))}\right)^{1/n(n-1)}\right]$$
$$=\frac{1}{p+q}\left[\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\gamma_i^{\omega_i n(1+T(\gamma_i))/\sum_{t=1}^{n}\omega_t(1+T(\gamma_t))}\right.\right. \tag{65}$$
$$\left.\left.\oplus q\gamma_j^{\omega_j n(1+T(\gamma_j))/\sum_{t=1}^{n}\omega_t(1+T(\gamma_t))}\right)^{1/n(n-1)}\right]$$

so $\mathrm{LIFGPBM}^{p,q}(\gamma_1',\gamma_2',\ldots,\gamma_n') = \mathrm{LIFGPBM}^{p,q}(\gamma_1,\gamma_2,\ldots,\gamma_n)$ which ends the proof of this theorem. □

Theorem 29 (boundedness). *Let γ_i $(i = 1,2,\ldots,n)$ be a collection of LIFNs, $\widetilde{\gamma} = \min(\gamma_1,\gamma_2,\ldots,\gamma_n) = (s_{\widetilde{\alpha}}, s_{\widetilde{\beta}})$, $\widehat{\gamma} = \max(\gamma_1,\gamma_2,\ldots,\gamma_n) = (s_{\widehat{\alpha}}, s_{\widehat{\beta}})$, and the $LIFPBM^{p,q}$ operator lies at*

$$\widetilde{m} \leq LIFGPBM^{p,q}(\gamma_1,\gamma_2,\ldots,\gamma_n) \leq \widehat{m}, \tag{66}$$

where

$$\widetilde{m}=\left(s_{t-t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(\widetilde{\alpha}/t)^{u_i})^p(1-(\widetilde{\alpha}/t)^{u_j})^q]^{1/n(n-1)}]^{1/(p+q)}},\right.$$
$$\left.s_{t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(1-\widetilde{\beta}/t)^{u_i})^p(1-(1-\widetilde{\beta}/t)^{u_j})^q]^{1/n(n-1)}]^{1/(p+q)}}\right)$$
$$\widehat{m}=\left(s_{t-t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(\widehat{\alpha}/t)^{u_i})^p(1-(\widehat{\alpha}/t)^{u_j})^q]^{1/n(n-1)}]^{1/(p+q)}},\right. \tag{67}$$
$$\left.s_{t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(1-\widehat{\beta}/t)^{u_i})^p(1-(1-\widehat{\beta}/t)^{u_j})^q]^{1/n(n-1)}]^{1/(p+q)}}\right)$$
$$u_i = \frac{n(1+T(\gamma_i))}{\sum_t^n(1+T(\gamma_t))}.$$

Proof. Since $1-(\widetilde{\alpha}/t)^{u_i} \geq 1-(\alpha_i/t)^{u_i}$,

$$1-\left(1-\left(\frac{\widetilde{\alpha}}{t}\right)^{u_i}\right)^p\left(1-\left(\frac{\widetilde{\alpha}}{t}\right)^{u_j}\right)^q$$
$$\leq 1-\left(1-\left(\frac{\alpha_i}{t}\right)^{u_i}\right)^p\left(1-\left(\frac{\alpha_j}{t}\right)^{u_j}\right)^q \tag{68}$$

and then

$$1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left[1-\left(1-\left(\frac{\widehat{\alpha}}{t}\right)^{u_i}\right)^{p}\left(1-\left(\frac{\widehat{\alpha}}{t}\right)^{u_j}\right)^{q}\right]^{1/n(n-1)} \geq 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left[1-\left(1-\left(\frac{\alpha_i}{t}\right)^{u_i}\right)^{p}\left(1-\left(\frac{\alpha_j}{t}\right)^{u_j}\right)^{q}\right]^{1/n(n-1)}. \quad (69)$$

Further, we have

$$t-t\left[1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left[1-\left(1-\left(\frac{\widehat{\alpha}}{t}\right)^{u_i}\right)^{p}\left(1-\left(\frac{\widehat{\alpha}}{t}\right)^{u_j}\right)^{q}\right]^{1/n(n-1)}\right]^{1/(p+q)} \leq t-t\left[1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left[1-\left(1-\left(\frac{\alpha_i}{t}\right)^{u_i}\right)^{p}\left(1-\left(\frac{\alpha_j}{t}\right)^{u_j}\right)^{q}\right]^{1/n(n-1)}\right]^{1/(p+q)}. \quad (70)$$

Similarly, we have

$$t\left[1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left[1-\left(1-\left(1-\frac{\widetilde{\beta}}{t}\right)^{u_i}\right)^{p}\cdot\left(1-\left(1-\frac{\widetilde{\beta}}{t}\right)^{u_j}\right)^{q}\right]^{1/n(n-1)}\right]^{1/(p+q)} \leq t\left[1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left[1-\left(1-\left(1-\frac{\beta_i}{t}\right)^{u_i}\right)^{p}\cdot\left(1-\left(1-\frac{\beta_j}{t}\right)^{u_j}\right)^{q}\right]^{1/n(n-1)}\right]^{1/(p+q)}. \quad (71)$$

Finally, we have

$$\widetilde{m} \leq \mathrm{LIFGPBM}^{p,q}\left(\gamma_1,\gamma_2,\ldots,\gamma_n\right). \quad (72)$$

In the same way, we can get

$$\mathrm{LIFGPBM}^{p,q}\left(\gamma_1,\gamma_2,\ldots,\gamma_n\right) \leq \widehat{m}. \quad (73)$$

So this theorem is proved. □

In the LIFGPBM operator, we think all arguments have the same importance. In order to consider the different weights of input arguments, the linguistic intuitionistic fuzzy weighted geometric power BM (LIFWGPBM) operator can be defined as follows.

Definition 30. Let $\gamma_i=(s_{\alpha_i},s_{\beta_i})$ $(i=1,2,\ldots,n)$ be a set of LIFNs, and $p,q\geq 0$; $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ is the associated weight of γ_i. If

$$\mathrm{LIFWGPBM}^{p,q}\left(\gamma_1,\gamma_2,\ldots,\gamma_n\right) = \frac{1}{p+q}\left[\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\gamma_i^{\omega_i n(1+T(\gamma_i))/\sum_{t=1}^{n}\omega_t(1+T(\gamma_t))} \oplus q\gamma_j^{\omega_j n(1+T(\gamma_j))/\sum_{t=1}^{n}\omega_t(1+T(\gamma_t))}\right)^{1/n(n-1)}\right] \quad (74)$$

then $\mathrm{LIFWGPBM}^{p,q}$ is called the linguistic intuitionistic fuzzy weighted geometric power BM operator.

Theorem 31. *Let $\gamma_i=(s_{\alpha_i},s_{\beta_i})$ $(i=1,2,\ldots,n)$ be a set of LIFNs, $p,q\geq 0$, and $LIFNs^n\rightarrow LIFNs$. Then the aggregated result according to Definition 30 is still a LIFN, and it can be denoted by*

$$LIFWGPBM^{p,q}\left(\gamma_1,\gamma_2,\ldots,\gamma_n\right) = \left(s_{t-t[1-\prod_{i,j=1,i\neq j}^{n}[1-(1-(\alpha_i/t)^{u_i})^{p}(1-(\alpha_j/t)^{u_j})^{q}]^{1/n(n-1)}]^{1/(p+q)}},\; s_{t[1-\prod_{i,j=1,i\neq j}^{n}[1-(1-(1-\beta_i/t)^{u_i})^{p}(1-(1-\beta_j/t)^{u_j})^{q}]^{1/n(n-1)}]^{1/(p+q)}}\right), \quad (75)$$

where $T(\gamma_i)=\sum_{j=1,i\neq j}^{n}\sup(\gamma_i,\gamma_j)$, $u_i=\omega_i n(1+T(\gamma_i))/\sum_{t=1}^{n}\omega_t(1+T(\gamma_t))$ $(i=1,2,\ldots,n)$.

The LIFWGPBM operator still has the character of boundedness; it is omitted here.

Next we will discuss some special cases of the LIFWGPBM operator about different parameters p and q.

(1) If $q=0$, then

$$\mathrm{LIFWGPBM}^{p,0}\left(\gamma_1,\gamma_2,\ldots,\gamma_n\right) = \left(s_{t-t[1-\prod_{i}^{n}[1-(1-(\alpha_i/t)^{u_i})^{p}]^{1/n(n-1)}]^{1/p}},\; s_{t[1-\prod_{i}^{n}[1-(1-(1-\beta_i/t)^{u_i})^{p}]^{1/n(n-1)}]^{1/p}}\right). \quad (76)$$

(2) If $p = 1$ and $q = 0$, then

$$\begin{aligned} &\text{LIFWGPBM}^{1,0}(\gamma_1,\gamma_2,\ldots,\gamma_n) \\ &\quad = \left(s_{t\prod_i^n(\alpha_i/t)^{u_i/n(n-1)}}, s_{t-t\prod_i^n(1-\beta_i/t)^{u_i/n(n-1)}}\right). \end{aligned} \tag{77}$$

(3) If $p = 1/2$ and $q = 1/2$, then

$$\begin{aligned} &\text{LIFWGPBM}^{1/2,1/2}(\gamma_1,\gamma_2,\ldots,\gamma_n) \\ &= \left(s_{t-t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(\alpha_i/t)^{u_i})^{1/2}(1-(\alpha_j/t)^{u_j})^{1/2}]^{1/n(n-1)}]},\right. \\ &\left. s_{t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(1-\beta_i/t)^{u_i})^{1/2}(1-(1-\beta_j/t)^{u_j})^{1/2}]^{1/n(n-1)}]}\right). \end{aligned} \tag{78}$$

(4) If $p = 1$ and $q = 1$, then

$$\begin{aligned} &\text{LIFWGPBM}^{1,1}(\gamma_1,\gamma_2,\ldots,\gamma_n) \\ &= \left(s_{t-t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(\alpha_i/t)^{u_i})(1-(\alpha_j/t)^{u_j})]^{1/n(n-1)}]^{1/2}},\right. \\ &\left. s_{t[1-\prod_{i,j=1, i\neq j}^{n}[1-(1-(1-\beta_i/t)^{u_i})(1-(1-\beta_j/t)^{u_j})]^{1/n(n-1)}]^{1/2}}\right). \end{aligned} \tag{79}$$

4. The MAGDM Method Based on LIFNs

This section proposes one method to cope with MAGDM problems with LIFNs by the LIFWGPBM operator and the LIFWPBM operator.

For a MAGDM problem with LIFNs, let $X = \{x_1, x_2, \ldots, x_m\}$ be the collection of alternatives, $A = \{a_1, a_2, \ldots, a_n\}$ be a discrete set of n attributes, and $D = \{d_1, d_2, \ldots, d_t\}$ be a set of t decision makers, and expert d_k provides his/her assessment information of an alternative x_i on an attribute a_j as a LIFN γ_{ij}^k $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$ according to a given linguistic term set S. In consequence, the decision matrices can be expressed as $R_k = (\gamma_{ij}^k)_{m\times n}$ $(k = 1,2,\ldots,t)$. Suppose that $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of attributes with $\omega_j \geq 0$, $\sum_{j=1}^n \omega_j = 1$ and $w = (w_1, w_2, \ldots, w_t)^T$ is the weight vector of decision makers such that $w_j \geq 0$, $\sum_{j=1}^t w_j = 1$. The aim of this MAGDM problem is to select the most desirable alternative based on above information.

The approach based on the proposed operators can be given as follows.

Step 1. Calculate $T(\gamma_{ij}^k)$ and weight vector of each decision maker u_{ij}^k.

Step 2. Aggregate all individual decision matrices $R_k = (\gamma_{ij}^k)_{m\times n}$ $(k = 1,2,\ldots,t)$ into a collective decision matrix $R = (\gamma_{ij})_{m\times n}$ on the basis of LIFWGPBM operator or LIFWPBM operator; that is,

$$\begin{aligned} \gamma_{ij} &= \text{LIFWGPBM}^{p,q}(\gamma_{ij}^1,\gamma_{ij}^2,\ldots,\gamma_{ij}^t) \\ \text{or } \gamma_{ij} &= \text{LIFWPBM}^{p,q}(\gamma_{ij}^1,\gamma_{ij}^2,\ldots,\gamma_{ij}^t). \end{aligned} \tag{80}$$

Step 3. Calculate $T(\gamma_{ij})$ and weight vector u_{ij} according to $R = (\gamma_{ij})_{m\times n}$ obtained in Step 2.

Step 4. Derive the collective overall linguistic intuitionistic fuzzy value γ_i of the alternative x_i $(i = 1,2,3,4)$ based on the LIFWGPBM operator or LIFWPBM operator; that is,

$$\begin{aligned} \gamma_i &= \text{LIFWGPBM}^{p,q}(\gamma_{i1},\gamma_{i2},\ldots,\gamma_{in}) \\ \text{or } \gamma_i &= \text{LIFWPBM}^{p,q}(\gamma_{i1},\gamma_{i2},\ldots,\gamma_{in}). \end{aligned} \tag{81}$$

Step 5. By using (21), we can figure out the score function $Ls(\gamma_i)$ as well as the accuracy function $Lh(\gamma_i)$ of the LIFN γ_i $(i = 1,2,3,4)$.

Step 6. Rank all the alternatives $\{x_1, x_2, x_3, x_4\}$ by values $Ls(\gamma_i)$ and $Lh(\gamma_i)$ in descending order, and then the most desirable alternative is obtained.

5. A Numerical Example

In this section, a numerical example of seeking for the best global supplier [13] is used to illustrate the proposed method to the MAGDM problem with the LIFNs.

A manufacturing company plans to search for the best global supplier to buy one of its most core products used in assembling process. Suppose that $X = \{x_1, x_2, x_3, x_4\}$ is the set of four potential global suppliers, and $A = \{a_1, a_2, a_3, a_4, a_5\}$ is a set of attributes whose weight vector is $w = (0.25, 0.2, 0.15, 0.18, 0.22)$. Those attributes are a_1: overall production cost; a_2: production quality; a_3: supplier's service performance; a_4: supplier's profile; a_5: supplier's risk. Four alternatives x_i $(i = 1,\ldots,4)$ are evaluated by a predefined linguistic term set: $S = \{s_0$ = extremely poor, s_1 = very poor, s_2 = poor, s_3 = slightly poor, s_4 = fair, s_5 = slightly good, s_6 = good, s_7 = very good, s_8 = extermely good$\}$ by four decision makers d_k $(k = 1,\ldots,4)$ under the above five attributes, and the weight vector of decision makers is $\omega = (0.25, 0.3, 0.2, 0.25)^T$. The decision matrices $R_k = (\gamma_{ij}^k)_{4\times 5}$ $(k = 1,2,3,4)$ are listed in Table 1, respectively.

5.1. Evaluation Steps by the LIFWGPBM Operator

Step 1. Calculate $T(\gamma_{ij}^k)$ and weight vector of each decision maker u_{ij}^k; we can get the results shown in Tables 2 and 3.

Step 2. Aggregate all individual decision matrices $R_k = (\gamma_{ij}^k)_{m\times n}$ $(k = 1,2,\ldots,t)$ into a collective decision matrix $R = (\gamma_{ij})_{m\times n}$ on the basis of LIFWGPBM operator proposed in Definition 25, and we can get the results shown in Table 4.

Table 1: Decision matrix R_i $(i = 1, 2, 3, 4)$.

R_1	a_1	a_2	a_3	a_4	a_5
x_1	(s_7, s_1)	(s_6, s_2)	(s_4, s_3)	(s_7, s_1)	(s_5, s_2)
x_2	(s_6, s_2)	(s_5, s_2)	(s_6, s_1)	(s_6, s_2)	(s_7, s_1)
x_3	(s_6, s_1)	(s_5, s_3)	(s_7, s_1)	(s_5, s_1)	(s_3, s_4)
x_4	(s_5, s_2)	(s_7, s_1)	(s_4, s_3)	(s_6, s_1)	(s_4, s_4)
R_2	a_1	a_2	a_3	a_4	a_5
x_1	(s_7, s_1)	(s_4, s_4)	(s_6, s_2)	(s_5, s_2)	(s_3, s_5)
x_2	(s_7, s_1)	(s_5, s_1)	(s_6, s_1)	(s_5, s_2)	(s_4, s_3)
x_3	(s_5, s_2)	(s_6, s_1)	(s_7, s_1)	(s_5, s_3)	(s_4, s_4)
x_4	(s_6, s_2)	(s_4, s_3)	(s_5, s_2)	(s_7, s_1)	(s_5, s_3)
R_3	a_1	a_2	a_3	a_4	a_5
x_1	(s_6, s_1)	(s_5, s_2)	(s_3, s_4)	(s_7, s_1)	(s_5, s_2)
x_2	(s_7, s_1)	(s_6, s_2)	(s_7, s_1)	(s_6, s_2)	(s_5, s_1)
x_3	(s_6, s_1)	(s_5, s_3)	(s_7, s_1)	(s_5, s_1)	(s_3, s_4)
x_4	(s_5, s_2)	(s_7, s_1)	(s_4, s_3)	(s_6, s_1)	(s_4, s_4)
R_4	a_1	a_2	a_3	a_4	a_5
x_1	(s_5, s_3)	(s_4, s_4)	(s_7, s_1)	(s_5, s_1)	(s_4, s_2)
x_2	(s_6, s_1)	(s_7, s_1)	(s_6, s_1)	(s_5, s_2)	(s_6, s_1)
x_3	(s_5, s_2)	(s_3, s_4)	(s_6, s_2)	(s_3, s_3)	(s_5, s_2)
x_4	(s_4, s_3)	(s_5, s_1)	(s_4, s_2)	(s_6, s_2)	(s_5, s_2)

Table 2: $T(\gamma_{ij}^k)$ of different decision maker.

$T(\gamma_{ij}^1)$	a_1	a_2	a_3	a_4	a_5
x_1	2.688	2.438	2.375	2.688	2.625
x_2	2.688	2.688	2.938	2.875	2.5
x_3	2.563	2.563	2.813	2.438	2.5
x_4	2.75	2.563	2.625	2.75	2.563
$T(\gamma_{ij}^2)$	a_1	a_2	a_3	a_4	a_5
x_1	2.688	2.563	2.375	2.563	2.125
x_2	2.813	2.688	2.938	2.875	2.25
x_3	2.813	2.313	2.813	2.688	2.5
x_4	2.75	2.188	2.75	2.625	2.813
$T(\gamma_{ij}^3)$	a_1	a_2	a_3	a_4	a_5
x_1	2.688	2.563	2.125	2.688	2.625
x_2	2.813	2.688	2.813	2.875	2.625
x_3	2.688	2.563	2.813	2.688	2.375
x_4	2.75	2.563	2.625	2.625	2.813
$T(\gamma_{ij}^4)$	a_1	a_2	a_3	a_4	a_5
x_1	2.313	2.563	2.125	2.688	2.625
x_2	2.813	2.563	2.938	2.875	2.625
x_3	2.813	2.188	2.688	2.563	2.375
x_4	2.5	2.563	2.75	2.75	2.688

Step 3. Calculate $T(\gamma_{ij})$ and weight vector u_{ij}, and we can get the results shown in Tables 5 and 6.

Table 3: Weight vector u_{ij}^k of each decision maker.

u_{ij}^1	a_1	a_2	a_3	a_4	a_5
x_1	1.026	0.973	1.034	1.01	1.043
x_2	0.975	1.009	1.006	1	1.005
x_3	0.956	1.05	1.008	0.957	1.016
x_4	1.017	1.033	0.981	1.017	0.958
u_{ij}^2	a_1	a_2	a_3	a_4	a_5
x_1	1.231	1.168	1.241	1.212	1.252
x_2	1.17	1.21	1.208	1.2	1.206
x_3	1.148	2.26	1.21	1.148	1.22
x_4	1.22	1.239	1.178	1.22	1.15
u_{ij}^3	a_1	a_2	a_3	a_4	a_5
x_1	0.821	0.779	0.828	0.808	0.835
x_2	0.78	0.807	0.805	0.8	0.804
x_3	0.765	0.84	0.807	0.765	0.813
x_4	0.814	0.826	0.785	0.814	0.766
u_{ij}^4	a_1	a_2	a_3	a_4	a_5
x_1	1.026	0.973	1.034	1.01	1.043
x_2	0.975	1.009	1.006	1	1.005
x_3	0.956	1.05	1.008	0.957	1.016
x_4	1.017	1.033	0.981	1.017	0.958

Table 4: Integrated decision matrix $R = (\gamma_{ij})_{m\times n}$.

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_{7.63}, s_{5.32})$	$(s_{6.72}, s_{2.91})$	$(s_{7.08}, s_{4.14})$	$(s_{7.48}, s_{5.77})$	$(s_{6.11}, s_{3.63})$
x_2	$(s_{7.72}, s_{5.59})$	$(s_{7.4}, s_{5.4})$	$(s_{7.58}, s_{6.13})$	$(s_{7.19}, s_{4.49})$	$(s_{7.27}, s_{5.41})$
x_3	$(s_{7.13}, s_{4.46})$	$(s_{6.57}, s_{3.83})$	$(s_{7.73}, s_{5.66})$	$(s_{6.35}, s_{3.65})$	$(s_{5.8}, s_{3.52})$
x_4	$(s_{6.98}, s_{4.14})$	$(s_{7.34}, s_{5.51})$	$(s_{6.47}, s_{4.42})$	$(s_{7.54}, s_{5.35})$	$(s_{6.76}, s_{3.01})$

Table 5: $T(\gamma_{ij})$ under every attribute.

T_{ij}	a_1	a_2	a_3	a_4	a_5
x_1	3.446	3.383	3.572	3.389	3.403
x_2	3.782	3.828	3.708	3.642	3.823
x_3	3.572	3.646	3.232	3.621	3.494
x_4	3.646	3.552	3.595	3.545	3.42

Table 6: Weight vector u_{ij} of each attribute.

u_{ij}	a_1	a_2	a_3	a_4	a_5
x_1	1.234	1.012	0.77	0.892	1.091
x_2	1.239	1.013	0.747	0.888	1.112
x_3	1.266	1.027	0.703	0.922	1.081
x_4	1.268	1.005	0.751	0.893	1.083

Step 4. Derive the collective overall linguistic intuitionistic fuzzy value γ_i of the alternative x_i $(i = 1, 2, 3, 4)$ based on the LIFWGPBM operator, and get

$$\gamma_i = (s_{4.613}, s_{2.712}) \quad (s_{5.39}, s_{1.803}) \quad (s_{4.139}, s_{2.749}) \quad (s_{4.621}, s_{2.555}). \tag{82}$$

Step 5. Calculate the score function $Ls(\gamma_i)$ and accuracy function $Lh(\gamma_i)$ of the LIFN γ_i $(i = 1, 2, 3, 4)$, and we have

$$\begin{aligned} Ls(\gamma_i) &= 1.901\ 3.587\ 1.391\ 2.066 \\ Lh(\gamma_i) &= 7.326\ 7.193\ 6.888\ 7.176. \end{aligned} \quad (83)$$

Step 6. Rank all the alternatives $\{x_1, x_2, x_3, x_4\}$ by values $Ls(\gamma_i)$ and $Lh(\gamma_i)$ in descending order, and then we can get the ranking results as follows:

$$x_2 \succ x_4 \succ x_1 \succ x_3. \quad (84)$$

So the best choice is alternative 2.

5.2. Evaluation Steps by the LIFWPBM Operator. We use the LIFWPBM operator to solve this example; there are the same steps as Section 5.1, and we can get

$$\begin{aligned} \gamma_i = &\left(s_{6.253}, s_{1.214}\right) \ \left(s_{6.801}, s_{0.672}\right) \ \left(s_{5.896}, s_{1.296}\right) \\ &\left(s_{6.208}, s_{1.133}\right). \end{aligned} \quad (85)$$

Then, the score function $Ls(\gamma_i)$ and the accuracy function $Lh(\gamma_i)$ are

$$\begin{aligned} Ls(\gamma_i) &= 5.039\ 6.129\ 4.6\ 5.075 \\ Lh(\gamma_i) &= 7.467\ 7.472\ 7.192\ 7.341. \end{aligned} \quad (86)$$

So, we can get the ranking results as follows:

$$x_2 \succ x_4 \succ x_1 \succ x_3. \quad (87)$$

The result reveals that those two operators have the same ranking results, and the best alternative is x_2.

5.3. Analyze the Effect of Factors p, q. In this part, in order to analyze the influence of parameters p, q on decision making results, we input different values p and q in LIFWGPBM operator to check if the ranking will have a difference. Since p and q are interconvertibility, we only discuss $p = 1, q = 0$; the condition of $p = 0, q = 1$ is omitted and similarly for others. The ordering results are shown in Table 7.

From the results in Table 7, we can conclude:

(1) Whatever values of p and q are, the best choice is always x_2.

(2) Different parameters p and q will influence the orders of alternatives. So decision makers can select different values p and q according to their interest to obtain a different optimal alternative. In practical applications, we may use the parameters $p = q = 1$, which is not only easy and intuitive but also a full capture to the correlations between criteria.

5.4. Compare with the Exiting Method. Chen et al. [13] proposed linguistic intuitionistic fuzzy weighted averaging (LIFWA) operator; we can use it to deal with this example.

The LIFWA operator is expressed by

$$\mathrm{LIFWA}_{\omega}\left(\gamma_1, \gamma_2, \ldots, \gamma_n\right) = \omega_1\gamma_1 \oplus \omega_2\gamma_2 \oplus \cdots \oplus \omega_n\gamma_n, \quad (88)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of γ_i $(i = 1, 2, \ldots, n)$ satisfying $0 \le \omega_i \le 1$ and $\sum_{i=1}^{n} \omega_i = 1$.

So we can get the aggregated LIFN γ_i $(i = 1, 2, 3, 4)$ as follows:

$$\begin{aligned} \gamma_1 &= \left(s_{5.61}, s_{1.921}\right), \\ \gamma_2 &= \left(s_{6.092}, s_{1.31}\right), \\ \gamma_3 &= \left(s_{5.082}, s_{2.004}\right), \\ \gamma_4 &= \left(s_{5.372}, s_{1.92}\right). \end{aligned} \quad (89)$$

Then we can get the score function $Ls(\gamma_i)$ as follows:

$$\begin{aligned} Ls(\gamma_1) &= 3.689, \\ Ls(\gamma_2) &= 4.782, \\ Ls(\gamma_3) &= 3.078, \\ Ls(\gamma_4) &= 3.452. \end{aligned} \quad (90)$$

So the ranking is

$$x_2 \succ x_1 \succ x_4 \succ x_3. \quad (91)$$

Obviously, these methods produce the same best alternative. However, the ranking is a little different. The reason causing this difference maybe is that (1) the proposed method in this paper considered the power weight which can relieve the influence of unreasonable big or small data, or (2) the proposed method in this paper considered the interrelationship of individual input arguments. So the results produced by the proposed method in this paper are more reasonable than that by Chen et al. [13].

6. Conclusion

The LIFNs, in which membership degree and the nonmembership degree were expressed by linguistic terms, can better express fuzzy evaluation information. In this paper, we firstly introduce the concept of the LIFNs. In addition, as we all know, the Bonferroni mean operator takes the relationship between attribute values into consideration, and the power operator has the advantage of relieving the effect of too great or too little data by power weighting vectors produced from input arguments. So we innovatively combine these two operators and extend them to the LIFNs and proposed some linguistic intuitionistic fuzzy power Bonferroni mean operators as a new tool to process linguistic intuitionistic fuzzy information. Afterwards, we proposed an approach on the base of newly developed LIFWPBM operator and LIFWGPBM operator. Finally, a numerical example is given to reveal the practicability of this new method. The significance of the paper is that we combine the power operator and the BM operator to cope with the MAGDM problems under the circumstance of LIFNs, and the developed method can relieve the effect of too great or too little data and considered the relationship between attributes. For further research, other aggregation operators can be applied to combine with linguistic intuitionistic fuzzy numbers to obtain the best alternative.

Table 7: Ranking according to different values of p, q.

p, q	Ls	Ranking
$p = 0, q = 0.1$	$Ls(\gamma_1) = -4.315$ $Ls(\gamma_2) = -2.712$ $Ls(\gamma_3) = -4.544$ $Ls(\gamma_4) = -4.096$	$x_2 \succ x_4 \succ x_1 \succ x_3$
$p = 1, q = 0$	$Ls(\gamma_1) = 0.097$ $Ls(\gamma_2) = 2.353$ $Ls(\gamma_3) = -0.238$ $Ls(\gamma_4) = 0.581$	$x_2 \succ x_4 \succ x_1 \succ x_3$
$p = 1, q = 6$	$Ls(\gamma_1) = 0.6$ $Ls(\gamma_2) = 2.91$ $Ls(\gamma_3) = 0.643$ $Ls(\gamma_4) = 1.519$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$p = 2, q = 0$	$Ls(\gamma_1) = 0.537$ $Ls(\gamma_2) = 2.865$ $Ls(\gamma_3) = 0.327$ $Ls(\gamma_4) = 1.218$	$x_2 \succ x_4 \succ x_1 \succ x_3$
$p = 2, q = 8$	$Ls(\gamma_1) = 0.497$ $Ls(\gamma_2) = 2.889$ $Ls(\gamma_3) = 0.556$ $Ls(\gamma_4) = 1.489$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$p = 3, q = 0$	$Ls(\gamma_1) = 0.408$ $Ls(\gamma_2) = 2.814$ $Ls(\gamma_3) = 0.35$ $Ls(\gamma_4) = 1.275$	$x_2 \succ x_4 \succ x_1 \succ x_3$
$p = 3, q = 9$	$Ls(\gamma_1) = 0.555$ $Ls(\gamma_2) = 2.96$ $Ls(\gamma_3) = 0.543$ $Ls(\gamma_4) = 1.509$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$p = 5, q = 0$	$Ls(\gamma_1) = -0.116$ $Ls(\gamma_2) = 2.397$ $Ls(\gamma_3) = 0.091$ $Ls(\gamma_4) = 1.067$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$p = 0, q = 10$	$Ls(\gamma_1) = -1.177$ $Ls(\gamma_2) = 1.497$ $Ls(\gamma_3) = -0.567$ $Ls(\gamma_4) = 0.514$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$p = 1, q = 10$	$Ls(\gamma_1) = -0.424$ $Ls(\gamma_2) = 2.161$ $Ls(\gamma_3) = -0.015$ $Ls(\gamma_4) = 0.97$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$p = 2, q = 10$	$Ls(\gamma_1) = 0.059$ $Ls(\gamma_2) = 2.564$ $Ls(\gamma_3) = 0.257$ $Ls(\gamma_4) = 1.234$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$p = 3, q = 10$	$Ls(\gamma_1) = 0.359$ $Ls(\gamma_2) = 2.818$ $Ls(\gamma_3) = 0.405$ $Ls(\gamma_4) = 1.392$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$p = 4, q = 10$	$Ls(\gamma_1) = 0.538$ $Ls(\gamma_2) = 2.981$ $Ls(\gamma_3) = 0.493$ $Ls(\gamma_4) = 1.485$	$x_2 \succ x_4 \succ x_1 \succ x_3$
$p = 10, q = 10$	$Ls(\gamma_1) = 0.688$ $Ls(\gamma_2) = 3.224$ $Ls(\gamma_3) = 0.539$ $Ls(\gamma_4) = 1.481$	$x_2 \succ x_4 \succ x_1 \succ x_3$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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