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Research Article

Chaos Control and Synchronization via Switched Output Control Strategy

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This paper investigates the control and synchronization of a class of chaotic systems with switched output which is assumed to be switched between the first and the second state variables of chaotic system. Some novel and yet simple criteria for the control and synchronization of a class of chaotic systems are proposed via the switched output. The generalized Lorenz chaotic system is taken as an example to show the feasibility and efficiency of theoretical results.

1. Introduction

Chaotic phenomenon occurs naturally in many engineering, physical, biological, and social systems [1]. Although chaotic phenomenon could be beneficial in some applications, it is undesirable in many physical applications and should therefore be controlled in order to improve the system performance. For a quite long period of time, due to the high sensitivity of a chaotic system to its initial condition, people thought that chaos was not controllable, and two chaotic systems could not be synchronized. However, the OGY method [2] developed by Ott et al. and in particular the concept of the synchronization proposed by Pecora and Carroll [3] in 1990 have completely changed the situation. Control and synchronization of chaotic systems have many potential applications in physical systems, lasers, circuits, chemical reactor, ecological systems, and secure communication [1, 4]. For example, in the field of secure communication many chaotic systems, such as the logistic map, the Hénon map, and the piecewise linear chaotic map [5], have been used to develop chaotic ciphers due to the sensitivity to initial conditions, ergodicity, and pseudorandom behavior of chaotic systems satisfies the analogous requirements for a good cryptosystem [6]. Recently, spatiotemporal chaos has been employed for hash functions [7] and Chebyshev maps have been used for key agreement protocols [8].

Owing to the potential applications of chaos control and synchronization, many efforts have been devoted by researchers to achieve the goals of chaos control and synchronization in the last two decades and, as a result, a wide variety of approaches have been proposed for the control and synchronization of chaotic systems which include adaptive control [9], active control [10], integral control [11], impulsive control [12], backstepping control [13], sampled-data control [14], and so forth.

In the literature there are many results concerning the control and synchronization of chaotic system [3, 9–21]. However, most of the existing works dealing with controlling chaos and chaos synchronization are based on the same assumption that the state variables of chaotic systems are all available for designing the controller. As it is well known that for most nonlinear systems the state variables are often unavailable in practice. For example, in the input-output system only the output is available which means that the above requirement is not very reasonable. Thus, the investigation of chaos control and synchronization with only output states available becomes an important topic. On the other hand, the transmitted signal may be switched between different signals and even interrupted for a variety of reasons. Therefore, it is necessary and important to investigate the control and synchronization of chaotic system with switched output. So far as we know, less attention has been paid to this issue.

With the above motivations, our main aim in this paper is to investigate the control and synchronization of a class of chaotic systems with switched output. The chaotic systems

are assumed that only the output variable is available and the output may be switched between the first and the second state variables. Some novel criteria for the control and synchronization of a class of chaotic systems are proposed via the switched output. The generalized Lorenz chaotic system is taken as an example to show the feasibility and efficiency of theoretical results.

The paper is organized as follows. First, a brief description of a class of chaotic systems is introduced in Section 2. The control and the synchronization schemes are discussed in Sections 3 and 4, respectively. Section 5 includes several numerical examples to demonstrate the effectiveness of the proposed approach. Finally, conclusion remarks are presented in Section 6.

2. System Description

Consider the following chaotic system:

$$\begin{aligned}
&\dot{x}_1 = a\left(x_2 - x_1\right),\\
&\dot{x}_2 = bx_1 + cx_2 - x_1x_3,\\
&\dot{x}_3 = x_1x_2 - dx_3,\\
&x_{\text{out}} = k_1(t)\,x_1 + k_2(t)\,x_2,\\
&k_1(t)\\
&\quad = \begin{cases} 1, & t \in \left[t_{2m}, t_{2m+1}\right),\ t_0 = 0,\ m = 0,1,2,\ldots,\\ 0, & t \in \left[t_{2m+1}, t_{2m+2}\right),\ m = 0,1,2,\ldots,\end{cases}\\
&k_2(t)\\
&\quad = \begin{cases} 0, & t \in \left[t_{2m}, t_{2m+1}\right),\ t_0 = 0,\ m = 0,1,2,\ldots,\\ 1, & t \in \left[t_{2m+1}, t_{2m+2}\right),\ m = 0,1,2,\ldots,\end{cases}
\end{aligned} \tag{1}$$

where $x = (x_1, x_2, x_3)^T \in R^{3\times 1}$ is the state vector of system (1) and $a(\neq 0)$, $b, c(< 0)$, and $d(> 0)$ are the system's parameters, which are known in advance. x_{out} denotes the output variable of system (1).

Remark 1. It is well known that some chaotic systems such as the Lorenz system [22], the generalized Lorenz system [23], and the unified chaotic system [24] can be written in the form of system (1).

Let $|t_it_{i+1}|$ denote the length of interval $[t_i, t_{i+1})$ or $[t_i, t_{i+1})$. Furthermore, we suppose that $\alpha_m = |t_{2m}t_{2m+1}|$, $\beta_m = |t_{2m+1}t_{2m+2}|$, $\alpha_{\min} = \min\{\alpha_m\}$, $\alpha_{\max} = \max\{\alpha_m\}$, $\beta_{\min} = \min\{\beta_m\}$, and $\beta_{\max} = \max\{\beta_m\}$, where $m = 0, 1, 2, \ldots$, $0 \le \alpha_{\max} < \infty$, $0 \le \beta_{\max} < \infty$, and $\alpha_{\max}$, $\beta_{\max}$ are not both zero.

Remark 2. It is easy to see that if $x_{\text{out}} = x_1$ or $x_{\text{out}} = x_2$, then the output x_{out} of system (1) is continuous variable. Otherwise, the output x_{out} is switched variable which is discontinuous.

3. The Control Scheme of a Class of Chaotic Systems

In this section, we investigate the stabilization of system (1) at origin. For the purpose of forcing the states to converge to origin, we add two controllers to system (1). The controlled system (1) with a specified output is given as

$$\begin{aligned}
&\dot{x}_1 = a\left(x_2 - x_1\right) + u_1,\\
&\dot{x}_2 = bx_1 + cx_2 - x_1x_3 + u_2,\\
&\dot{x}_3 = x_1x_2 - dx_3,\\
&x_{\text{out}} = k_1(t)\,x_1 + k_2(t)\,x_2,\\
&k_1(t)\\
&\quad = \begin{cases} 1, & t \in \left[t_{2m}, t_{2m+1}\right),\ t_0 = 0,\ m = 0,1,2,\ldots,\\ 0, & t \in \left[t_{2m+1}, t_{2m+2}\right),\ m = 0,1,2,\ldots,\end{cases}\\
&k_2(t)\\
&\quad = \begin{cases} 0, & t \in \left[t_{2m}, t_{2m+1}\right),\ t_0 = 0,\ m = 0,1,2,\ldots,\\ 1, & t \in \left[t_{2m+1}, t_{2m+2}\right),\ m = 0,1,2,\ldots,\end{cases}
\end{aligned} \tag{2}$$

where u_1 and u_2 are the controllers to be designed later.

Our work in this section is to design controller $u(t) = u(x_{\text{out}}(t))$ to make system (2) be stabilized at origin; that is, $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$.

Assumption 3. Suppose the state variables of system (2) are bounded, which means that there exists nonnegative constant M such that $|a(bx_1 + cx_2 - x_1x_3) + a(1-a)(x_2 - x_1) - \ddot{x}_1 + x_1| \le M$.

Theorem 4. *For system (2), suppose*

$$\begin{aligned}
u_1 &= -ak_2(t)\,x_{\text{out}},\\
u_2 &= -k_1(t)\,\operatorname{sgn}\left(x_{\text{out}}\right)\frac{M}{a};
\end{aligned} \tag{3}$$

then we have the following:

(1) *If* $\alpha_{\max} = \infty$, *then* $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$.

(2) *If* $\beta_{\max} = \infty$ *and* $a > 0$, *then* $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$.

(3) *If* $\alpha_{\max}\beta_{\max} \neq \infty$, $a < 0$, *and there exists constant* $\varepsilon > 0$ *such that* $\alpha_{\min} + a\beta_{\max} \ge \varepsilon$, *then* $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$.

(4) *If* $\alpha_{\max}\beta_{\max} \neq \infty$, $a > 0$, *then* $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$.

Proof. Note that $c < 0$; from the second equation of system (2) and the expression of u_1 and u_2 it is easy to see that if $\lim_{t\to+\infty}x_1 = 0$ then $\lim_{t\to+\infty}x_2 = 0$. Keep in mind that $d > 0$; from the third equation of system (2) it is obvious that if $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = 0$ then $\lim_{t\to+\infty}x_3 = 0$. Thus, in

order to prove $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$ we only need to show that $\lim_{t\to+\infty}x_1 = 0$. Two cases are discussed in the following according to t.

Case 1. If $t \in [t_{2m}, t_{2m+1})$, $m = 0, 1, 2, \ldots$, then $k_1(t) = 1$ and $k_2(t) = 0$. From the first equation of system (2), we have

$$\begin{aligned}
&\dot{x}_1 = a(x_2 - x_1)\\
&\ddot{x}_1 = a(\dot{x}_2 - \dot{x}_1) = a((bx_1 + cx_2 - x_1x_3 + u_2) - a(x_2\\
&\quad - x_1))\\
&\dot{x}_1 + x_1 = a(\dot{x}_2 - \dot{x}_1) - \ddot{x}_1 + \dot{x}_1 + x_1 = a(bx_1 + cx_2\\
&\quad - x_1x_3 + u_2) + a(1-a)(x_2 - x_1) - \ddot{x}_1 + x_1\\
&\quad = a(bx_1 + cx_2 - x_1x_3) + a(1-a)(x_2 - x_1) - \ddot{x}_1\\
&\quad + x_1 + au_2\\
&2x_1(\dot{x}_1 + x_1) = 2x_1(a(bx_1 + cx_2 - x_1x_3) + a(1-a)\\
&\quad \cdot (x_2 - x_1) - \ddot{x}_1 + x_1 + au_2) = 2x_1(a(bx_1 + cx_2\\
&\quad - x_1x_3) + a(1-a)(x_2 - x_1) - \ddot{x}_1 + x_1 - \operatorname{sgn}(x_1)\\
&\quad \cdot M) = 2x_1\operatorname{sgn}(x_1)(\operatorname{sgn}(x_1)(a(bx_1 + cx_2 - x_1x_3)\\
&\quad + a(1-a)(x_2 - x_1) - \ddot{x}_1 + x_1) - M) \le 0.
\end{aligned} \tag{4}$$

That is,

$$\dot{x}_1^2 + 2x_1^2 \le 0, \quad \text{for } t \in [t_{2m}, t_{2m+1}),\ m = 0, 1, 2, \ldots. \tag{5}$$

Therefore, one gets

$$x_1^2 \le e^{-2(t-t_{2m})}x_1^2(t_{2m}) \tag{6}$$

Thus, we have

$$|x_1| \le e^{-(t-t_{2m})}|x_1(t_{2m})|. \tag{7}$$

From inequality (7), it is easy to see that if $\alpha_{\max} = \infty$, then $|x_1| \to 0$ as $t \to +\infty$. Therefore we have $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$ which means that conclusion (1) of Theorem 4 is correct.

Case 2. If $t \in [t_{2m+1}, t_{2m+2})$, $m = 0, 1, 2, \ldots$, then $k_2(t) = 1$ and $k_1(t) = 0$. From the first equation of system (2), we have

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + u_1 = a(x_2 - x_1) - ak_2(t)x_2\\
&= -ax_1.
\end{aligned} \tag{8}$$

That is

$$\dot{x}_1 = -ax_1 \quad \text{for } t \in [t_{2m+1}, t_{2m+2}),\ m = 0, 1, 2, \ldots. \tag{9}$$

Therefore, one gets

$$|x_1| = e^{-a(t-t_{2m+1})}|x_1(t_{2m+1})|. \tag{10}$$

From inequality (10), it is obvious that if $\beta_{\max} = \infty$ and $a > 0$, then $|x_1| \to 0$ as $t \to +\infty$. Therefore we obtain $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$ which implies that conclusion (2) of Theorem 4 is correct.

Now, we assume that $\alpha_{\max}\beta_{\max} \ne \infty$, $a < 0$.

For $t \in [t_0, t_1)$, based on inequality (7), one gets

$$|x_1(t)| \le e^{-t}|x_1(0)|, \tag{11}$$

which leads to

$$|x_1(t_1)| \le e^{-\alpha_1}|x_1(0)|. \tag{12}$$

For $t \in [t_1, t_2)$, based on inequality (10), we have

$$|x_1(t)| = e^{-a(t-t_1)}|x_1(t_1)|, \tag{13}$$

which implies

$$|x_1(t_2)| \le e^{-\alpha_1 - a\beta_1}|x_1(0)|. \tag{14}$$

In general, for $t \in [t_{2m}, t_{2m+1})$ one obtains that

$$\begin{aligned}
&|x_1(t)|\\
&\quad \le e^{-(\alpha_1+\alpha_2+\cdots+\alpha_{m-1})}e^{-a(\beta_1+\beta_2+\cdots+\beta_{m-1})}e^{-(t-t_{2m})}|x_1(0)|\\
&\quad \le e^{-(m-1)\alpha_{\min}}e^{-a(m-1)\beta_{\max}}e^{-(t-t_{2m})}|x_1(0)|\\
&\quad = e^{-(m-1)(\alpha_{\min}+a\beta_{\max})}e^{-(t-t_{2m})}|x_1(0)|\\
&\quad \le e^{-(m-1)(\alpha_{\min}+a\beta_{\max})}|x_1(0)|.
\end{aligned} \tag{15}$$

Since $\alpha_{\max}\beta_{\max} \ne \infty$, it is easy to see that $t \to \infty$ implies that $m \to \infty$. Note that $\alpha_{\min} + a\beta_{\max} \ge \varepsilon > 0$; thus we have $-(m-1)(\alpha_{\min} + a\beta_{\max}) \to -\infty$ as $t \to +\infty$ which implies that $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$.

For $t \in [t_{2m+1}, t_{2m+2})$ one gets that

$$\begin{aligned}
&|x_1(t)|\\
&\quad \le e^{-(\alpha_1+\alpha_2+\cdots+\alpha_m)}e^{-a(\beta_1+\beta_2+\cdots+\beta_m)}e^{-a(t-t_{2m+1})}|x_1(0)|.
\end{aligned} \tag{16}$$

Note that $a < 0$, and we obtain that

$$|x_1(t)| \le e^{-m(\alpha_{\min}+a\beta_{\max})}e^{-a\beta_{\max}}|x_1(0)|. \tag{17}$$

Since $m \to \infty$ and $\alpha_{\min} + a\beta_{\max} \ge \varepsilon > 0$, we have $-m(\alpha_{\min} + a\beta_{\max}) \to -\infty$ as $t \to +\infty$ which implies that $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$. Therefore, conclusion (3) of Theorem 4 is correct.

In the following, we assume that $\alpha_{\max}\beta_{\max} \ne \infty$, $a > 0$.

For $t \in [t_{2m}, t_{2m+1})$, based on inequality (15), one gets

$$\begin{aligned}
&|x_1(t)|\\
&\quad \le e^{-(\alpha_1+\alpha_2+\cdots+\alpha_{m-1})}e^{-a(\beta_1+\beta_2+\cdots+\beta_{m-1})}e^{-(t-t_{2m})}|x_1(0)|\\
&\quad \le e^{-(m-1)\alpha_{\min}}|x_1(0)|.
\end{aligned} \tag{18}$$

Note that in this case $\alpha_{\min} > 0$ and $m \to +\infty$ as $t \to +\infty$; we know that $|x_1(t)| \to 0$ as $t \to +\infty$ which implies that $\lim_{t\to+\infty}x_1 = \lim_{t\to+\infty}x_2 = \lim_{t\to+\infty}x_3 = 0$.

For $t \in [t_{2m+1}, t_{2m+2})$, based on inequality (16), one gets

$$|x_1(t)| \le e^{-m\alpha_{\min}} |x_1(0)|. \tag{19}$$

Note that in this case $\alpha_{\min} > 0$ and $m \to +\infty$ as $t \to +\infty$; we obtain that $|x_1(t)| \to 0$ as $t \to +\infty$ which means that $\lim_{t\to+\infty} x_1 = \lim_{t\to+\infty} x_2 = \lim_{t\to+\infty} x_3 = 0$. Therefore, conclusion (4) of Theorem 4 is correct. This completes the proof of Theorem 4. □

Remark 5. It is well known that the states of chaotic systems are bounded which means that x_1, x_2, and x_3 are all bounded.

In order to prove $|a(bx_1 + cx_2 - x_1x_3) + a(1-a)(x_2 - x_1) - \ddot{x}_1 + x_1| \le M$ we only need to show that $\ddot{x}_1$ is bounded. In the following a simple proof shows that $\ddot{x}_1$ is bounded.

From system (2) and Theorem 4 we know

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + u_1, \\ \dot{x}_2 &= bx_1 + cx_2 - x_1x_3 + u_2, \end{aligned} \tag{20}$$

where

$$\begin{aligned} u_1 &= -ak_2(t)(k_1(t)x_1 + k_2(t)x_2), \\ u_2 &= -k_1(t)\operatorname{sgn}(k_1(t)x_1 + k_2(t)x_2)\frac{M}{a}. \end{aligned} \tag{21}$$

Since x_1, x_2, and x_3 are all bounded, based on (20)–(21), one can easily derive that $\dot{x}_1$, $\dot{x}_2$, and u_2 are bounded. Furthermore, by (21) we can obtain that $\dot{u}_1$ is bounded.

Now, in view of

$$\begin{aligned} \ddot{x}_1 &= a(\dot{x}_2 - \dot{x}_1) + \dot{u}_1 \\ &= a((bx_1 + cx_2 - x_1x_3 + u_2) - a(x_2 - x_1)) + \dot{u}_1 \end{aligned} \tag{22}$$

we conclude that $\ddot{x}_1$ is bounded which implies that Assumption 3 is reasonable.

4. The Synchronization of a Class of Chaotic Systems

Suppose system (1) is the drive system. In order to synchronize system (1), we introduce the following response system:

$$\begin{aligned} \hat{x}_1 &= k_1(t)w_{11} + k_1(t)x_{\text{out}} + k_2(t)w_{21} \\ &\quad + \frac{k_2(t)(l_{21} - a)}{b}x_{\text{out}}, \\ \dot{w}_{11} &= -l_{11}(\hat{x}_1 - x_{\text{out}}), \\ \dot{w}_{21} &= -l_{21}\hat{x}_1 - \frac{(l_{21} - a)c - ab}{b}\hat{x}_2 + \frac{l_{21} - a}{b}\hat{x}_1\hat{x}_3, \\ \hat{x}_2 &= k_1(t)w_{12} + k_1(t)\frac{l_{12} + c}{a}x_{\text{out}} + k_2(t)w_{22} \\ &\quad + k_2(t)\frac{l_{22} + c}{c}x_{\text{out}}, \\ \dot{w}_{12} &= -l_{12}w_{12} - \frac{l_{12}(l_{12} + c)}{a}x_{\text{out}} + (l_{12} + c)x_{\text{out}} \\ &\quad + b\hat{x}_1 - \hat{x}_1\hat{x}_3, \\ \dot{w}_{22} &= -\frac{l_{22}b}{c}\hat{x}_1 - l_{22}\hat{x}_2 + \frac{l_{22}}{c}\hat{x}_1\hat{x}_3, \\ \dot{\hat{x}}_3 &= \hat{x}_1\hat{x}_2 - d\hat{x}_3, \end{aligned} \tag{23}$$

where $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ are the estimated values of x_1, x_2, and x_3, respectively. w_{ij}, $i, j = 1, 2$, are intermediate variables. l_{i1} and l_{i2}, $i = 1, 2$, are positive numbers to be designed later.

Prior to going any further, we make the following assumption.

Assumption 6. Suppose the state variables of systems (1) and (23) are bounded, which means that there exists nonnegative constant M such that $|x_i| \le M$, $|\hat{x}_i| \le M$, $i = 1, 2, 3$.

Let λ_1, λ_2 denote the minimum eigenvalues of matrix Q_1, Q_2, respectively, where

$$Q_1 = \begin{pmatrix} l_1 & -\frac{1}{2}(|b| + M) & -\frac{1}{2}M \\ -\frac{1}{2}(|b| + M) & l_2 & 0 \\ -\frac{1}{2}M & 0 & d \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} l_{21} - \left|\frac{l_{21} - a}{b}\right|M & -\frac{1}{2}\left(\left|\frac{(l_{21} - a)c - ab}{b} + \frac{l_{22}b}{c}\right| + \left|\frac{l_{22}}{c}\right|M\right) & -\frac{1}{2}\left(\left|\frac{l_{21-a}}{b}\right| + 1\right)M \\ -\frac{1}{2}\left(\left|\frac{(l_{21} - a)c - ab}{b} + \frac{l_{22}b}{c}\right| + \left|\frac{l_{22}}{c}\right|M\right) & l_{22} & -\frac{1}{2}\left(\left|\frac{l_{22}}{c}\right| + 1\right)M \\ -\frac{1}{2}\left(\left|\frac{l_{21} - a}{b}\right| + 1\right)M & -\frac{1}{2}\left(\left|\frac{l_{22}}{c}\right| + 1\right)M & d \end{pmatrix}. \tag{24}$$

Then we have the following theorem.

Theorem 7. *(1) If $\alpha_{\max} = \infty$ and there exist nonnegative constants l_{11} and l_{12} such that $Q_1 > 0$, then system (23) can synchronize system (1) in the sense of $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \lim_{t\to+\infty} e_3 = 0$.*

(2) If $\beta_{\max} = \infty$ and there exist nonnegative constants l_{21} and l_{22} such that $Q_2 > 0$, then system (23) can synchronize system (1) in the sense of $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \lim_{t\to+\infty} e_3 = 0$.

(3) For $\alpha_{\max}\beta_{\max} \neq \infty$.

(a) If $\lambda_1 > 0$ and there exists constant $\varepsilon > 0$ such that $\lambda_1\alpha_{\min} + \lambda_2\beta_{\max} \geq \varepsilon$, then system (23) can synchronize system (1) in the sense of $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \lim_{t\to+\infty} e_3 = 0$.

(b) If $\lambda_2 > 0$ and there exists constant $\varepsilon > 0$ such that $\lambda_2\beta_{\min} + \lambda_1\alpha_{\max} \geq \varepsilon$, then system (23) can synchronize system (1) in the sense of $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \lim_{t\to+\infty} e_3 = 0$, where $e_1 = \widehat{x}_1 - x_1$, $e_2 = \widehat{x}_2 - x_2$, and $e_3 = \widehat{x}_3 - x_3$. λ_1 and λ_2 are the minimum eigenvalues of matrixes Q_1 and Q_2, respectively.

Proof. Let $e_1 = \widehat{x}_1 - x_1$, $e_2 = \widehat{x}_2 - x_2$, and $e_3 = \widehat{x}_3 - x_3$. Two cases are discussed in the following.

Case 1. If $t \in [t_{2m}, t_{2m+1})$, $m = 0, 1, 2, \ldots$, then $k_1(t) = 1$, $k_2(t) = 0$, and $x_{\text{out}} = x_1$. In this case from systems (1) and (23) we have

$$\begin{aligned}
\dot{e}_1 &= \dot{\widehat{x}}_1 - \dot{x}_1 \\
&= -l_{11}\left(\widehat{x}_1 - x_{\text{out}}\right) + a\left(x_2 - x_1\right) - a\left(x_2 - x_1\right) \\
&= -l_{11}e_1 \\
\dot{\widehat{x}}_2 &= \dot{w}_{12} + \frac{l_{12}+c}{a}\dot{x}_{\text{out}} \\
&= -l_{12}w_{12} - \frac{l_{12}\left(l_{12}+c\right)}{a}x_{\text{out}} + \left(l_{12}+c\right)x_{\text{out}} + b\widehat{x}_1 \\
&\quad - \widehat{x}_1\widehat{x}_3 + \frac{l_{12}+c}{a}a\left(x_2 - x_1\right) \\
&= -l_{12}\left(\widehat{x}_2 - \frac{l_{12}+c}{a}x_{\text{out}}\right) - \frac{l_{12}\left(l_{12}+c\right)}{a}x_{\text{out}} \\
&\quad + \left(l_{12}+c\right)x_{\text{out}} + b\widehat{x}_1 - \widehat{x}_1\widehat{x}_3 \\
&\quad + \frac{l_{12}+c}{a}a\left(x_2 - x_1\right) \\
&= -l_{12}\widehat{x}_2 + \left(l_{12}+c\right)x_{\text{out}} + b\widehat{x}_1 - \widehat{x}_1\widehat{x}_3 \\
&\quad + \left(l_{12}+c\right)\left(x_2 - x_1\right) \\
&= -l_{12}\widehat{x}_2 + b\widehat{x}_1 - \widehat{x}_1\widehat{x}_3 + \left(l_{12}+c\right)x_2, \\
\dot{e}_2 &= -l_{12}\widehat{x}_2 + b\widehat{x}_1 - \widehat{x}_1\widehat{x}_3 + \left(l_{12}+c\right)x_2 - bx_1 - cx_2 \\
&\quad + x_1x_3 = -l_{12}e_2 + be_1 - x_1e_3 - \widehat{x}_3e_1 \\
&= -l_{12}e_2 + \left(b - \widehat{x}_3\right)e_1 - x_1e_3, \\
\dot{e}_{3x} &= \widehat{x}_1\widehat{x}_2 - d\widehat{x}_3 - \left(x_1x_2 - dx_3\right) \\
&= x_1e_2 + \widehat{x}_2e_1 - de_3.
\end{aligned} \tag{25}$$

Thus, we get the error dynamical system:

$$\begin{aligned}
\dot{e}_1 &= -l_{11}e_1, \\
\dot{e}_2 &= -l_{12}e_2 + \left(b - \widehat{x}_3\right)e_1 - x_1e_3, \\
\dot{e}_3 &= x_1e_2 + \widehat{x}_2e_1 - de_3.
\end{aligned} \tag{26}$$

For system (26), if we take Lyapunov candidate

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2\right), \tag{27}$$

then the derivative of (27) is

$$\begin{aligned}
\dot{V} &= e_1\left(-l_{11}e_1\right) + e_2\left(-l_{12}e_2 + \left(b - \widehat{x}_3\right)e_1 - x_1e_3\right) \\
&\quad + e_3\left(x_1e_2 + \widehat{x}_2e_1 - de_3\right) \\
&= -l_{11}e_1^2 - l_{12}e_2^2 + \left(b - \widehat{x}_3\right)e_1e_2 - x_1e_3e_2 + x_1e_2e_3 \\
&\quad + \widehat{x}_2e_1e_3 - de_3^2 \\
&\leq -l_{11}e_1^2 - l_{12}e_2^2 + \left(|b| + M\right)\left|e_1e_2\right| + M\left|e_1e_3\right| - de_3^2 \\
&= -\left(\left|e_{1x}\right|, \left|e_{2x}\right|, \left|e_{3x}\right|\right)Q_1\left(\left|e_{1x}\right|, \left|e_{2x}\right|, \left|e_{3x}\right|\right)^T \\
&\leq -2\lambda_1V.
\end{aligned} \tag{28}$$

Thus, we have

$$\left|V(t)\right| \leq e^{-2\lambda_1(t-t_{2m})}\left|V\left(t_{2m}\right)\right|. \tag{29}$$

From inequality (29), it is easy to see that if $\alpha_{\max} = \infty$, then $|V| \to 0$ as $t \to +\infty$. Therefore we have $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \lim_{t\to+\infty} e_3 = 0$ which means that conclusion (1) of Theorem 7 is correct.

Case 2. If $t \in [t_{2m+1}, t_{2m+2})$, $m = 0, 1, 2, \ldots$, then $k_2(t) = 1$, $k_1(t) = 0$, and $x_{\text{out}} = x_2$. In this case from systems (1) and (23) we have

$$\begin{aligned}
\dot{e}_1 &= -l_{21}\widehat{x}_1 - \frac{\left(l_{21}-a\right)c - ab}{b}\widehat{x}_2 + \frac{l_{21}-a}{b}\widehat{x}_1\widehat{x}_3 \\
&\quad + \left(l_{21}-a\right)x_1 + \frac{\left(l_{21}-a\right)c}{b}x_2 - \frac{l_{21}-a}{b}x_1x_3 \\
&\quad - ax_2 + ax_1 \\
&= -l_{21}\widehat{x}_1 - \frac{\left(l_{21}-a\right)c - ab}{b}\widehat{x}_2 + \frac{l_{21}-a}{b}\widehat{x}_1\widehat{x}_3 \\
&\quad + l_{21}x_1 + \frac{\left(l_{21}-a\right)c}{b}x_2 - \frac{l_{21}-a}{b}x_1x_3 - ax_2 \\
&= -l_{21}e_1 - \frac{\left(l_{21}-a\right)c - ab}{b}e_2 \\
&\quad + \frac{l_{21}-a}{b}\left(\widehat{x}_1\widehat{x}_3 - x_1x_3\right).
\end{aligned}$$

$$\begin{aligned}
\dot{e}_2 &= \dot{w}_{22} + \frac{l_{22}+c}{c}\left(bx_1 + cx_2 - x_1x_3\right) \\
&\quad - \left(bx_1 + cx_2 - x_1x_3\right) \\
&= -\frac{l_{22}b}{c}\widehat{x}_1 - l_{22}\widehat{x}_2 + \frac{l_{22}}{c}\widehat{x}_1\widehat{x}_3 + \frac{l_{22}b}{c}x_1 + l_{22}x_2 \\
&\quad - \frac{l_{22}}{c}x_1x_3 \\
&= -\frac{l_{22}b}{c}e_1 - l_{22}e_2 + \frac{l_{22}}{c}\left(\widehat{x}_1\widehat{x}_3 - x_1x_3\right). \\
\dot{e}_3 &= \left(\widehat{x}_1\widehat{x}_2 - x_1x_2\right) - de_3.
\end{aligned} \tag{30}$$

Thus, we get the error dynamical system:

$$\begin{aligned}
\dot{e}_1 &= -l_{21}e_1 - \frac{\left(l_{21}-a\right)c - ab}{b}e_2 \\
&\quad + \frac{l_{21}-a}{b}\left(\widehat{x}_1\widehat{x}_3 - x_1x_3\right), \\
\dot{e}_2 &= -\frac{l_{22}b}{c}e_1 - l_{22}e_2 + \frac{l_{22}}{c}\left(\widehat{x}_1\widehat{x}_3 - x_1x_3\right), \\
\dot{e}_3 &= \left(\widehat{x}_1\widehat{x}_2 - x_1x_2\right) - de_3.
\end{aligned} \tag{31}$$

For system (31), if we take Lyapunov candidate

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2\right), \tag{32}$$

then the derivative of (32) is

$$\begin{aligned}
\dot{V} &= e_1\left(-l_{21}e_1 - \frac{\left(l_{21}-a\right)c - ab}{b}e_2\right. \\
&\quad \left. + \frac{l_{21}-a}{b}\left(\widehat{x}_1\widehat{x}_3 - x_1x_3\right)\right) + e_2\left(-\frac{l_{22}b}{c}e_1 - l_{22}e_2\right. \\
&\quad \left. + \frac{l_{22}}{c}\left(\widehat{x}_1\widehat{x}_3 - x_1x_3\right)\right) + e_3\left(\widehat{x}_1\widehat{x}_2 - x_1x_2 - de_3\right) \\
&= -l_{21}e_1^2 - \frac{\left(l_{21}-a\right)c - ab}{b}e_1e_2 + \frac{l_{21}-a}{b}x_3e_1^2 \\
&\quad + \frac{l_{21}-a}{b}\widehat{x}_1e_1e_3 - \frac{l_{22}b}{c}e_1e_2 - l_{22}e_2^2 + \frac{l_{22}}{c}\widehat{x}_1e_2e_3 \\
&\quad + \frac{l_{22}}{c}x_3e_1e_2 + \widehat{x}_1e_2e_3 + x_2e_1e_3 - de_3^2 = -\left(l_{21}\right. \\
&\quad \left. - \frac{l_{21}-a}{b}x_3\right)e_1^2 - \left(\frac{\left(l_{21}-a\right)c - ab}{b} + \frac{l_{22}b}{c}\right. \\
&\quad \left. - \frac{l_{22}}{c}x_3\right)e_1e_2 + \left(\frac{l_{21}-a}{b}\widehat{x}_1 + x_2\right)e_1e_3 - l_{22}e_2^2 \\
&\quad + \left(\frac{l_{22}}{c}\widehat{x}_1 + \widehat{x}_1\right)e_2e_3 - de_3^2 \le -\left(l_{21} - \left|\frac{l_{21}-a}{b}\right|M\right) \\
&\quad \cdot e_1^2 + \left(\left|\frac{\left(l_{21}-a\right)c - ab}{b} + \frac{l_{22}b}{c}\right| + \left|\frac{l_{22}}{c}\right|M\right)\left|e_1e_2\right| \\
&\quad + \left(\left|\frac{l_{21}-a}{b}\right| + 1\right)M\left|e_1e_3\right| - l_{22}e_2^2 + \left(\left|\frac{l_{22}}{c}\right| + 1\right) \\
&\quad \cdot M\left|e_2e_3\right| - de_3^2 = -\left(\left|e_1\right|, \left|e_2\right|, \left|e_3\right|\right) \\
&\quad \cdot Q_2\left(\left|e_1\right|, \left|e_2\right|, \left|e_3\right|\right)^T \le -2\lambda_2 V.
\end{aligned} \tag{33}$$

Thus, we have

$$\left|V(t)\right| \le e^{-2\lambda_2(t-t_{2m+1})}\left|V\left(t_{2m+1}\right)\right|. \tag{34}$$

From inequality (34), it is easy to see that if $\beta_{\max} = \infty$, then $|V| \to 0$ as $t \to +\infty$. Therefore we have $\lim_{t\to+\infty}e_1 = \lim_{t\to+\infty}e_2 = \lim_{t\to+\infty}e_3 = 0$ which means that conclusion (2) of Theorem 7 is correct.

In the following, we assume that $\alpha_{\max}\beta_{\max} \neq \infty$. Suppose $\lambda_1 > 0$.

For $t \in [t_0, t_1)$, based on inequality (29) one gets

$$V(t) \le e^{-2\lambda_1 t}V(0), \tag{35}$$

which leads to

$$V\left(t_1\right) \le e^{-2\lambda_1\alpha_1}V(0). \tag{36}$$

For $t \in [t_1, t_2)$, based on inequality (34) we have

$$\left|V(t)\right| \le e^{-2\lambda_2(t-t_1)}\left|V\left(t_1\right)\right|, \tag{37}$$

which implies

$$\left|V\left(t_2\right)\right| \le e^{-2\lambda_2\beta_1}\left|V\left(t_1\right)\right|. \tag{38}$$

In general, for $t \in [t_{2m}, t_{2m+1})$ one obtains that

$$\begin{aligned}
&\left|V(t)\right| \\
&\le e^{-2\lambda_1(\alpha_1+\alpha_2+\cdots+\alpha_{m-1})}e^{-2\lambda_2(\beta_1+\beta_2+\cdots+\beta_{m-1})}e^{-(t-t_{2m})}\left|V(0)\right| \\
&\le e^{-2\lambda_1(m-1)\alpha_{\min}}e^{-2\lambda_2(m-1)\beta_{\max}}e^{-(t-t_{2m})}\left|V(0)\right| \\
&= e^{-2(m-1)(\lambda_1\alpha_{\min}+\lambda_2\beta_{\max})}e^{-(t-t_{2m})}\left|V(0)\right| \\
&\le e^{-2(m-1)(\lambda_1\alpha_{\min}+\lambda_2\beta_{\max})}\left|V(0)\right|.
\end{aligned} \tag{39}$$

Since $\alpha_{\max}\beta_{\max} \neq \infty$, it is easy to see that $t \to \infty$ implies that $m \to \infty$. Note that $\lambda_1\alpha_{\min} + \lambda_2\beta_{\max} \ge \varepsilon > 0$; thus we have $-(m-1)(\lambda_1\alpha_{\min} + \lambda_2\beta_{\max}) \to -\infty$ as $t \to +\infty$ which implies that $\lim_{t\to+\infty}e_1 = \lim_{t\to+\infty}e_2 = \lim_{t\to+\infty}e_3 = 0$.

For $t \in [t_{2m+1}, t_{2m+2})$ one gets that

$$\begin{aligned}
&\left|V(t)\right| \\
&\le e^{-2\lambda_1(\alpha_1+\alpha_2+\cdots+\alpha_m)}e^{-2\lambda_2(\beta_1+\beta_2+\cdots+\beta_m)}e^{-2\lambda_2(t-t_{2m+1})}\left|V(0)\right|.
\end{aligned} \tag{40}$$

If $\lambda_2 \le 0$, from the inequality (40), we obtain

$$|V(t)| \le e^{-2m(\lambda_1\alpha_{\min}+\lambda_2\beta_{\max})} e^{-2\lambda_2\beta_{\max}} |V(0)|. \quad (41)$$

Since $m \to \infty$ and $\lambda_1\alpha_{\min} + \lambda_2\beta_{\max} \ge \varepsilon > 0$, we have $-2m(\lambda_1\alpha_{\min} + \lambda_2\beta_{\max}) \to -\infty$ as $t \to +\infty$ which implies that $\lim_{t\to+\infty} e^{-2m(\lambda_1\alpha_{\min}+\lambda_2\beta_{\max})} = 0$. Note that $e^{-2\lambda_2\beta_{\max}}|V(0)|$ is a constant; we derive $\lim_{t\to+\infty} e^{-2m(\lambda_1\alpha_{\min}+\lambda_2\beta_{\max})} e^{-2\lambda_2\beta_{\max}}|V(0)| = 0$ which means that $\lim_{t\to+\infty}|V(t)| = 0$. In view of $V = (1/2)(e_1^2 + e_2^2 + e_3^2)$ we have $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \lim_{t\to+\infty} e_3 = 0$.

If $\lambda_2 > 0$, from the inequality (40), we get

$$|V(t)| \le e^{-2m(\lambda_1\alpha_{\min}+\lambda_2\beta_{\max})} |V(0)|. \quad (42)$$

Along the same lines, one gets $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \lim_{t\to+\infty} e_3 = 0$. Therefore, conclusion (3) (a) of Theorem 7 is correct. In the same way we can prove that conclusion (3) (b) of Theorem 7 is correct. This completes the proof of Theorem 7.

$$\begin{aligned} |V(t)| &\le e^{-2m(\lambda_1\alpha_{\min}+\lambda_2\beta_{\max})} e^{-2\lambda_2\beta_{\max}} |V(0)| \\ &\le e^{-2m(\lambda_1\alpha_{\min}+\lambda_2\beta_{\max})} |V(0)|. \end{aligned} \quad (43)$$

Since $m \to \infty$ and $\lambda_1\alpha_{\min} + \lambda_2\beta_{\max} \ge \varepsilon > 0$, we have $-2m(\lambda_1\alpha_{\min} + \lambda_2\beta_{\max}) \to -\infty$ as $t \to +\infty$ which implies that $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \lim_{t\to+\infty} e_3 = 0$. Therefore, conclusion (3) (a) of Theorem 7 is correct. In the same way we can proof that conclusion (3) (b) of Theorem 7 is correct. This completes the proof of Theorem 7. □

Remark 8. It is easy to see that in our synchronization scheme we do not need to restrict $c < 0$.

5. Numerical Simulations

In this section, we will take the generalized Lorenz chaotic system as an example to illustrate the effectiveness of the proposed methods. The generalized Lorenz chaotic system is given by the following set of differential equations [17]:

$$\begin{aligned} \dot{x}_1 &= \left(10 + \frac{25}{29}\mu\right)(x_2 - x_1), \\ \dot{x}_2 &= \left(28 - \frac{35}{29}\mu\right)x_1 + (\mu - 1)x_2 - x_1x_3, \\ \dot{x}_3 &= \left(-\frac{8}{3} - \frac{1}{87}\mu\right)x_3 + x_1x_2, \end{aligned} \quad (44)$$

where $x = (x_1, x_2, x_3)^T \in R^{3\times1}$ is the state vector of system (44), and the system parameter μ satisfies $\mu \in \{\mu \mid -232 < \mu < -11.6\} \cup \{\mu \mid \mu > -11.6\}$. System (44) is chaotic for each $0 \le \mu \le 29$. Since in our control and synchronization schemes we need $\mu - 1 < 0$, we set $\mu = 1/10$. The dynamic behavior with $\mu = 1/10$ and $(x_1(0), x_2(0), x_3(0)) = (6, 8, 10)$ is shown in Figure 1.

System (44) with output variable can be rewritten as

$$\begin{aligned} \dot{x}_1 &= \left(10 + \frac{2.5}{29}\right)(x_2 - x_1), \\ \dot{x}_2 &= \left(28 - \frac{3.5}{29}\right)x_1 - 0.9x_2 - x_1x_3, \\ \dot{x}_3 &= \left(-\frac{8}{3} - \frac{0.1}{87}\right)x_3 + x_1x_2, \\ x_{\text{out}} &= k_1(t)x_1 + k_2(t)x_2, \\ k_1(t) &= \begin{cases} 1, & t \in [t_{2m}, t_{2m+1}),\ t_0 = 0,\ m = 0, 1, 2, \ldots, \\ 0, & t \in [t_{2m+1}, t_{2m+2}),\ m = 0, 1, 2, \ldots, \end{cases} \\ k_2(t) &= \begin{cases} 0, & t \in [t_{2m}, t_{2m+1}),\ t_0 = 0,\ m = 0, 1, 2, \ldots, \\ 1, & t \in [t_{2m+1}, t_{2m+2}),\ m = 0, 1, 2, \ldots. \end{cases} \end{aligned} \quad (45)$$

By comparing with system (1) it is easy to see that $a = 10 + 2.5/29$, $b = 28 - 3.5/29$, $c = -0.9$, and $d = 8/3 + 0.1/87$.

Example 9 (the control of the generalized Lorenz chaotic system). In this simulation process, we suppose that $M = 100a$ and the controllers u_1 and u_2 are taken as that of Theorem 4. Then, according to Theorem 4 we have $\lim_{t\to+\infty} x_1 = \lim_{t\to+\infty} x_2 = \lim_{t\to+\infty} x_3 = 0$. The simulation result with $x_1(0) = 3$, $x_2(0) = 5$, and $x_3(0) = -4$ is shown in Figure 2. It can be seen that the orbits of system (45) approach the origin asymptotically.

In order to show the robust of our control scheme to parameter uncertainties and external disturbances, we add some parameter uncertainties and external disturbances to system (45). Thus the controlled system (45) is rewritten as

$$\begin{aligned} \dot{x}_1 &= \left(10 + \frac{2.5}{29} + 0.1\right)(x_2 - x_1) - 0.2x_1x_2 + u_1, \\ \dot{x}_2 &= \left(28 - \frac{3.5}{29} - 0.1\right)x_1 - (0.9 - 0.1)x_2 - x_1x_3 \\ &\quad + 0.1x_2x_3 + u_2, \\ \dot{x}_3 &= \left(-\frac{8}{3} - \frac{0.1}{87} - 0.1\right)x_3 + x_1x_2 - 0.1x_1x_3, \\ x_{\text{out}} &= k_1(t)x_1 + k_2(t)x_2, \\ k_1(t) &= \begin{cases} 1, & t \in [t_{2m}, t_{2m+1}),\ t_0 = 0,\ m = 0, 1, 2, \ldots, \\ 0, & t \in [t_{2m+1}, t_{2m+2}),\ m = 0, 1, 2, \ldots, \end{cases} \end{aligned}$$

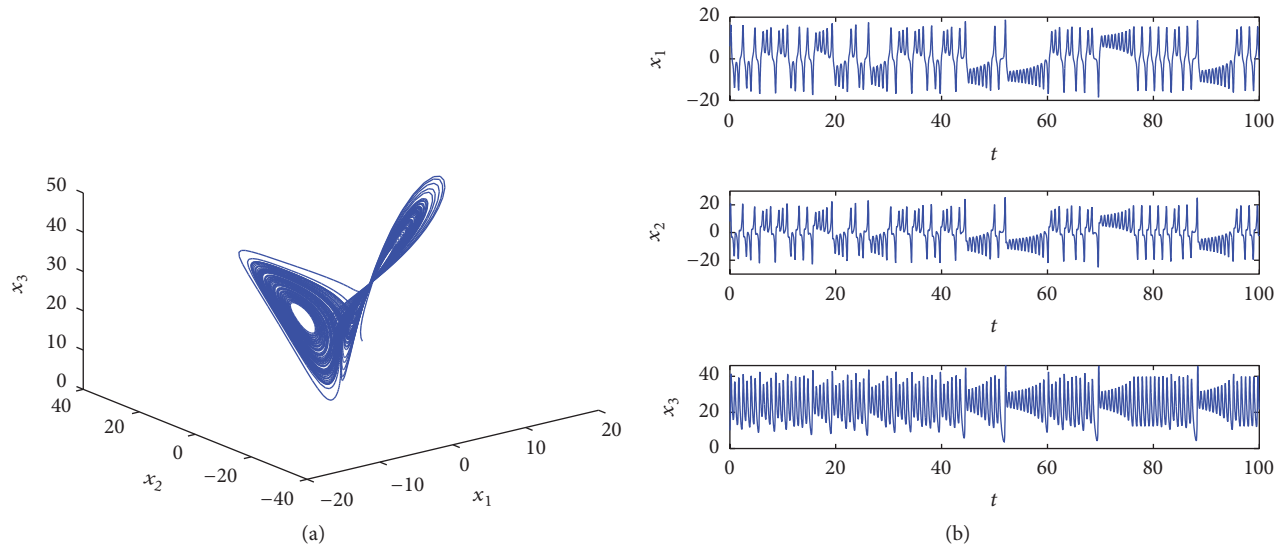

FIGURE 1: The dynamic behavior of system (44) with $\mu = 1/10$ and $(x_1(0), x_2(0), x_3(0)) = (6, 8, 10)$. (a) The chaotic attractor of system (44). (b) The state trajectories of system (44).

$$k_2(t) = \begin{cases} 0, & t \in [t_{2m}, t_{2m+1}),\ t_0 = 0,\ m = 0, 1, 2, \ldots, \\ 1, & t \in [t_{2m+1}, t_{2m+2}),\ m = 0, 1, 2, \ldots, \end{cases} \tag{46}$$

where $0.1(x_2 - x_1)$, $-0.1x_1$, and $-0.1x_3$ denote the parameter uncertainties, while $-0.2x_1x_2$, $0.1x_2x_3$, and $-0.1x_1x_3$ represent the external disturbances. The simulation result with the same controller and initial conditions as in Figure 2 is given in Figure 3. From Figure 3 one can easily see that the asymptotically stable of the origin of system (46) is actually achieved.

Example 10 (the synchronization scheme of the generalized Lorenz chaotic system). Based on Figure 1, the upper bound M is chosen as $M = 45$. If we take $l_{11} = 400$ and $l_{12} = 100$, then one can easily check that Q_1 is a positive definite matrix. Its eigenvalues are easy to obtain which are 1.3535, 95.7129, and 405.6014, respectively. Thus we can take $\lambda_1 = 1.3535$. For matrix Q_2, we suppose that $l_{21} = a = 10 + 2.5/29$ and $l_{22} = 0$. The eigenvalues of Q_2 are -30.8391, 9.8910, and 33.7021, respectively, which means that $\lambda_2 = -30.8391$. For convenience, we let $\alpha_{\min} = \alpha_{\max} = 0.08$ and $\beta_{\min} = \beta_{\max} = 0.003$; then it is easy to check that $\lambda_1\alpha_{\min} + \lambda_2\beta_{\max} = 0.0158 > 0$ which implies that condition (3) (a) of Theorem 7 is satisfied. According Theorem 7, we know that system (23) can synchronize system (45) in the sense of $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \lim_{t\to+\infty} e_3 = 0$.

The simulation results with $x_1(0) = 6$, $x_2(0) = 8$, $x_3(0) = 10$, $\hat{x}_1(0) = 2$, $w(0) = 6$, and $\hat{x}_3(0) = 13$ are presented in Figure 4. From Figure 4, it is easy to see that the drive system (45) and the response system (23) are synchronized within a few seconds.

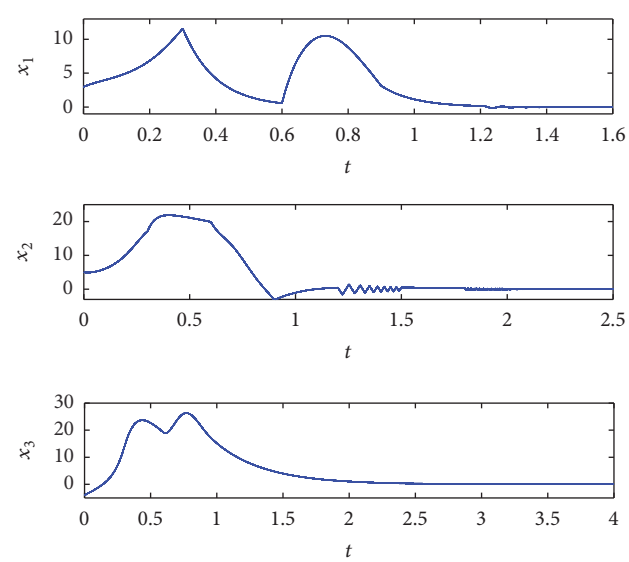

FIGURE 2: Orbits of controlled system (45) with $x_1(0) = 3$, $x_2(0) = 5$, and $x_3(0) = -4$.

For the sake of showing the robust of our synchronization scheme to parameter uncertainties and external disturbances, we add some parameter uncertainties and external disturbances to system (45). Thus system (45) is rewritten as

$$\dot{x}_1 = \left(10 + \frac{2.5}{29} + 0.1\right)(x_2 - x_1) - 0.2x_1x_2,$$

$$\dot{x}_2 = \left(28 - \frac{3.5}{29} - 0.1\right)x_1 - (0.9 - 0.1)\,x_2 - x_1x_3,$$

$$\dot{x}_3 = \left(-\frac{8}{3} - \frac{0.1}{87}\right)x_3 + x_1x_2,$$

$$x_{\text{out}} = k_1(t)\,x_1 + k_2(t)\,x_2,$$

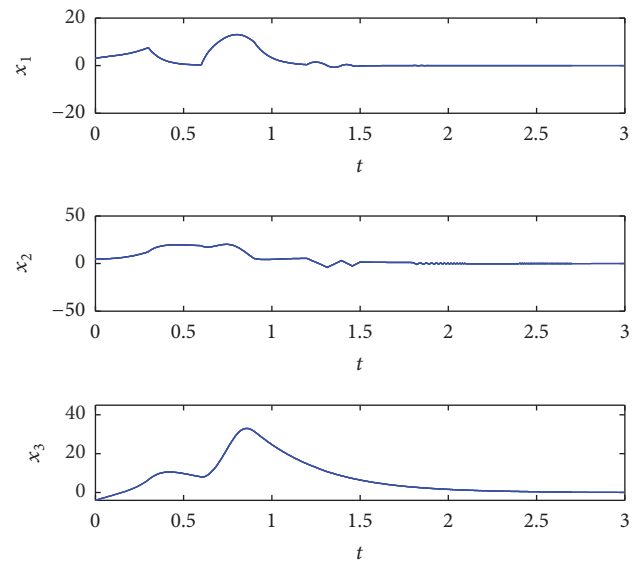

FIGURE 3: Orbits of controlled system (46) with $x_1(0) = 3$, $x_2(0) = 5$, and $x_3(0) = -4$.

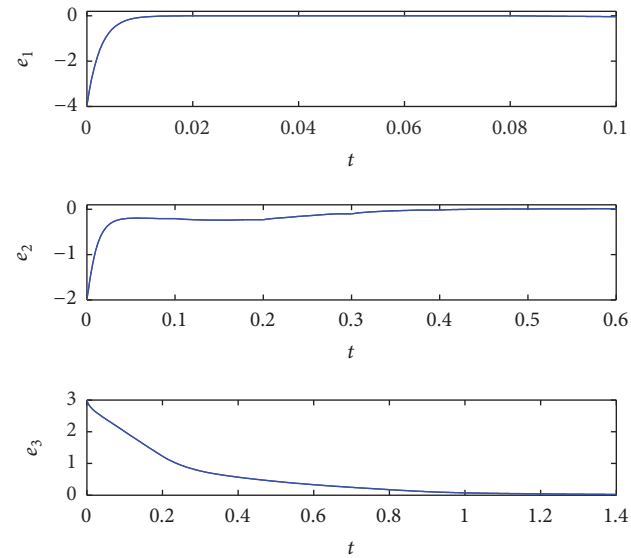

FIGURE 4: Plots of synchronization errors between systems (23) and (45).

$$k_1(t) = \begin{cases} 1, & t \in [t_{2m}, t_{2m+1}),\ t_0 = 0,\ m = 0, 1, 2, \ldots, \\ 0, & t \in [t_{2m+1}, t_{2m+2}),\ m = 0, 1, 2, \ldots, \end{cases}$$
$$k_2(t) = \begin{cases} 0, & t \in [t_{2m}, t_{2m+1}),\ t_0 = 0,\ m = 0, 1, 2, \ldots, \\ 1, & t \in [t_{2m+1}, t_{2m+2}),\ m = 0, 1, 2, \ldots, \end{cases} \tag{47}$$

where $0.1(x_2 - x_1)$ and $-0.1x_1$ denote the parameter uncertainties, while $-0.2x_1x_2$ represents the external disturbances. The synchronization simulation between systems (47) and (23) with the same initial conditions as in Figure 4 is presented in Figure 5. From Figure 5 it is easy to see that although in Figure 5 the synchronization between systems (47) and (23) is reached, its speed of synchronization is slower than that shown in Figure 4.

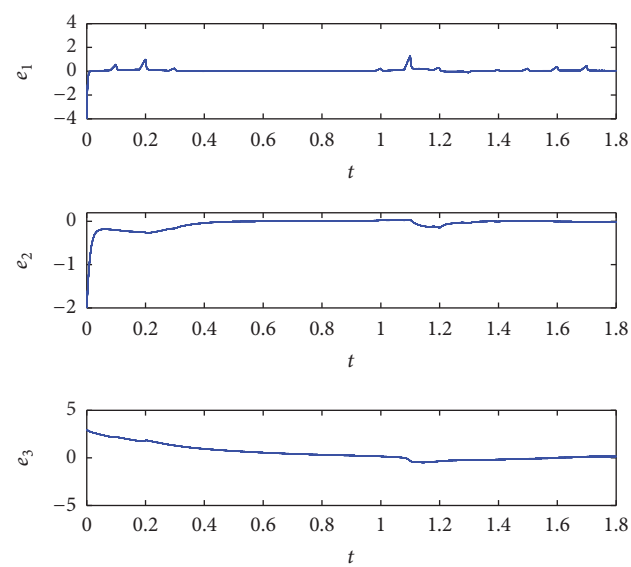

FIGURE 5: Plots of synchronization errors between systems (47) and (23).

Remark 11. From Figures 3 and 5 one can conclude that the presented control and synchronization schemes are robust to some special parameter uncertainties and external disturbances. However, it should be pointed out that since in our approaches the adaptive control method is not used our schemes may not be robust to arbitrary uncertainties and external disturbances.

6. Conclusion

The control and synchronization of a class of chaotic systems with switched output is investigated in this paper. By using the switched output, which is assumed to be switched between the first and the second state variables, some novel criteria for the control and synchronization of a class of chaotic systems are proposed. The generalized Lorenz chaotic system is taken as an example to demonstrate the efficiency of the proposed approach.

It is not difficult to see that our paper has two contributions. First, we present a new model which has switched output. As it is well known that in real-life situations the transmission signals may be interrupted for various reasons which means that the output should be discontinuous variable. Since the switched output is discontinuous variable, our model is closer to the actual situation than that having continuous output variable.

Second, the proposed model is especially suitable for secure communication. In a typical chaotic synchronization communication scheme the transmission signal is carried from the transmitter to the receiver by a chaotic signal through an analog channel. In order to mask the contents of transmission signal using chaotic signals different methods have been developed [25, 26]. However, it has been shown that most of these approaches are not secure or have a low level of security because one can extract the encoded message signal from the transmitted chaotic signal by using different unmasking techniques [27, 28]. Using our proposed synchronization scheme to transmit signal, one can add the transmitted signal $s(t)$ to x_{out} and derive

$$\begin{aligned} x_{\text{out}} &= k_1(t)\left(x_1 + \alpha_1 s(t)\right) + k_2(t)\left(x_2 + \alpha_2 s(t)\right), \\ k_1(t) &= \begin{cases} 1, & t \in \left[t_{2m}, t_{2m+1}\right),\ t_0 = 0,\ m = 0, 1, 2, \ldots, \\ 0, & t \in \left[t_{2m+1}, t_{2m+2}\right),\ m = 0, 1, 2, \ldots, \end{cases} \\ k_2(t) &= \begin{cases} 0, & t \in \left[t_{2m}, t_{2m+1}\right),\ t_0 = 0,\ m = 0, 1, 2, \ldots, \\ 1, & t \in \left[t_{2m+1}, t_{2m+2}\right),\ m = 0, 1, 2, \ldots, \end{cases} \end{aligned} \tag{48}$$

where $\alpha_1\alpha_2 \neq 0$. From (48) it is easy to see that the transmission signal can be alternately sent to the response system which can enhance the complexity and decrease the correlation of the signals being transmitted and thereby improving the antiattack ability and antitranslated capability of the transmitted signals. Thus, our research may have significant meanings in secure communication.

Despite its advantages, our approach is not without limitations. For example, we have not taken into account the disturbances in our presented chaotic systems. It is well known that in practical applications some dynamical systems are inevitably disturbed by the noises from external circumstance. Thus, the control and synchronization of switched output chaotic systems with external disturbances is an important issue. This issue will be our research emphasis in the future.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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