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Research Article

A Gyrostatic Low-Order Model for the El Ñino-Southern Oscillation

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The paper demonstrates that, similar to other important models of atmospheric phenomena, beginning with the celebrated Lorenz model of the Rayleigh-Bénard convection, the Vallis low-order model (LOM) of the El Ñino-Southern Oscillation admits a gyrostatic form and discusses how gyrostatic LOMs may offer a general framework for deriving effective physically sound models for atmospheric dynamics and time series analysis. Any such model has a quadratic integral of motion (interpreted as some form of energy), which eliminates unphysical behaviors that have often plagued atmospheric LOMs and paves way for developing Hamiltonian LOMs. Restricting LOMs to a gyrostatic form also helps to design LOMs of optimum size and provides a modular construction of LOMs using gyrostats as elementary building blocks.

1. Introduction

Atmospheric and climate dynamics exhibit complex behavior reflected in a hierarchy of models from "simple" nonlinear 3-mode ordinary differential equations (ODEs) to full-fledged climate-system models involving nonlinear partial differential equations (PDEs). The need for such hierarchy was argued by Held [1] who noted that "we typically gain some understanding of a complex system by relating its behavior to that of other, especially simpler, systems," and asked "what does it mean, after all, to understand a system as complex as the climate, when we cannot fully understand idealized nonlinear systems with only a few degrees of freedom?"

An effective way to deal with formidable mathematical difficulties posed by the PDEs of fluid dynamics via approximating them with finite systems of nonlinear ODEs (the so-called low-order models (LOMs)) has been established in pioneering work by Kolmogorov (described in [2, 3]), Lorenz [4, 5], and Obukhov [6, 7]. LOMs are commonly derived from the PDEs via the Galerkin method: fluid dynamical fields are expanded into infinite series in time-independent basis functions (commonly Fourier modes); then the series are truncated and substituted into the PDEs yielding a finite system of ODEs (the LOM) for the time evolution of the coefficients in the truncated expansions.

Both Lorenz and Obukhov insisted that LOMs should retain conservation properties of the original PDEs. Arbitrary truncations in the Galerkin method, however, lead to models that may lack the fundamental physical properties of the original equations, such as energy conservation (here and throughout the paper understood as conservation in the limit of no damping and forcing). For example, the celebrated Lorenz model [4] of the two-dimensional Rayleigh-Bénard convection (RBC)

$$\begin{aligned}\dot{x} &= \sigma(y - x),\\ \dot{y} &= -xz + rx - y,\\ \dot{z} &= xy - bz\end{aligned} \tag{1}$$

has revolutionized understanding of randomness in nature, but attempts to extend it to larger, more realistic models of atmospheric dynamics have sometimes led to LOMs exhibiting unphysical behaviors. The problem was addressed [8, 9] by developing the so-called gyrostatic LOMs, or *G-models* (see Section 2, in particular, the simplest such model in a forced regime proved equivalent to the Lorenz model (1)), and via extended Nambu or Lie-Poisson formalisms [10–17].

RBC is the most carefully studied example of nonlinear systems exhibiting self-organization and transition to chaos. It also promotes understanding of many real-world fluid flows in the atmosphere (being the principal mechanism of mesoscale shallow convection), liquid core of the Earth, and astrophysics.

The El Ñino-Southern Oscillation (ENSO) is also one of the most important and longest-studied phenomena as it affects the global atmospheric circulation and therefore temperature and precipitation across the globe. What is demonstrated in this paper is that the well-known Vallis LOM [18, 19] of ENSO

$$\begin{aligned} \dot{u} &= \frac{B\left(T_e - T_w\right)}{2\Delta x} - C\left(u - u^*\right), \\ \dot{T}_w &= \frac{u\left(\overline{T} - T_e\right)}{2\Delta x} - A\left(T_w - T^*\right), \\ \dot{T}_e &= \frac{u\left(T_w - \overline{T}\right)}{2\Delta x} - A\left(T_e - T^*\right) \end{aligned} \qquad (2)$$

is a G-model as well.

G-models are defined and briefly described in Section 2, and Section 3 presents the derivation of the G-model form for the Vallis model and a general discussion of G-models as a novel tool in atmospheric dynamics and time series analysis, with conclusions in Section 4.

2. G-Models

The basic G-model is the Volterra gyrostat [20, 21], a classical system, which admits various mechanical and fluid dynamical interpretations and can be written [22] as

$$\begin{aligned} \dot{X}_1 &= pX_2X_3 + bX_3 - cX_2, \\ \dot{X}_2 &= qX_2X_3 + cX_1 - aX_3, \\ \dot{X}_3 &= rX_2X_3 + aX_2 - bX_1, \end{aligned} \qquad (3)$$

where $p + q + r = 0$. Note that, unlike linear friction terms, linear terms in (3) *(linear gyrostatic terms)* do not affect the conservation of energy nor the conservation of phase space volume.

The simplest Volterra gyrostat ($r = b = c = 0$ in (3)) in a forced regime, that is, with added constant forcing and linear friction,

$$\begin{array}{r|r|l} \dot{X}_1 = & -X_2X_3 \quad\quad & -\alpha_1 X_1 + F, \\ \dot{X}_2 = & X_3X_1 - X_3 & -\alpha_2 X_2, \\ \dot{X}_3 = & X_2 & -\alpha_3 X_3, \end{array} \qquad (4)$$

was proved [8] to be equivalent to the Lorenz model (1) (for this reason we call it the *Lorenz gyrostat*). In (4) and others below, separate Volterra gyrostats are shown within vertical bars, variables are denoted by X_i, friction coefficients by α_i, forces by F or F_i, and the overdot means differentiation with respect to dimensionless time τ.

It was also found [23, 24] that effective LOMs for atmospheric circulations and turbulence could be developed as systems of *coupled* gyrostats (3). For example, the following system of two coupled Lorenz gyrostats (4) in a forced regime

$$\begin{array}{r|r|r|l} \dot{X}_1 = & -X_2X_3 \quad\quad & -X_4X_5 \quad\quad & -\alpha_1 X_1 + F, \\ \dot{X}_2 = & X_3X_1 - X_3 & & -\alpha_2 X_2, \\ \dot{X}_3 = & X_2 & & -\alpha_3 X_3, \\ \dot{X}_4 = & & X_5X_1 - X_5 & -\alpha_4 X_4, \\ \dot{X}_5 = & & X_4 & -\alpha_5 X_5 \end{array} \qquad (5)$$

provides an analog of the Lorenz model the *three*-dimensional RBC, where the Lorenz gyrostats describe the dynamics in two perpendicular planes [25].

Finally, similar to the Arnold's definition of the n-dimensional rigid body [26], the *n-dimensional gyrostat* was introduced [8, 9] as the n-dimensional analog of the Volterra equations (3); the latter recovered at $n = 3$. It has turned out that the 6-mode extension of the Lorenz model (1) recently suggested as a more appropriate minimal model of the two-dimensional RBC [12] could also be treated in terms of G-models, namely, as a 4-dimensional gyrostat [27].

G-models are all of the above gyrostatic LOMs: Volterra gyrostats, coupled Volterra gyrostats, and n-dimensional gyrostats. Any G-model has a quadratic integral of motion (interpreted as some form of energy), which eliminates unphysical behaviors that have often plagued LOMs obtained through ad hoc Galerkin truncations. This integral is often a good candidate for a Hamiltonian function, thus paving way for developing Hamiltonian LOMs [9], which is important since the conservative part of various atmospheric models (the primitive equations, shallow water equations, and quasi-geostrophic equations) is Hamiltonian (e.g., [28]).

3. Results and Discussion

3.1. G-Model Form of the Vallis Model. After the linear change of variables

$$\begin{aligned} u &\longrightarrow \sqrt{B\overline{T}}\left(X_3 - 1\right), \\ T_w &\longrightarrow \overline{T}X_2, \\ T_e &\longrightarrow \overline{T}X_2, \\ t &\longrightarrow \frac{2\Delta x}{\sqrt{B\overline{T}}}, \end{aligned} \qquad (6)$$

(2) take the form of gyrostat (3) in a forced regime

$$\begin{array}{r|r|r|l} \dot{X}_1 = & -X_2X_3 \quad\quad & +X_2 \quad -X_3 & -\alpha_1 X_1 + F_1, \\ \dot{X}_2 = & X_3X_1 - X_3 & -X_1 \quad\quad\quad & -\alpha_2 X_2 + F_2, \\ \dot{X}_3 = & X_2 & +X_1 & -\alpha_3 X_3 + F_3, \end{array} \qquad (7)$$

where

$$
\begin{aligned}
F_1 &= 1 + \frac{2\Delta x A T^*}{\sqrt{B\overline{T}^3}}, \\
F_2 &= 1 - \frac{2\Delta x A T^*}{\sqrt{B\overline{T}^3}}, \\
F_3 &= C\left(\sqrt{B\overline{T}} + u^*\right), \\
\alpha_1 &= \alpha_2 = \frac{2\Delta x A}{\sqrt{B\overline{T}}}, \\
\alpha_3 &= \frac{2\Delta x C}{\sqrt{B\overline{T}}}.
\end{aligned}
\tag{8}
$$

In comparison to the general form (3), the gyrostat in (7) has only two nonlinear terms, but it has all three pairs of linear gyrostatic terms unlike the Lorenz gyrostat in (4) that has only one such pair. In addition, the external force in (7) has three components versus one in (4).

These differences between the Lorenz model (see (1) and (4)) and the Vallis model (see (2) and (7)) could perhaps be clarified using a mechanical interpretation of the original version of (3) [20, 21],

$$
\begin{aligned}
I_1\dot{\omega}_1 &= \left(I_2 - I_3\right)\omega_2\omega_3 + h_2\omega_3 - h_3\omega_2, \\
I_2\dot{\omega}_2 &= \left(I_3 - I_1\right)\omega_3\omega_1 + h_3\omega_1 - h_1\omega_3, \\
I_3\dot{\omega}_3 &= \left(I_1 - I_2\right)\omega_1\omega_2 + h_1\omega_2 - h_2\omega_1,
\end{aligned}
\tag{9}
$$

as a rigid body containing an axisymmetric rotor that rotates with a constant angular velocity about an axis fixed in the carrier body. In (9) then I_i are the principal moments of inertia of the gyrostat, $\boldsymbol{\omega}$ is the angular velocity of the carrier, and $\mathbf{h}$ is the fixed angular momentum caused by the relative motion of the rotor (the gyrostatic motion). For the Lorenz gyrostat (4), this means [22] that the rotor is rotating around its principal axis 1 ($h_2 = h_3 = 0$) with a constant external force directed along this axis, while its ellipsoid of inertia is the ellipsoid of rotation around another principal axis, axis 3 ($I_1 = I_2$). The latter is also true for the gyrostat in G-model (7), but in this case both $\mathbf{h}$ and $\mathbf{F}$ have three nonzero components.

Of particular importance for using the Vallis model (see (2) and (7)) in ENSO studies is that, similar to the Lorenz model (see (1) and (4)), it exhibits both stable and chaotic behaviors [18, 19, 29].

3.2. G-Models as Effective LOMs for Atmospheric Studies. The Vallis model augments the list of phenomena described by G-models. Among them are shell models of turbulence [22, 24], models of convection in rotating fluid [30, 31], of a barotropic atmosphere with topography and of the thermal convection with shear [32], and Hamiltonian LOMs [9]. In fact, we have found [33] that all physically sound LOMs of 2D RBC that have appeared in recent publications are equivalent to G-models, while the LOMs lacking this form exhibit violations of energy conservation.

A new promising application of G-models is their use as novel atmospheric time series models [33], thereby utilizing both their deterministic and probabilistic facets. It was motivated by current problems with handling atmospheric data and by recent progress in statistical properties of dynamical systems. In particular, it has been proved that the flow of the Lorenz model (1) possesses a physical ergodic invariant probability measure [34] and satisfies the central limit theorem [35, 36]; that is, series of observations on this model may exhibit statistics of sequences of random variables. In contrast to common models (borrowed from traditional time series analysis and having little to do with the atmosphere per se), G-models are derived from the underlying equations, and so their statistical behavior is in better agreement with reality.

LOMs reveal basic mechanisms and their interplay through the focus on key elements and retaining only minimal number of degrees of freedom. Any G-model, as mentioned above, has a quadratic integral of motion, which eliminates certain unphysical behaviors, which often plague other LOMs. Another attractive feature of G-models is that increasing the order of approximation in the Galerkin procedure results in adding to the system new gyrostats (or blocks of gyrostats) whose linear terms represent various effects important in atmospheric dynamics, such as stratification, rotation, or topography. In this way, larger G-models become even more useful as they provide increasingly better approximations to the original system. This is because the dynamics generated by fundamental mathematical models of fluid flows are in a sense asymptotically finite-dimensional [37].

For all these reasons, G-models may probably offer a general framework for deriving effective LOMs of atmospheric dynamics and atmospheric time series analysis.

4. Conclusions

We have shown that the Vallis model of ENSO is a G-model and argued advantages of this type of LOMs. Even with ever-increasing computer power, LOMs remain important since, as noted by Smith [38], "although it is unreasonable to expect solutions to low-dimensional problems to generalize to a million dimensional spaces, so too it is unlikely that problems identified in the simplified models will vanish in operational models."

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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