Research Article

Finite-Time H_2/H_∞ Control for Linear Itô Stochastic Systems with (x, u, v) -Dependent Noise

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This paper deals with the problem of the H_2/H_∞ control based on finite-time boundedness for linear stochastic systems. The motivation for investigating this problem comes from one observation that the H_2/H_∞ control does not involve systems' transient performance. To express this problem clearly, a concept called finite-time H_2/H_∞ control is introduced. Moreover, state feedback and observer-based finite-time H_2/H_∞ controllers are designed, which guarantee finite-time boundedness, H_2 performance index, and H_∞ performance index of the closed-loop systems. Furthermore, an optimization algorithm on the finite-time H_2/H_∞ control is presented to obtain the minimum values of the H_2 index and H_∞ index. Finally, we use an example to show the validity of the obtained results.

1. Introduction

Recently, stochastic systems have been receiving more and more attention, which is due to their applications in many practical fields, such as finance systems [1] and power systems [2]. A lot of results on stochastic systems have been obtained. For example, [3] gave some Razumikhin-type theorems on the p th moment input-to-state stability for impulsive stochastic delayed systems by utilizing the Razumikhin technique and average dwell-time approach. By using the interval matrix transformation method, the problems of finite-time dissipative control for stochastic interval systems are investigated in [4]. The literature [5] investigated the state prediction problem for nonlinear stochastic differential systems by utilizing the Carleman embedding technique. In addition, many other nice results on stochastic systems have also been obtained; see, e.g., Lyapunov stability conditions [6], reliable output feedback control [7] and distributed containment control [8], and the references therein.

On the other hand, one of the important control methods in dealing with robust control problems is the H_2/H_∞ control method. Its concrete implication is to seek a controller which not only minimizes the H_2 cost function but also restrains the effect of external disturbance. The H_2/H_∞ control method has been applied to many fields, such as communication systems [9] and synthetic gene network design [10]. Meanwhile, H_2/H_∞ control problems have been extended to many control system models, and many nice results have been obtained, such as stochastic systems [11], descriptor systems [12, 13], Markovian jump systems [14], and the recent monograph [15]. Nevertheless, most of the results on H_2/H_∞ control problems in existing literature are based on Lyapunov's asymptotic stability, which do not reflect the transient performance of the systems. In some cases, large transient performance has a negative effect on the practical systems. For example, in power systems, a large transient current cannot be permitted, because it can damage the system [16]. In order to describe the phenomenon precisely, finite-

time (FT) stability was introduced in the literature [17–19]. In the sequel, considering external disturbance, [20] extended FT stability to FT boundedness. Currently, the issues on FT stability and FT boundedness have been extensively studied for many kinds of system models; see, e.g. [21–30] and the references therein. Taking into account the advantages of FT stability and H_2/H_∞ control, the finite-time H_2/H_∞ control problem is proposed in this paper, which not only guarantees FTB but also satisfies the H_2/H_∞ performance indices. Up to date, there is almost no literature to consider the FT H_2/H_∞ control of stochastic systems.

This paper will study the problems of FT H_2/H_∞ control for linear Itô stochastic systems. Because of the complexity of the problems, the appropriate controller design will be more difficult. By using the method of stochastic analysis, state feedback and observer-based FT H_2/H_∞ controllers are designed. The main contributions of this paper are as follows: (i) The definition of the FT H_2/H_∞ control for linear stochastic systems with (x, u, v) -dependent noise (which is called state-, control-, and disturbance-dependent noise) is first given, which simultaneously presents FT boundedness, H_2 performance index, and H_∞ performance index, respectively, of the closed-loop system. (ii) The two new sufficient conditions for the existence of the state feedback controller and observer-based controller are obtained in the form of linear matrix inequalities (LMIs). (iii) A parameter optimization algorithm is given to obtain the minimum

values of the H_2 performance index and H_∞ performance index, simultaneously.

The organization of this paper is summarized as follows: In Section 2, we give some preliminaries. In Section 3, the state feedback FT H_2/H_∞ controller is designed. In Section 4, the observer-based FT H_2/H_∞ controller is designed. In Section 5, an optimization algorithm is provided for obtaining the minimum values of the H_2 performance index and H_∞ performance index, simultaneously. In Section 6, a numerical example is discussed, and some remarks are concluded in Section 7.

Notation. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$: a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. M' : transpose of a matrix M . $M > 0$: M is a positive definite symmetric matrix. $I_{n \times n}$: $n \times n$ identity matrix. $\mathcal{L}_{\mathcal{F}}^2(\mathcal{R}^+, \mathcal{R}^l)$: the space of nonanticipative stochastic process $\xi(t)$ with respect to filtration $\mathcal{F}_t$ satisfying $\|\xi(t)\|_{L_2}^2 := E \int_0^\infty \|\xi(t)\|^2 dt < \infty$. $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$: the maximum and minimum eigenvalue of matrix M . $E[\cdot]$ is the mathematical expectation of the stochastic process. The asterisk “*” in a matrix stands for the symmetry term. $\text{diag}\{\dots\}$ represents a diagonal matrix. The “wrt” denotes “with respect to.”

2. Preliminaries

We consider the linear stochastic system as follows:

$$\begin{cases} dx(t) = [A_1 x(t) + B_1 u(t) + G_1 v(t)]dt + [A_2 x(t) + B_2 u(t) + G_2 v(t)]dw(t), \\ y(t) = C_1 x(t), \\ z(t) = C_2 x(t) + D_1 u(t) + D_2 v(t), \\ x(0) = x_0 \in R^n, \end{cases} \quad (1)$$

where $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, G_1$, and G_2 are constant matrices with appropriate dimensions. $x(t) \in R^n$, $u(t) \in R^m$, $v(t) \in R^p$, $y(t) \in R^q$, and $z(t) \in R^l$ are called state, control input, external disturbance, measurement output, and controlled output, respectively. x_0 is the initial state. $w(t)$ is a one-dimensional Wiener process, which is defined as $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. $v(t)$ is the disturbance and satisfies the following set:

$$\Omega = \left\{ v(t): E \int_0^t v'(s)v(s)ds < h, h > 0, t \in [0, T] \right\}. \quad (2)$$

Next, the definition of FT stochastic boundedness for system (1) is introduced, which is the generalization of FT boundedness in [20, 31–33].

Definition 1. System (1) with $u(t) = 0$ is FT stochastically bounded wrt (d_1, d_2, T, R, h) , if

$$x'(0)Rx(0) \leq d_1 \Rightarrow E \left[x'(t)Rx(t) \right] < d_2, \quad \forall t \in [0, T], \forall v(t) \in \Omega, \quad (3)$$

where $0 < d_1 < d_2$, $T > 0$, and $R > 0$.

For the subsequent analysis, the following lemmas are useful for the FT H_2/H_∞ controller design.

Lemma 1 [27]. If $\rho(t) > 0$, $f \geq 0$, and $g \geq 0$ satisfy

$$\rho(t) \leq f + g \int_0^t \rho(s)ds, \quad t \in [0, T], \quad (4)$$

then we obtain

$$\rho(t) \leq f \exp(gt). \quad (5)$$

Lemma 2 [34]. Let $V(t, x) \in C^{1,2}(R^+, R^+ \times R^n)$ be a scalar function and $V(t, x) > 0$. For the following stochastic system

$$dx(t) = a(x)dt + b(x)dw(t), \quad (6)$$

the Itô formula of $V(t, x)$ is given as follows:

$$dV(t, x) = \mathcal{L}V(t, x)dt + \frac{\partial V(t, x)}{\partial x} b(x)dw(t), \quad (7)$$

where

$$\begin{aligned} \mathcal{L}V(t, x) = & \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} a(x) \\ & + \frac{1}{2} b'(x) \frac{\partial^2 V(t, x)}{\partial x^2} b(x). \end{aligned} \quad (8)$$

3. State Feedback Finite-Time H_2/H_∞ Controller Design

This section gives a problem formulation of state-feedback finite-time (SFFT) H_2/H_∞ control for system (1). And also, the two sufficient conditions are given for the existence of a SF controller.

Considering the following linear state-feedback (SF) controller

$$u(t) = Kx(t), \quad (9)$$

where K is the feedback gain matrix.

Substituting the controller (9) into system (1) leads to the closed-loop system

$$\begin{cases} dx(t) = [\tilde{A}_1 x(t) + G_1 v(t)] dt + [\tilde{A}_2 x(t) + G_2 v(t)] dw(t), \\ y(t) = C_1 x(t), \\ z(t) = \tilde{C} x(t) + D_2 v(t), \\ x(0) = x_0 \in R^n, \end{cases} \quad (10)$$

where $\tilde{A}_1 = A_1 + B_1 K$, $\tilde{A}_2 = A_2 + B_2 K$, and $\tilde{C} = C_2 + D_1 K$.

Associated with system (1), the following cost function is provided:

$$J_s(x(t), u(t)) = E \int_0^T [x'(t) P_1 x(t) + u'(t) P_2 u(t)] dt, \quad (11)$$

where $P_1 > 0$ and $P_2 > 0$ are the given positive scalars or given weighting matrices.

Also, by substituting (9) into (11), the corresponding performance function (11) becomes

$$J_s(x(t)) = E \int_0^T [x'(t) P_1 x(t) + x'(t) K' P_2 K x(t)] dt. \quad (12)$$

For a given $\gamma > 0$, under the condition of zero initial value, the discretional nonzero disturbance $v(t)$ and the control output $z(t)$ satisfy the following form:

$$E \int_0^T z'(t) z(t) dt < \gamma^2 E \int_0^T v'(t) v(t) dt. \quad (13)$$

Based on the above preparations, we are in a position to give the definition of the state-feedback finite-time H_2/H_∞ control.

Definition 2. For some given positive scalars γ , d_1 , d_2 , T , R , and h , considering system (1), cost function (11), and the inequality (13), if there exist a positive scalar J_s^* and a SF controller (9) such that

- (i) system (10) is FT stochastically bounded wrt (d_1, d_2, T, R, h)
- (ii) H_2 cost function (12) satisfies $J_s(x(t)) \leq J_s^*$ under the condition of $v(t) = 0$
- (iii) for any nonzero disturbance $v(t)$, under the zero initial condition, the inequality (13) is satisfied

then (9) is a state-feedback finite-time H_2/H_∞ controller of the stochastic system (1).

Remark 1. Definition 2 considers three aspects of actual systems: FT boundedness of systems' states, minimum performance cost, and ability to suppress interference of the closed-loop systems, which is more complex than only considering the problems of the H_2 performance index or H_∞ performance index. In addition, these three aspects involved by the FT H_2/H_∞ control are often required by the actual systems. For example, in the power systems, a large transient current cannot be permitted, electric energy consumption is expected to be minimum, and the system is expected to have better ability to suppress interference [16].

A sufficient condition for the existence of the SFFT H_2/H_∞ controller will be given by the following theorem.

Theorem 1. For the given positive scalars γ , d_1 , d_2 , T , R , and h , if there exist a matrix $M > 0$, a matrix K , and a scalar $\alpha \geq 0$ such that

$$\begin{bmatrix} \mathcal{T}_1 & G_1 + \tilde{M} \tilde{C}' D_2 & \tilde{M} \tilde{A}_2' \\ * & -\gamma^2 I + D_2' D_2 & G_2' \\ * & * & -\tilde{M} \end{bmatrix} < 0, \quad (14)$$

$$\begin{bmatrix} \mathcal{T}_2 & \tilde{M}\tilde{A}_2' \\ * & -\tilde{M} \end{bmatrix} < 0, \quad (15)$$

$$\frac{d_1}{\lambda_{\min}(M)} + h\gamma^2 < \frac{d_2}{\lambda_{\max}(M)} e^{-\alpha T}, \quad (16)$$

hold, where $\tilde{M} = R^{-1/2}MR^{-1/2}$, $\mathcal{T}_1 = \tilde{A}_1\tilde{M} + \tilde{M}\tilde{A}_1' - \alpha\tilde{M} + \tilde{M}\tilde{C}'\tilde{C}\tilde{M}$, and $\mathcal{T}_2 = \tilde{A}_1\tilde{M} + \tilde{M}\tilde{A}_1' - \alpha\tilde{M} + \tilde{M}P_1\tilde{M} + \tilde{M}K'P_2K\tilde{M}$, then $u(t) = Kx(t)$ is a SFFT H_2/H_∞ controller of system (1) and the upper bound of the H_2 index can be given as $J_{\text{state}}^* = \lambda_{\max}(M^{-1})d_1e^{\alpha T}$.

Proof. The proof will be divided into three steps.

Step 1. Prove that the condition (i) is satisfied.

It is noticed that

$$\begin{bmatrix} \tilde{M}\tilde{C}'\tilde{C}\tilde{M} & \tilde{M}\tilde{C}'D_2 & 0 \\ D_2'\tilde{C}\tilde{M} & D_2'D_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{M}\tilde{C}' \\ D_2' \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{M}\tilde{C}' \\ D_2' \\ 0 \end{bmatrix}' \geq 0. \quad (17)$$

Therefore, the condition (14) implies that

$$\begin{bmatrix} \tilde{A}_1\tilde{M} + \tilde{M}\tilde{A}_1' - \alpha\tilde{M} & G_1 & \tilde{M}\tilde{A}_2' \\ * & -\gamma^2 I & G_2' \\ * & * & -\tilde{M} \end{bmatrix} < 0. \quad (18)$$

Let $\mathcal{V}(x(t)) = x'(t)\tilde{M}^{-1}x(t)$, the infinitesimal operator $\mathcal{L}_1\mathcal{V}(x(t))$ of system (10) is defined as follows:

$$\begin{aligned} \mathcal{L}_1\mathcal{V}(x(t)) &= [(A_1 + B_1K)x(t) + G_1v(t)]'\tilde{M}^{-1}x(t) \\ &\quad + x'(t)\tilde{M}^{-1}[(A_1 + B_1K)x(t) + G_1v(t)] \\ &\quad + [(A_2 + B_2K)x(t) + G_2v(t)]'\tilde{M}^{-1} \\ &\quad \cdot [(A_2 + B_2K)x(t) + G_2v(t)] \\ &= \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \begin{bmatrix} \bar{Z}_1 & \tilde{A}_1'\tilde{M}^{-1}G_2 + \tilde{M}^{-1}G_1 \\ * & G_2'\tilde{M}^{-1}G_2 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}, \end{aligned} \quad (19)$$

where $\bar{Z}_1 = \tilde{A}_1'\tilde{M}^{-1} + \tilde{M}^{-1}\tilde{A}_1 + \tilde{A}_2'\tilde{M}^{-1}\tilde{A}_2$.

Pre- and postmultiplying the inequality (18) by $\text{diag}\{\tilde{M}^{-1}, I, \tilde{M}^{-1}\}$, it leads to the following inequality:

$$\begin{bmatrix} \tilde{A}_1'\tilde{M}^{-1} + \tilde{M}^{-1}\tilde{A}_1 - \alpha\tilde{M}^{-1} & \tilde{M}^{-1}G_1 & \tilde{A}_2'\tilde{M}^{-1} \\ * & -\gamma^2 I & G_2'\tilde{M}^{-1} \\ * & * & -\tilde{M}^{-1} \end{bmatrix} < 0. \quad (20)$$

According to the Schur complement, (20) is equivalent to

$$\begin{bmatrix} \tilde{A}_1'\tilde{M}^{-1} + \tilde{M}^{-1}\tilde{A}_1 + \tilde{A}_2'\tilde{M}^{-1}\tilde{A}_2 - \alpha\tilde{M}^{-1} & \tilde{M}^{-1}G_1 + \tilde{A}_2'\tilde{M}^{-1}G_2 \\ * & G_2'\tilde{M}^{-1}G_2 - \gamma^2 I \end{bmatrix} < 0. \quad (21)$$

Considering the conditions (19) and (21), it follows that

$$\mathcal{L}_1\mathcal{V}(x(t)) < \alpha\mathcal{V}(x(t)) + \gamma^2 v'(t)v(t). \quad (22)$$

Integrating (22) from 0 to t and then taking the expectation, it follows that

$$\begin{aligned} E\mathcal{V}(x(t)) &< \mathcal{V}(x(0)) + \alpha \int_0^t E\mathcal{V}(x(s))ds \\ &\quad + \gamma^2 \int_0^t Ev'(s)v(s)ds. \end{aligned} \quad (23)$$

By Lemma 1, we get

$$E\mathcal{V}(x(t)) < \mathcal{V}(x(0))e^{\alpha t} + \gamma^2 e^{\alpha t} \int_0^t Ev'(s)v(s)ds. \quad (24)$$

According to known conditions, it yields

$$\begin{aligned} E\mathcal{V}(x(t)) &= E\left[x'(t)R^{1/2}M^{-1}R^{1/2}x(t)\right] \\ &\geq \lambda_{\min}(M^{-1})E\left[x'(t)Rx(t)\right], \end{aligned} \quad (25)$$

$$\begin{aligned} \mathcal{V}(x(0))e^{\alpha t} &= x'(0)R^{1/2}M^{-1}R^{1/2}x(0)e^{\alpha t} \\ &\leq \lambda_{\max}(M^{-1})x'(0)Rx(0)e^{\alpha t} \\ &\leq \lambda_{\max}(M^{-1})d_1e^{\alpha T}, \end{aligned} \quad (26)$$

$$\gamma^2 e^{\alpha t} \int_0^t Ev'(s)v(s)ds < e^{\alpha T}h\gamma^2. \quad (27)$$

From (24), (25), (26), and (27), it is easy to obtain

$$E\left[x'(t)Rx(t)\right] < \lambda_{\max}(M)e^{\alpha T} \left[\frac{d_1}{\lambda_{\min}(M)} + h\gamma^2 \right]. \quad (28)$$

According to condition (16), it follows that (28) leads to $E[x'(t)Rx(t)] < d_2$ for all $t \in [0, T]$. So,

the closed-loop system (10) is FT stochastically bounded wrt (d_1, d_2, T, R, h) .

Step 2. Prove that the condition (ii) is satisfied.

Under the condition of $v(t) = 0$, the infinitesimal operator $\mathcal{L}_2 \mathcal{V}(x(t))$ of system (10) is as follows:

$$\begin{aligned} \mathcal{L}_2 \mathcal{V}(x(t)) &= \left[\tilde{A}_1 x(t) \right]' \tilde{M}^{-1} x(t) + x'(t) \tilde{M}^{-1} \tilde{A}_1 x(t) \\ &\quad + \left[\tilde{A}_2 x(t) \right]' \tilde{M}^{-1} \tilde{A}_2 x(t) \\ &= x'(t) \left[\tilde{A}_1' \tilde{M}^{-1} + \tilde{M}^{-1} \tilde{A}_1 + \tilde{A}_2' \tilde{M}^{-1} \tilde{A}_2 \right] x(t). \end{aligned} \quad (29)$$

By Schur's complement, it can be seen that (15) is equivalent to

$$\begin{aligned} \tilde{M} \tilde{A}_1' + \tilde{A}_1 \tilde{M} + \tilde{M} \tilde{A}_2' \tilde{M}^{-1} \tilde{A}_2 \tilde{M} + \tilde{M} P_1 \tilde{M} \\ + \tilde{M} K' P_2 K \tilde{M} - \alpha \tilde{M} < 0. \end{aligned} \quad (30)$$

Premultiplying and postmultiplying (30) by $\tilde{M}^{-1}$, we obtain

$$\begin{aligned} \tilde{A}_1' \tilde{M}^{-1} + \tilde{M}^{-1} \tilde{A}_1 + \tilde{A}_2' \tilde{M}^{-1} \tilde{A}_2 + P_1 \\ + K' P_2 K - \alpha \tilde{M}^{-1} < 0. \end{aligned} \quad (31)$$

According to (29) and (31), we get

$$\begin{aligned} \mathcal{L}_2 \mathcal{V}(x(t)) - \alpha \mathcal{V}(x(t)) \\ + x'(t) (P_1 + K' P_2 K) x(t) < 0. \end{aligned} \quad (32)$$

Integrating (32) from 0 to t , $t \in [0, T]$, and taking the expectation, it is obtained that

$$\begin{aligned} E \int_0^t x'(s) (P_1 + K' P_2 K) x(s) ds + E \mathcal{V}(x(t)) \\ < \mathcal{V}(x(0)) + \alpha E \int_0^t \mathcal{V}(x(s)) ds. \end{aligned} \quad (33)$$

From (33), we get

$$E \mathcal{V}(x(t)) < \mathcal{V}(x(0)) + \alpha E \int_0^t \mathcal{V}(x(s)) ds, \quad (34)$$

$$J_s(x(t)) < \alpha E \int_0^t \mathcal{V}(x(s)) ds + \mathcal{V}(x(0)). \quad (35)$$

From (34), by Lemma 1, we obtain

$$E \mathcal{V}(x(t)) < \mathcal{V}(x(0)) e^{\alpha t}. \quad (36)$$

According to (35) and (36), it is obtained that

$$\begin{aligned} J_s(x(t)) &< \alpha E \int_0^t x'(s) R^{1/2} M^{-1} R^{1/2} x(s) e^{\alpha s} ds + x'(0) \tilde{M}^{-1} x(0) \\ &= x'(0) R^{1/2} M^{-1} R^{1/2} x(0) e^{\alpha t} < \lambda_{\max}(M^{-1}) d_1 e^{\alpha T} = J_s^*. \end{aligned} \quad (37)$$

Step 3. Prove that the condition (iii) is satisfied.

Premultiplying and postmultiplying (14) by $\text{diag}\{\tilde{M}^{-1}, I, \tilde{M}^{-1}\}$, and according to the Schur complement, we have

$$\begin{bmatrix} \mathcal{T}_3 & \tilde{A}_1' \tilde{M}^{-1} G_2 + \tilde{M}^{-1} G_1 + \tilde{C}' D_2 \\ * & -\gamma^2 I + G_2' \tilde{M}^{-1} G_2 + D_2' D_2 \end{bmatrix} < 0, \quad (38)$$

where $\mathcal{T}_3 = \tilde{A}' \tilde{M}^{-1} + \tilde{M}^{-1} \tilde{A} + \tilde{A}_1' \tilde{M}^{-1} \tilde{A}_1 - \alpha \tilde{M}^{-1} + P_1 + K' P_2 K + \tilde{C}' \tilde{C}$.

According to (19) and (38), we have

$$\mathcal{L}_1 \mathcal{V}(x(t)) < \alpha \mathcal{V}(x(t)) + \gamma^2 v'(t) v(t) - z'(t) z(t). \quad (39)$$

Premultiplying and postmultiplying (39) by $e^{-\alpha t}$, we have

$$\begin{aligned} e^{-\alpha t} \mathcal{L}_1 \mathcal{V}(x(t)) &< \alpha e^{-\alpha t} \mathcal{V}(x(t)) \\ &\quad + e^{-\alpha t} [\gamma^2 v'(t) v(t) - z'(t) z(t)]. \end{aligned} \quad (40)$$

By applying Lemma 2, we can obtain that

$$\mathcal{L}_1 [e^{-\alpha t} \mathcal{V}(x(t))] = -\alpha e^{-\alpha t} \mathcal{V}(x(t)) + e^{-\alpha t} \mathcal{L}_1 \mathcal{V}(x(t)). \quad (41)$$

According to (40) and (41), we get

$$\mathcal{L}_1 [e^{-\alpha t} \mathcal{V}(x(t))] < e^{-\alpha t} [\gamma^2 v'(t) v(t) - z'(t) z(t)]. \quad (42)$$

Under the condition of zero initial value, integrating both sides of (42) from 0 to t , $t \in [0, T]$, taking the expectation, and after some calculations, we have

$$\begin{aligned} e^{-\alpha T} E \mathcal{V}(x(t)) &< e^{-\alpha T} \left[E \int_0^T (\gamma^2 v'(s) v(s) ds - z'(s) z(s)) ds \right] \\ &< \gamma^2 E \int_0^T v'(s) v(s) ds - E \int_0^T z'(s) z(s) ds. \end{aligned} \quad (43)$$

Because $e^{-\alpha T} E \mathcal{V}'(x(t)) > 0$, we can get

$$E \int_0^T z'(s)z(s)ds < \gamma^2 E \int_0^T v'(s)v(s)ds. \quad (44)$$

This completes the proof.

It should be noted that (14), (15), and (16) are not linear matrix inequalities and difficult to solve. To overcome the above difficulties, the following theorem is obtained.

Theorem 2. For the given positive scalars γ , d_1 , d_2 , T , R , and h , if there exist a matrix $M > 0$, a matrix Y , a scalar $\lambda > 0$, and a scalar $\alpha \geq 0$ such that

$$\begin{bmatrix} \mathcal{T}_4 & G_1 & \tilde{M}A_2' + Y'B_2' & \tilde{M}C_2' + Y'D_1' \\ * & -\gamma^2 I & G_2' & D_2 \\ * & * & -\tilde{M} & 0 \\ * & * & * & -I \end{bmatrix} < 0, \quad (45)$$

$$\begin{bmatrix} \mathcal{T}_4 & \tilde{M}A_2' + Y'B_2' & \tilde{M} & Y' \\ * & -\tilde{M} & 0 & 0 \\ * & * & -P_1^{-1} & 0 \\ * & * & * & -P_2^{-1} \end{bmatrix} < 0, \quad (46)$$

$$\begin{bmatrix} h\gamma^2 - d_2 e^{-\alpha T} & \sqrt{d_1} \\ * & -\lambda \end{bmatrix} < 0, \quad (47)$$

$$\lambda I < M < I, \quad (48)$$

hold, where $\mathcal{T}_4 = A_1 \tilde{M} + \tilde{M}A_1' - \alpha \tilde{M} + B_1 Y + Y'B_1'$, then $u(t) = Kx(t) = Y\tilde{M}^{-1}x(t)$ is a SFFT H_2/H_∞ controller and the upper bound of the H_2 index can be given as $J_{\text{state}}^* = \lambda^{-1} d_1 e^{\alpha T}$.

Proof. Letting $Y = K\tilde{M}$, (45) and (46) can lead to inequalities (14) and (15), respectively, and it is easy to check that (16) in Theorem 1 can be guaranteed by inequalities (47) and (48). The proof is completed.

Remark 2. It is noted that the inequalities in Theorem 2 are not LMIs, but if α is fixed, (45), (46), (47), and (48) can be regarded as LMIs.

4. Observer-Based Finite-Time H_2/H_∞ Controller Design

In the above Section 3, the SFFT H_2/H_∞ control problem of system (1) has been discussed. However, in many practical cases, it is difficult to measure the total states. Therefore, considering the problem of the observer-based finite-time (OBFT) H_2/H_∞ controller design is necessary. Typically, an observer-based (OB) controller is provided as follows:

$$\begin{cases} d\hat{x}(t) = [A_1 \hat{x}(t) + B_1 u(t) + L(y(t) - C_1 \hat{x}(t))]dt, \\ u(t) = K\hat{x}(t), \\ \hat{x}(0) = 0, \end{cases} \quad (49)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimation of $x(t)$ and L is the estimator gain.

Substituting (49) into system (1), we will obtain the closed-loop system

$$\begin{cases} d\tilde{x}(t) = [\bar{A}_1 \tilde{x}(t) + W_1 v(t)]dt + [\bar{A}_2 \tilde{x}(t) + W_2 v(t)]dw(t), \\ \tilde{z}(t) = T_1 \tilde{x}(t) + D_2 v(t), \end{cases} \quad (50)$$

and the corresponding closed-loop cost function

$$J_o(\tilde{x}(t)) = E \int_0^T \tilde{x}'(t) \mathcal{T} \tilde{x}(t) dt, \quad (51)$$

where

$$\begin{aligned} \tilde{x}(t) &= \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \\ \bar{A}_1 &= \begin{bmatrix} A_1 & B_1 K \\ LC_1 & A_1 + B_1 K - LC_1 \end{bmatrix}, \\ W_1 &= \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \end{aligned} \quad (52)$$

$$\begin{aligned} \bar{A}_2 &= \begin{bmatrix} A_2 & B_2 K \\ 0 & 0 \end{bmatrix}, \\ W_2 &= \begin{bmatrix} G_2 \\ 0 \end{bmatrix}, \\ T_1 &= \begin{bmatrix} C_2' \\ K'D_1' \end{bmatrix}, \\ \mathcal{T} &= \begin{bmatrix} P_1 & 0 \\ 0 & K'P_2 K \end{bmatrix}. \end{aligned} \quad (53)$$

For a given $\gamma > 0$, under the condition of zero initial value, the discretonal nonzero disturbance $v(t)$ and the control output $z(t)$ satisfy the following form:

$$E \int_0^T \tilde{z}'(t)\tilde{z}(t)dt < \gamma^2 E \int_0^T v'(t)v(t)dt. \quad (54)$$

Then, the definition of the observer-based finite-time H_2/H_∞ control is given as follows.

Definition 3. For the given positive scalars γ , d_1 , d_2 , T , R , and h , considering system (1), cost function (11), and the

inequality (13), if there exist an OB controller (49) and a positive scalar J_o^* such that

- (i) system (50) is FT stochastically bounded wrt (d_1, d_2, T, R, h)
- (ii) H_2 cost function (51) satisfies $J(\tilde{x}(t)) \leq J_o^*$ under the condition of $v(t) = 0$
- (iii) for any nonzero disturbance, under the zero initial condition, the inequality (54) is satisfied

then (49) is said to be an observer-based finite-time H_2/H_∞ controller for the stochastic system (1).

Remark 3. From Definition 3, it can be seen that the existence of the observer-based finite-time H_2/H_∞ controller (49) implies the existence of the SFFT H_2/H_∞ controller.

By the results in Theorem 2, we have designed a state-feedback finite-time H_2/H_∞ controller $u(t) = Kx(t)$. Next, a design condition of the observer-based finite-time H_2/H_∞ controller for the stochastic system (1) will be given.

Theorem 3. For the given positive scalars γ, d_1, d_2, T, R , and h , if there exist a positive matrix Q and a scalar $\beta \geq 0$ such that

$$\begin{bmatrix} \mathcal{T}_5 & \tilde{Q}W_1 + T_1'D_2 & \bar{A}_2'\tilde{Q} \\ * & -\gamma^2 I + D_2'D_2 & W_2'\tilde{Q} \\ * & * & -\tilde{Q} \end{bmatrix} < 0, \quad (55)$$

$$\bar{A}_1'\tilde{Q} + \tilde{Q}\bar{A}_1 + \bar{A}_2'\tilde{Q}\bar{A}_2 - \beta\tilde{Q} + \mathcal{T} < 0, \quad (56)$$

$$\lambda_{\max}(Q)d_1 + h\gamma^2 < \lambda_{\min}(Q)d_2e^{-\beta T}, \quad (57)$$

hold, where $\tilde{Q} = R^{1/2}QR^{1/2}$ and $\mathcal{T}_5 = \bar{A}_1'\tilde{Q} + \tilde{Q}\bar{A}_1 - \beta\tilde{Q} + T_1'T_1$, then (49) is an OBFT H_2/H_∞ controller and the upper bound of the H_2 index can be given as $J_o^* = \lambda_{\max}(Q)d_1e^{\beta T}$.

Proof. The proof will be divided into three steps.

Step 1. Prove that the condition (i) is satisfied.

Let $\mathcal{V}(\tilde{x}(t)) = \tilde{x}'(t)\tilde{Q}\tilde{x}(t)$ with $\tilde{Q} > 0$ be solutions to (55), (56), and (57). The infinitesimal operator $\mathcal{L}_3\mathcal{V}(\tilde{x}(t))$ of system (50) is defined as follows:

$$\begin{aligned} \mathcal{L}_3\mathcal{V}(\tilde{x}(t)) &= [\bar{A}_1\tilde{x}(t) + W_1v(t)]'\tilde{Q}\tilde{x}(t) \\ &\quad + \tilde{x}'(t)\tilde{Q}[\bar{A}_1\tilde{x}(t) + W_1v(t)] \\ &\quad + [\bar{A}_2\tilde{x}(t) + W_2v(t)]'\tilde{Q}[\bar{A}_2\tilde{x}(t) + W_2v(t)] \end{aligned}$$

$$\begin{aligned} &= \tilde{x}'(t) \begin{bmatrix} \bar{A}_1'\tilde{Q} + \tilde{Q}\bar{A}_1 + \bar{A}_2'\tilde{Q}\bar{A}_2 \\ W_1'\tilde{Q} + W_2'\tilde{Q}\bar{A}_2 \\ \tilde{Q}W_1 + \bar{A}_2'\tilde{Q}W_2 \end{bmatrix} \tilde{x}(t) \\ &\quad + v'(t) \begin{bmatrix} W_1'\tilde{Q} + W_2'\tilde{Q}\bar{A}_2 \\ \tilde{Q}W_1 + \bar{A}_2'\tilde{Q}W_2 \end{bmatrix} \tilde{x}(t) \\ &\quad + \tilde{x}'(t) \begin{bmatrix} \tilde{Q}W_1 + \bar{A}_2'\tilde{Q}W_2 \\ W_1'\tilde{Q} + W_2'\tilde{Q}\bar{A}_2 \end{bmatrix} v(t) \\ &\quad + v'(t) \begin{bmatrix} W_1'\tilde{Q} + W_2'\tilde{Q}\bar{A}_2 \\ \tilde{Q}W_1 + \bar{A}_2'\tilde{Q}W_2 \end{bmatrix} v(t) \\ &= \begin{bmatrix} \tilde{x}(t) \\ v(t) \end{bmatrix}' \begin{bmatrix} \bar{A}_1'\tilde{Q} + \tilde{Q}\bar{A}_1 + \bar{A}_2'\tilde{Q}\bar{A}_2 & \tilde{Q}W_1 + \bar{A}_2'\tilde{Q}W_2 \\ * & W_1'\tilde{Q} + W_2'\tilde{Q}\bar{A}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ v(t) \end{bmatrix}. \end{aligned} \quad (58)$$

Note that

$$\begin{bmatrix} T_1'T_1 & T_1'D_2 & 0 \\ D_2'T_1 & D_2'D_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1' \\ D_2' \\ 0 \end{bmatrix} \begin{bmatrix} T_1 \\ D_2 \\ 0 \end{bmatrix}' \geq 0. \quad (59)$$

Therefore, the inequality (55) implies that

$$\begin{bmatrix} \bar{A}_1'\tilde{Q} + \tilde{Q}\bar{A}_1 - \beta\tilde{Q} & \tilde{Q}W_1 & \bar{A}_2'\tilde{Q} \\ * & -\gamma^2 I & W_2'\tilde{Q} \\ * & * & -\tilde{Q} \end{bmatrix} < 0. \quad (60)$$

According to the Schur complement, (60) can be transformed into

$$\begin{bmatrix} \bar{A}_1'\tilde{Q} + \tilde{Q}\bar{A}_1 + \bar{A}_2'\tilde{Q}\bar{A}_2 - \beta\tilde{Q} & \tilde{Q}W_1 + \bar{A}_2'\tilde{Q}W_2 \\ * & -\gamma^2 I + W_2'\tilde{Q}W_2 \end{bmatrix} < 0. \quad (61)$$

According to (58) and (61), we have

$$\mathcal{L}_3\mathcal{V}(\tilde{x}(t)) < \beta\mathcal{V}(\tilde{x}(t)) + \gamma^2 v'(t)v(t). \quad (62)$$

Integrating (62) from 0 to t and then taking the expectation, it follows that

$$\begin{aligned} E\mathcal{V}(\tilde{x}(t)) &< \mathcal{V}(\tilde{x}(0)) + \beta \int_0^t E\mathcal{V}(\tilde{x}(s))ds \\ &\quad + \gamma^2 \int_0^t E v'(s)v(s)ds. \end{aligned} \quad (63)$$

According to Lemma 1, it is obtained that

$$E\mathcal{V}(\tilde{x}(t)) < \mathcal{V}(\tilde{x}(0))e^{\beta t} + \gamma^2 e^{\beta t} \int_0^t E v'(s) v(s) ds. \quad (64)$$

According to known conditions, it yields that

$$\begin{aligned} E\mathcal{V}(\tilde{x}(t)) &= E \left[\tilde{x}'(t) R^{1/2} Q R^{1/2} \tilde{x}(t) \right] \\ &\geq \lambda_{\min}(Q) E \left[\tilde{x}'(t) R \tilde{x}(t) \right], \end{aligned} \quad (65)$$

$$\begin{aligned} \mathcal{V}(\tilde{x}(0))e^{\beta t} &= \tilde{x}'(0) R^{1/2} Q R^{1/2} \tilde{x}(0) e^{\beta t} \\ &\leq \lambda_{\max}(Q) \tilde{x}'(0) R \tilde{x}(0) e^{\beta t} \\ &\leq \lambda_{\max}(Q) d_1 e^{\beta T}, \end{aligned} \quad (66)$$

$$\gamma^2 e^{\beta t} \int_0^t E v'(s) v(s) ds < e^{\beta T} h \gamma^2. \quad (67)$$

From (64), (65), (66), and (67), it is easy to obtain that

$$E \left[\tilde{x}'(t) R \tilde{x}(t) \right] < \frac{\lambda_{\max}(Q) d_1 e^{\beta T} + h \gamma^2 e^{\beta T}}{\lambda_{\min}(Q)}. \quad (68)$$

According to condition (57), it follows that (68) leads to $E[\tilde{x}'(t) R \tilde{x}(t)] < d_2$ for all $t \in [0, T]$. So, the closed-loop system (50) is FT stochastically bounded wrt (d_1, d_2, T, R, h) .

Step 2. Prove that the condition (ii) is satisfied.

Under the condition of $v(t) = 0$, the infinitesimal operator $\mathcal{L}_4 \mathcal{V}(\tilde{x}(t))$ of system (50) is defined as follows:

$$\begin{aligned} \mathcal{L}_4 \mathcal{V}(\tilde{x}(t)) &= [\bar{A}_1 \tilde{x}(t)]' \tilde{Q} \tilde{x}(t) + \tilde{x}'(t) \tilde{Q} [\bar{A}_1 \tilde{x}(t)] \\ &\quad + [\bar{A}_2 \tilde{x}(t)]' \tilde{Q} [\bar{A}_2 \tilde{x}(t)] \\ &= \tilde{x}'(t) [\bar{A}_1' \tilde{Q} + \tilde{Q} \bar{A}_1 + \bar{A}_2' \tilde{Q} \bar{A}_2] \tilde{x}(t). \end{aligned} \quad (69)$$

According to (56), we get

$$\mathcal{L}_4 \mathcal{V}(\tilde{x}(t)) - \beta \mathcal{V}(\tilde{x}(t)) + \tilde{x}'(t) \mathcal{T} \tilde{x}(t) < 0. \quad (70)$$

Integrating (70) from 0 to t and taking the expectation, the following results are obtained:

$$\begin{aligned} E\mathcal{V}(\tilde{x}(t)) &+ E \int_0^t \tilde{x}'(s) \mathcal{T} \tilde{x}(s) ds \\ &< \mathcal{V}(\tilde{x}(0)) + \beta E \int_0^t \mathcal{V}(\tilde{x}(s)) ds. \end{aligned} \quad (71)$$

From (71), we have

$$E\mathcal{V}(\tilde{x}(t)) < \mathcal{V}(\tilde{x}(0)) + \beta E \int_0^t \mathcal{V}(\tilde{x}(s)) ds, \quad (72)$$

$$E \int_0^t \tilde{x}'(s) \mathcal{T} \tilde{x}(s) ds < \mathcal{V}(\tilde{x}(0)) + \beta E \int_0^t \mathcal{V}(\tilde{x}(s)) ds. \quad (73)$$

From (72), by Lemma 1, we obtain

$$E\mathcal{V}(\tilde{x}(t)) < \mathcal{V}(\tilde{x}(0))e^{\beta t}. \quad (74)$$

From (73) and (74), we have

$$\begin{aligned} J_o(\tilde{x}(t)) &< \beta E \int_0^t \mathcal{V}(\tilde{x}(s)) ds + \mathcal{V}(\tilde{x}(0)) \\ &< \beta E \int_0^t \mathcal{V}(\tilde{x}(0)) e^{\beta s} ds + \mathcal{V}(\tilde{x}(0)) \\ &= \mathcal{V}(\tilde{x}(0)) e^{\beta t} < \lambda_{\max}(Q) d_1 e^{\beta T} = J_o^*. \end{aligned} \quad (75)$$

Step 3. Prove that the condition (iii) is satisfied.

According to the Schur complement, it can be seen that (55) is equivalent to

$$\begin{bmatrix} \bar{A}_1' \tilde{Q} + \tilde{Q} \bar{A}_1 - \beta \tilde{Q} + T_1' T_1 + \bar{A}_2' \tilde{Q} \bar{A}_2 & \tilde{Q} W_1 + T_1' D_2 + \bar{A}_2' \tilde{Q} W_2 \\ * & W_2' \tilde{Q} W_2 + D_2' D_2 - \gamma^2 I \end{bmatrix} < 0. \quad (76)$$

According to (58) and (76), we get

$$\mathcal{L}_3 \mathcal{V}(\tilde{x}(t)) < \beta \mathcal{V}(\tilde{x}(t)) + \gamma^2 v'(t) v(t) - \tilde{z}'(t) \tilde{z}(t). \quad (77)$$

Using the similar procedure to prove Step 3 in Theorem 3, we can obtain

$$E \int_0^T \tilde{z}'(s) \tilde{z}(s) ds < \gamma^2 E \int_0^T v'(s) v(s) ds. \quad (78)$$

This completes the proof.

We find that inequalities (55), (56), and (57) in Theorem 3 are not easy to solve, so the following theorem will transform (55), (56), and (57) into LMIs.

Theorem 4. For the given positive scalars γ , d_1 , d_2 , T , R , and h , if there exist a scalar $\beta > 0$, two matrices $\tilde{Q}_{11} > 0$ and $\tilde{Q}_{22} > 0$, a matrix N , and a scalar $\eta > 0$ such that

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ * & \Sigma_{22} & \Sigma_{23} \\ * & * & \Sigma_{33} \end{bmatrix} < 0, \quad (79)$$

$$\begin{bmatrix} \mathcal{E}_{11} - \beta \tilde{Q}_{11} & \mathcal{E}_{12} \\ * & \mathcal{E}_{22} - \beta \tilde{Q}_{22} \end{bmatrix} < 0, \quad (80)$$

$$e^{\beta T} (\eta d_1 + h\gamma^2) - d_2 < 0, \quad (81)$$

$$I < \text{diag} \{ \tilde{Q}_{11}, \tilde{Q}_{22} \} < \eta I, \quad (82)$$

hold, where $\Sigma_{11} = A_1' \tilde{Q}_{11} + \tilde{Q}_{11} A_1 + A_2' \tilde{Q}_{11} A_2 + C_2' C_2 - \beta \tilde{Q}_{11}$, $\Sigma_{12} = C_1' N' + \tilde{Q}_{11} B_1 K + A_2' \tilde{Q}_{11} B_2 K + C_2' D_1 K$, $\Sigma_{22} = (A_1 + B_1 K)' \tilde{Q}_{22} + \tilde{Q}_{22} (A_1 + B_1 K) + K' B_2' \tilde{Q}_{11} B_2 K - N C_1 - C_1' N' + K' D_1' D_1 K - \beta \tilde{Q}_{22}$, $\Sigma_{13} = \tilde{Q}_{11} G_1 + C_2' D_2 + A_2' \tilde{Q}_{11} G_2$, $\Sigma_{23} = K' D_1' D_2 + K' B_2' \tilde{Q}_{11} G_2$, $\Sigma_{33} = G_2' \tilde{Q}_{11} G_2 + D_2' D_2 - \gamma^2 I$, $\mathcal{E}_{11} = A_1' \tilde{Q}_{11} + \tilde{Q}_{11} A_1 + A_2' \tilde{Q}_{11} A_2 + P_1$, $\mathcal{E}_{12} = C_1' N' + \tilde{Q}_{11} B_1 K + A_2' \tilde{Q}_{11} B_2 K$, and $\mathcal{E}_{22} = (A_1 + B_1 K)' \tilde{Q}_{22} + \tilde{Q}_{22} (A_1 + B_1 K) + K' B_2' \tilde{Q}_{11} B_2 K - N C_1 - C_1' N' + K' P_2 K$, then (49) is an OBFT H_2/H_∞ controller and the upper bound of the H_2 index can be given as $J_o^* = \eta d_1 e^{\beta T}$. In this case, an appropriate estimator gain matrix is given by $L = \tilde{Q}_{22}^{-1} N$.

Proof. Setting $\tilde{Q} = \text{diag} \{ \tilde{Q}_{11}, \tilde{Q}_{22} \}$, substituting (52) and (53) into (55), and letting $N = \tilde{Q}_{22} L$, (79) can be obtained from (55). It is obvious that (80) can be obtained from (56). And then, (81) and (82) imply (57). The proof is completed.

5. Algorithm

In this section, an optimization algorithm on the FT H_2/H_∞ control is presented. The following algorithm is given for Theorem 2. Similarly, the algorithm can be utilized in Theorem 4.

Analysis. By Theorem 2, let $J_s^* < \xi$, and then minimize ξ , subject to (45), (46), (47), and (48), $M > 0$ and

$$\frac{M e^{-\alpha T}}{d_1} - \xi^{-1} I > 0, \quad (83)$$

where $0 < d_1 < d_2$, $T > 0$, $\alpha \geq 0$, and $\lambda > 0$.

By analyzing (45), (46), (47), and (48) in Theorem 2, we find the following: if (45), (46), (47), and (48) have no feasible solutions at the case of $\alpha = 0$, then (45), (46), (47), and (48) will have no feasible solutions for all $\alpha > 0$. Therefore, we seek α that makes (45), (46), (47), and (48) have feasible solutions from $\alpha = 0$ by linear searching. The detailed algorithm will be given as follows. One can use this algorithm to obtain the minimum value of ξ and γ^2 .

By running Algorithm 1, we get the following results:

- (1) If $U(i) = \text{empty}$, $F(i) = \text{empty}$, and $H(i) = \text{empty}$, then it shows that the above optimization problem (OP) is not solvable.
- (2) If $U(i) \neq \text{empty}$, $F(i) \neq \text{empty}$, and $H(i) \neq \text{empty}$, then it shows that the above optimization problem is solvable. Moreover, we can respectively plot the curves of $(U(i), F(i))$, $(U(i), H(i))$, and $(F(i),$

$H(i))$. The three curves can help us choose a suitable finite-time H_2/H_∞ controller.

- (3) This algorithm can find the most suitable α to decrease the conservativeness of the conditions of Theorem 2.

6. Examples

This section provides an example to show the effectiveness of the obtained results. System (10) with parameters is as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} -10.1 & -10.8 \\ 10.7 & -10.2 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -0.8 & 1 \\ 0.4 & -1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} -2.1 & 5 \\ -4.1 & 3 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} -0.8 & 1.2 \\ -1 & 1.5 \end{bmatrix}, \\ G_1 &= \begin{bmatrix} -0.5 & 0.4 \\ -0.1 & -0.5 \end{bmatrix}, \\ G_2 &= \begin{bmatrix} 0.2 & -0.1 \\ 0.4 & 0.5 \end{bmatrix}, \\ P_1 &= \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} -0.8 & -1.5 \\ 1.1 & -3 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} -1.9 & -0.1 \\ 1.5 & -2.5 \end{bmatrix}, \\ x(0) &= [-0.7 \quad 0.7]'. \end{aligned} \quad (84)$$

and $d_1 = 1$, $d_2 = 5$, $D_1 = 2$, $D_2 = 0.2$, $T = 1$, $R = I$, and $h = 0.5$.

6.1. SFFT H_2/H_∞ Controller Design. By running Algorithm 1, the relations of α and ξ (Figure 1), α and γ (Figure 2), and γ and ξ (Figure 3) are obtained. From Figure 1, it can be seen that the bigger the number of α , the bigger ξ is. Also, the minimum value of the H_2 performance index is $\xi = 1$ when $\alpha = 0$, and the maximum value of the H_2 performance index is $\xi = 4.4817$ when $\alpha = 1.5$. The range of α is $[0, 1.5]$, and the range of ξ is $[1, 4.4817]$.

From Figure 2, it can be seen that the bigger the number of α , the smaller γ is. Also, the maximum value of γ is

Step 1: Given d_1, d_2, T, R, h , and a step size d_α for α .

Step 2: Set $i = 1$ and take $\alpha_i = 0$.

Step 3: If α_i makes the following optimization problem

$$\begin{aligned} & \min_{\xi} \\ & \text{s.t. (45), (46), (47), (48), (83), } M > 0 \\ & \min_{\gamma^2} \\ & \text{s.t. (45), (46), (47), (48), (83), } M > 0 \end{aligned}$$

solvable, then store α_i into $U(i)$, $\xi_{\min}$ into $F(i)$, and $\gamma_{\min}$ into $H(i)$ and let $\alpha_{i+1} = \alpha_i + d_\alpha$, loop.

Otherwise, go to Step 4.

Step 4: Exit.

ALGORITHM 1

FIGURE 1: ξ versus α .

FIGURE 3: γ versus ξ .

FIGURE 2: γ versus α .

0.6757 when $\alpha = 0$, and the minimum value of γ is 0.6734 when $\alpha = 1.5$. The range of α is $[0, 1.5]$ and the range of γ is $[1, 4.4817]$.

In fact, Figures 1 and 2 imply Figure 3. To show the relations between γ and ξ clearly, Figure 3 is also drawn. From Figure 3, it can be seen that the bigger the number of ξ , the

smaller γ is. Figures 1, 2, and 3 can provide us a method for how to select the suitable SFFT H_2/H_∞ controller. If we are mainly concerned with the H_2 performance, a smaller α can be selected. If we are mainly concerned with the H_∞ performance, a bigger α can be selected.

Next, substituting $\alpha = 0$ into Theorem 2 and then solving (45), (46), (47), and (48), we have

$$\begin{aligned} M &= \begin{bmatrix} 0.6528 & 0.0810 \\ 0.0810 & 0.6935 \end{bmatrix}, \\ Y &= \begin{bmatrix} 0.0746 & -0.0635 \\ 0.5236 & 0.3750 \end{bmatrix}, \\ \lambda &= 0.3980. \end{aligned} \quad (85)$$

Therefore, we get the following controller gain matrix:

$$K = YM^{-1} = \begin{bmatrix} 0.1275 & -0.1064 \\ 0.7460 & 0.4536 \end{bmatrix}. \quad (86)$$

Suppose $v(t) = \sin t (\int_0^1 \sin^2 t dt = 0.275 < h = 0.5)$ uses the Euler-Maruyama method to simulate the standard Brownian motion, one can obtain the simulated curves

FIGURE 4: The response for $E[x'(t)Rx(t)]$. FIGURE 5: ξ versus β .

of $x'(t)Rx(t)$ (30 curves) and $E[x'(t)Rx(t)]$ of system (10) in Figure 4. From Figure 4, we see that system (10) is FT stochastically bounded wrt $(1, 5, 1, I, 0.5)$.

6.2. OBFT H_2/H_∞ Controller Design. On the basis of the SF case, Algorithm 1 can also be used to solve the similar OP for the OBFT H_2/H_∞ control. The relations of β and ξ (Figure 5), β and γ (Figure 6), and γ and ξ (Figure 7) are obtained. From Figure 5, it can be seen that the bigger the number of β , the bigger ξ is. Also, the minimum value of the H_2 performance index is $\xi = 1$ when $\beta = 0$, and the maximum value of the H_2 performance index is $\xi = 4.0552$ when $\beta = 1.4$. The range of α is $[0, 1.4]$, and the range of ξ is $[1, 4.0552]$.

From Figure 6, it can be seen that the bigger the number of β , the smaller γ is. Also, the maximum value of γ is 0.8656 when $\beta = 0$, and the minimum value of γ is 0.8413 when $\beta = 1.4$. The range of β is $[0, 1.4]$, and the range of γ is $[0.8413, 0.8656]$.

In fact, Figures 5 and 6 also imply Figure 7. To show the relations between γ and ξ clearly, Figure 7 is also drawn. From Figure 7, it can be seen that the bigger the number of

FIGURE 6: γ versus β . FIGURE 7: γ versus ξ .

ξ , the smaller γ is. Figures 5–7 can provide us a method for how to select the suitable OBFT H_2/H_∞ controller. If we are mainly concerned with the H_2 performance, a smaller β can be selected. If we are mainly concerned with the H_∞ performance, a bigger β can be selected.

Substituting $\beta = 0$ into Theorem 4 and solving (79), (80), (81), and (82) yields

$$\begin{aligned} Q_{11} &= \begin{bmatrix} 1.9421 & -0.0442 \\ -0.0442 & 2.2252 \end{bmatrix}, \\ Q_{22} &= \begin{bmatrix} 3.1913 & 0.0900 \\ 0.0900 & 2.2214 \end{bmatrix}, \\ N &= \begin{bmatrix} 4.2749 & -2.9218 \\ 1.6587 & 1.1962 \end{bmatrix}, \\ \eta &= 3.6737 \end{aligned} \quad (87)$$

FIGURE 8: The response for $E[\tilde{x}'(t)R\tilde{x}(t)]$.

Then, the corresponding observer gain matrix is obtained:

$$L = Q_{22}^{-1}N = \begin{bmatrix} 1.3200 & -0.9318 \\ 0.6932 & 0.5762 \end{bmatrix}. \quad (88)$$

Suppose $v(t) = \sin t (\int_0^1 \sin^2 t dt = 0.275 < h)$ and uses the Euler-Maruyama method to simulate the standard Brownian motion, one can obtain the simulated curves of $\tilde{x}'(t)R\tilde{x}(t)$ (30 curves) and $E[\tilde{x}'(t)R\tilde{x}(t)]$ of system (50) in Figure 8. From Figure 8, we see that system (10) is FT stochastically bounded wrt $(1, 5, 1, I, 0.5)$.

7. Conclusions

In this study, state-feedback and observer-based finite-time H_2/H_∞ control problems have been investigated. The problems have been transformed into optimization problems with the constraint of matrix inequalities, and a detailed optimization algorithm has been given to obtain the minimum values of the H_2 index and H_∞ index. An example has been given to demonstrate the results obtained.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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