Hindawi
Complexity
Volume 2018, Article ID 1984348, 8 pages
https://doi.org/10.1155/2018/1984348

Research Article

The Robust Control and Synchronization of a Class of Fractional-Order Chaotic Systems with External Disturbances via a Single Output

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Received 11 July 2018; Revised 19 October 2018; Accepted 28 October 2018; Published 8 November 2018

Academic Editor: Basil M. Al-Hadithi

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This paper investigates the stabilization and synchronization of a class of fractional-order chaotic systems which are affected by external disturbances. The chaotic systems are assumed that only a single output can be used to design the controller. In order to design the proper controller, some observer systems are proposed. By using the observer systems some sufficient conditions for achieving chaos control and synchronization of fractional-order chaotic systems are derived. Numerical examples are presented by taking the fractional-order generalized Lorenz chaotic system as an example to show the feasibility and validity of the proposed method.

1. Introduction

Chaos control and synchronization have attracted a great deal of attention since the innovative works proposed by Huber, Pecora, and Carroll in 1990 [1]. Nowadays, owing to their potential applications in many areas such as in chemical reactions, power converters, information processing, and secure communications, various types of synchronization phenomena have been discovered, such as complete synchronization [2], combination synchronization [3], and equal combination synchronization [4].

Fractional-order calculus is a branch of mathematics that deals with derivatives and integrals of non-integer orders. It has been shown that the models presented by fractional-order systems are more adequate than that described by integer order systems. Many systems such as viscoelastic systems, dielectric polarization, and electromagnetic waves [5] are known to display fractional-order dynamics. In recent years, a number of fractional-order chaotic systems have been investigated, such as the fractional-order economical system [6] and the fractional-order Lorenz system [7].

Similarly to integer order chaotic systems, the control and synchronization of fractional-order chaotic systems has become an active research field [8–15]. It is not difficult to see that in papers [8–15] the authors have used all state variables to design controllers. However, in the real situation it is well known that only part of the variables can be used in many nonlinear systems. Therefore, it is necessary to investigate the control and synchronization of fractional-order chaotic system with a single output.

Motivated by the above discussion, in this paper we consider the stabilization and synchronization of a class of fractional-order chaotic systems via a single output. Some sufficient conditions for achieving chaos control and synchronization of fractional-order chaotic systems are derived via the observer systems. The fractional-order generalized Lorenz chaotic system is taken as an example to show the feasibility and validity of the proposed method.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries, including some definitions, lemmas, and the general form description of a class of fractional-order chaotic systems. The control and synchronization schemes of a class of fractional-order chaotic systems via a single output are presented in Sections 3 and 4, respectively. In Section 5, numerical simulations results are shown. Some conclusions are drawn in Section 6.

2. Preliminaries and System Description

2.1. Fractional-Order Integral and Derivative. In this subsection, some basic definitions with respect to Caputo's fractional derivative are introduced. Also, some useful lemmas are proposed.

Definition 1 (see [16]). Caputo's fractional derivative of function $f(t) \in C^n([t_0, \infty), R)$ is defined by

$$D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau, \quad (1)$$

where $0 < \alpha \leq 1$.

Definition 2 (see [16]). The Mittag-Leffler function with two parameters is given by

$$E_{\alpha,\beta}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(p\alpha + \beta)}, \quad (2)$$

where $\alpha > 0, \beta > 0$ and $z \in C$.

If $\beta = 1$, then $E_{\alpha,1}(z) = E_{\alpha}(z) = \sum_{p=0}^{\infty}(z^p/\Gamma(p\alpha + 1))$.

Let $\mathscr{L}(f(t))$ denote the Laplace transform of a function $f(t)$. Based on the definition of Laplace transform: $F(s) = \mathscr{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt$, we have

$$\mathscr{L}\left(D_0^{\alpha} f(t)\right) = s^{\alpha} F(s) - s^{\alpha-1} f(0), \quad (0 < \alpha \leq 1). \quad (3)$$

The Laplace transform of Mittag-Leffler function with two parameters is

$$\mathscr{L}\left(t^{\beta-1} E_{\alpha,\beta}(-\lambda t^{\alpha})\right) = \frac{s^{\alpha-\beta}}{s^{\alpha} + \lambda}, \quad \left(Re(s) > |\lambda|^{1/\alpha}\right). \quad (4)$$

Lemma 3 (see [17]). *If $\lambda > 0$, then the origin of system*

$$D^{\alpha} x(t) = -\lambda x(t), \quad 0 < \alpha \leq 1 \quad (5)$$

is globally asymptotically stable.

Lemma 4 (see [18]). *If $\lambda > 0$ and $\lim_{t \to \infty} \eta(t) = 0$, then the origin of system*

$$D^{\alpha} x(t) = -\lambda x(t) + \eta(t), \quad 0 < \alpha < 1 \quad (6)$$

is globally asymptotically stable.

If $x(t) \geq 0$, then based on Lemma 4 we can easily get the following Corollary 5.

Corollary 5. *Suppose $x(t) \geq 0$. If $\lambda > 0$ and $\lim_{t \to \infty} \eta(t) = 0$, then the origin of system*

$$D^{\alpha} x(t) \leq -\lambda x(t) + \eta(t), \quad 0 < \alpha < 1 \quad (7)$$

is globally asymptotically stable.

Lemma 6. *Consider the following two systems:*

$$D^{\alpha} x = a(t) x + b(t), \quad (0 < \alpha < 1). \quad (8)$$

and

$$D^{\alpha} y = \overline{a} y + B(t), \quad (0 < \alpha < 1). \quad (9)$$

where $x \in R, y \in R, a(t) \in R, b(t) \in R$ and $a(t) \leq \overline{a} < 0, |b(t)| \leq B(t)$, and then we have

$$|x| \leq y + E_{\alpha}\left(\overline{a} t^{\alpha}\right)\left(|x(0)| - y(0)\right), \quad (10)$$

where $x(0)$, $y(0)$ are initial values of x and y, respectively.

Proof. Based on system (8), we obtain

$$\begin{aligned} D^{\alpha}|x| &= D^{\alpha}\left(x \operatorname{sign}(x)\right) = \operatorname{sign}(x) D^{\alpha}(x) \\ &= \operatorname{sign}(x)\left(a(t) x + b(t)\right) \leq \overline{a}|x| + B(t). \end{aligned} \quad (11)$$

The Laplace transform of (11) gives

$$s^{\alpha}|x(s)| - s^{\alpha-1}|x(0)| \leq \overline{a}|x(s)| + B(s), \quad (s \in R). \quad (12)$$

The Laplace transform of (9) is

$$s^{\alpha} y(s) - s^{\alpha-1} y(0) = \overline{a} y(s) + B(s). \quad (13)$$

Subtracting (13) from (12) one has

$$\begin{aligned} &s^{\alpha}\left(|x(s)| - y(s)\right) - s^{\alpha-1}\left(|x(0)| - y(0)\right) \\ &\quad \leq \overline{a}\left(|x(s)| - y(s)\right). \end{aligned} \quad (14)$$

The above inequality is equivalent to

$$|x(s)| - y(s) \leq \left(s^{\alpha} - \overline{a}\right)^{-1}\left(|x(0)| - y(0)\right) s^{\alpha-1}. \quad (15)$$

Taking the inverse Laplace transform of (15) yields

$$|x(t)| - y(t) \leq \left(|x(0)| - y(0)\right) E_{\alpha}\left(\overline{a} t^{\alpha}\right). \quad (16)$$

It implies that

$$|x(t)| \leq y(t) + \left(|x(0)| - y(0)\right) E_{\alpha}\left(\overline{a} t^{\alpha}\right). \quad (17)$$

This concludes the proof of Lemma 6. □

Remark 7. System (9) is called as the observer of system (8).

2.2. System Description. In this paper we consider the class of chaotic systems which are described by

$$\begin{aligned} D^{\alpha} x_1 &= a(t) x_1 + a_1 x_2 + a_2 x_2^2, \\ D^{\alpha} x_2 &= b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_1 x_3 + d(t), \\ D^{\alpha} x_3 &= c(t) x_3 + c_1 x_1 x_2 + c_2 x_2^2, \\ x_{out} &= x_2, \end{aligned} \quad (18)$$

where $x = (x_1, x_2, x_3)^T \in R^{3\times 1}$ is the state vector of system (18) and $\alpha \in (0,1)$ is the order of fractional derivatives. $a(t)(< 0), c(t)(< 0)$ and a_j, c_j, b_i $(i = 1, \cdots, 4, j = 1, 2)$ are system's parameters. $d(t)$ represents the model uncertainty or the external disturbance. x_{out} is the measured output signal which can be used to design the controller.

3. The Control Scheme

In this section, the stabilization of system (18) is investigated. In order to force the states of system (18) to its origin, the control input u is added to the second state equation. Thus, system (18) can be rewritten as

$$\begin{aligned} D^{\alpha} x_1 &= a(t) x_1 + a_1 x_2 + a_2 x_2^2, \\ D^{\alpha} x_2 &= b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_1 x_3 + d(t) + u, \\ D^{\alpha} x_3 &= c(t) x_3 + c_1 x_1 x_2 + c_2 x_2^2, \\ x_{out} &= x_2, \end{aligned} \tag{19}$$

where u is a controller to be designed later.

Now, some Assumptions are introduced.

Assumption 8. Suppose $d(t)$ is bounded which means that there exists constants $\delta > 0$ such that $|d(t)| \le \delta$.

Assumption 9. Suppose there exist two constants $\overline{a}, \overline{c}$ such that $a(t) \le \overline{a} < 0, c(t) \le \overline{c} < 0$.

Since $a_1 x_2 + a_2 x_2^2 \le |a_1 x_2| + |a_2 x_2^2|$ and $a(t) \le \overline{a}$, by Lemma 6 the observer of system

$$D^{\alpha} x_1 = a(t) x_1 + a_1 x_2 + a_2 x_2^2 \tag{20}$$

is

$$D^{\alpha} \widehat{x}_1 = \overline{a} \widehat{x}_1 + |a_1 x_2| + |a_2 x_2^2|. \tag{21}$$

Furthermore, by Lemma 6 we have

$$|x_1| \le \widehat{x}_1 + (|x_1(0)| - \widehat{x}_1(0)) E_{\alpha}(\overline{a} t^{\alpha}). \tag{22}$$

Thus, we get

$$\begin{aligned} c_1 x_1 x_2 &\le |c_1| |x_1| |x_2| \\ &\le |c_1| |x_2| (\widehat{x}_1 + E_{\alpha}(\overline{a} t^{\alpha}) (|x_1(0)| - \widehat{x}_1(0))). \end{aligned} \tag{23}$$

In view of $c_1 x_1 x_2 + c_2 x_2^2 \le |c_1||x_1||x_2| + |c_2 x_2^2| \le |c_1||x_2|(\widehat{x}_1 + E_{\alpha}(\overline{a} t^{\alpha})(|x_1(0)| - \widehat{x}_1(0))) + |c_2 x_2^2|$ and $c(t) \le \overline{c}$, then by Lemma 6 we know the observer of system

$$D^{\alpha} x_3 = c(t) x_3 + c_1 x_1 x_2 + c_2 x_2^2 \tag{24}$$

is

$$\begin{aligned} D^{\alpha} \widehat{x}_3 &= \overline{c} \widehat{x}_3 \\ &\quad + |c_1| |x_2| (\widehat{x}_1 + E_{\alpha}(\overline{a} t^{\alpha}) (|x_1(0)| - \widehat{x}_1(0))) \\ &\quad + |c_2 x_2^2|. \end{aligned} \tag{25}$$

Thus, in order to design proper controller u, we can propose the following observer system for variables x_1 and x_3:

$$\begin{aligned} D^{\alpha} \widehat{x}_1 &= \overline{a} \widehat{x}_1 + |a_1 x_2| + |a_2 x_2^2|, \\ D^{\alpha} \widehat{x}_3 &= \overline{c} \widehat{x}_3 \\ &\quad + |c_1| |x_2| (\widehat{x}_1 + E_{\alpha}(\overline{a} t^{\alpha}) (|x_1(0)| - \widehat{x}_1(0))) \\ &\quad + |c_2 x_2^2|, \end{aligned} \tag{26}$$

where $\widehat{x}_1$ and $\widehat{x}_3$ are the estimated values x_1 and x_3, respectively.

Theorem 10. *Suppose Assumptions 8 and 9 are satisfied. For system (19), if we choose u such that*

$$\begin{aligned} u = &-(|b_1| \widehat{x}_1 + |b_3| \widehat{x}_3 + |b_4| \widehat{x}_1 \widehat{x}_3 + \delta + k|x_2| + 1) \\ &\cdot \operatorname{sign}(x_2) - b_2 x_2, \end{aligned} \tag{27}$$

then the origin of system (19) is stable in the sense of $\lim_{t\to\infty} x_1 = \lim_{t\to\infty} x_2 = \lim_{t\to\infty} x_3 = 0$, where $\widehat{x}_1, \widehat{x}_3$ are defined by (26) and $k > 0$.

Proof. The proof of Theorem 10 is divided into two steps. In the first step, we shall show that $\lim_{t\to\infty} x_2 = 0$.

To this purpose, let us consider the following Lyapunov function candidate:

$$V = \frac{1}{2} x_2^2. \tag{28}$$

The time derivative of (28) is

$$\begin{aligned} D^{\alpha} V &\le x_2 D^{\alpha} x_2 \\ &= x_2 (b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_1 x_3 + d(t) + u) \\ &\le (|b_1| |x_1| + |b_3| |x_3| + |b_4| |x_1| |x_3| + |d(t)|) |x_2| \\ &\quad + (b_2 x_2 + u) x_2. \end{aligned} \tag{29}$$

By Lemma 6, we obtain

$$\begin{aligned} |x_1| &\le \widehat{x}_1 + \rho_{1x}, \\ |x_3| &\le \widehat{x}_3 + \rho_{3x}, \end{aligned} \tag{30}$$

where $\rho_{1x} = E_{\alpha}(\overline{a} t^{\alpha})(|x_1(0)| - \widehat{x}_1(0)), \rho_{3x} = E_{\alpha}(\overline{c} t^{\alpha})(|x_3(0)| - \widehat{x}_3(0))$.

Substituting inequality (30) into (29), we have

$$\begin{aligned} D^{\alpha} V &\le (|b_1| (\widehat{x}_1 + \rho_{1x}) + |b_3| (\widehat{x}_3 + \rho_{3x}) \\ &\quad + |b_4| (\widehat{x}_1 + \rho_{1x}) (\widehat{x}_3 + \rho_{3x}) + |d(t)|) |x_2| + (b_2 x_2 \\ &\quad + u) x_2 = (|b_1| \widehat{x}_1 + |b_3| \widehat{x}_3 + |b_4| \widehat{x}_1 \widehat{x}_3 + |d(t)|) |x_2| \\ &\quad + (b_2 x_2 + u) x_2 + (|b_1| \rho_{1x} + |b_3| \rho_{3x} + |b_4| \rho_{1x} \widehat{x}_3 \\ &\quad + |b_4| \rho_{3x} \widehat{x}_1 + |b_4| \rho_{1x} \rho_{3x}) |x_2|. \end{aligned} \tag{31}$$

By using (27), we have

$$\begin{aligned} D^{\alpha} V \le &-(1 - |b_1| \rho_{1x} - |b_3| \rho_{3x} - |b_4| \rho_{1x} \widehat{x}_3 \\ &- |b_4| \rho_{3x} \widehat{x}_1 - |b_4| \rho_{1x} \rho_{3x} + k|x_2|) |x_2|. \end{aligned} \tag{32}$$

Note that $\lim_{t\to\infty} \rho_{1x} = \lim_{t\to\infty} \rho_{3x} = 0$. Therefore, we have $\lim_{t\to\infty}(|b_1|\rho_{1x} + |b_3|\rho_{3x} + |b_4|\rho_{1x}\widehat{x}_3 + |b_4|\rho_{3x}\widehat{x}_1 + |b_4|\rho_{1x}\rho_{3x}) = 0$. Thus, there exists a finite time t_0 such that when $t \ge t_0$ we get $|b_1|\rho_{1x} + |b_3|\rho_{3x} + |b_4|\rho_{1x}\widehat{x}_3 + |b_4|\rho_{3x}\widehat{x}_1 + |b_4|\rho_{1x}\rho_{3x} - 1 < 0$. Thus, when $t \ge t_0$ we obtain

$$D^{\alpha} V \le -k |x_2|^2 = -2kV. \tag{33}$$

It should be noted that $V \geq 0$ and $k > 0$. Thus, based on Lemma 3 we have $\lim_{t\longrightarrow\infty} x_2 = 0$.

Now, in the second step we prove that $\lim_{t\longrightarrow\infty} x_1 = \lim_{t\longrightarrow\infty} x_3 = 0$. By using the results obtained in the proof of step 1, it is obvious that $\lim_{t\longrightarrow\infty}(a_1 x_2 + a_2 x_2^2) = 0$. From the first equation of system (19) we derive that

$$D^{\alpha} x_1 = a(t) x_1 + a_1 x_2 + a_2 x_2^2. \tag{34}$$

Since $a(t) < 0$, according to Lemma 4 we know that $\lim_{t\longrightarrow\infty} x_1 = 0$. In the same way we have $\lim_{t\longrightarrow\infty} x_3 = 0$. This completes the proof of Theorem 10. □

Remark 11. The constant 1 in controller (27) is designed to eliminate the affect caused by $|b_1|\rho_{1x} + |b_3|\rho_{3x} + |b_4|\rho_{1x}\hat{x}_3 + |b_4|\rho_{3x}\hat{x}_1 + |b_4|\rho_{1x}\rho_{3x}$. Note that $\lim_{t\longrightarrow\infty}(|b_1|\rho_{1x} + |b_3|\rho_{3x} + |b_4|\rho_{1x}\hat{x}_3 + |b_4|\rho_{3x}\hat{x}_1 + |b_4|\rho_{1x}\rho_{3x}) = 0$, and therefore the constant 1 can be replaced by any positive number. Since $D^{\alpha}V \leq -2kV$, thus the factor k in controller (27) determines the speed of convergence. In general, the larger the number k the faster the rate of the convergence.

4. The Synchronization Scheme

In this section the synchronization scheme of a class of fractional-order chaotic systems is presented via the observer based method.

Suppose system (18) is the drive system; in order to synchronize system (18) the corresponding response system with controller u is constructed as

$$\begin{aligned} D^{\alpha} y_1 &= a(t) y_1 + a_1 y_2 + a_2 y_2^2, \\ D^{\alpha} y_2 &= b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_1 y_3 + d_1(t) + u, \\ D^{\alpha} y_3 &= c(t) y_3 + c_1 y_1 y_2 + c_2 y_2^2, \\ y_{out} &= y_2, \end{aligned} \tag{35}$$

where $y = (y_1, y_2, y_3)^T \in R^{3\times 1}$ is the state vector of system (35), $\alpha \in (0,1)$ is the order of fractional derivatives. $d_1(t)$ represents the model uncertainty or the external disturbance. y_{out} is the measured output signal which can be used to design the controller. u is the controller to be designed later.

Let us define the synchronization error as $e = y - x$, then the dynamics of synchronization error between systems (18) and (35) can be described by

$$\begin{aligned} D^{\alpha} e_1 &= a(t) e_1 + a_1 y_2 + a_2 y_2^2 - \left(a_1 x_2 + a_2 x_2^2\right), \\ D^{\alpha} e_2 &= b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_1 y_3 + d_1(t) \\ &\quad - \left(b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_1 x_3 + d(t)\right) + u, \\ D^{\alpha} e_3 &= c(t) e_3 + c_1 y_1 y_2 + c_2 y_2^2 - \left(c_1 x_1 x_2 + c_2 x_2^2\right), \end{aligned} \tag{36}$$

Assumption 12. Suppose $d(t), d_1(t)$ are all bounded which means that there exists constants $\delta > 0$ such that $|d_1(t) - d(t)| \leq \delta$.

The objective of the current synchronization problem is to design an appropriate control signal u such that, for any initial conditions of the drive and response systems, the synchronization errors converge to zero. For this end, similar to system (26) we proposed the following observer for variables x_1, x_3 and y_1, y_3:

$$\begin{aligned} D^{\alpha} \hat{x}_1 &= \overline{a}\hat{x}_1 + |a_1 x_2| + \left|a_2 x_2^2\right|, \\ D^{\alpha} \hat{x}_3 &= \overline{c}\hat{x}_3 \\ &\quad + |c_1| |x_2| \left(\hat{x}_1 + E_{\alpha}\left(\overline{a}t^{\alpha}\right)\left(|x_1(0)| - \hat{x}_1(0)\right)\right) \\ &\quad + \left|c_2 x_2^2\right|, \\ D^{\alpha} \hat{y}_1 &= \overline{a}\hat{y}_1 + |a_1 y_2| + \left|a_2 y_2^2\right|, \\ D^{\alpha} \hat{y}_3 &= \overline{c}\hat{y}_3 \\ &\quad + |c_1| |y_2| \left(\hat{y}_1 + E_{\alpha}\left(\overline{a}t^{\alpha}\right)\left(|y_1(0)| - \hat{y}_1(0)\right)\right) \\ &\quad + \left|c_2 y_2^2\right|, \end{aligned} \tag{37}$$

where $\hat{x}_1, \hat{x}_3, \hat{y}_1, \hat{y}_3$ are the estimated values x_1, x_3 and y_1, y_3, respectively.

Theorem 13. *Suppose Assumptions 9 and 12 are satisfied. If we choose u such that*

$$\begin{aligned} u = &-\left(|b_1|\left(\hat{y}_1 + \hat{x}_1\right) + |b_3|\left(\hat{y}_3 + \hat{x}_3\right)\right. \\ &\left. + |b_4|\left(\hat{y}_1\hat{y}_3 + \hat{x}_1\hat{x}_3\right) + \delta + k|e_2| + 1\right)\operatorname{sign}(e_2) \\ &- b_2 e_2, \end{aligned} \tag{38}$$

then the synchronization between drive system (18) and response system (35) will occur in the sense of $\lim_{t\longrightarrow\infty} e_1 = \lim_{t\longrightarrow\infty} e_2 = \lim_{t\longrightarrow\infty} e_3 = 0$, *where* $\hat{x}_1, \hat{x}_3, \hat{y}_1, \hat{y}_3$ *are defined by (37) and* $k > 0$.

Proof. The proof of Theorem 13 is similar to that of Theorem 10. In the first step, we shall show that

$$\lim_{t\longrightarrow\infty} e_2 = 0. \tag{39}$$

To this purpose, let us consider the following Lyapunov function candidate:

$$V = \frac{1}{2} e_2^2. \tag{40}$$

The time derivative of (40) is

$$\begin{aligned} D^{\alpha} V &\leq e_2 D^{\alpha} e_2 = e_2\left(b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_1 y_3\right. \\ &\quad + d_1(t) - \left(b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_1 x_3 + d(t)\right) \\ &\quad \left. + u\right) \leq \left(|b_1|\left(|y_1| + |x_1|\right) + |b_3|\left(|y_3| + |x_3|\right)\right. \\ &\quad \left. + |b_4|\left(|y_1||y_3| + |x_1||x_3|\right) + |d_1(t) - d(t)|\right)|e_2| \\ &\quad + \left(b_2 e_2 + u\right) e_2. \end{aligned} \tag{41}$$

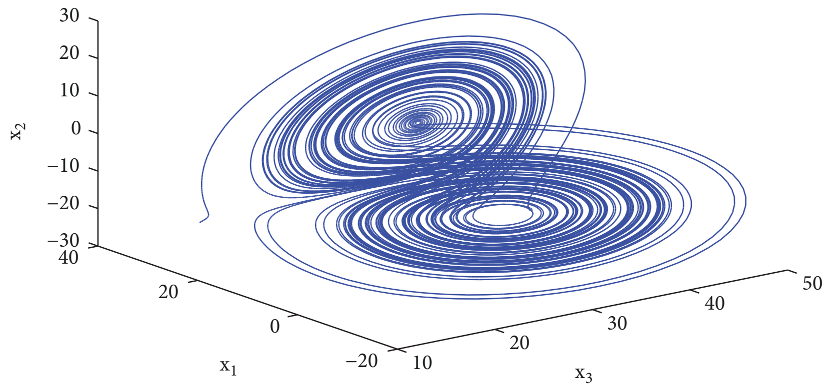

FIGURE 1: The chaotic trajectories of system (35) with $\alpha = 0.995, d(t) = 0, \theta = 2$ and $x_1(0) = 4, x_2(0) = -3, x_3(0) = 2$.

By Lemma 6, we obtain

$$\begin{aligned} |x_1| &\le \hat{x}_1 + \rho_{x1}, \\ |x_3| &\le \hat{x}_3 + \rho_{x3}, \\ |y_1| &\le \hat{y}_1 + \rho_{y1}, \\ |y_3| &\le \hat{y}_3 + \rho_{y3}, \end{aligned} \tag{42}$$

where $\rho_{x1} = E_\alpha(\overline{a}t^\alpha)(|x_1(0)| - \hat{x}_1(0)), \rho_{x3} = E_\alpha(\overline{c}t^\alpha)(|x_3(0)| - \hat{x}_3(0)), \rho_{y1} = E_\alpha(\overline{a}t^\alpha)(|y_1(0)| - \hat{y}_1(0)), \rho_{y3} = E_\alpha(\overline{c}t^\alpha)(|y_3(0)| - \hat{y}_3(0))$.

Substituting inequality (42) into (41), we have

$$\begin{aligned} D^\alpha V \le{} & \left(|b_1| \left(\hat{y}_1 + \rho_{y1} + \hat{x}_1 + \rho_{x1}\right) + |b_3| \left(\hat{y}_3 + \rho_{y3} \right.\right. \\ & \left. + \hat{x}_3 + \rho_{x3}\right) + |d_1(t) - d(t)| + |b_4| \left(\left(\hat{y}_1 + \rho_{y1}\right)\left(\hat{y}_3 \right.\right. \\ & \left.\left.\left. + \rho_{y3}\right) + \left(\hat{x}_1 + \rho_{x1}\right)\left(\hat{x}_3 + \rho_{x3}\right)\right)\right) |e_2| + (b_2 e_2 + u) e_2 \\ ={} & \left(|b_1| (\hat{y}_1 + \hat{x}_1) + |b_3| (\hat{y}_3 + \hat{x}_3) + |d_1(t) - d(t)| \right. \\ & \left. + |b_4| (\hat{y}_1 \hat{y}_3 + \hat{x}_1 \hat{x}_3)\right) |e_2| + (b_2 e_2 + u) e_2 + \left(|b_1| \right. \\ & \cdot \left(\rho_{y1} + \rho_{x1}\right) + |b_3| \left(\rho_{y3} + \rho_{x3}\right) + |b_4| \left(\hat{y}_1 \rho_{y3} \right. \\ & \left.\left. + \rho_{y1} \hat{y}_3 + \rho_{y1} \rho_{y3} + \hat{x}_1 \rho_{x3} + \rho_{x1} \hat{x}_3 + \rho_{x1} \rho_{x3}\right)\right) |e_2|. \end{aligned} \tag{43}$$

By using (38), we have

$$\begin{aligned} D^\alpha V \le{} & -\left(1 - |b_1| \left(\rho_{y1} + \rho_{x1}\right) - |b_3| \left(\rho_{y3} + \rho_{x3}\right) \right. \\ & - |b_4| \left(\hat{y}_1 \rho_{y3} + \rho_{y1} \hat{y}_3 + \rho_{y1} \rho_{y3} + \hat{x}_1 \rho_{x3} + \rho_{x1} \hat{x}_3 \right. \\ & \left.\left. + \rho_{x1} \rho_{x3}\right) + k |e_2|\right) |e_2|. \end{aligned} \tag{44}$$

The following proof is similar to that of Theorem 10 and omitted here. This ends the proof of Theorem 13. □

5. Numerical Simulations

In this section we take the fractional-order generalized Lorenz chaotic systems as an example to verify and demonstrate the effectiveness of the proposed control scheme.

The integer-order generalized Lorenz chaotic systems can be described as [19]

$$\begin{aligned} \dot{x}_1 &= \left(10 + \frac{25}{29}\theta\right)(x_2 - x_1), \\ \dot{x}_2 &= \left(28 - \frac{35}{29}\theta\right)x_1 + (\theta - 1)x_2 - x_1 x_3, \\ \dot{x}_3 &= \left(-\frac{8}{3} - \frac{1}{87}\theta\right)x_3 + x_1 x_2, \end{aligned} \tag{45}$$

where $x = (x_1, x_2, x_3)^T \in R^{3\times 1}$ is the state vector of system (32) and θ is the system parameter which satisfies $\theta \in \{\theta \mid -232 < \theta < -11.6\} \cup \{\theta \mid \theta > -11.6\}$. It is well known that the systems (32) display chaotic behavior for each $0 \le \theta \le 29$ [19].

Base on system (45), the fractional-order generalized Lorenz chaotic systems are given as

$$\begin{aligned} D^\alpha x_1 &= \left(10 + \frac{25}{29}\theta\right)(x_2 - x_1), \\ D^\alpha x_2 &= \left(28 - \frac{35}{29}\theta\right)x_1 + (\theta - 1)x_2 - x_1 x_3 + d(t), \\ D^\alpha x_3 &= \left(-\frac{8}{3} - \frac{1}{87}\theta\right)x_3 + x_1 x_2. \end{aligned} \tag{46}$$

where $d(t)$ represents the model uncertainty or the external disturbance. The chaos attractor with $\alpha = 0.995, d(t) = 0, \theta = 2$ is shown in Figure 1.

By comparing system (46) with system (18), it yields that $a(t) = -(10 + (25/29)\theta), a_1 = (10 + (25/29)\theta), b_1 = (28 - (35/29)\theta), b_2 = \theta - 1, b_4 = -1, c(t) = -8/3 - (1/87)\theta, c_1 = 1, a_2 = b_3 = c_2 = 0$. Since systems (32) display chaotic behavior for each $0 \le \theta \le 29$, so we suppose $0 \le \theta \le 29$ in systems (46). Thus, it is easy to see that $\overline{a} = -(10 +$

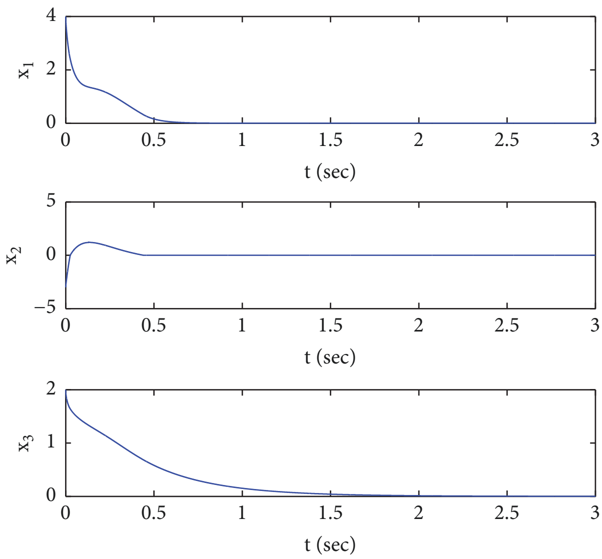

Figure 2: The time response of states x_1, x_2, x_3 of system (47) with controller (27).

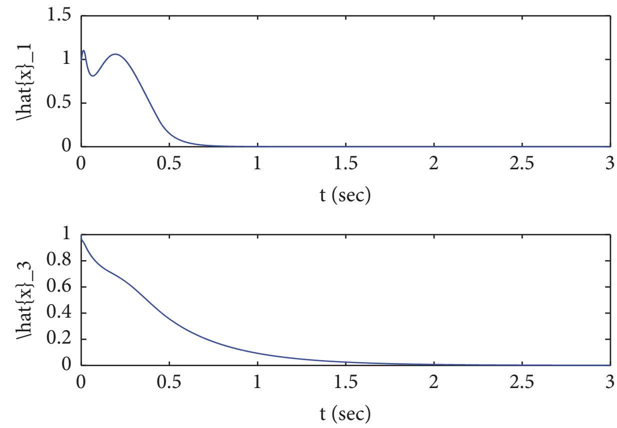

Figure 3: The time response of states $\hat{x}_1, \hat{x}_3$ of system (26) with $\hat{x}_1(0) = 1, \hat{x}_3(0) = 1$.

$(25/29)\theta), \bar{c} = -8/3 - (1/87)\theta$. In the simulation process, we take $\alpha = 0.995, \theta = 2$ such that system (46) is chaotic.

Example 1. The control of the fractional-order generalized Lorenz chaotic systems.

The controlled system, based on system (46), is given as

$$\begin{aligned} D^{\alpha} x_1 &= \left(10 + \frac{25}{29}\theta\right)(x_2 - x_1), \\ D^{\alpha} x_2 &= \left(28 - \frac{35}{29}\theta\right) x_1 + (\theta - 1) x_2 - x_1 x_3 + d(t) \\ &\quad + u, \\ D^{\alpha} x_3 &= \left(-\frac{8}{3} - \frac{1}{87}\theta\right) x_3 + x_1 x_2. \end{aligned} \tag{47}$$

where u is the controller to be designed later.

Let $d(t) = 2\sin(t)$; then we get $\delta = 2, \bar{a} = -(10 + 25/29), \bar{c} = (-8/3 - 1/87)$. The parameter k is taken as $k = 3$. Suppose the observer system is (26) and the controller u is selected as (27); then according to Theorem 10 we know that the origin of system (47) is stable in the sense of $\lim_{t\longrightarrow\infty} x_1 = \lim_{t\longrightarrow\infty} x_2 = \lim_{t\longrightarrow\infty} x_3 = 0$. The simulation results with $x_1(0) = 4, x_2(0) = -3, x_3(0) = 2$ and $\hat{x}_1(0) = \hat{x}_3(0) = 1$ are shown in Figures 2 and 3. Figure 2 is the time response of states x_1, x_2, x_3 of system (47) and Figure 3 is the time response of states $\hat{x}_1, \hat{x}_3$ of system (26). From Figure 2 it is easy to see that although there is disturbance in system (47) the states x_1, x_2, x_3 of system (47) approach 0 as $t \longrightarrow +\infty$, respectively, which means that the origin of system (47) is asymptotic stable. Figure 3 shows that $\lim_{t\longrightarrow\infty} \hat{x}_1 = \lim_{t\longrightarrow\infty} \hat{x}_3 = 0$ which means that in this case $\hat{x}_1$ and $\hat{x}_3$ can recover the information of x_1 and x_3.

Example 2. The synchronization between two identical fractional-order generalized Lorenz chaotic systems.

Let system (46) be the drive system; then the corresponding response system is given as

$$\begin{aligned} D^{\alpha} y_1 &= \left(10 + \frac{25}{29}\theta\right)(y_2 - y_1), \\ D^{\alpha} y_2 &= \left(28 - \frac{35}{29}\theta\right) y_1 + (\theta - 1) y_2 - y_1 y_3 + d_1(t) \\ &\quad + u, \\ D^{\alpha} y_3 &= \left(-\frac{8}{3} - \frac{1}{87}\theta\right) y_3 + y_1 y_2. \end{aligned} \tag{48}$$

where u is the controller to be designed later and $d_1(t)$ is the model uncertainty or the external disturbance.

In our simulation, we take $d(t) = 2\sin(t), d_1(t) = 2\cos(t)$, thus $|d(t) - d_1(t)| \leq 4$ which means that $\delta = 4$. The parameter k is taken as $k = 30$. Suppose the observer system is (37) and the controller u is designed as (38); then by Theorem 13 we know the synchronization between the drive system (46) and the response system (48) will be achieved. Numerical simulations with $x(0) = (4, -3, 2)^T, y(0) = (2, 1, 6)^T$, and $\hat{x}_1(0) = \hat{x}_3(0) = \hat{y}_1(0) = \hat{y}_3(0) = 1$ are shown in Figure 4. Figure 4 is the time evolution of errors e_1, e_2, e_3 between system (46) and system (48) with controller (38). From Figure 4 it is easy to see that although there are disturbances both in the drive system and the response system, the synchronization errors converge quickly to zero which means that the synchronization between system (46) and response system (48) is reached.

6. Conclusions

The observer-chaos-based stabilization and synchronization of a class of fractional-order chaotic systems with a single output are investigated in this paper. In the literature, there are some papers that consider the stabilization and synchronization of chaotic systems via the observer -based method. There are two main differences between our paper and the published papers: (1) it is easy to see that in

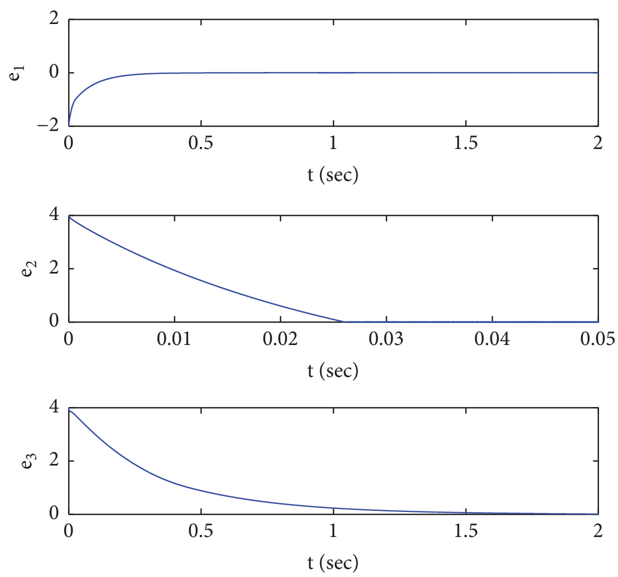

Figure 4: The time response of synchronization errors e_1, e_2, e_3 between system (46) and response system (48).

the published paper the constructed observer system can exactly recover the information of the unavailable states. One can use the recovery states to design the controller easily. However, the observer system presented in this paper cannot recover the information of the unavailable states exactly; it just provides the upper bound of the unavailable states. Thus, to design the proper controller is a more difficult task. (2) In the literature, most of the papers that concerned the observer-based scheme do not consider the effects of external disturbances. However, the external disturbances are taken into consideration in this paper. By using the observer system, some sufficient conditions for achieving chaos control and synchronization of fractional-order chaotic systems are derived. The fractional-order generalized Lorenz chaotic system is taken as an example to show the feasibility of the designed method.

Data Availability

If necessary, the numerical simulation codes can be uploaded at any time.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was jointly supported by the National Natural Science Foundation of China under Grant Nos. 11761050 and 11361043, the Natural Science Foundation of Jiangxi Province under Grant No. 20161BAB201008, and the Graduate Innovative Foundation of Jiangxi Province under Grant No. YC2017-S059.

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