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Research Article

Fuzzy Entropy for Pythagorean Fuzzy Sets with Application to Multicriterion Decision Making

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The concept of Pythagorean fuzzy sets (PFSs) was initially developed by Yager in 2013, which provides a novel way to model uncertainty and vagueness with high precision and accuracy compared to intuitionistic fuzzy sets (IFSs). The concept was concretely designed to represent uncertainty and vagueness in mathematical way and to furnish a formalized tool for tackling imprecision to real problems. In the present paper, we have used both probabilistic and nonprobabilistic types to calculate fuzzy entropy of PFSs. Firstly, a probabilistic-type entropy measure for PFSs is proposed and then axiomatic definitions and properties are established. Secondly, we utilize a nonprobabilistic-type with distances to construct new entropy measures for PFSs. Then a min–max operation to calculate entropy measures for PFSs is suggested. Some examples are also used to demonstrate suitability and reliability of the proposed methods, especially for choosing the best one/ones in structured linguistic variables. Furthermore, a new method based on the chosen entropies is presented for Pythagorean fuzzy multicriterion decision making to compute criteria weights with ranking of alternatives. A comparison analysis with the most recent and relevant Pythagorean fuzzy entropy is conducted to reveal the advantages of our developed methods. Finally, this method is applied for ranking China-Pakistan Economic Corridor (CPEC) projects. These examples with applications demonstrate practical effectiveness of the proposed entropy measures.

1. Introduction

The concept of fuzzy sets was first proposed by Zadeh [1] in 1965. With a widely spread use in various fields, fuzzy sets not only provide broad opportunity to measure uncertainties in more powerful and logical way, but also give us a meaningful way to represent vague concepts in natural language. It is known that most systems based on 'crisp set theory' or 'two-valued logics' are somehow difficult for handling imprecise and vague information. In this sense, fuzzy sets can be used to provide better solutions for more real world problems. Moreover, to treat more imprecise and vague information in daily life, various extensions of fuzzy sets are suggested by researchers, such as interval-valued fuzzy set [2], type-2 fuzzy sets [3], fuzzy multiset [4], intuitionistic fuzzy sets [5], hesitant fuzzy sets [6, 7], and Pythagorean fuzzy sets [8, 9].

Since fuzzy sets were based on membership values or degrees between 0 and 1, in real life setting it may not be always true that nonmembership degree is equal to (1-membership). Therefore, to get more purposeful reliability and applicability, Atanassov [5] generalized the concept of 'fuzzy set theory' and proposed Intuitionistic fuzzy sets (IFSs) which include both membership degree and nonmembership degree and degree of nondeterminacy or uncertainty where degree of uncertainty = (1- (degree of membership + non-membership degree)). In IFSs, the pair of membership grades is denoted by (μ, ν) satisfying the condition of $\mu + \nu \le 1$. Recently, Yager and Abbasov [8] and Yager [9] extended the condition $\mu + \nu \le 1$ to $\mu^2 + \nu^2 \le 1$ and then introduced a class of Pythagorean fuzzy sets (PFSs) whose membership values are ordered pairs (μ, ν) that fulfills the required condition of $\mu^2 + \nu^2 \le 1$ with different aggregation operations and applications in multicriterion decision making. According to Yager and Abbasov [8] and Yager [9], the space of all intuitionistic membership values (IMVs) is also Pythagorean membership values (PMVs), but PMVs are not necessary to be IMVs. For instance, for the situation when the numbers $\mu = \sqrt{3}/2$ and $\nu = 1/2$, we can use PFSs, but IFSs cannot be used since $\mu + \nu > 1$, but $\mu^2 + \nu^2 \leq 1$. PFSs are wider than IFSs so that they can tackle more daily life problems under imprecision and uncertainty cases.

More researchers are actively engaged in the development of PFSs properties. For example, Yager [10] gave Pythagorean membership grades in multicriterion decision making. Extensions of technique for order preference by similarity to an ideal solution (TOPSIS) to multiple criteria decision making with Pythagorean and hesitant fuzzy sets were proposed by Zhang and Xu [11]. Zhang [12] considered a novel approach based on similarity measure for Pythagorean fuzzy multicriteria group decision making. Pythagorean fuzzy TODIM approach to multicriterion decision making was given by Ren et al. [13]. Pythagorean fuzzy Choquet integral based MABAC method for multiple attribute group decision making was developed by Peng and Yang [14]. Zhang [15] gave a hierarchical QUALIFLEX approach. Peng et al. [16] investigated Pythagorean fuzzy information measures. Zhang et al. [17] proposed generalized Pythagorean fuzzy Bonferroni mean aggregation operators. Liang and Xu [18] extended TOPSIS to hesitant Pythagorean fuzzy sets. Pérez-Domínguez et al. [19] gave MOORA under Pythagorean fuzzy sets. Recently, Pythagorean fuzzy LINMAP method based on the entropy for railway project investment decision making was proposed by Xue et al. [20]. Zhang and Meng [21] proposed an approach to interval-valued hesitant fuzzy multiattribute group decision making based on the generalized Shapley-Choquet integral. Pythagorean fuzzy (R, S) – norm information measure for multicriteria decision making problem was presented by Guleria and Bajaj [22]. Furthermore, Yang and Hussain [23] proposed distance and similarity measures of hesitant fuzzy sets based on Hausdorff metric with applications to multicriteria decision making and clustering. Hussain and Yang [24] gave entropy for hesitant fuzzy sets based on Hausdorff metric with construction of hesitant fuzzy TOPSIS.

The entropy of fuzzy sets is a measure of fuzziness between fuzzy sets. De Luca and Termini [25] first introduced the axiom construction for entropy of fuzzy sets with reference to Shannon's probability entropy. Yager [26] defined fuzziness measures of fuzzy sets in terms of a lack of distinction between the fuzzy set and its negation based on Lp norm. Kosko [27] provided a measure of fuzziness between fuzzy sets using a ratio of distance between the fuzzy set and its nearest set to the distance between the fuzzy set and its farthest set. Liu [28] gave some axiom definitions of entropy and also defined a σ-entropy. Pal and Pal [29] proposed exponential entropies. While Fan and Ma [30] gave some new fuzzy entropy formulas. Some extended entropy measures for IFS were proposed by Burillo and Bustince [31], Szmidt and Kacprzyk [32], Szmidt and Baldwin [33], and Hung and Yang [34].

In this paper, we propose new entropies of PFS based on probability-type, distance, Pythagorean index, and min–max operation. We also extend the concept to σ-entropy and then apply it to multicriteria decision making. This paper is organized as follows. In Section 2, we review some definitions of IFSs and PFSs. In Section 3, we propose several new entropies of PFSs and then construct an axiomatic definition of entropy for PFSs. Based on the definition of entropy for PFSs, we find that the proposed nonprobabilistic entropies of PFSs are σ-entropy. In Section 4, we exhibit some examples for comparisons and also use structured linguistic variables to validate our proposed methods. In Section 5, we construct a new Pythagorean fuzzy TOPSIS based on the proposed entropy measures. A comparison analysis of the proposed Pythagorean fuzzy TOPSIS with the recently developed entropy of PFS [20] is shown. We then apply the proposed method to multicriterion decision making for ranking China-Pakistan Economic Corridor projects. Finally, we state our conclusion in Section 6.

2. Intuitionistic and Pythagorean Fuzzy Sets

In this section, we give a brief review for intuitionistic fuzzy sets (IFSs) and Pythagorean fuzzy sets (PFSs).

Definition 1. An intuitionistic fuzzy set (IFS) $\widetilde{M}$ in X is defined by Atanassov [5] with the following form:

$$\widetilde{M} = \{(x, \mu_{\widetilde{M}}(x), \nu_{\widetilde{M}}(x)) : x \in X\} \tag{1}$$

where $0 \leq \mu_{\widetilde{M}}(x) + \nu_{\widetilde{M}}(x) \leq 1$, $\forall x \in X$, and the functions $\mu_{\widetilde{M}}(x) : X \longrightarrow [0,1]$ denotes the degree of membership of x in $\widetilde{M}$ and $\nu_{\widetilde{M}}(x) : X \longrightarrow [0,1]$ denotes the degree of nonmembership of x in $\widetilde{M}$. The degree of uncertainty (or intuitionistic index, or indeterminacy) of x to $\widetilde{M}$ is represented by $\pi_{\widetilde{M}}(x) = 1 - (\mu_{\widetilde{M}}(x) + \nu_{\widetilde{M}}(x))$.

For modeling daily life problems carrying imprecision, uncertainty, and vagueness more precisely and with high accuracy than IFSs, Yager [9, 10] presented Pythagorean fuzzy sets (PFSs), where PFSs are the generalizations of IFSs. Yager [9, 10] also validated that IFSs are contained in PFSs. The concept of Pythagorean fuzzy set was originally developed by Yager [8, 9], but the general mathematical form of Pythagorean fuzzy set was developed by Zhang and Xu [11].

Definition 2 (Zhang and Xu [11]). A Pythagorean fuzzy set (PFS) $\widetilde{P}$ in X proposed by Yager [8, 9] is mathematically formed as

$$\widetilde{P} = \{\langle x, \mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x)\rangle : x \in X\} \tag{2}$$

where the functions $\mu_{\widetilde{P}}(x) : X \longrightarrow [0,1]$ represent the degree of membership of x in $\widetilde{P}$ and $\nu_{\widetilde{P}}(x) : X \longrightarrow [0,1]$ represent the degree of nonmembership of x in $\widetilde{P}$. For every $x \in X$, the following condition should be satisfied:

$$0 \leq \mu_{\widetilde{P}}^2(x) + \nu_{\widetilde{P}}^2(x) \leq 1. \tag{3}$$

Definition 3 (Zhang and Xu [11]). For any PFS $\widetilde{P}$ in X, the value $\pi_{\widetilde{P}}(x)$ is called Pythagorean index of the element x in $\widetilde{P}$ with

$$\pi_{\widetilde{P}}(x) = \sqrt{1 - \left\{\mu_{\widetilde{P}}^2(x) + \nu_{\widetilde{P}}^2(x)\right\}}$$
$$\text{or } \pi_{\widetilde{P}}^2(x) = 1 - \mu_{\widetilde{P}}^2(x) - \nu_{\widetilde{P}}^2(x). \quad (4)$$

In general, $\pi_{\widetilde{P}}(x)$ is also called hesitancy (or indeterminacy) degree of the element x in $\widetilde{P}$. It is obvious that $0 \le \pi_{\widetilde{P}}^2(x) \le 1$, $\forall x \in X$. It is worthy to note, for a PFS $\widetilde{P}$, if $\mu_{\widetilde{P}}^2(x) = 0$ then $\nu_{\widetilde{P}}^2(x) + \pi_{\widetilde{P}}^2(x) = 1$, and if $\mu_{\widetilde{P}}^2(x) = 1$ then $\nu_{\widetilde{P}}^2(x) = 0$ and $\pi_{\widetilde{P}}^2(x) = 0$. Similarly, if $\nu_{\widetilde{P}}^2(x) = 0$ then $\mu_{\widetilde{P}}^2(x)+\pi_{\widetilde{P}}^2(x) = 1$. If $\nu_{\widetilde{P}}^2(x) = 1$ then $\mu_{\widetilde{P}}^2(x) = 0$ and $\pi_{\widetilde{P}}^2(x) = 0$. If $\pi_{\widetilde{P}}^2(x) = 0$ then $\mu_{\widetilde{P}}^2(x) + \nu_{\widetilde{P}}^2(x) = 1$. If $\pi_{\widetilde{P}}^2(x) = 1$ then $\mu_{\widetilde{P}}^2(x) = \nu_{\widetilde{P}}^2(x) = 0$. For convenience, Zhang and Xu [11] denoted the pair $(\mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x))$ as Pythagorean fuzzy number (PFN), which is represented by $\widetilde{p} = (\mu_{\widetilde{p}}, \nu_{\widetilde{p}})$.

Since PFSs are a generalized form of IFSs, we give the following definition for PFSs.

Definition 4. Let $\widetilde{P}$ be a PFS in X. $\widetilde{P}$ is called a completely Pythagorean if $\mu_{\widetilde{P}}^2(x) = \nu_{\widetilde{P}}^2(x) = 0$, $\forall x \in X$.

Peng et al. [16] suggested various mathematical operations for PFSs as follows:

Definition 5 (Peng et al. [16]). If $\widetilde{P}$ and $\widetilde{Q}$ are two PFSs in X, then

(i) $\widetilde{P} \le \widetilde{Q}$ if and only if $\forall x \in X$, $\mu_{\widetilde{P}}^2(x) \le \mu_{\widetilde{Q}}^2(x)$ and $\nu_{\widetilde{P}}^2(x) \ge \nu_{\widetilde{Q}}^2(x)$;

(ii) $\widetilde{P} = \widetilde{Q}$ if and only if $\forall x \in X$, $\mu_{\widetilde{P}}^2(x) = \mu_{\widetilde{Q}}^2(x)$ and $\nu_{\widetilde{P}}^2(x) = \nu_{\widetilde{Q}}^2(x)$;

(iii) $\widetilde{P} \cup \widetilde{Q} = \{\langle x, \max(\mu_{\widetilde{P}}^2(x), \mu_{\widetilde{Q}}^2(x)), \min(\nu_{\widetilde{P}}^2(x), \nu_{\widetilde{Q}}^2(x))\rangle : x \in X\}$;

(iv) $\widetilde{P} \cap \widetilde{Q} = \{\langle x, \min(\mu_{\widetilde{P}}^2(x), \mu_{\widetilde{Q}}^2(x)), \max(\nu_{\widetilde{P}}^2(x), \nu_{\widetilde{Q}}^2(x))\rangle : x \in X\}$;

(v) $\widetilde{P}^c = \{\langle x, \nu_{\widetilde{P}}^2(x), \mu_{\widetilde{Q}}^2(x)\rangle : x \in X\}$.

We next define more operations of PFS in X, especially about hedges of "very", "highly", "more or less", "concentration", "dilation", and other terms that are needed to represent linguistic variables. We first define the n power (or exponent) of PFS as follows.

Definition 6. Let $\widetilde{P} = \{\langle x, \mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x)\rangle : x \in X\}$ be a PFS in X. For any positive real number n, the n power (or exponent) of the PFS $\widetilde{P}$, denoted by $\widetilde{P}^n$, is defined as

$$\widetilde{P}^n = \left\{\left\langle x, \left(\mu_{\widetilde{P}}(x)\right)^n, \sqrt{1 - \left(1 - \nu_{\widetilde{P}}^2(x)\right)^n}\right\rangle : x \in X\right\}. \quad (5)$$

It can be easily verified that, for any positive real number n, $0 \le [\mu_{\widetilde{P}}(x)]^n + [\sqrt{1 - (1 - \nu_{\widetilde{P}}^2(x))^n}] \le 1$, $\forall x \in X$.

By using Definition 6, the concentration and dilation of a PFS $\widetilde{P}$ can be defined as follows.

Definition 7. The concentration $CON(\widetilde{P})$ of a PFS $\widetilde{P}$ in X is defined as

$$CON\left(\widetilde{P}\right) = \left\{\left\langle x, \mu_{CON(\widetilde{P})}(x), \nu_{CON(\widetilde{P})}(x)\right\rangle : x \in X\right\} \quad (6)$$

where $\mu_{CON(\widetilde{P})}(x) = [\mu_{\widetilde{P}}(x)]^2$ and $\nu_{CON(\widetilde{P})}(x) = \sqrt{1 - [1 - \nu_{\widetilde{P}}^2(x)]^2}$.

Definition 8. The dilation $DIL(\widetilde{P})$ of a PFS $\widetilde{P}$ in X is defined as

$$DIL\left(\widetilde{P}\right) = \left\{\left\langle x, \mu_{DIL(\widetilde{P})}(x), \nu_{DIL(\widetilde{P})}(x)\right\rangle : x \in X\right\} \quad (7)$$

where $\mu_{DIL(\widetilde{P})}(x) = [\mu_{\widetilde{P}}(x)]^{1/2}$ and $\nu_{DIL(\widetilde{P})}(x) = \sqrt{1 - [1 - \nu_{\widetilde{P}}^2(x)]^{1/2}}$.

In next section, we construct new entropy measures for PFSs based on probability-type, entropy induced by distance, Pythagorean index, and max-min operation. We also give an axiomatic definition of entropy for PFSs.

3. New Fuzzy Entropies for Pythagorean Fuzzy Sets

We first provide a definition of entropy for PFSs. De Luca and Termini [25] gave the axiomatic definition of entropy measure of fuzzy sets. Later on, Szmidt and Kacprzyk [32] extended it to entropy of IFS. Since PFSs developed by Yager [8, 9] are generalized forms of IFSs, we use similar notions as IFSs to give a definition of entropy for PFSs. Assume that $PFS(X)$ represents the set of all PFSs in X.

Definition 9. A real function $E : PFS(X) \longrightarrow [0, 1]$ is called an entropy on $PFS(X)$ if E satisfies the following axioms:

(*A0*) (*Nonnegativity*) $0 \le E(\widetilde{P}) \le 1$;

(*A1*) (*Minimality*) $E(\widetilde{P}) = 0$, *iff* $\widetilde{P}$ *is a crisp set*;

(*A2*) (*Maximality*) $E(\widetilde{P}) = 1$, *iff* $\mu_{\widetilde{P}}(x) = \nu_{\widetilde{P}}(x)$, $\forall x \in X$;

(*A3*) (*Resolution*) $E(\widetilde{P}) \le E(\widetilde{Q})$, *if* $\widetilde{P}$ *is crisper than* $\widetilde{Q}$, *i.e.*, $\forall x \in X$,

$\mu_{\widetilde{P}}(x) \le \mu_{\widetilde{Q}}(x)$ *and* $\nu_{\widetilde{P}}(x) \ge \nu_{\widetilde{Q}}(x)$ *for* $\mu_{\widetilde{Q}}(x) \le \nu_{\widetilde{Q}}(x)$ *or*

$\mu_{\widetilde{P}}(x) \ge \mu_{\widetilde{Q}}(x)$ *and* $\nu_{\widetilde{P}}(x) \le \nu_{\widetilde{Q}}(x)$ *for* $\mu_{\widetilde{Q}}(x) \ge \nu_{\widetilde{Q}}(x)$;

(*A4*) (*Symmetric*) $E(\widetilde{P}) = E(\widetilde{P}^c)$, *where* $\widetilde{P}^c$ *is the complement of* $\widetilde{P}$;

For probabilistic-type entropy, we need to omit the axiom (A0). On the other hand, because we take the three-parameter $\mu_{\widetilde{P}}^2$, $\nu_{\widetilde{P}}^2$, and $\pi_{\widetilde{P}}^2$ as a probability mass function $p = \{\mu_{\widetilde{P}}^2, \nu_{\widetilde{P}}^2, \pi_{\widetilde{P}}^2\}$, the probabilistic-type entropy $E(\widetilde{P})$ should attain a unique maximum at $\mu_{\widetilde{P}}(x) = \nu_{\widetilde{P}}(x) = \pi_{\widetilde{P}}(x) = 1/\sqrt{3}$, $\forall x \in X$. Therefore, for probabilistic-type entropy, we replace the axiom $(A2)$ with $(A2')$ and the axiom $(A3)$ with $(A3')$ as follows:

$(A2')$ (*Maximality*) *$E(\widetilde{P})$ attains a unique maximum at $\mu_{\widetilde{P}}(x) = \nu_{\widetilde{P}}(x) = \pi_{\widetilde{P}}(x) = 1/\sqrt{3}, \forall x \in X$.*

$(A3')$ (*Resolution*) *$E(\widetilde{P}) \le E(\widetilde{Q})$, if $\widetilde{P}$ is crisper than $\widetilde{Q}$, i.e.,, $\forall x \in X$, $\mu_{\widetilde{P}}(x) \le \mu_{\widetilde{Q}}(x)$ and $\nu_{\widetilde{P}}(x) \le \nu_{\widetilde{Q}}(x)$*

for $\max(\mu_{\widetilde{Q}}(x), \nu_{\widetilde{Q}}(x)) \le 1/\sqrt{3}$ *and* $\mu_{\widetilde{P}}(x) \ge \mu_{\widetilde{Q}}(x)$, $\nu_{\widetilde{P}}(x) \ge \nu_{\widetilde{Q}}(x)$ *for* $\min(\mu_{\widetilde{Q}}(x), \nu_{\widetilde{Q}}(x)) \ge 1/\sqrt{3}$.

In addition to the five axioms (A0)~(A4) in Definition 9, if we add the following axiom (A5), E is called σ- entropy:

$(A5)$ (*Valuation*) *$E(\widetilde{P}) + E(\widetilde{Q}) = E(\widetilde{P} \cup \widetilde{Q}) + E(\widetilde{P} \cap \widetilde{Q})$, if $\forall x \in X$*

$\mu_{\widetilde{P}}(x) \le \mu_{\widetilde{Q}}(x)$ *and* $\nu_{\widetilde{P}}(x) \ge \nu_{\widetilde{Q}}(x)$ *for* $\mu_{\widetilde{Q}}(x) \le \nu_{\widetilde{Q}}(x)$ or

$\mu_{\widetilde{P}}(x) \ge \mu_{\widetilde{Q}}(x)$ *and* $\nu_{\widetilde{P}}(x) \le \nu_{\widetilde{Q}}(x)$ *for* $\mu_{\widetilde{Q}}(x) \ge \nu_{\widetilde{Q}}(x)$.

We present a property for the axiom $(A3')$ when $\widetilde{P}$ is crisper than $\widetilde{Q}$.

Property 10. If $\widetilde{P}$ is crisper than $\widetilde{Q}$ in the axiom $(A3')$, then we have the following inequality:

$$\begin{aligned}&\left(\mu_{\widetilde{P}}(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \left(\nu_{\widetilde{P}}(x_i) - \frac{1}{\sqrt{3}}\right)^2 \\ &\quad + \left(\pi_{\widetilde{P}}(x_i) - \frac{1}{\sqrt{3}}\right)^2 \\ &\ge \left(\mu_{\widetilde{Q}}(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \left(\nu_{\widetilde{Q}}(x_k) - \frac{1}{\sqrt{3}}\right)^2 \\ &\quad + \left(\nu_{\widetilde{Q}}(x_k) - \frac{1}{\sqrt{3}}\right)^2, \quad \forall x_i\end{aligned} \tag{8}$$

Proof. If $\widetilde{P}$ is crisper than $\widetilde{Q}$, then $\forall x_i$, $\mu_{\widetilde{P}}(x) \le \mu_{\widetilde{Q}}(x)$ and $\nu_{\widetilde{P}}(x) \le \nu_{\widetilde{Q}}(x)$ for $\max(\mu_{\widetilde{Q}}(x), \nu_{\widetilde{Q}}(x)) \le 1/\sqrt{3}$. Therefore, we have that $\mu_{\widetilde{P}}(x) - 1/\sqrt{3} \le \mu_{\widetilde{Q}}(x) - 1/\sqrt{3} \le 0$ and $\nu_{\widetilde{P}}(x) - 1/\sqrt{3} \le \nu_{\widetilde{Q}}(x) - 1/\sqrt{3} \le 0$ and so $1 - \mu_{\widetilde{P}}^2(x) - \nu_{\widetilde{P}}^2(x) \ge 1 - \mu_{\widetilde{Q}}^2(x) - \nu_{\widetilde{Q}}^2(x) \ge 1/3$, i.e., $\pi_{\widetilde{P}}^2(x) \ge \pi_{\widetilde{Q}}^2(x) \ge 1/3$ and $\pi_{\widetilde{P}}(x) - 1/\sqrt{3} \ge \pi_{\widetilde{Q}}(x) - 1/\sqrt{3} \ge 0$. Thus, we have $(\mu_{\widetilde{P}}(x) - 1/\sqrt{3})^2 \ge (\mu_{\widetilde{Q}}(x) - 1/\sqrt{3})^2$, $(\nu_{\widetilde{P}}(x) - 1/\sqrt{3})^2 \ge (\nu_{\widetilde{Q}}(x) - 1/\sqrt{3})^2$, and $(\pi_{\widetilde{P}}(x) - 1/\sqrt{3})^2 \ge (\pi_{\widetilde{Q}}(x) - 1/\sqrt{3})^2$. This induces the inequality. Similarly, the part of $\mu_{\widetilde{P}}(x) \ge \mu_{\widetilde{Q}}(x)$ and $\nu_{\widetilde{P}}(x) \ge \nu_{\widetilde{Q}}(x)$ for $\min(\mu_{\widetilde{Q}}(x), \nu_{\widetilde{Q}}(x)) \ge 1/\sqrt{3}$ also induces the inequality. Hence, we prove the property. □

Since PFSs are generalized form of IFSs, the distances between PFSs need to be computed by considering all the three components $\mu_{\widetilde{P}}^2(x), \nu_{\widetilde{P}}^2(x)$ and $\pi_{\widetilde{P}}^2(x)$ in PFSs. The well-known distance between PFSs is Euclidean distance. Therefore, the inequality in Property 10 indicates that the Euclidean distance between $(\mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x), \pi_{\widetilde{P}}(x))$ and $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ is larger than the Euclidean distance between $(\mu_{\widetilde{Q}}(x), \nu_{\widetilde{Q}}(x), \pi_{\widetilde{Q}}(x))$ and $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. This manifests that $(\mu_{\widetilde{Q}}(x), \nu_{\widetilde{Q}}(x), \pi_{\widetilde{Q}}(x))$ is located more nearby to $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ than that of $(\mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x), \pi_{\widetilde{P}}(x))$. From a geometrical perspective, the axiom $(A3')$ is reasonable and logical because the closer PFS to the unique point $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ with maximum entropy reflects the greater entropy of that PFS.

We next construct entropies for PFSs based on a probability-type. To formulate the probability-type of entropy for PFSs, we use the idea of entropy $H^{\gamma}(p)$of Havrda and Charavát [35] to a probability mass function $p = \{p_1, \ldots, p_k\}$ with

$$H^{\gamma}(p) = \begin{cases} \dfrac{1}{\gamma - 1}\left(1 - \displaystyle\sum_{i=1}^{k} p_i^{\gamma}\right), & \gamma \ne 1\ (\gamma > 0) \\ -\displaystyle\sum_{i=1}^{k} p_i \log p_i, & \gamma = 1. \end{cases} \tag{9}$$

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourses. Thus, for a PFS $\widetilde{P}$ in X, we propose the following probability-type entropy for the PFS $\widetilde{P}$ with

$$e_{HC}^{\gamma}(\widetilde{P}) = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \dfrac{1}{\gamma - 1}\left[1 - \left(\left(\mu_{\widetilde{P}}^2(x_i)\right)^{\gamma} + \left(\nu_{\widetilde{P}}^2(x_i)\right)^{\gamma} + \left(\pi_{\widetilde{P}}^2(x_i)\right)^{\gamma}\right)\right], & \gamma \ne 1\ (\gamma > 0) \\ \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} -\left(\mu_{\widetilde{P}}^2(x_i)\log\mu_{\widetilde{P}}^2(x_i) + \nu_{\widetilde{P}}^2(x_i)\log\nu_{\widetilde{P}}^2(x_i) + \pi_{\widetilde{P}}^2(x_i)\log\pi_{\widetilde{P}}^2(x_i)\right), & \gamma = 1. \end{cases} \tag{10}$$

Apparently, one may ask a question: "Are these proposed entropy measures for PFSs are suitable and acceptable?" To answer this question, we present the following theorem.

Theorem 11. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourses. The proposed probabilistic-type entropy e_{HC}^{γ} for a PFS $\widetilde{P}$ satisfies the axioms $(A1), (A2'), (A3')$, and $(A4)$ in Definition 9.*

To prove the axioms $(A2')$ and $(A3')$ for Theorem 11, we need Lemma 12.

Lemma 12. *Let $\psi_{\gamma}(x)$, $0 < x < 1$ be defined as*

$$\psi_{\gamma}(x) = \begin{cases} \dfrac{1}{\gamma - 1}(x - x^{\gamma}), & \gamma \ne 1\ (\gamma > 0) \\ -x\log x, & \gamma = 1. \end{cases} \tag{11}$$

Then $\psi_{\gamma}(x)$ is a strictly concave function of x.

Proof. By twice differentiating $\psi_\gamma(x)$, we get $\psi''_\gamma(x) = -\gamma x^{\gamma-2} < 0$, for $\gamma \neq 1 (\gamma > 0)$. Then $\psi_\gamma(x)$ is a strictly concave function of x. Similarly, we can show that $\psi_{\gamma=1}(x) = x \log x$ is also a strictly concave function of x. □

Proof of Theorem 11. It is easy to check that e^γ_{HC} satisfies the axioms $(A1)$ and $(A4)$. We need only to prove that e^γ_{HC} satisfies the axioms $(A2')$ and $(A3')$. To prove e^γ_{HC} for the case $\gamma \neq 1 (\gamma > 0)$ that satisfies the axiom $(A2')$, we use Lagrange multipliers for e^γ_{HC} with $F(\mu^2_{\widetilde{P}}, \nu^2_{\widetilde{P}}, \pi^2_{\widetilde{P}}, \lambda) = (1/n)\sum_{i=1}^n (1/(\gamma - 1))[1 - ((\mu^2_{\widetilde{P}}(x_i))^\gamma + (\nu^2_{\widetilde{P}}(x_i))^\gamma + (\pi^2_{\widetilde{P}}(x_i))^\gamma)] + \sum_{i=1}^n \lambda_i(\mu^2_{\widetilde{P}}(x_i) + \nu^2_{\widetilde{P}}(x_i) + \pi^2_{\widetilde{P}}(x_i) - 1)$. By taking the derivative of $F(\mu^2_{\widetilde{P}}, \nu^2_{\widetilde{P}}, \pi^2_{\widetilde{P}}, \lambda)$ with respect to $\mu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{P}}(x_i), \pi^2_{\widetilde{P}}(x_i)$, and λ_i, we obtain

$$\begin{aligned}\frac{\partial F}{\partial \mu^2_{\widetilde{P}}(x_i)} &= \frac{-\gamma}{(\gamma-1)}\left(\mu^2_{\widetilde{P}}(x_i)\right)^{\gamma-1} + \lambda_i \overset{set}{=\!=} 0,\\ \frac{\partial F}{\partial \nu^2_{\widetilde{P}}(x_i)} &= \frac{-\gamma}{(\gamma-1)}\left(\nu^2_{\widetilde{P}}(x_i)\right)^{\gamma-1} + \lambda_i \overset{set}{=\!=} 0,\\ \frac{\partial F}{\partial \pi^2_{\widetilde{P}}(x_i)} &= \frac{-\gamma}{(\gamma-1)}\left(\pi^2_{\widetilde{P}}(x_i)\right)^{\gamma-1} + \lambda_i \overset{set}{=\!=} 0,\\ \frac{\partial F}{\partial \lambda_i} &= \mu^2_{\widetilde{P}}(x_i) + \nu^2_{\widetilde{P}}(x_i) + \pi^2_{\widetilde{P}}(x_i) - 1 \overset{set}{=\!=} 0.\end{aligned} \tag{12}$$

From above PDEs, we get $(\mu^2_{\widetilde{P}}(x_i))^{\gamma-1} = \lambda_i(\gamma-1)/\gamma$ and then $\mu^2_{\widetilde{P}}(x_i) = (\lambda_i(\gamma-1)/\gamma)^{1/(\gamma-1)}$; $(\nu^2_{\widetilde{P}}(x_i))^{\gamma-1} = \lambda_i(\gamma-1)/\gamma$ and $\nu^2_{\widetilde{P}}(x_i) = (\lambda_i(\gamma-1)/\gamma)^{1/(\gamma-1)}$; $(\pi^2_{\widetilde{P}}(x_i))^{\gamma-1} = \lambda_i(\gamma-1)/\gamma$ and $\pi^2_{\widetilde{P}}(x_i) = (\lambda_i(\gamma-1)/\gamma)^{1/(\gamma-1)}$; $\mu^2_{\widetilde{P}}(x_i) + \nu^2_{\widetilde{P}}(x_i) + \pi^2_{\widetilde{P}}(x_i) = 1$ and then $(\lambda_i(\gamma-1)/\gamma)^{1/(\gamma-1)} = 1/3$. Thus, we have $\lambda_i = (\gamma/(\gamma-1))(1/3)^{\gamma-1}$. We obtain $\mu^2_{\widetilde{P}}(x_i) = ((\gamma/(\gamma-1))(1/3)^{\gamma-1}(\gamma-1)/\gamma)^{1/(\gamma-1)}$, and then $\mu_{\widetilde{P}}(x_i) = 1/\sqrt{3}$. We also get $\nu_{\widetilde{P}}(x_i) = 1/\sqrt{3}$ and $\pi_{\widetilde{P}}(x_i) = 1/\sqrt{3}$. That is, $\mu_{\widetilde{P}}(x_i) = \nu_{\widetilde{P}}(x_i) = \pi_{\widetilde{P}}(x_i) = 1/\sqrt{3}$, $\forall i$. Similarly, we can show that the equation of e^γ_{HC} for the case $\gamma = 1$ also obtains $\mu_{\widetilde{P}}(x_i) = \nu_{\widetilde{P}}(x_i) = \pi_{\widetilde{P}}(x_i) = 1/\sqrt{3}$, $\forall i$. By Lemma 12, we learn that the function $\psi_\gamma(x)$ is a strictly concave function of x. We know that $e^\gamma_{HC}(\widetilde{P}) = (1/n)\sum_{i=1}^n(\psi_\gamma(\mu^2_{\widetilde{P}}(x_i)) + \psi_\gamma(\nu^2_{\widetilde{P}}((x_i))) + \psi_\gamma(\pi^2_{\widetilde{P}}(x_i)))$ and so e^γ_{HC} is also a strictly concave function. Therefore, it is proved that e^γ_{HC} attains a unique maximum at $\mu_{\widetilde{P}}(x_i) = \nu_{\widetilde{P}}(x_i) = \pi_{\widetilde{P}}(x_i) = 1/\sqrt{3}$, $\forall i$. We next prove that the probabilistic-type entropy e^γ_{HC} satisfies the axiom $(A3')$. If $\widetilde{P}$ is crisper than $\widetilde{Q}$, we notice that $\widetilde{P}$ is far away from $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ compared to $\widetilde{Q}$ according to Property 10. However, e^γ_{HC} is a strictly concave function and e^γ_{HC} attains a unique maximum at $\mu_{\widetilde{P}}(x_i) = \nu_{\widetilde{P}}(x_i) = \pi_{\widetilde{P}}(x_i) = 1/\sqrt{3}$, $\forall i$. From here, we obtain $e^\gamma_{HC}(\widetilde{P}) \le e^\gamma_{HC}(\widetilde{Q})$ if $\widetilde{P}$ is crisper than $\widetilde{Q}$. Thus, we prove the axiom $(A3')$. □

Concept to determine uncertainty from a fuzzy set and its complement was first given by Yager [26]. In this section, we first use the similar idea to measure uncertainty of PFSs in terms of the amount of distinction between a PFS $\widetilde{P}$ and its complement $\widetilde{P}^c$. However, various distance measures are made to express numerically the difference between two objects with high accuracy. Therefore, the distance between two fuzzy sets plays a vital role in theoretical and practical issues. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourses; we first define a Pythagorean normalized Euclidean (PNE) distance between two Pythagorean fuzzy sets $\widetilde{P}, \widetilde{Q} \in PFSs(X)$ as

$$\zeta_E(\widetilde{P}, \widetilde{Q}) = \left[\frac{1}{2n}\sum_{i=1}^n\left(\left(\mu^2_{\widetilde{P}}(x_i) - \mu^2_{\widetilde{Q}}(x_i)\right)^2 + \left(\nu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{Q}}(x_k)\right)^2 + \left(\pi^2_{\widetilde{P}}(x_i) - \pi^2_{\widetilde{Q}}(x_i)\right)^2\right)\right]^{1/2} \tag{13}$$

We next propose fuzzy entropy induced by the PNE distance ζ_E between the PFS $\widetilde{P}$ and its complement $\widetilde{P}^c$. Let $\widetilde{P} = \{\langle x_i, \mu_{\widetilde{P}}(x_i), \nu_{\widetilde{P}}(x_i)\rangle : x_i \in X\}$ be any Pythagorean fuzzy set on the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ with its complement $\widetilde{P}^c = \{\langle x, \nu_{\widetilde{P}}(x_i), \mu_{\widetilde{P}}(x_i)\rangle : x_i \in X\}$. The PNE distance between PFSs $\widetilde{P}$ and $\widetilde{P}^c$ will be $\zeta_E(\widetilde{P}, \widetilde{P}^c) = [(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i))^2]^{1/2}$. Thus, we define a new entropy e_E for the PFS $\widetilde{P}$ as

$$\begin{aligned} e_E(\widetilde{P}) &= 1 - \zeta_E(\widetilde{P}, \widetilde{P}^c)\\ &= 1 - \sqrt{\frac{1}{n}\sum_{i=1}^n\left(\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i)\right)^2}\end{aligned} \tag{14}$$

Theorem 13. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourse. The proposed entropy e_E for a PFS $\widetilde{P}$ satisfies the axioms (A0)~(A5) in Definition 9, and so e_E is a σ-entropy.*

Proof. We first prove the axiom (A0). Since the distance $\zeta_E(\widetilde{P}, \widetilde{P}^c)$ is between 0 and 1, $0 \le 1 - \zeta_E(\widetilde{P}, \widetilde{P}^c) \le 1$, and then $0 \le e_E(\widetilde{P}) \le 1$. The axiom (A0) is satisfied. For the axiom (A1), if $\widetilde{P}$ is crisp, i.e., $\mu^2_{\widetilde{P}}(x_i) = 0$, $\nu^2_{\widetilde{P}}(x_i) = 1$ or $\mu^2_{\widetilde{P}}(x_i) = 1$, $\nu^2_{\widetilde{P}}(x_i) = 0$, $\forall x_i \in X$, then $\zeta_E(\widetilde{P}, \widetilde{P}^c) = \sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i))^2} = 1$. Thus, we obtain $e_E(\widetilde{P}) = 1 - 1 = 0$. Conversely, if $e_E(\widetilde{P}) = 0$, then $\zeta_E(\widetilde{P}, \widetilde{P}^c) = 1 - e_E(\widetilde{P}) = 1$, i.e., $\sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i))^2} = 1$. Thus, $\mu^2_{\widetilde{P}}(x_i) = 1, \nu^2_{\widetilde{P}}(x_i) = 0$ or $\mu^2_{\widetilde{P}}(x_i) = 0$, $\nu^2_{\widetilde{P}}(x_i) = 1$ and so $\widetilde{P}$ is crisp. Thus, the axiom (A1) is satisfied. Now, we prove the axiom (A2). $\mu^2_{\widetilde{P}}(x_i) = \nu^2_{\widetilde{P}}(x_i)$, $\forall x_i \in X$, implies that $\zeta_E(\widetilde{P}, \widetilde{P}^c) = 0, e_E(\widetilde{P}) = 1$. Conversely, $e_E(\widetilde{P}) = 1$ implies $\zeta_E(\widetilde{P}, \widetilde{P}^c) = 0$. Thus, we have that $\mu^2_{\widetilde{P}}(x_i) = \nu^2_{\widetilde{P}}(x_i)$, $\forall x_i \in X$. For the axiom (A3), since $\mu^2_{\widetilde{P}}(x_i) \le \mu^2_{\widetilde{Q}}(x_i)$ and $\nu^2_{\widetilde{P}}(x_i) \ge \nu^2_{\widetilde{Q}}(x_i)$ for $\mu^2_{\widetilde{Q}}(x_i) \le \nu^2_{\widetilde{Q}}(x_i)$ imply that $\mu^2_{\widetilde{P}}(x_i) \le \mu^2_{\widetilde{Q}}(x_i) \le \nu^2_{\widetilde{Q}}(x_i) \le \nu^2_{\widetilde{P}}(x_i)$, then we have the distance $\forall x_i \in X$, $\sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i))^2} \ge \sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{Q}}(x_i) - \nu^2_{\widetilde{Q}}(x_i))^2}$. Again from the axiom (A3) of Definition 9, we have $\mu^2_{\widetilde{P}}(x_i) \ge \mu^2_{\widetilde{Q}}(x_i)$ and

$\nu^2_{\widetilde{P}}(x_i) \le \nu^2_{\widetilde{Q}}(x_i)$ for $\mu^2_{\widetilde{Q}}(x_i) \ge \nu^2_{\widetilde{Q}}(x_i)$ implies that $\nu^2_{\widetilde{P}}(x_i) \le \nu^2_{\widetilde{Q}}(x_i) \le \mu^2_{\widetilde{Q}}(x_i) \le \mu^2_{\widetilde{P}}(x_i)$, and then the distance $\forall x_i \in X$, $\sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i))^2} \ge \sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{Q}}(x_i) - \nu^2_{\widetilde{Q}}(x_i))^2}$. From inequalities $\sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i))^2} \ge \sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{Q}}(x_i) - \nu^2_{\widetilde{Q}}(x_i))^2}$ and $\sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i))^2} \ge \sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{Q}}(x_i) - \nu^2_{\widetilde{Q}}(x_i))^2}$, we have $\zeta_E(\widetilde{P}, \widetilde{P}^c) \ge \zeta_H(\widetilde{Q}, \widetilde{Q}^c)$, and then $1 - \zeta_E(\widetilde{P}, \widetilde{P}^c) \le 1 - \zeta_E(\widetilde{Q}, \widetilde{Q}^c)$. This concludes that $e_E(\widetilde{P}) \le e_E(\widetilde{Q})$. In this way the axiom (A3) is proved. Next, we prove the axiom (A4). Since $\forall x_i \in X$, $e_E(\widetilde{P}) = 1 - \zeta_E(\widetilde{P}, \widetilde{P}^c) = 1 - \sqrt{(1/n)\sum_{i=1}^n(\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i))^2} = 1 - \sqrt{(1/n)\sum_{i=1}^n(\nu^2_{\widetilde{P}}(x_i) - \mu^2_{\widetilde{P}}(x_i))^2}$, $1 - \zeta_E(\widetilde{P}^c, \widetilde{P}) = e_E(\widetilde{P}^c)$. Hence, the axiom (A4) is satisfied. Finally, for proving the axiom (A5), let $\widetilde{P}$ and $\widetilde{Q}$ be two PFSs. Then, we have

(i) $\mu^2_{\widetilde{P}}(x_i) \le \mu^2_{\widetilde{Q}}(x_i)$ *and* $\nu^2_{\widetilde{P}}(x_i) \ge \nu^2_{\widetilde{Q}}(x_i)$ *for* $\mu^2_{\widetilde{Q}}(x_i) \le \nu^2_{\widetilde{Q}}(x_i)$, $\forall x_i \in X$, or

(ii) $\mu^2_{\widetilde{P}}(x_i) \ge \mu^2_{\widetilde{Q}}(x_i)$ *and* $\nu^2_{\widetilde{P}}(x_i) \le \nu^2_{\widetilde{Q}}(x_i)$ *for* $\mu^2_{\widetilde{Q}}(x_i) \ge \nu^2_{\widetilde{Q}}(x_i)$, $\forall x_i \in X$.

From (i), we have $\forall x_i \in X$, $\mu^2_{\widetilde{P}}(x_i) \le \mu^2_{\widetilde{Q}}(x_i) \le \nu^2_{\widetilde{Q}}(x_i) \le \nu^2_{\widetilde{P}}(x_i)$, then $\max(\mu^2_{\widetilde{P}}(x_i), \mu^2_{\widetilde{Q}}(x_i)) = \mu^2_{\widetilde{Q}}(x_i)$ and $\min(\nu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{Q}}(x_i)) = \nu^2_{\widetilde{Q}}(x_i)$. That is, $(\widetilde{P} \cup \widetilde{Q}) = (\mu^2_{\widetilde{Q}}(x_i), \nu^2_{\widetilde{Q}}(x_i)) = \widetilde{Q}$ which implies that $e_E(\widetilde{P} \cup \widetilde{Q}) = e_E(\mu^2_{\widetilde{Q}}(x_i), \nu^2_{\widetilde{Q}}(x_i)) = e_E(\widetilde{Q})$. Also, $\forall x_i \in X$, $\min(\mu^2_{\widetilde{P}}(x_i), \mu^2_{\widetilde{Q}}(x_i)) = \mu^2_{\widetilde{P}}(x_i)$ and $\max(\nu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{Q}}(x_i)) = \nu^2_{\widetilde{P}}(x_i)$, then $(\widetilde{P} \cap \widetilde{Q}) = (\mu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{P}}(x_i)) = \widetilde{P}$ implies $e_E(\widetilde{P} \cap \widetilde{Q}) = e_E(\mu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{P}}(x_i)) = e_E(\widetilde{P})$. Hence, $e_E(\widetilde{P}) + e_E(\widetilde{Q}) = e_E(\widetilde{P} \cap \widetilde{Q}) + e_E(\widetilde{P} \cup \widetilde{Q})$. Again, from (ii), we have $\forall x_i \in X$, $\nu^2_{\widetilde{P}}(x_i) \le \nu^2_{\widetilde{Q}}(x_i) \le \mu^2_{\widetilde{Q}}(x_i) \le \mu^2_{\widetilde{P}}(x_i)$. Then, $\max(\mu^2_{\widetilde{P}}(x_i), \mu^2_{\widetilde{Q}}(x_i)) = \mu^2_{\widetilde{P}}(x_i)$ and $\min(\nu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{Q}}(x_i)) = \nu^2_{\widetilde{P}}(x_i)$, and so $(\widetilde{P} \cup \widetilde{Q}) = (\mu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{P}}(x_i)) = \widetilde{P}$ which implies that $e_E(\widetilde{P} \cup \widetilde{Q}) = e_E(\mu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{P}}(x_i)) = e_E(\widetilde{P})$. Also, $\forall x_i \in X$, $\min(\mu^2_{\widetilde{P}}(x_i), \mu^2_{\widetilde{Q}}(x_i)) = \mu^2_{\widetilde{Q}}(x_i)$ and $\max(\nu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{Q}}(x_i)) = \nu^2_{\widetilde{Q}}(x_i)$, then $(\widetilde{P} \cap \widetilde{Q}) = (\mu^2_{\widetilde{Q}}(x_i), \nu^2_{\widetilde{Q}}(x_i)) = \widetilde{Q}$ implies that $e_E(\widetilde{P} \cap \widetilde{Q}) = e_E(\mu^2_{\widetilde{Q}}(x_i), \nu^2_{\widetilde{Q}}(x_i)) = e_E(\widetilde{Q})$. Hence, $e_E(\widetilde{P}) + e_E(\widetilde{Q}) = e_E(\widetilde{P} \cap \widetilde{Q}) + e_E(\widetilde{P} \cup \widetilde{Q})$. Thus, we complete the proof of Theorem 13. □

Burillo and Bustince [31] gave fuzzy entropy of intuitionistic fuzzy sets by using intuitionistic index. Now, we modify and extend the similar concept to construct the new entropy measure of PFSs by using Pythagorean index as follows.

Let $\widetilde{P}$ be a PFS on $X = \{x_1, x_2, \ldots, x_n\}$. We define an entropy e_{PI} of $\widetilde{P}$ using Pythagorean index as

$$e_{PI}(\widetilde{P}) = \frac{1}{n}\sum_{k=1}^{n}\left(1 - \mu^2_{\widetilde{P}}(x_k) - \nu^2_{\widetilde{P}}(x_k)\right) = \frac{1}{n}\sum_{k=1}^{n}\pi^2_{\widetilde{P}}(x_k) \quad (15)$$

Theorem 14. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourse. The proposed entropy e_{PI} for the PFS $\widetilde{P}$ using Pythagorean index satisfies the axioms (A0)~(A4) and (A5) in Definition 9, and so it is a σ- entropy.*

Proof. Similar to Theorem 13. □

We next propose a new and simple method to calculate fuzzy entropy of PFSs by using the ratio of min and max operations. All three components $(\mu^2_{\widetilde{P}}, \nu^2_{\widetilde{P}}, \pi^2_{\widetilde{P}})$ of a PFS $\widetilde{P}$ are given equal importance to make the results more authentic and reliable. The new entropy is easy to be computed. Let $\widetilde{P}$ be a PFS on $X = \{x_1, x_2, \ldots, x_n\}$, and we define a new entropy for the PFS $\widetilde{P}$ as

$$e_{\min/\max}(\widetilde{P}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\min\left(\mu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{P}}(x_i), \pi^2_{\widetilde{P}}(x)\right)}{\max\left(\mu^2_{\widetilde{P}}(x_i), \nu^2_{\widetilde{P}}(x_i), \pi^2_{\widetilde{P}}(x)\right)} \quad (16)$$

Theorem 15. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourse. The proposed entropy $e_{\min/\max}$ for a PFS $\widetilde{P}$ satisfies the axioms (A0)~(A4) and (A5) in Definition 9, and so it is a σ-entropy.*

Proof. Similar to Theorem 13. □

Recently, Xue at el. [20] developed the entropy of Pythagorean fuzzy sets based on the similarity part and the hesitancy part that reflect fuzziness and uncertainty features, respectively. They defined the following Pythagorean fuzzy entropy for a PFS in a finite universe of discourses $X = \{x_1, x_2, \ldots, x_n\}$. Let $\widetilde{P} = \{\langle x_i, \mu_{\widetilde{P}}(x_i), \nu_{\widetilde{P}}(x_i)\rangle : x_i \in X\}$ be a PFS in X. The Pythagorean fuzzy entropy, $E_{Xue}(\widetilde{P})$, proposed by Xue at el. [20], is defined as

$$E_{Xue}(\widetilde{P}) = \frac{1}{n}\sum_{i=1}^{n}\left[1 - \left(\mu^2_{\widetilde{P}}(x_i) + \nu^2_{\widetilde{P}}(x_i)\right)\left|\mu^2_{\widetilde{P}}(x_i) - \nu^2_{\widetilde{P}}(x_i)\right|\right]. \quad (17)$$

The entropy $E_{Xue}(\widetilde{P})$ will be compared and exhibited in next section.

4. Examples and Comparisons

In this section, we present simple examples to observe behaviors of our proposed fuzzy entropies for PFSs. To make it mathematically sound and practically acceptable as well as choose better entropy by comparative analysis, we give an example involving linguistic hedges. By considering linguistic example, we use various linguistic hedges like "more or less large", "quite large" "very large", "very very large", etc. in the problems under Pythagorean fuzzy environment to select

Table 1: Comparison of the degree of fuzziness with different entropy measures of PFSs.

PFSs	e_{HC}^1	e_{HC}^2	e_E	e_{PI}	$e_{\min/\max}$
$\widetilde{P}$	0.4378	0.2368	0.2576	0.0126	0.0146
$\widetilde{Q}$	0.8131	0.5310	0.5700	0.5041	0.0643
$\widetilde{R}$	1.0363	0.6276	0.7225	0.3527	0.3999

Table 2: Degree of fuzziness from entropies of different PFSs.

PFSs	e_{HC}^1	e_{HC}^2	e_E	e_{PI}	$e_{\min/\max}$
$\widetilde{P}^{1/2}$	0.6407	0.3923	0.3490	0.2736	0.2013
$\widetilde{P}$	0.6066	0.3762	0.3386	0.3100	0.0957
$\widetilde{P}^{3/2}$	0.5375	0.3311	0.2969	0.3053	0.0542
$\widetilde{P}^{2}$	0.4734	0.2908	0.2638	0.2947	0.0233
$\widetilde{P}^{5/2}$	0.4231	0.2625	0.2414	0.2843	0.0108
$\widetilde{P}^{3}$	0.3848	0.2425	0.2267	0.2763	0.0050

better entropy. We check the performance and behaviors of the proposed entropy measures in the environment of PFSs by exhibiting its simple intuition as follows.

Example 1. Let $\widetilde{P}, \widetilde{Q}$, and $\widetilde{R}$ be singleton element PFSs in the universe of discourse $X = \{x_1\}$ defined as $\widetilde{P} = \{\langle x_1, 0.93, 0.35, 0.1122\rangle\}$, $\widetilde{Q} = \{\langle x_1, 0.68, 0.18, 0.7100\rangle\}$, and $\widetilde{R} = \{\langle x_1, 0.68, 0.43, 0.5939\rangle\}$. The numerical simulation results of entropy measures $e_{HC}^1, e_{HC}^2, e_E, e_{PI}$, and $e_{\min/\max}$ are shown in Table 1 for the purpose of numerical comparison. From Table 1, we can see the entropy measures of $\widetilde{R}$ almost have larger entropy than $\widetilde{P}$ and $\widetilde{Q}$ without any conflict, except e_{PI}. That is, the degree of uncertainty of $\widetilde{R}$ is greater than that of $\widetilde{P}$ and $\widetilde{Q}$. Furthermore, the behavior and performance of all entropies are analogous to each other, except e_{PI}. Apparently, all entropies e_{HC}^1, e_{HC}^2, e_E, and $e_{\min/\max}$ behave well, except e_{PI}.

However, in Example 1, it seems to be difficult for choosing appropriate entropy that may provide a better way to decide the fuzziness of PFSs. To overcome it, we give an example with structured linguistic variables to further analyze and compare these entropy measures in Pythagorean fuzzy environment. Thus, the following example with linguistic hedges is presented to further check behaviors and performance of the proposed entropy measures.

Example 2. Let $\widetilde{P}$ be a PFS in a universe of discourse $X = \{1, 3, 5, 7, 9\}$ defined as $\widetilde{P} = \{\langle 1, 0.1, 0.8\rangle, \langle 3, 0.4, 0.7\rangle, \langle 5, 0.5, 0.3\rangle, \langle 7, 0.9, 0.0\rangle, \langle 9, 1.0, 00\rangle\}$. By Definitions 6, 7, and 8 where the concentration and dilation of $\widetilde{P}$ are defined as Concentration: $CON(\widetilde{P}) = \widetilde{P}^2$, Dilation: $DIL(\widetilde{P})^{1/2}$. By considering the characterization of linguistic variables, we use the PFS $\widetilde{P}$ to define the strength of the structural linguistic variable $\widetilde{P}$ in $X = \{1, 3, 5, 7, 9\}$. Using above defined operators, we consider the following:

$\widetilde{P}^{1/2}$ is regarded as "More or less LARGE"; $\widetilde{P}$ is regarded as "LARGE";

$\widetilde{P}^{3/2}$ is regarded as "Quite LARGE"; $\widetilde{P}^2$ is regarded as "Very LARGE";

$\widetilde{P}^{5/2}$ is regarded as "Quite very LARGE"; $\widetilde{P}^3$ is regarded as "Very very LARGE".

We use above linguistic hedges for PFSs to compare the entropy measures $e_{HC}^1, e_{HC}^2, e_E, e_{PI}$, and $e_{\min/\max}$, respectively. From intuitive point of view, the following requirement of (18) for a good entropy measure should be followed:

$$e\left(\widetilde{P}^{1/2}\right) > e\left(\widetilde{P}\right) > e\left(\widetilde{P}^{3/2}\right) > e\left(\widetilde{P}^{2}\right) > e\left(\widetilde{P}^{5/2}\right) > e\left(\widetilde{P}^{3}\right). \tag{18}$$

After calculating these entropy measures $e_{HC}^1, e_{HC}^2, e_E, e_{PI}$, and $e_{\min/\max}$ for these PFSs, the results are shown in Table 2. From Table 2, it can be seen that these entropy measures e_{HC}^1, e_{HC}^2, e_E, and $e_{\min/\max}$ satisfy the requirement of (18), but e_{PI} fails to satisfy (18) that has $e_{PI}(\widetilde{P}) > e_{PI}(\widetilde{P}^{3/2}) > e_{PI}(\widetilde{P}^2) > e_{PI}(\widetilde{P}^{5/2}) > e_{PI}(\widetilde{P}^3) > e_{PI}(\widetilde{P}^{1/2})$. Therefore, we say that the behaviors of e_{HC}^1, e_{HC}^2, e_E, and $e_{\min/\max}$ are good, but e_{PI} is not.

In order to make more comparisons of entropy measures, we shake the degree of uncertainty of the middle value "5" in X. We decrease the degree of uncertainty for the middle point in X, and then we observe the amount of changes and also the impact of entropy measures when the degree of uncertainty of the middle value in X is decreasing. To observe how different PFS "LARGE" in X affects entropy measures, we modify $\widetilde{P}$ as

$$\text{"LARGE"} = \widetilde{P}_1 = \{\langle 1, 0.1, 0.8\rangle, \langle 3, 0.4, 0.7\rangle, \langle 5, 0.6, 0.5\rangle, \langle 7, 0.9, 0.0\rangle, \langle 9, 1.0, 00\rangle\} \tag{19}$$

Again, we use PFSs $\widetilde{P}_1^{1/2}, \widetilde{P}_1, \widetilde{P}_1^{3/2}, \widetilde{P}_1^{2}, \widetilde{P}_1^{5/2}$, and $\widetilde{P}_1^{3}$ with linguistic hedges to compare and observe behaviors of entropy measures. The results of degree of fuzziness for different PFSs from entropy measures are shown in Table 3. From Table 3, we can see that these entropies e_{HC}^1, e_E, and $e_{\min/\max}$ satisfy the requirement of (18), but e_{HC}^2 and e_{PI} could not fulfill the requirement of (18) with $e_{HC}^2(\widetilde{P}) > e_{HC}^2(\widetilde{P}^{1/2}) > e_{HC}^2(\widetilde{P}^{3/2}) > e_{HC}^2(\widetilde{P}^{2}) > e_{HC}^2(\widetilde{P}^{5/2}) > e_{HC}^2(\widetilde{P}^{3})$ and $e_{PI}(\widetilde{P}) > (\widetilde{P}^{3/2}) > e_{PI}(\widetilde{P}^{1/2}) > e_{PI}(\widetilde{P}^{2}) > e_{PI}(\widetilde{P}^{5/2}) > e_{PI}(\widetilde{P}^{3})$. Therefore, the

Table 3: Degree of fuzziness from entropies of different PFSs.

PFSs	e_{HC}^1	e_{HC}^2	e_E	e_{PI}	$e_{\min/\max}$
$\widetilde{P}_1^{1/2}$	0.6569	0.3952	0.3473	0.2360	0.2276
$\widetilde{P}_1^{1}$	0.6554	0.4086	0.3406	0.2560	0.1966
$\widetilde{P}_1^{3/2}$	0.5997	0.3757	0.2944	0.2440	0.1201
$\widetilde{P}_1^{2}$	0.5351	0.3356	0.2526	0.2281	0.0662
$\widetilde{P}_1^{5/2}$	0.4753	0.2993	0.2209	0.2145	0.0329
$\widetilde{P}_1^{3}$	0.4233	0.2682	0.1976	0.2038	0.0171

Table 4: Degree of fuzziness from entropies of different PFSs.

PFSs	E_{Xue}	e_{HC}^1	e_E	$e_{\min/\max}$
$\widetilde{P}$	0.4718	0.7532	0.3603	0.1270
$\widetilde{Q}$	0.4718	0.6886	0.3066	0.0470
$\widetilde{R}$	0.4718	0.7264	0.4241	0.1111

performance of e_{HC}^1, e_E, and $e_{\min/\max}$ is good, and e_{HC}^2 is not good, but e_{PI} presents very poor. We see that a little change in uncertainty for the middle value in X did not affect entropies e_{HC}^1, e_E, and $e_{\min/\max}$, and it brings a slight change in e_{HC}^2, but it gives an absolute big effect for entropy e_{PI}.

In viewing the results from Tables 1, 2, and 3, we may say that entropy measures e_{HC}^1, e_E, and $e_{\min/\max}$ present better performance. On the other hand, from the viewpoint of structured linguistic variables, we see that entropy measures e_{HC}^1, e_E, and $e_{\min/\max}$ are more suitable, reliable, and well suited in Pythagorean fuzzy environment for exhibiting the degree of fuzziness of PFS. We, therefore, recommend these entropies e_{HC}^1, e_E, and $e_{\min/\max}$ in a subsequent application involving multicriteria decision making.

In the following example, we conduct the comparison analysis of proposed entropies e_{HC}^1, e_E, and $e_{\min/\max}$ with the entropy $E_{Xue}(\widetilde{P})$, developed by Xue at el. [20], to demonstrate the advantages of our developed entropies e_{HC}^1, e_E, and $e_{\min/\max}$.

Example 3. Let $\widetilde{P}, \widetilde{Q}$, and $\widetilde{R}$ be PFSs in the singleton universe set $X = \{x_1\}$ as

$$\widetilde{P} = \{\langle x_1, 0.305, 856, 0.4174\rangle\},$$
$$\widetilde{Q} = \{\langle x_1, 0.1850, 0.8530, 0.4880\rangle\}, \quad (20)$$
$$\widetilde{R} = \{\langle x_1, 0.4130, 0.8640, 0.2880\rangle\}.$$

The degrees of entropy for different PFSs between the proposed entropies $e_{HC}^1, e_E, e_{\min/\max}$ and the entropy $E_{Xue}(\widetilde{P})$ by Xue at el. [20] are shown in Table 4. As can be seen from Table 4, we find that despite having three different PFSs, the entropy measure E_{Xue} could not distinguish the PFSs $\widetilde{P}, \widetilde{Q}$, and $\widetilde{R}$. However, the proposed entropy measures e_{HC}^1, e_E, and $e_{\min/\max}$ can actually differentiate these different PFSs $\widetilde{P}, \widetilde{Q}$, and $\widetilde{R}$.

5. Pythagorean Fuzzy Multicriterion Decision Making Based on New Entropies

In this section, we construct a new multicriterion decision making method. Specifically, we extend the technique for order preference by similarity to an ideal solution (TOPSIS) to multicriterion decision making, based on the proposed entropy measures for PFSs. Impreciseness and vagueness is a reality of daily life which requires close attentions in the matters of management and decision. In real life setting with decision making process, information available is often uncertain, vague, or imprecise. PFSs are found to be a powerful tool to solve decision making problems involving uncertain, vague, or imprecise information with high precision. To display practical reasonability and validity, we apply our proposed new entropies e_{HC}^1, e_E, and $e_{\min/\max}$ in a multicriteria decision making problem, involving unknown information about criteria weights for alternatives in Pythagorean fuzzy environment.

We formalize the problem in the form of decision matrix in which it lists various project alternatives. We assume that there are m project alternatives and we want to compare them on n various criteria $C_j, j = 1, 2, \ldots, n$. Suppose, for each criterion, we have an evaluation value. For instance, the first project on the first criterion has an evaluation x_{11}. The first project on the second criterion has an evaluation x_{12}, and the first project on nth criterion has an evaluation x_{1n}. Our objective is to have these evaluations on individual criteria and come up with a consolidated value for the project 1 and do something similar to the project 2 and so on. We then ultimately obtain a value for each of the projects. Finally, we can rank the projects with selecting the best one among all projects.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of alternatives, and let the set of criteria for the alternatives $A_i, i = 1, 2, \ldots, m$ be represented by $C_j, j = 1, 2, \ldots, n$. The aim is to choose the best alternative out of the n alternatives. The construction steps for the new Pythagorean fuzzy TOPSIS based on the proposed entropy measures are as follows.

Step 1 (construction of Pythagorean fuzzy decision matrix). Consider that the alternative A_i acting on the criteria C_j is represented in terms of Pythagorean fuzzy value $\widetilde{b}_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij})_p$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, where μ_{ij} denotes the degree of fulfillment, ν_{ij} represents the degree of not fulfillment, and π_{ij} represents the degree of hesitancy against the alternative A_i to the criteria C_j with the following conditions: $0 \le \mu_{ij}^2 \le 1$, $0 \le \nu_{ij}^2 \le 1$, $0 \le \pi_{ij}^2 \le 1$, and $\mu_{ij}^2+\nu_{ij}^2+\pi_{ij}^2 = 1$. The decision matrix $\widetilde{D} = (\widetilde{b}_{ij})_{m\times n}$ is constructed to handle the problems involving multicriterion decision making, where the decision matrix $\widetilde{D} = (\widetilde{b}_{ij})_{m\times n}$ can be constructed as follows:

$$\widetilde{D} = \left(\widetilde{b}_{ij}\right)_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} (\mu_{11},\nu_{11},\pi_{11})_p & (\mu_{12},\nu_{12},\pi_{12})_p & \cdots & (\mu_{1n},\nu_{1n},\pi_{1n})_p \\ (\mu_{21},\nu_{21},\pi_{21})_p & (\mu_{22},\nu_{22},\pi_{22})_p & \cdots & (\mu_{2n},\nu_{2n},\pi_{2n})_p \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m1},\nu_{m1},\pi_{m1})_p & (\mu_{m2},\nu_{m2},\pi_{m2})_p & \cdots & (\mu_{mn},\nu_{mn},\pi_{mn})_p \end{bmatrix} \end{array} \tag{21}$$

Step 2 (determination of the weights of criteria). In this step, the crux to the problem is that weights to criteria have to be identified. The weights or priorities can be obtained by different ways. Suppose that the criteria information weights are unknown and therefore, the weights w_j, $j = 1,2,3,\ldots,n$ of criteria for Pythagorean fuzzy entropy measures can be obtained by using e_{HC}^1, e_E, and $e_{\min/\max}$, respectively. Suppose the weights of criteria C_j, $j = 1,2,\ldots,n$ are w_j, $j = 1,2,3,\ldots,n$ with $0 \le w_j \le 1$ and $\sum_{j=1}^n w_j = 1$. Since weights of criteria are completely unknown, we propose a new entropy weighting method based on the proposed Pythagorean fuzzy entropy measures as follows:

$$w_j = \frac{\widetilde{E}_j}{\sum_{j=1}^n \widetilde{E}_j} \tag{22}$$

where the weights of the criteria C_j is calculated with $\widetilde{E}_j = (1/m)\sum_{i=1}^m \widetilde{b}_{ij}$.

Step 3 (Pythagorean fuzzy positive-ideal solution (PFPIS) and Pythagorean fuzzy negative-ideal solution (PFNIS)). In general, it is important to determine the positive-ideal solution (PIS) and negative-ideal solution (NIS) in a TOPSIS method. Since the evaluation criteria can be categorized into two categories, benefit and cost criteria in TOPSIS, let M_1 and M_2 be the sets of benefit criteria and cost criteria in criteria C_j, respectively. According to Pythagorean fuzzy sets and the principle of a TOPSIS method, we define a Pythagorean fuzzy PIS (PFPIS) as follows:

$$\begin{aligned} A^+ &= \left\{\left\langle C_j, \left(\mu_j^+, \nu_j^+, \pi_j^+\right)\right\rangle\right\}, \\ \text{where } &\left(\mu_j^+, \nu_j^+, \pi_j^+\right) = (1,0,0), \\ &\left(\mu_j^-, \nu_j^-, \pi_j^-\right) = (0,1,0), \\ & j \in M_1 \end{aligned} \tag{23}$$

Similarly, a Pythagorean fuzzy NIS (PFNIS) is defined as

$$\begin{aligned} A^- &= \left\{\left\langle C_j, \left(\mu_j^-, \nu_j^-, \pi_j^-\right)\right\rangle\right\}, \\ \text{where } &\left(\mu_j^-, \nu_j^-, \pi_j^-\right) = (0,1,0), \\ &\left(\mu_j^+, \nu_j^+, \pi_j^+\right) = (1,0,0), \\ & j \in M_2 \end{aligned} \tag{24}$$

Step 4 (calculation of distance measures from PFPIS and PFNIS). In this step, we need to use a distance between two PFSs. Following the similar idea from the previously defined PNE distance ζ_E between two PFSs, we define a Pythagorean weighted Euclidean (PWE) distance for any two PFSs $\widetilde{P}, \widetilde{Q} \in PFS(X)$ as

$$\begin{aligned} \zeta_{wE}\left(\widetilde{P},\widetilde{Q}\right) = \Bigg[\frac{1}{2}\sum_{i=1}^n w_i \bigg(&\left(\mu_{\widetilde{P}}^2(x_i) - \mu_{\widetilde{Q}}^2(x_i)\right)^2 \\ &+ \left(\nu_{\widetilde{P}}^2(x_i) - \nu_{\widetilde{Q}}^2(x_i)\right)^2 + \left(\pi_{\widetilde{P}}^2(x_i) - \pi_{\widetilde{Q}}^2(x_i)\right)^2\bigg)\Bigg]^{1/2} \end{aligned} \tag{25}$$

We next use the PWE distance ζ_{wE} to calculate the distances $\widetilde{D}^+(A_i)$ and $\widetilde{D}^-(A_i)$ of each alternative A_i from PFPIS and PFNIS, respectively, as follows:

$$\begin{aligned} \widetilde{D}^+(A_i) &= \widetilde{d}_E\left(A_i, A^+\right) \\ &= \sqrt{\frac{1}{2}\sum_{j=1}^n w_j\left[\left(1-\mu_{ij}^2\right)^2 + \left(\nu_{ij}^2\right)^2 + \left(1-\mu_{ij}^2-\nu_{ij}^2\right)^2\right]} \\ \widetilde{D}^-(A_i) &= \widetilde{d}_E\left(A_i, A^-\right) \\ &= \sqrt{\frac{1}{2}\sum_{j=1}^n w_j\left[\left(\mu_{ij}^2\right)^2 + \left(1-\nu_{ij}^2\right)^2 + \left(1-\mu_{ij}^2-\nu_{ij}^2\right)^2\right]} \end{aligned} \tag{26}$$

Step 5 (calculation of relative closeness degree and ranking of alternatives). The relative closeness degree $\widetilde{N}(A_i)$ of each

Table 5: Criteria to evaluate an audit company.

Criteria	Description of criteria
Required experience and capability to make independent decision (C_1)	Certification and required knowledge on accounting business and taxation law, understanding of management system, auditor should not be trembled or influence by anyone, actions, decision and report should be based on careful analysis.
The capability to comprehend different business needs (C_2)	Ability to work with different companies setups, analytical ability, planning and strategy.
Effective communication skills (C_3)	Mastered excellent communication skills, well versed in compelling report, convincing skills to present their reports, should be patient enough to elaborate points to the entire satisfaction of the auditee.

Table 6: Pythagorean fuzzy decision matrices.

Alternatives	Criteria Evalution*		
	C_1	C_2	C_3
A_1	(0.5500, 0.4130, 0.7259)	(0.6030, 0.51400, 0.6101)	(0.5000, 0.1000, 0.8602)
A_2	(0.4000, 0.2000, 0.8944)	(0.4000, 0.2000, 0.8944)	(0.4500, 0.5050, 0.7365)
A_3	(0.5000, 0.1000, 0.8602)	(0.6030, 0.51400, 0.6101)	(0.5500, 0.4130, 0.7259)

Table 7: Weight of criteria.

	w_1	w_2	w_3
E_{Xue}	0.3333	0.3333	0.3333
e^1_{HC}	0.2912	0.3646	0.3442
$e_{\min/\max}$	0.2209	0.4640	0.3151

alternative A_i with respect to PFPIS and PFNIS is obtained by using the following expression:

$$\widetilde{N}(A_i) = \frac{\widetilde{D}^-(A_i)}{\widetilde{D}^-(A_i) + \widetilde{D}^+(A_i)} \tag{27}$$

Finally, the alternatives are ordered according to the relative closeness degrees. The larger value of the relative closeness degrees reflects that an alternative is closer to PFPIS and farther from PFNIS, simultaneously. Therefore, the ranking order of all alternatives can be determined according to ascending order of the relative closeness degrees. The most preferred alternative is the one with the highest relative closeness degree.

In the next example, we present a comparison between the proposed entropies e^1_{HC} and $e_{\min/\max}$ with the entropy E_{Xue} by Xue at el. [20] based on PFSs for multicriteria decision making problem. The prime objective of decision makers is to select a best alternative from a set of available alternatives according to some criteria in multicriteria decision making process. Corruption is the misuse and mishandle of public power and resources for private and individual interest and benefits, usually in the form of bribery and favouritism. In addition, corruption twists and manipulates the basis of competitions by misallocating resources and slowing economic activity (Wikipedia).

Example 4. In this example, a real world problem on selection of well renowned national or international audit company is taken into account to demonstrate the comparison analysis among the proposed probabilistic entropy e^1_{HC} and non-probabilistic entropy $e_{\min/\max}$ with the entropy E_{Xue} [20]. To ensure the transparency and accountability of state run intuitions, the ministry of finance of a developing country offers quotations to select a renowned audit company to get unbiased and fair audit report to keep on tract the economic development of a state. The quotations which are gone through scrutiny process and found successful by a committee are comprised of experts. The quotations which are found successfully by the committee are called eligible while the rest are rejected. A commission of experts is invited to rank the audit companies $\{A_1, A_2, A_3\}$ and to select the best one on the basis of set criteria $\{C_1, C_2, C_3\}$. The descriptions about criteria are given in Table 5, and Pythagorean fuzzy decision matrices are presented in Table 6. The obtained weights of criteria from the entropies e^1_{HC}, $e_{\min/\max}$, and E_{Xue} are shown in Table 7. From Table 7, it is seen that the weights of criteria obtained by the entropy E_{Xue} are always the same despite having different alternatives. However, the proposed entropies e^1_{HC} and $e_{\min/\max}$ correctly differentiate the weights of criteria for each alternative A_i. The weights of criteria in Table 7 are also used to calculate the distances $\widetilde{D}^+(A_i)$ and $\widetilde{D}^-(A_i)$ of each alternative A_i from PFPIS and PFNIS, respectively, where the results are shown in Table 8. Furthermore, the relative closeness degrees of each alternative to ideal solution are shown in Table 9. It can be seen that the relative closeness degrees obtained by the proposed entropies e^1_{HC} and $e_{\min/\max}$ are different for different alternativesA_i, but the relative closeness degrees obtained by the entropy E_{Xue} [20] could not differentiate different

Table 8: Distance for each alternative.

E_{Xue}	$D^{-}(A_i)$	$D^{+}(A_i)$	e_{HC}^{1}	$D^{-}(A_i)$	$D^{+}(A_i)$	$e_{\min/\max}$	$D^{-}(A_i)$	$D^{+}(A_i)$
A_1	0.7593	0.6476	A_1	0.7587	0.6468	A_1	0.7455	0.6363
A_2	0.8230	0.7842	A_2	0.8208	0.7830	A_2	0.8269	0.7862
A_3	0.7593	0.6476	A_3	0.7493	0.6403	A_3	0.7284	0.6244

Table 9: Degree of relative closeness.

E_{Xue}	$N(A_i)$	e_{HC}^{1}	$N(A_i)$	$e_{\min/\max}$	$N(A_i)$
A_1	0.5397	A_1	0.5398	A_1	0.5395
A_2	0.5121	A_2	0.5118	A_2	0.5126
A_3	0.5397	A_3	0.5392	A_3	0.5384

Table 10: Ranking of alternatives by different methods.

Method	Ranking	Best alternative
E_{Xue}	$A_2 \prec A_1 = A_3$	None
e_{HC}^{1}	$A_2 \prec A_3 \prec A_1$	A_1
$e_{\min/\max}$	$A_2 \prec A_3 \prec A_1$	A_1

alternatives so that it gives bias ranking of alternatives. These ranking results of different alternatives by the entropies e_{HC}^{1}, $e_{\min/\max}$ and E_{Xue} are shown in Table 10. As can be seen, the ranking of alternatives by the proposed entropies e_{HC}^{1} and $e_{\min/\max}$ is well; however, the entropy E_{Xue} [20] could not rank the alternative A_1 and A_3. It is found that there is no conflict in ranking alternatives by using the proposed Pythagorean fuzzy TOPSIS method based on the proposed entropies e_{HC}^{1} and $e_{\min/\max}$. Totally, the comparative analysis shows that the best alternative is A_1.

We next apply the constructed Pythagorean fuzzy TOPSIS in multicriterion decision making for China-Pakistan Economic Corridor projects.

Example 5. A case study in ranking China-Pakistan Economic Corridor projects on priorities basis in the light of related experts' opinions is used in order to demonstrate the efficiency of the proposed Pythagorean fuzzy TOPSIS being applied to multicriterion decision making. China-Pakistan Economic Corridor (CPEC) is a collection of infrastructure projects that are currently under construction throughout in Pakistan. Originally valued at \$46 billion, the value of CPEC projects is now worth \$62 billion. CPEC is intended to rapidly modernize Pakistani infrastructure and strengthen its economy by the construction of modern transportation networks, numerous energy projects, and special economic zones. CPEC became partly operational when Chinese cargo was transported overland to Gwadar Port for onward maritime shipment to Africa and West Asia (see Wikipedia). It is not only to benefit China and Pakistan, but also to have positive impact on other countries and regions. Under CPEC projects, it will have more frequent and free exchanges of growth, people to people contacts, and integrated region of shared destiny by enhancing understanding through academic, cultural, regional knowledge, and activity of higher volume of flows in trades and businesses. The enhancement of geographical linkages and cooperation by a win-win model will result in improving the life standard of people, road, rail, and air transportation systems and also sustainable and perpetual development in China and Pakistan.

Now suppose the concern and relevant experts are allowed to rank the CPEC projects according to needs of both countries on priorities basis. Assume that initially there are five mega projects which are Gwadar Port (A_1), Infrastructure (A_2), Economic Zones (A_3), Transportation and Energy (A_4), and Social Sector Development (A_5), according to the following four criteria: time frame and infrastructural improvement (C_1), maintenance and sustainability (C_2), socioeconomic development (C_3), and eco-friendly (C_4). A detailed description of such criteria is displayed in Table 11. Consider a decision organization with the five concerns, where relevant experts are authorized to rank the satisfactory degree of an alternative with respect to the given criterion, which is represented by a Pythagorean fuzzy value (PFV). The evaluation values of the five alternatives A_i, $i = 1, 2, \ldots, 5$ with Pythagorean fuzzy decision matrix are given in Table 12.

We next use entropy measures e_{HC}^{1}, e_E, and $e_{\min/\max}$ based on better performance in numerical analysis to calculate the criteria weights w_j using (22). These results are shown in Table 13. From Table 13, we can see that the criteria weights and ranking of weights obtained by each entropy measure are different. We find the distances $\widetilde{D}^{+}$ and $\widetilde{D}^{-}$ of each alternative from PFPIS and PFNIS using (23) and (24). The results are shown in Table 14. We also calculate the relative closeness degrees $\widetilde{N}$ of alternatives using (27). The results are shown in Table 15. From Table 15, it can be clearly seen that, under different entropy measures, the relative closeness degrees of alternatives obtained are different, but the gap between these values are considerably small. Thus, the ranking of alternatives is almost the same. The final ranking results from different entropies are shown in Table 16. From Table 16, it can be identified that there is no conflict in selecting the best alternative among alternatives by using the proposed Pythagorean fuzzy TOPSIS method based on the entropies e_{HC}^{1}, e_E, and $e_{\min/\max}$. There is only one conflict to be found in deciding the preference ordering of alternatives A_4 and A_5 in e_E. Hence, the results of ranking of alternatives according to

Table 11: Criteria to assess CPEC projects.

Criterion	Description of criterion
Time frame and infrastructural improvement C_1	Roadmap to ensure timely completion of project without any interruption and hurdle and play a vital role in making infrastructural improvement and development
Maintenance and sustainability C_2	The maintenance, repair, reliability and sustainability of the project
Socioeconomic development C_3	Bring visible development and improvement in GDP, economy stability and prosperity, life expectancy, education, health, employment, personal dignity, personal safety and freedom
Eco – friendly C_4	Not harmful to the environment, contributes to green living, practices that help conserve natural resources and prevent contribution to air, water and land pollution.

Table 12: Pythagorean fuzzy decision matrix.

Alternatives	Criteria Evaluation			
	C_1	C_2	C_3	C_4
A_1	(0.60, 0.50, 0.6245)	(0.65, 0.45, 0.6124)	(0.35, 0.70, 0.6225)	(0.50, 0.70, 0.5099)
A_2	(0.80, 0.40, 0.4472)	(0.80, 0.40, 0.4472)	(0.70, 0.30, 0.6481)	(0.60, 0.30, 0.7416)
A_3	(0.60, 0.50, 0.6245)	(0.70, 0.50, 0.5099)	(0.70, 0.35, 0.6225)	(0.40, 0.20, 0.8944)
A_4	(0.90, 0.30, 0.3162)	(0.80, 0.35, 0.4873)	(0.50, 0.30, 0.8124)	(0.20, 0.50, 0.8426)
A_5	(0.80, 0.40, 0.4472)	(0.50, 0.30, 0.8124)	(0.70, 0.50, 0.5099)	(0.60, 0.50, 0.6245)

Table 13: Entropies and weights of the criteria.

	w_1	w_2	w_3	w_4
e_{HC}^1	0.2487	0.2563	0.2585	0.2366
e_E	0.2063	0.2411	0.2539	0.2987
$e_{\min/\max}$	0.3048	0.2524	0.2141	0.2288

Table 14: Distance for each alternative.

e_{HC}^1	$D^-(A_i)$	$D^+(A_i)$	e_E	$D^-(A_i)$	$D^+(A_i)$	$e_{\min/\max}$	$D^-(A_i)$	$D^+(A_i)$
A_1	0.5732	0.6297	A_1	0.5608	0.6354	A_1	0.5825	0.6196
A_2	0.7757	0.4381	A_2	0.7776	0.4557	A_2	0.7741	0.4294
A_3	0.7445	0.5848	A_3	0.7592	0.6055	A_3	0.7377	0.5865
A_4	0.8012	0.5819	A_4	0.7944	0.6163	A_4	0.8049	0.5582
A_5	0.7252	0.5268	A_5	0.7180	0.5331	A_5	0.7302	0.5195

Table 15: Degree of relative closeness for each alternative.

e_{HC}^1	$N(A_i)$	e_E	$N(A_i)$	$e_{\min/\max}$	$N(A_i)$
A_1	0.4765	A_1	0.4688	A_1	0.4846
A_2	0.6391	A_2	0.6305	A_2	0.6432
A_3	0.5601	A_3	0.5563	A_3	0.5571
A_4	0.5793	A_4	0.5631	A_4	0.5905
A_5	0.5792	A_5	0.5739	A_5	0.5843

the closeness degrees are made in an increasing order. Therefore, our analysis shows that the most feasible alternative is A_2 which is unanimously chosen by all proposed entropy measures.

6. Conclusions

In this paper, we have proposed new fuzzy entropy measures for PFSs based on probabilistic-type, distance, Pythagorean

Table 16: Ranking of alternative for different entropies.

Method	Ranking	Best alternative
e_{HC}^1	$A_1 \prec A_3 \prec A_5 \prec A_4 \prec A_2$	A_2
e_E	$A_1 \prec A_3 \prec A_4 \prec A_5 \prec A_2$	A_2
$e_{\min/\max}$	$A_1 \prec A_3 \prec A_5 \prec A_4 \prec A_2$	A_2

index, and min–max operator. We have also extended non-probabilistic entropy to σ-entropy for PFSs. The entropy measures are considered especially for PFSs on finite universes of discourses. As these are not only used in purposes of computing environment, but also used in more general cases for large universal sets. Structured linguistic variables are used to analyze and compare behaviors and performance of the proposed entropies for PFSs in different Pythagorean fuzzy environments. We have examined and analyzed these comparison results obtained from these entropy measures and then selected appropriate entropies which can be useful and also be helpful to decide fuzziness of PFSs more clearly and efficiently. We have utilized our proposed methods to perform comparison analysis with the most recently developed entropy measure for PFSs. In this connection, we have demonstrated a simple example and a problem involving MCDM to show the advantages of our suggested methods. Finally, the proposed entropy measures of PFSs are applied in an application to multicriterion decision making for ranking China-Pakistan Economic Corridor projects. Based on obtained results, we conclude that the proposed entropy measures for PFSs are reasonable, intuitive, and well suited in handling different kinds of problems, involving linguistic variables and multicriterion decision making in Pythagorean fuzzy environment.

Data Availability

All data are included in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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