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Research Article

A Novel Approach to a Time-Dependent-Coefficient WBK System: Doubly Periodic Waves and Singular Nonlinear Dynamics

Bo Xu[1] **and Sheng Zhang**[2,3,4]

[1]School of Education and Sports, Bohai University, Jinzhou 121013, China
[2]School of Mathematics and Physics, Bohai University, Jinzhou 121013, China
[3]School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China
[4]Department of Mathematics, Hohhot Minzu College, Hohhot 010051, China

Correspondence should be addressed to Sheng Zhang; szhangchina@126.com

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Under investigation in this paper is a more general time-dependent-coefficient Whitham-Broer-Kaup (tdcWBK) system, which includes some important models as special cases, such as the approximate equations for long water waves, the WBK equations in shallow water, the Boussinesq-Burgers equations, and the variant Boussinesq equations. To construct doubly periodic wave solutions, we extend the generalized F-expansion method for the first time to the tdcWBK system. As a result, many new Jacobi elliptic doubly periodic solutions are obtained; the limit forms of which are the hyperbolic function solutions and trigonometric function solutions. It is shown that the original F-expansion method cannot derive Jacobi elliptic doubly periodic solutions of the tdcWBK system, but the novel approach of this paper is valid. To gain more insight into the doubly periodic waves contained in the tdcWBK system, we simulate the dynamical evolutions of some obtained Jacobi elliptic doubly periodic solutions. The simulations show that the doubly periodic waves possess time-varying amplitudes and velocities as well as singularities in the process of propagations.

1. Introduction

Nonlinear complex phenomena in natural world, for example, solitons first observed by Russell in 1834 [1], are often described by nonlinear PDEs. Usually, people restore to exact solutions of nonlinear PDEs to gain more insight into the essence behind these nonlinear phenomena for further applications. In the past several decades, many effective methods for exactly solving nonlinear PDEs have been presented like those in [2–22]. In 2003, Zhou et al. proposed the so-called F-expansion method [22] to construct different Jacobi elliptic doubly periodic solutions of nonlinear PDEs in a uniform way, which can be thought of as an overall generalization of the Jacobi elliptic function expansion method [23]. The F-expansion method has been widely used to a great many of nonlinear PDEs [24–26] and was improved in different manners [27–30]. In 2006, Zhang and Xia [30] generalized the F-expansion method by introducing a new and more general ansätz. The present paper is motivated by the desire to extend the generalized F-expansion method [30] to the new and more general tcdWBK system [31, 32]:

$$\begin{aligned} u_t + \gamma_1 u u_x + \gamma_2 v_x + \gamma_3 u_{xx} &= 0, \\ v_t + \gamma_4 u_x v + \gamma_4 u v_x - \gamma_5 v_{xx} + \gamma_6 u_{xxx} &= 0, \end{aligned} \tag{1}$$

where $\gamma_i (i = 1, 2, \ldots, 6)$ are arbitrary smooth functions of t, which represent different dispersion and dissipation forces. Clearly, (1) includes some existing well-known important equations as special cases; they are the approximate equations for long water waves [33]:

$$\begin{aligned} u_t - u u_x - v_x + \frac{1}{2} u_{xx} &= 0, \\ v_t - (uv)_x - \frac{1}{2} v_{xx} &= 0, \end{aligned} \tag{2}$$

the WBK equations in shallow water [34]:

$$\begin{aligned} &u_t + uu_x + v_x + \beta u_{xx} = 0, \\ &v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0, \end{aligned} \tag{3}$$

the Boussinesq-Burgers (BB) equations [35]:

$$\begin{aligned} &u_t + 2uu_x - \frac{1}{2} v_x = 0, \\ &v_t + 2(uv)_x - \frac{1}{2} u_{xxx} = 0, \end{aligned} \tag{4}$$

the variant Boussinesq equations [34]:

$$\begin{aligned} &u_t + uu_x + v_x = 0, \\ &v_t + (uv)_x + u_{xxx} = 0. \end{aligned} \tag{5}$$

It should be pointed out that the F-expansion method [22] cannot derive Jacobi elliptic doubly periodic solutions of (1). To be specific, according to the F-expansion method [22], we first suppose that (1) has exact solutions of the forms:

$$\begin{aligned} u &= a_0 + \sum_{i=1}^{n} \left(a_i F^i(\xi) + b_i F^{-i}(\xi) \right), \\ v &= A_0 + \sum_{i=1}^{m} \left(A_i F^i(\xi) + B_i F^{-i}(\xi) \right), \end{aligned} \tag{6}$$

where $\xi = kx + \eta$, the integers n and m and the constant k are to be determined, while $a_0 = a_0(t)$, $a_i = a_i(t)$, $b_i = b_i(t)$, $A_0 = A_0(t)$, $A_i = A_i(t)$, $B_i = B_i(t)$, and $\eta = \eta(t)$ are all undetermined functions of the indicated variables; $F(\xi)$ satisfies

$$F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R, \tag{7}$$

and hence holds

$$\begin{aligned} &F''(\xi) = 2PF^3(\xi) + QF(\xi), \\ &F^{(3)}(\xi) = \left(6PF^2(\xi) + Q \right) F'(\xi), \\ &F^{(4)}(\xi) = 24P^2F^5(\xi) + 20PQF^3(\xi) + \left(Q^2 + 12PR \right) F(\xi), \dots, \end{aligned} \tag{8}$$

where P, Q, and R are parameters. In [30], Jacobi elliptic function solutions and their degenerated solutions of (7) are listed, which depend on the values of parameters P, Q, and R. Secondly, substituting (6) along with (7) and (8) into (1) and then balancing the highest order partial derivative uu_x and the highest order nonlinear term u_{xx} yield the integer $2n + 1 = n + 2$ which gives $n = 1$. Similarly, we determine the integer $m = 2$ by balancing uv_x and u_{xxx}. Thirdly, we substitute (6) given the values of $n = 1$ and $m = 2$ along with (7) and (8) into (1) and collect all terms with the same order of $F^j(\xi)F'^s(\xi)$ $(j = 0, \pm 1, \pm 2, \dots, s = 0, 1)$ together, we have

$$\begin{aligned}
&-k\left(b_1^2\gamma_1 + 2B_2\gamma_2\right)F'(\xi)F^{-3}(\xi) \\
&\quad - \left(ka_0b_1\gamma_1 + kB_1\gamma_2 + b_1\eta'\right)F'(\xi)F^{-2}(\xi) \\
&\quad + \left(ka_0a_1\gamma_1 + kA_1\gamma_2 + a_1\eta'\right)F'(\xi) \\
&\quad + k\left(a_1^2\gamma_1 + 2A_2\gamma_2\right)F'(\xi)F(\xi) \\
&\quad + 2k^2Rb_1\gamma_3F^{-3}(\xi) + \left(k^2Qb_1\gamma_3 + b_1'\right)F^{-1}(\xi) \\
&\quad + a_0' + \left(k^2Qa_1\gamma_3 + a_1'\right)F(\xi) \\
&\quad + 2k^2Pa_1\gamma_3F^3(\xi) = 0, \\
&-3kb_1\left(B_2\gamma_4 + 2k^2R\gamma_6\right)F'(\xi)F^{-4}(\xi) \\
&\quad - 2\left(kb_1B_1\gamma_4 + ka_0B_2\gamma_4 + B_2\eta'\right)F'(\xi)F^{-3}(\xi) \\
&\quad - \left(kA_0b_1\gamma_4 + ka_0B_1\gamma_4 + ka_1B_2\gamma_4 + k^3Qb_1\gamma_6 + B_1\eta'\right)F' \\
&\quad \cdot (\xi)F^{-2}(\xi) + \left(kA_0a_1\gamma_4 + ka_0A_1\gamma_4 + kA_2b_1\gamma_4\right. \\
&\qquad\qquad \left. + k^3Qa_1\gamma_6 + A_1\eta'\right)F'(\xi) \\
&\quad + 2\left(ka_1A_1\gamma_4 + ka_0A_2\gamma_4 + A_2\eta'\right)F'(\xi)F(\xi) \\
&\quad + 3ka_1\left(A_2\gamma_4 + 2k^2P\gamma_6\right)F'(\xi)F^2(\xi) \\
&\quad - 6k^2RB_2\gamma_5F^{-4}(\xi) - 2k^2RB_1\gamma_5F^{-3}(\xi) \\
&\quad - \left(4k^2QB_2\gamma_5 - B_2'\right)F^{-2}(\xi) - \left(k^2QB_1\gamma_5 - B_1'\right)F^{-1}(\xi) \\
&\quad - 2k^2RA_2\gamma_5 - 2k^2PB_2\gamma_5 + A_0' - \left(k^2QA_1\gamma_5 - A_1'\right)F(\xi) \\
&\quad - \left(4k^2QA_2\gamma_5 - A_2'\right)F^2(\xi) - 2k^2PA_1\gamma_5F^3(\xi) \\
&\quad - 6k^2PA_2\gamma_5F^4(\xi) = 0.
\end{aligned} \tag{9}$$

Since $PQR \neq 0$ is the necessary condition that (7) exists Jacobi elliptic doubly periodic, without loss of generality, we have

$$a_0' = a_1 = b_1 = A_0' = A_1 = A_2 = B_1 = B_2 = 0, \tag{10}$$

when setting each coefficient of $F^j(\xi)F'^s(\xi)$ of (9) to zeros. This tells that (6) let (1) has only constant solutions but not Jacobi elliptic doubly periodic solutions as expected.

The present paper is motivated by the desire to investigate Jacobi elliptic doubly periodic solutions of (1). The rest of the paper is organized as follows. In Section 2, the generalized F-expansion method [30] is extended to (1) for constructing Jacobi elliptic doubly periodic solutions. In Section 3, we simulate the dynamical evolutions of some obtained Jacobi elliptic doubly periodic solutions to gain more insight into the doubly periodic waves contained in (1). In Section 4, we conclude this paper.

2. Doubly Periodic Waves

To extend the generalized F-expansion method [30] to (1), in this section, we suppose that (1) has Jacobi elliptic doubly periodic solutions of the forms:

$$\begin{aligned} u &= a_0 + a_1 F(\xi) + b_1 F^{-1}(\xi) + c_1 F'(\xi) \\ &\quad + d_1 F^{-1}(\xi) F'(\xi), \\ v &= A_0 + A_1 F(\xi) + A_2 F^2(\xi) + B_1 F^{-1}(\xi) \\ &\quad + B_2 F^{-2}(\xi) + C_1 F'(\xi) + C_2 F(\xi) F'(\xi) \\ &\quad + D_1 F^{-1}(\xi) F'(\xi) + D_2 F^{-2}(\xi) F'(\xi). \end{aligned} \tag{11}$$

Substituting (11) along with (7) and (8) into (1) and collecting all terms with the same order of $F^{j}(\xi)F'^{s}(\xi)(j = 0, \pm 1, \pm 2, \ldots, s = 0, 1)$ together, we derive a set of nonlinear PDEs for $a_0, a_1, b_1, c_1, d_1, A_0, A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, \eta$, and k. Solving this set of nonlinear PDEs, we have three cases.

Case 1.

$$\begin{aligned} a_1 &= \pm \frac{2k\sqrt{P(\gamma_3^2 + c_0\gamma_1\gamma_6)}}{\gamma_1}, \\ b_1 &= \pm \frac{2k\sqrt{R(\gamma_3^2 + c_0\gamma_1\gamma_6)}}{\gamma_1}, \\ a_0 &= A_0 = \text{const.}, \\ A_2 &= -\frac{2k^2 P\gamma_3(\gamma_3^2 + c_0\gamma_1\gamma_6)}{c_0\gamma_1^2}, \\ B_2 &= -\frac{2k^2 R\gamma_3(\gamma_3^2 + c_0\gamma_1\gamma_6)}{c_0\gamma_1^2}, \\ C_1 &= \mp \frac{2k^2\gamma_3\sqrt{P(\gamma_3^2 + c_0\gamma_1\gamma_6)}}{c_0\gamma_1^2}, \\ D_2 &= \pm \frac{2k^2\gamma_3\sqrt{R(\gamma_3^2 + c_0\gamma_1\gamma_6)}}{c_0\gamma_1^2}, \\ c_1 &= d_1 = A_1 = B_1 = C_2 = D_1 = 0, \\ \eta &= -ka_0 \int \gamma_1 \mathrm{d}t, \end{aligned} \tag{12}$$

where k is an arbitrary constant; the signs "±" and "∓" in (12) and (13) mean that all possible combinations of "+" and "−" can be taken. If it is taken the same sign in a_1 and b_1, then it must be taken "−" in γ_3' and γ_6'. If it is taken the different signs in a_1 and b_1, then it must be taken "+" in γ_3' and γ_6'. At the same time, it must be taken the different signs in a_1 and C_1 and the same sign in b_1 and D_2. While $\gamma_i (i = 1, 3, 6)$ in (12) satisfy the constraints:

$$\begin{aligned} \gamma_4 &= \gamma_1, \\ \gamma_5 &= \gamma_3, \\ \gamma_2 &= c_0\gamma_1, \\ \gamma_3' &= c_0A_0\gamma_1^2 + k^2\left(Q \mp 2\sqrt{P}\right)(\gamma_3^2 + c_0\gamma_1\gamma_6) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 + k^2\left(Q \mp 2\sqrt{P}\right)(\gamma_3^2 + c_0\gamma_1\gamma_6) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \end{aligned} \tag{13}$$

Case 2.

$$\begin{aligned} a_1 &= \pm \frac{2k\sqrt{P(\gamma_3^2 + c_0\gamma_1\gamma_6)}}{\gamma_1}, \\ a_0 &= A_0 = \text{const.}, \end{aligned} \tag{14}$$

$$\begin{aligned} A_2 &= -\frac{2k^2 P\gamma_3(\gamma_3^2 + c_0\gamma_1\gamma_6)}{c_0\gamma_1^2}, \\ C_1 &= \mp \frac{2k^2\gamma_3\sqrt{P(\gamma_3^2 + c_0\gamma_1\gamma_6)}}{c_0\gamma_1^2}, \end{aligned} \tag{15}$$

$$\begin{aligned} b_1 &= c_1 = d_1 = A_1 = B_1 = B_2 = C_2 = D_1 = D_2 = 0, \\ \eta &= -ka_0 \int \gamma_1 \mathrm{d}t, \end{aligned} \tag{16}$$

where k is an arbitrary constant; the signs "±" and "∓" in (14) and (15) mean that it must be taken the different signs in a_1

and C_1, while $\gamma_i (i = 1, 3, 6)$ in (14), (15), and (16) satisfy the constraints:

$$
\begin{aligned}
&\gamma_4 = \gamma_1, \\
&\gamma_5 = \gamma_3, \\
&\gamma_2 = c_0\gamma_1, \\
&\gamma_3' = c_0A_0\gamma_1^2 + k^2Q\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\
&\gamma_6' = \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 + k^2Q\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}.
\end{aligned}
\tag{17}
$$

Case 3.

$$
\begin{aligned}
&b_1 = \pm\frac{2k\sqrt{R\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{\gamma_1}, \\
&a_0 = A_0 = \text{const.},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&B_2 = -\frac{2k^2R\gamma_3\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}, \\
&D_2 = \mp\frac{2k^2\gamma_3\sqrt{R\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{c_0\gamma_1^2},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&a_1 = c_1 = d_1 = A_1 = A_2 = B_1 = C_1 = C_2 = D_1 = 0, \\
&\eta = -ka_0\int \gamma_1 \mathrm{d}t,
\end{aligned}
\tag{20}
$$

where k is an arbitrary constant; the signs "±" and "∓" in (18) and (19) mean that it must be taken the same sign in b_1 and D_2, while $\gamma_i (i = 1, 3, \ldots, 6)$ in (18), (19), and (20) satisfy the same constraints as Case 2 in (17).

From Cases 1–3, we obtain three formulae of fundamental solutions of (1) as follows:

$$
\begin{aligned}
u = {} & a_0 \pm \frac{2k\sqrt{P\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{\gamma_1}F(\xi) \pm \frac{2k\sqrt{R\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{\gamma_1}F^{-1}(\xi), \\
v = {} & A_0 - \frac{2k^2P\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}F^2(\xi) - \frac{2k^2R\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}F^{-2}(\xi) \mp \frac{2k^2\gamma_3\sqrt{P\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{c_0\gamma_1^2}F'(\xi) \\
& \pm \frac{2k^2\gamma_3\sqrt{R\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{c_0\gamma_1^2}F^{-2}(\xi)F'(\xi),
\end{aligned}
\tag{21}
$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0\int\gamma_1 \mathrm{d}t$, and $\gamma_i (i = 1, 2, 3, 6)$ in (21) satisfy the constraints in (13).

$$
\begin{aligned}
u = {} & a_0 \pm \frac{2k\sqrt{P\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{\gamma_1}F(\xi), \\
v = {} & A_0 - \frac{2k^2P\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}F^2(\xi) \\
& \mp \frac{2k^2\gamma_3\sqrt{P\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{c_0\gamma_1^2}F'(\xi),
\end{aligned}
\tag{22}
$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0\int\gamma_1 \mathrm{d}t$, and $\gamma_i (i = 1, 3, 6)$ in (22) satisfy the constraints in (17).

$$
\begin{aligned}
u = {} & a_0 \pm \frac{2k\sqrt{R\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{\gamma_1}F^{-1}(\xi), \\
v = {} & A_0 - \frac{2k^2R\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}F^{-2}(\xi) \\
& \pm \frac{2k^2\gamma_3\sqrt{R\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{c_0\gamma_1^2}F^{-2}(\xi)F'(\xi),
\end{aligned}
\tag{23}
$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 dt$, and $\gamma_i (i = 1, 2, 3, 6)$ in (24) and (25) satisfy the constraints in (17).

With the help of (22), (23), (24), and (25) and (Appendices A, B, and C [30]), we obtain many exact solutions of (1). For example, selecting $P = 1$, $Q = -(1 + m^2)$, $R = m^2$, and $F(\xi) = \mathrm{ns}\xi$, from (21), we obtain new Jacobi elliptic doubly periodic solutions of (1):

$$u = a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{\gamma_1}\mathrm{ns}\xi \pm \frac{2km\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{\gamma_1}\mathrm{sn}\xi, \tag{24}$$

$$\begin{aligned} v = A_0 &- \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}\mathrm{ns}^2\xi \\ &- \frac{2k^2m^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}\mathrm{sn}^2\xi \\ &\pm \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{c_0\gamma_1^2}\mathrm{cs}\xi\mathrm{ds}\xi \\ &\mp \frac{2k^2m\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{c_0\gamma_1^2}\mathrm{sn}^2\xi\mathrm{cs}\xi\mathrm{ds}\xi, \end{aligned} \tag{25}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 dt$, and $\gamma_i (i = 1, 3, 6)$ in (24) and (25) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0A_0\gamma_1' + k^2\left(-1 - m^2 \mp 2\right)\left(\gamma_3' + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 + k^2\left(-1 - m^2 \mp 2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \end{aligned} \tag{26}$$

Selecting $P = -1$, $Q = 2 - m^2$, $R = m^2 - 1$, and $F(\xi) = \mathrm{dn}\xi$, from (22), we obtain new Jacobi elliptic doubly periodic solutions of (1):

$$\begin{aligned} u &= a_0 \pm \frac{2k\sqrt{-\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{\gamma_1}\mathrm{dn}\xi, \\ v &= A_0 + \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}\mathrm{dn}^2(\xi) \\ &\quad \pm \frac{2k^2m^2\gamma_3\sqrt{-\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{c_0\gamma_1^2}\mathrm{sn}\xi\mathrm{cn}\xi, \end{aligned} \tag{27}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 dt$, and $\gamma_i (i = 1, 3, 6)$ in (27) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0A_0\gamma_1^2 + k^2\left(2 - m^2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 + k^2\left(2 - m^2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \end{aligned} \tag{28}$$

Selecting $P = 1$, $Q = -(1 + m^2)$, $R = m^2$, and $F(\xi) = \mathrm{ns}\xi$, from (22), we obtain new Jacobi elliptic doubly periodic solutions of (1):

$$\begin{aligned} u &= a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{\gamma_1}\mathrm{ns}\xi, \\ v &= A_0 - \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}\mathrm{ns}^2(\xi) \pm \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{c_0\gamma_1^2}\mathrm{cs}\xi\mathrm{ds}\xi, \end{aligned} \tag{29}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \, dt$, and $\gamma_i (i = 1, 3, 6)$ in (29) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0A_0\gamma_1^2 - k^2\left(1 + m^2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 - k^2\left(1 + m^2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \end{aligned} \tag{30}$$

Selecting $P = 1 - m^2$, $Q = 2 - m^2$, $R = 1$, and $F(\xi) = \mathrm{sc}\xi$, from (22), we obtain new Jacobi elliptic doubly periodic solutions of (1):

$$\begin{aligned} u &= a_0 \pm \frac{2k\sqrt{\left(1 - m^2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{\gamma_1}\mathrm{sc}\xi, \\ v &= A_0 - \frac{2k^2\left(1 - m^2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}\mathrm{sc}^2(\xi) \mp \frac{2k^2\gamma_3\sqrt{\left(1 - m^2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{c_0\gamma_1^2}\mathrm{dc}\xi\mathrm{nc}\xi, \end{aligned} \tag{31}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \, dt$, and $\gamma_i (i = 1, 3, 6)$ in (31) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0A_0\gamma_1^2 + k^2\left(2 - m^2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 + k^2\left(2 - m^2\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \end{aligned} \tag{32}$$

Selecting $P = 1$, $Q = 2m^2 - 1$, $R = -m^2(1 - m^2)$, and $F(\xi) = \mathrm{ds}\xi$, from (22), we obtain new Jacobi elliptic doubly periodic solutions of (1):

$$\begin{aligned} u &= a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{\gamma_1}\mathrm{ds}\xi, \\ v &= A_0 - \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2}\mathrm{ds}^2\xi \pm \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{c_0\gamma_1^2}\mathrm{cs}\xi\mathrm{ns}\xi, \end{aligned} \tag{33}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \, dt$, and $\gamma_i (i = 1, 3, 6)$ in (33) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\gamma_3' = c_0A_0\gamma_1^2 + k^2\left(2m^2 - 1\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \tag{34}$$

$$\gamma_6' = \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 + k^2\left(2m^2 - 1\right)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \tag{35}$$

In the limits at $m \to 1$ and $m \to 0$, the above obtained Jacobi elliptic doubly periodic solutions degenerate into hyperbolic function solutions and trigonometric function solutions, respectively. When $m \to 1$, the Jacobi elliptic doubly periodic solutions (24) and (25) degenerate into hyperbolic function solutions:

$$\begin{aligned} u &= a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{\gamma_1} \coth \xi \\ &\quad \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{\gamma_1} \tanh \xi, \\ v &= A_0 - \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2} \coth^2\xi \\ &\quad - \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2} \tanh^2\xi \\ &\quad \pm \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{c_0\gamma_1^2} \operatorname{csch}^2\xi \\ &\quad \mp \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{c_0\gamma_1^2} \operatorname{sech}^2\xi, \end{aligned} \tag{36}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \, dt$, and $\gamma_i (i = 1, 3, 6)$ in (36) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0A_0\gamma_1^2 + k^2(-2 \mp 2)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 + k^2(-2 \mp 2)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \end{aligned} \tag{37}$$

When $m \to 0$, the Jacobi elliptic doubly periodic solutions (24) and (25) degenerate into trigonometric function solutions:

$$\begin{aligned} u &= a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{\gamma_1} \csc \xi, \\ v &= A_0 - \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2} \csc^2\xi \\ &\quad \pm \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{c_0\gamma_1^2} \cot \xi \csc \xi, \end{aligned} \tag{38}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \, dt$, and $\gamma_i (i = 1, 3, 6)$ in (38) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0A_0\gamma_1^2 + k^2(-1 \mp 2)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 + k^2(-1 \mp 2)\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \end{aligned} \tag{39}$$

When $m \to 1$, the Jacobi elliptic doubly periodic solutions (27) degenerate into hyperbolic function solutions:

$$\begin{aligned} u &= a_0 \pm \frac{2k\sqrt{-\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{\gamma_1} \operatorname{sech} \xi, \\ v &= A_0 - \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2} \operatorname{sech}^2(\xi) \\ &\quad \pm \frac{2k^2\gamma_3\sqrt{-\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}}{c_0\gamma_1^2} \tanh \xi \operatorname{sech} \xi, \end{aligned} \tag{40}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \mathrm{d}t$, and $\gamma_i (i = 1, 3, 6)$ in (40) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, and

$$\begin{aligned} \gamma_3' &= c_0 A_0 \gamma_1^2 + k^2 \left(\gamma_3^2 + c_0 \gamma_1 \gamma_6\right) + \frac{\gamma_3 \gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0 \gamma_1 \gamma_1' \gamma_3^2 + c_0^2 \gamma_1^2 \gamma_1' \gamma_6 - 2c_0 \gamma_1^2 \gamma_3 \left[c_0 A_0 \gamma_1^2 + k^2 \left(\gamma_3^2 + c_0 \gamma_1 \gamma_6\right) - 2c_0 \gamma_1 \gamma_1' \gamma_3\right]}{c_0^2 \gamma_1^3}. \end{aligned} \tag{41}$$

When $m \to 1$, the Jacobi elliptic doubly periodic solutions (29) degenerate into hyperbolic function solutions:

$$u = a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0 \gamma_1 \gamma_6}}{\gamma_1} \coth \xi, \tag{42}$$

$$\begin{aligned} v &= A_0 - \frac{2k^2 \left(\gamma_3^2 + c_0 \gamma_1 \gamma_6\right)}{c_0 \gamma_1^2} \coth^2(\xi) \\ &\quad \pm \frac{2k^2 \gamma_3 \sqrt{\gamma_3^2 + c_0 \gamma_1 \gamma_6}}{c_0 \gamma_1^2} \operatorname{csch}^2 \xi, \end{aligned} \tag{43}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \mathrm{d}t$, and $\gamma_i (i = 1, 3, 6)$ in (42) and (43) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0 \gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0 A_0 \gamma_1^2 - 2k^2 \left(\gamma_3^2 + c_0 \gamma_1 \gamma_6\right) + \frac{\gamma_3 \gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0 \gamma_1 \gamma_1' \gamma_3^2 + c_0^2 \gamma_1^2 \gamma_1' \gamma_6 - 2c_0 \gamma_1^2 \gamma_3 \left[c_0 A_0 \gamma_1^2 - 2k^2 \left(\gamma_3^2 + c_0 \gamma_1 \gamma_6\right) - 2c_0 \gamma_1 \gamma_1' \gamma_3\right]}{c_0^2 \gamma_1^3}. \end{aligned} \tag{44}$$

When $m \to 0$, the Jacobi elliptic doubly periodic solutions (31) degenerate into trigonometric function solutions:

$$\begin{aligned} u &= a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0 \gamma_1 \gamma_6}}{\gamma_1} \tan \xi, \\ v &= A_0 - \frac{2k^2 \left(\gamma_3^2 + c_0 \gamma_1 \gamma_6\right)}{c_0 \gamma_1^2} \tan^2(\xi) \\ &\quad \mp \frac{2k^2 \gamma_3 \sqrt{\gamma_3^2 + c_0 \gamma_1 \gamma_6}}{c_0 \gamma_1^2} \sec^2 \xi, \end{aligned} \tag{45}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \mathrm{d}t$, and $\gamma_i (i = 1, 3, 6)$ in (45) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0 \gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0 A_0 \gamma_1^2 + 2k^2 \left(\gamma_3^2 + c_0 \gamma_1 \gamma_6\right) + \frac{\gamma_3 \gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0 \gamma_1 \gamma_1' \gamma_3^2 + c_0^2 \gamma_1^2 \gamma_1' \gamma_6 - 2c_0 \gamma_1^2 \gamma_3 \left[c_0 A_0 \gamma_1^2 + 2k^2 \left(\gamma_3^2 + c_0 \gamma_1 \gamma_6\right) - 2c_0 \gamma_1 \gamma_1' \gamma_3\right]}{c_0^2 \gamma_1^3}. \end{aligned} \tag{46}$$

When $m \to 1$, the Jacobi elliptic doubly periodic solutions (33) degenerate into trigonometric function solutions:

$$u = a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{\gamma_1} \operatorname{csch} \xi,$$

$$v = A_0 - \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2} \operatorname{csch}^2\xi \pm \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{c_0\gamma_1^2} \coth \xi \operatorname{csch} \xi, \tag{47}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \, dt$, and $\gamma_i (i = 1, 3, 6)$ in (47) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0A_0\gamma_1^2 + k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 + k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \end{aligned} \tag{48}$$

When $m \to 0$, the Jacobi elliptic doubly periodic solutions (33) degenerate into the following trigonometric function solutions which have the same expressions as solutions (38) but with different constraints (50):

$$u = a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{\gamma_1} \csc \xi,$$

$$v = A_0 - \frac{2k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right)}{c_0\gamma_1^2} \csc^2\xi \pm \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_1\gamma_6}}{c_0\gamma_1^2} \cot \xi \csc \xi, \tag{49}$$

where a_0, A_0, and k are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 \, dt$, and $\gamma_i (i = 1, 3, 6)$ in (49) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\begin{aligned} \gamma_3' &= c_0A_0\gamma_1^2 - k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) + \frac{\gamma_3\gamma_1'}{\gamma_1}, \\ \gamma_6' &= \frac{2c_0\gamma_1\gamma_1'\gamma_3^2 + c_0^2\gamma_1^2\gamma_1'\gamma_6 - 2c_0\gamma_1^2\gamma_3\left[c_0A_0\gamma_1^2 - k^2\left(\gamma_3^2 + c_0\gamma_1\gamma_6\right) - 2c_0\gamma_1\gamma_1'\gamma_3\right]}{c_0^2\gamma_1^3}. \end{aligned} \tag{50}$$

3. Singular Nonlinear Dynamics

In this section, we further investigate the nonlinear dynamics of (1) by means of Jacobi elliptic doubly periodic solutions. Firstly, we consider solutions (24) and (25). To determine γ_3 and γ_6 with the sign "−" in (26), we select $\gamma_1 = e^t$ and then have

$$\begin{aligned} \gamma_3 &= e^{-\left(e^{-2t}/2A_0c_0\right)}\left(c_1 \int e^{e^{-2t}/2A_0c_0} dt + c_2\right), \\ \gamma_6 &= \frac{-e^{-\left(e^{-2t}/2A_0c_0\right)-t}\left[A_0c_0c_1\left(-1 + m^2\right)^2 e^{\left(e^{-2t}/2A_0c_0\right)+2t} + k^2\left(-1 + m^2\right)^2 + k^2(1 - m)^2\left(c_1 \int e^{e^{-2t}/2A_0c_0} dt + c_2\right)^2\right]}{c_0k^2\left(-1 + m^2\right)^2}, \end{aligned} \tag{51}$$

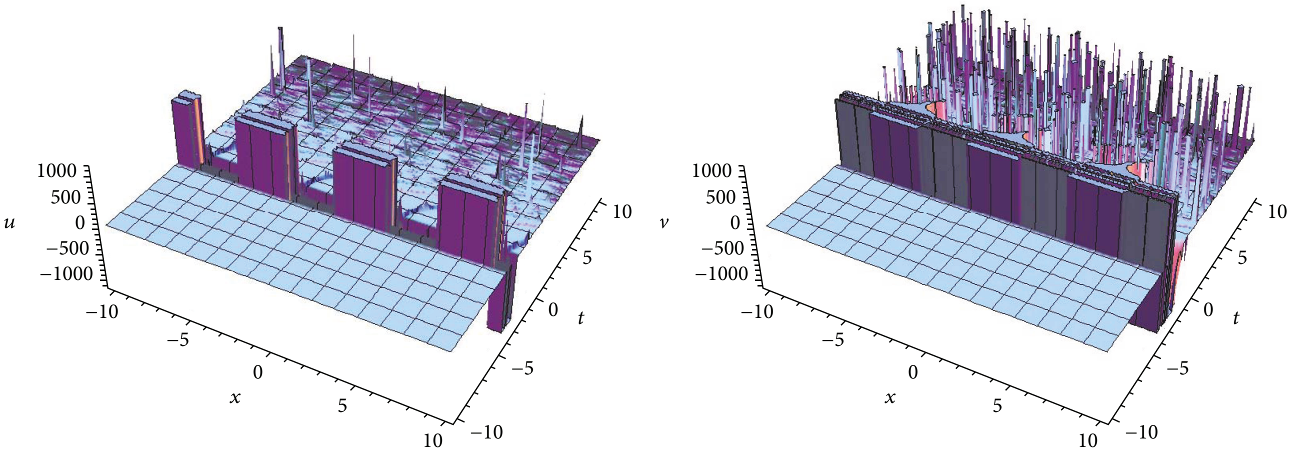


Figure 1: Spatial structures of solutions (24) and (25) determined by (51).

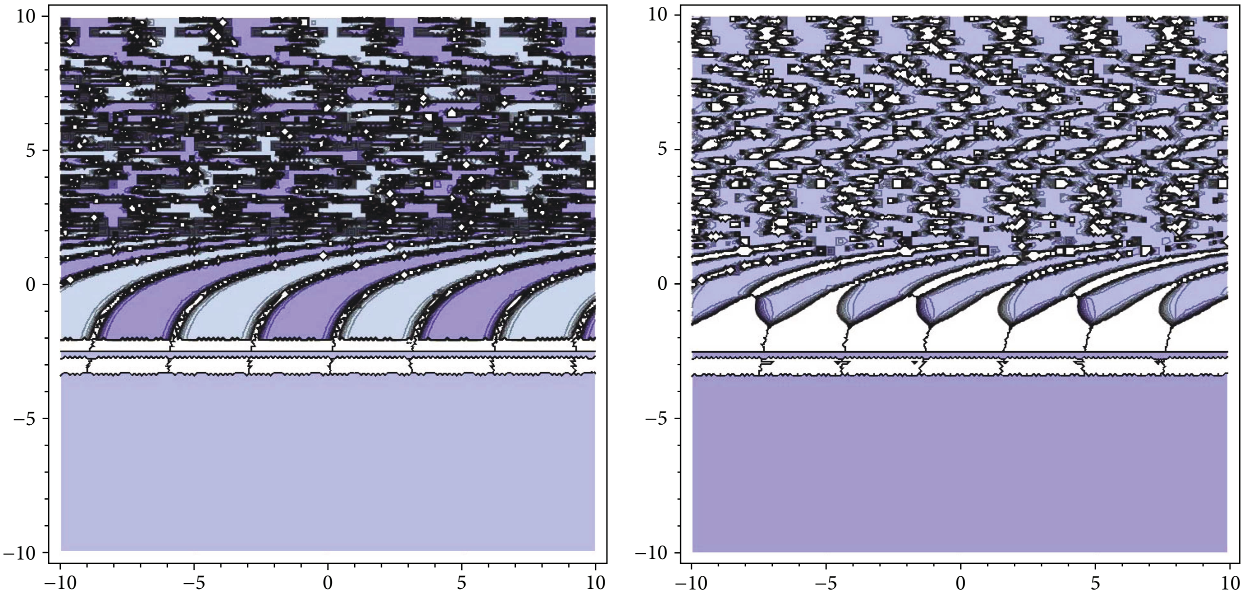
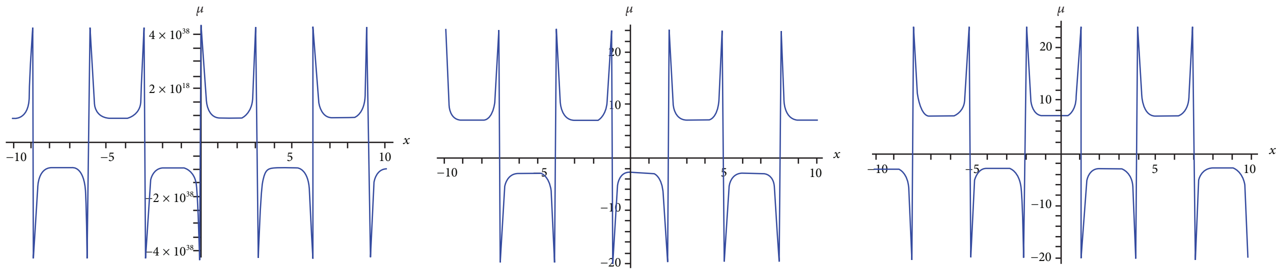


Figure 2: Contour lines of solutions (24) and (25) determined by (51).



(a) $t = -3$

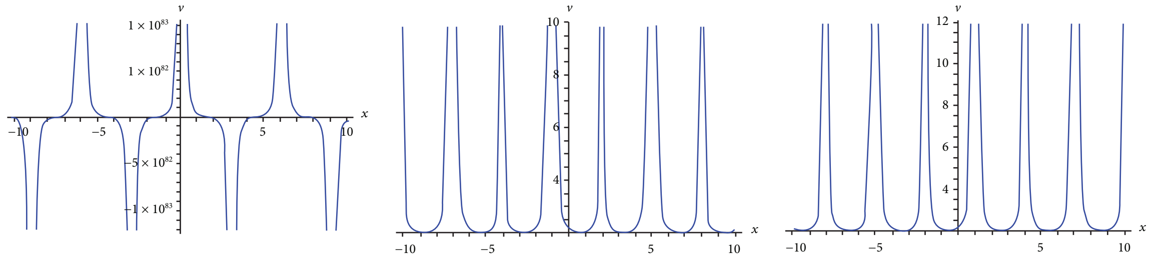
(b) $t = 0$

(c) $t = 3$

Figure 3: Nonlinear dynamical evolutions of solution (24) determined by (51).

where c_1 and c_2 are two integration constants.

In Figures 1 and 2, the spatial structures and contour lines of solutions (24) and (25) determined by (51) are shown by selecting the parameters as $a_0 = 2$, $A_0 = 0.5$, $c_0 = -4$, $c_1 = 3$, $c_2 = -1$, $k = 1.5$, and $m = 0.8$, respectively. We shown the nonlinear dynamical evolutions of solutions (24) and (25) in Figures 3 and 4. It is easy to see from Figures 1–4 that the doubly periodic waves determined by solutions (24) and



(a) $t = -3$

(b) $t = 0$

(c) $t = 3$

FIGURE 4: Nonlinear dynamical evolutions of solution (25) determined by (51).

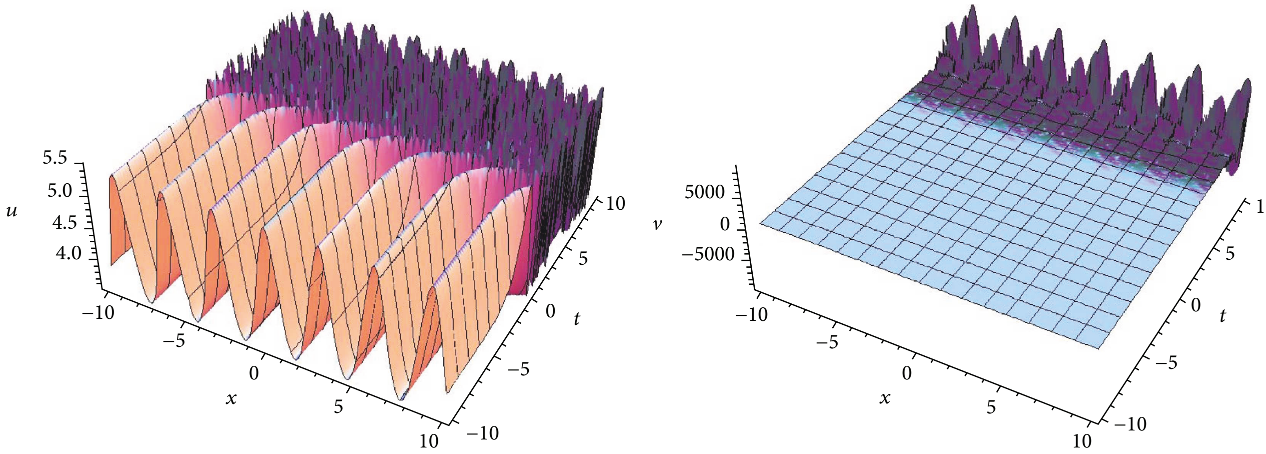


FIGURE 5: Spatial structures of solutions (27) determined by (52).

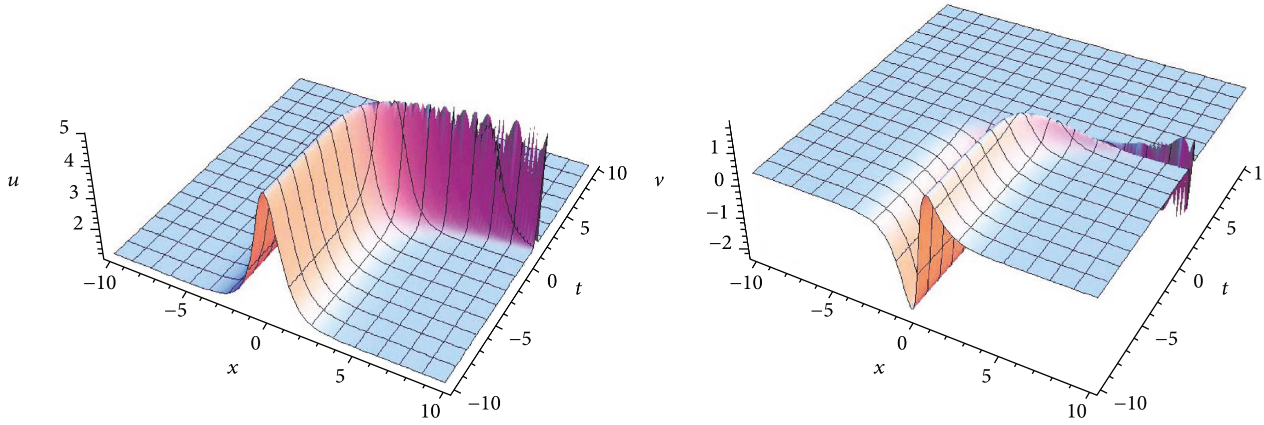


FIGURE 6: Spatial structures of solutions (40) determined by (52) given $m = 1$.

(25) possess time-varying amplitudes and velocities as well as singularities in the process of propagations.

Secondly, we consider solutions (27). To determine γ_3 and γ_6 in (28), we let $\gamma_1 = \mathrm{e}^t$ and then have

$$\begin{aligned} \gamma_3 &= c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t}, \\ \gamma_6 &= \frac{\mathrm{e}^t \left[-A_0 c_0 + c_2 - k^2 \left(2 - m^2\right) \left(c_1 + c_2 \mathrm{e}^t\right)^2 \right]}{c_0 k^2 \left(2 - m^2\right)}, \end{aligned} \tag{52}$$

where c_1 and c_2 are two integration constants.

In Figure 5, the spatial structures of solutions (27) determined by (52) are shown by selecting the parameters as $a_0 = 2$, $A_0 = 0.5$, $c_0 = -4$, $c_1 = 3$, $c_2 = -1$, $k = 1.5$, and $m = 0.8$, respectively. We shown the spatial structures of solutions (40) in Figure 6. It is easy to see from Figures 5 and 6 that both the doubly periodic waves determined by solutions (27) and the hyperbolic function solutions (40) possess time-varying amplitudes and velocities as well as singularities in the process of propagations.

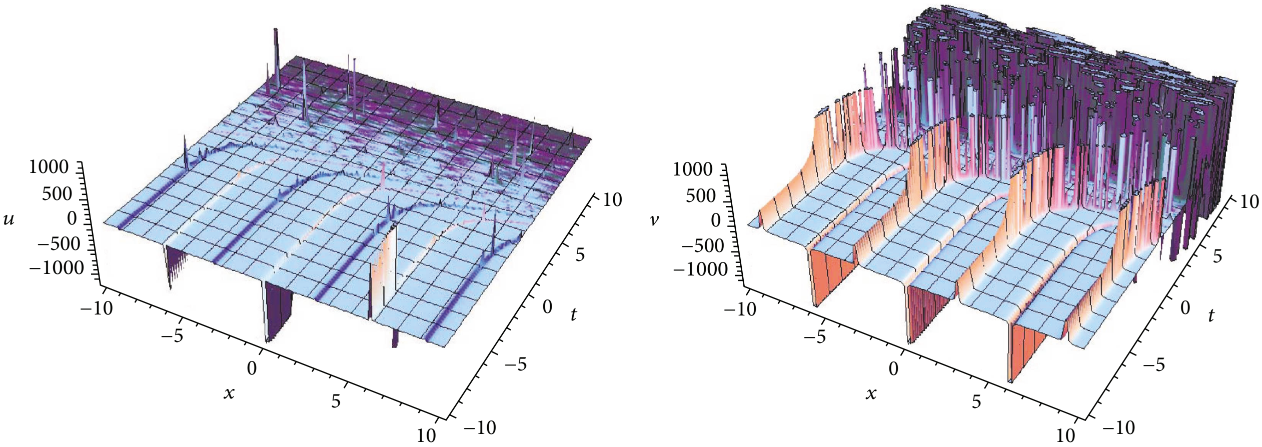


FIGURE 7: Spatial structures of solutions (33) determined by (52) given $m = 1$.

Finally, we consider solutions (33). To determine γ_3 and γ_6 in (34), (35), we let $\gamma_1 = e^t$ and then have

$$\gamma_3 = c_1 e^t + c_2 e^{2t},$$
$$\gamma_6 = \frac{e^t \left[-A_0 c_0 - c_2 - k^2 \left(1 - m^2 \right) \left(c_1 + c_2 e^t \right)^2 \right]}{c_0 k^2 \left(1 - m^2 \right)}, \tag{53}$$

where c_1 and c_2 are two integration constants.

In Figure 7, the spatial structures of solutions (33) determined by (53) are shown by selecting the parameters as $a_0 = 2$, $A_0 = 0.5$, $c_0 = -4$, $c_1 = 3$, $c_2 = -1$, $k = 1.5$, and $m = 0.8$, respectively. We can see from Figure 7 that the doubly periodic waves determined by solutions (33) possess time-varying amplitudes and velocities as well as singularities in the process of propagations.

4. Conclusion

In summary, new and more general Jacobi elliptic doubly periodic solutions of the tdcWBK system have been obtained, which degenerate into the hyperbolic function solutions and trigonometric function solutions in the limit cases. To the best of our knowledge, the obtained Jacobi elliptic doubly periodic solutions have not been reported in literatures. It is shown that the original F-expansion method cannot derive Jacobi elliptic doubly periodic solutions of the tdcWBK system but the novel approach of this paper is valid. In this sense, we would like to conclude that a novel approach of the generalized F-expansion method is extended to the tdcWBK system. The simulations show that the doubly periodic waves possess time-varying amplitudes and velocities as well as singularities in the process of propagations. Recently, fractional-order differential calculus and its applications have attached much attention [36–51]. Constructing Jacobi elliptic doubly periodic solutions of nonlinear PDEs with fractional derivatives is worthy of the study. At the same time, constructing multisoliton solutions via the Riemann-Hilbert approach, for example, see Kang et al.'s meaningful work [52, 53], has been a hot topic. Dealing with initial-boundary problems of nonlinear PDEs by means of the Riemann-Hilbert approach is also worthy of the study.

Data Availability

The data in the manuscript are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgments

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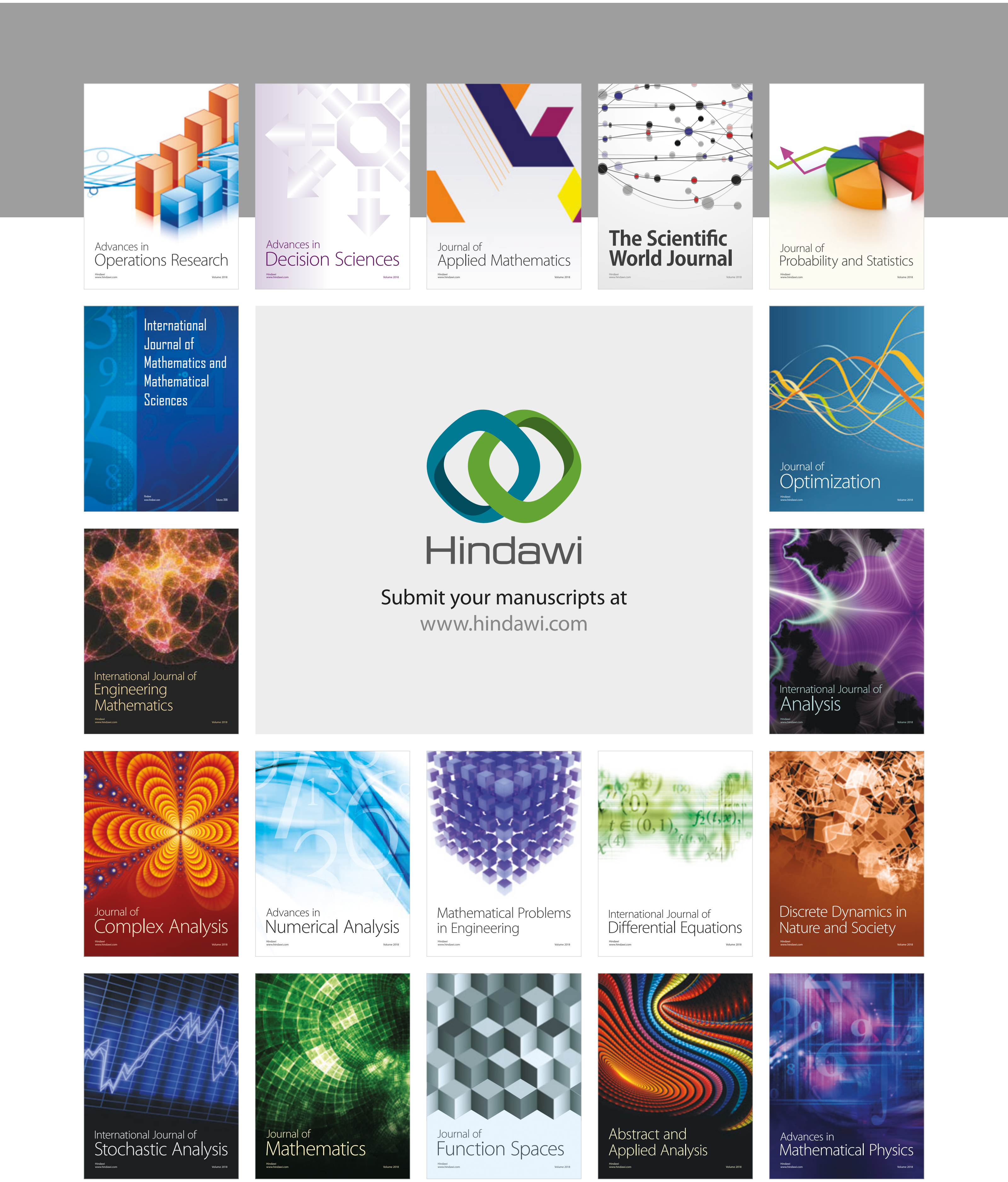
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