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Research Article

Pythagorean Fuzzy Interaction Partitioned Bonferroni Mean Operators and Their Application in Multiple-Attribute Decision-Making

Wei Yang,[1] **Jiarong Shi**,[1] **Yong Liu**,[1] **Yongfeng Pang**,[1] **and Ruiyue Lin**[2]

[1]*Department of Mathematics, School of Science, Xi'an University of Architecture and Technology, Xi'an, Shaanxi 710055, China*
[2]*Department of Mathematics, Wenzhou University, Higher Education Zone, Wenzhou, Zhejiang 325035, China*

Correspondence should be addressed to Wei Yang; yangweipyf@163.com

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The aim of this paper is to develop partitioned Pythagorean fuzzy interaction Bonferroni mean operators based on the Pythagorean fuzzy set, Bonferroni mean, and interaction between membership and nonmembership. Several new aggregation operators are developed including the Pythagorean fuzzy interaction partitioned Bonferroni mean (PFIPBM) operator, the Pythagorean fuzzy weighted interaction partitioned Bonferroni mean (PFWIPBM) operator, the Pythagorean fuzzy interaction partitioned geometric Bonferroni mean (PFIPGBM) operator, and the Pythagorean fuzzy weighted interaction partitioned geometric Bonferroni mean (PFWIPGBM) operator. Some main properties and some special particular cases of the new operators are studied. Many existing operators are the special cases of new aggregation operators. Moreover, a multiple-attribute decision-making method based on the proposed operator has been developed and the investment company selection problem is presented to illustrate feasibility and practical advantages of the new method.

1. Introduction

Pythagorean fuzzy set was first developed by Yager [1, 2], which is the extension of intuitionistic fuzzy set [3–5].

In Pythagorean fuzzy set, the square sum of membership and nonmembership is no more than 1, which can lead to larger feasible space than that of intuitionistic fuzzy set. Hence, comparing with the existing tools to model fuzzy and uncertain information, the Pythagorean fuzzy set is more powerful and flexible. In the literature, many studies have been conducted for decision-making problems with complex uncertainty in Pythagorean fuzzy environment [6–12].

Yager [12] developed the Pythagorean fuzzy weighted averaging (PFWA) operator and Pythagorean fuzzy weighted geometric averaging (PFWGA) operator. Garg proposed some Pythagorean fuzzy Einstein aggregation operations in [13] and Pythagorean fuzzy Einstein geometric aggregation operators using t-norm and t-conorm in [14]. Some Pythagorean fuzzy interaction weighted geometric aggregation operators were proposed in [15]. Yang and Pang [16] developed some Pythagorean fuzzy interaction Maclaurin symmetric mean operators. Peng and Yang [17] defined the Pythagorean fuzzy Choquet integral aggregation operator. Zhang et al. [18] proposed generalized Pythagorean fuzzy Bonferroni mean operator. Liang et al. [19] developed Pythagorean fuzzy geometric Bonferroni mean and weighted Pythagorean fuzzy geometric Bonferroni mean operator. Wei [20] presented some Pythagorean fuzzy interaction aggregation operators. Wei and Lu proposed some Pythagorean fuzzy power aggregation operators in [21] and presented some Pythagorean fuzzy Maclaurin symmetric mean operators in [22]. Zeng [23] developed Pythagorean fuzzy probabilistic ordered weighted averaging operator by considering probabilistic information in aggregating Pythagorean fuzzy values. Garg [24] proposed some probabilistic Pythagorean fuzzy aggregation operators by considering probabilistic information and decision maker's attitudinal character. Peng and Dai [25] proposed Pythagorean fuzzy stochastic decision-making method based on the prospect theory and regret theory. Some Pythagorean fuzzy multiple-attribute decision-making methods have been developed including the TOPSIS [26], QUALIFLEX [27], clustering analysis [28], TODIM [29], and VIKOR [30]. Pythagorean fuzzy set has been further extended to accommodate interval values [31, 32], linguistic variables [33] and so on.

Though several studies have been conducted in Pythagorean fuzzy environments, interaction between membership and nonmembership is considered less in existing studies and partitioned Pythagorean fuzzy values to be aggregated are rarely considered yet.

Bonferroni mean (BM) was introduced by Bonferroni [34], which has the capability of capturing interrelationship among arguments to be aggregated by considering conjunction among each pair of aggregated arguments. Yager [35] provided an interpretation of Bonferroni mean as involving a product of each argument with the average of the other arguments. Beliakov et al. [36] developed generalized Bonferroni mean to extend the Bonferroni mean in a more general form. Beliakov and James [37] extended the generalized Bonferroni mean to intuitionistic fuzzy environment. Xu and Yager [38] extended the Bonferroni mean to accommodate intuitionistic fuzzy values. Zhu and Xu [39] developed hesitant fuzzy Bonferroni mean operator and weighted hesitant fuzzy Bonferroni mean operator. Zhu et al. [40] explored the geometric Bonferroni mean under hesitant fuzzy environment. Xia et al. [41] introduced the Bonferroni geometric mean and further developed intuitionistic fuzzy geometric Bonferroni mean operator. Blanco-Mesa et al. [42] developed Bonferroni ordered weighted averaging index of maximum and minimum level operators by using Bonferroni mean, OWA operators, and some distance measures. Liang et al. [43] proposed the Pythagorean fuzzy Bonferroni mean and the weighted Pythagorean fuzzy Bonferroni mean. Dutta and Guha [44] presented the partitioned Bonferroni mean for 2-tuple linguistic information by considering the partitioned attribute class. Z. Liu and P. Liu [45] developed intuitionistic uncertain linguistic partitioned Bonferroni mean.

In some cases, the interrelationship does not exist in the whole attributes, but in some of the attributes. For example, consider a candidate selection problem for research sector in a university where the best candidate is selected among several candidates based on the following attributes: management skill (A_1), interpersonal relationship (A_2), research ability (A_3), and grant (A_4). The attributes should be partitioned into two classes $P_1 = \{A_1, A_2\}$ and $P_2 = \{A_3, A_4\}$. Obviously, A_1 and A_2 are interrelated and they belong to P_1. A_3 and A_4 are interrelated and they belong to P_2. But there is no interrelation between P_1 and P_2. Hence, there is a need to partition attributes into several classes when there is no interrelationship among all the attributes but there is interrelationship among parts of the attributes. Though many useful Bonferroni mean operators have been developed in various environments, the partitioned aggregation operators in Pythagorean fuzzy environment have not been considered yet. Moreover, interaction between the membership and nonmembership of Pythagorean fuzzy values should be considered in the partitioned Pythagorean fuzzy aggregation operator. Hence in this paper, based on the partitioned Bonferroni mean operator, we develop Pythagorean fuzzy interaction partitioned Bonferroni mean operators by considering partitioned values and interaction between membership and nonmembership. Then, we give a new multiple-attribute decision-making method based on the partitioned Bonferroni mean operators. Comparing with the existing methods based on the Bonferroni mean operators, our proposed method is a good complement to the existing work and it can be used to solve more complex multiple-attribute decision-making problems.

The structure of the paper is as follows. In Section 2, some basic concepts on Pythagorean fuzzy set and Bonferroni mean have been reviewed. In Section 3, some Bonferroni mean operators in Pythagorean fuzzy environments considering interaction have been developed including the Pythagorean fuzzy interaction partitioned Bonferroni mean (PFIPBM) operator, the Pythagorean fuzzy weighted interaction partitioned Bonferroni mean (PFWIPBM) operator, the Pythagorean fuzzy interaction partitioned geometric Bonferroni mean (PFIPGBM) operator, and the Pythagorean fuzzy weighted interaction partitioned geometric Bonferroni mean (PFWIPGBM) operator. Some properties and some special cases of the new aggregation operators have been studied. In Section 4, a new multiple-attribute group decision-making method based on the new proposed operators has been proposed. In Section 5, the problem of investment company selection has been presented to illustrate the new method and some comparisons with other methods have been conducted. Conclusions have been given in the last section.

2. Preliminaries

Pythagorean fuzzy set [1, 2] is the extension of fuzzy set and intuitionistic fuzzy set. We review some concepts of Pythagorean fuzzy set and their operations in the following.

Definition 1 (see [26]). Let X be a fixed set. A Pythagorean fuzzy set P on X can be represented as follows:

$$P = \{<x, (\mu_P(x), \nu_P(x)) > | x \in X\}, \quad (1)$$

where $\mu_P(x)$: $X \rightarrow [0, 1]$ is the membership function and $\nu_P(x)$: $X \rightarrow [0, 1]$ is the nonmembership function. For each $x \in X$, it satisfies the following condition $0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$. $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}$ is the indeterminacy degree of x to X. For simplicity, $(\mu_P(x), \nu_P(x))$ is called a Pythagorean fuzzy number (PFN), denoted by (μ_P, ν_P), where $\mu_P, \nu_P \in [0, 1]$, $\pi_P = \sqrt{1 - (\mu_P)^2 - (\nu_P)^2}$, and $0 \leq (\mu_P)^2 + (\nu_P)^2 \leq 1$.

Definition 2 (see [26]). Let $\alpha = (\mu_\alpha, \nu_\alpha)$, $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$, and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be three PFNs. The operations are as follows:

(1) $\alpha_1 \oplus \alpha_2 = \left(\sqrt{\mu_{\alpha_1}^2 + \mu_{\alpha_2}^2 - \mu_{\alpha_1}^2 \mu_{\alpha_2}^2}, \nu_{\alpha_1} \nu_{\alpha_2}\right)$.

(2) $\alpha_1 \otimes \alpha_2 = \left(\mu_{\alpha_1}\mu_{\alpha_2}, \sqrt{\nu_{\alpha_1}^2 + \nu_{\alpha_2}^2 - \nu_{\alpha_1}^2\nu_{\alpha_2}^2}\right)$.

(3) $k\alpha = \left(\sqrt{1-(1-\mu_\alpha^2)^k}, (\nu_\alpha)^k\right), k \geq 0$.

(4) $\alpha^k = \left(\mu_\alpha^k, \sqrt{1-(1-\nu_\alpha^2)^k}\right), k \geq 0$.

Let $\alpha = (\mu_\alpha, \nu_\alpha)$ be a PFN; the score function [26] is defined as

$$S(\alpha) = (\mu_\alpha)^2 - (\nu_\alpha)^2. \quad (2)$$

The accuracy function [46] is defined as

$$A(\alpha) = (\mu_\alpha)^2 + (\nu_\alpha)^2. \quad (3)$$

Definition 3 (see [2]). Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be two PFNs. Yager and Abbasov defined the following method to compare two PFNs:

(1) If $S(\alpha_1) < S(\alpha_2)$, then $\alpha_1 < \alpha_2$.

(2) If $S(\alpha_1) = S(\alpha_2)$,

 (i) If $A(\alpha_1) < A(\alpha_2)$, then $\alpha_1 < \alpha_2$.

 (ii) If $A(\alpha_1) = A(\alpha_2)$, then $\alpha_1 = \alpha_2$.

Example 1. Suppose $\alpha_1 = (0.6{,}0.4)$, $\alpha_2 = (0.5{,}0.6)$, and $\alpha_3 = (0.7{,}0.0)$, the corresponding weight vector is $(0.25{,}0.4{,}0.35)$, then $\alpha = w_1\alpha_1 \oplus w_2\alpha_2 \oplus w_3\alpha_3 = (0.6045{,}0)$. This means that nonmemberships have no effects on the overall results, which is not reasonable. In order to overcome this shortcoming, some new operational laws on Pythagorean fuzzy set were developed.

Definition 4 (see [20]). Let $\alpha = (\mu_\alpha, \nu_\alpha)$, $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$, and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be three PFNs. The operation laws can be defined as follows:

(1) $\alpha_1 \oplus \alpha_2 = \left(\sqrt{\mu_{\alpha_1}^2 + \mu_{\alpha_2}^2 - \mu_{\alpha_1}^2\mu_{\alpha_2}^2}, \sqrt{\nu_{\alpha_1}^2 + \nu_{\alpha_2}^2 - \nu_{\alpha_1}^2\nu_{\alpha_2}^2 - \mu_{\alpha_1}^2\nu_{\alpha_2}^2 - \nu_{\alpha_1}^2\mu_{\alpha_2}^2}\right)$.

(2) $\alpha_1 \otimes \alpha_2 = \left(\sqrt{\mu_{\alpha_1}^2 + \mu_{\alpha_2}^2 - \mu_{\alpha_1}^2\mu_{\alpha_2}^2 - \nu_{\alpha_1}^2\mu_{\alpha_2}^2 - \mu_{\alpha_1}^2\nu_{\alpha_2}^2}, \sqrt{\nu_{\alpha_1}^2 + \nu_{\alpha_2}^2 - \nu_{\alpha_1}^2\nu_{\alpha_2}^2}\right)$.

(3) $\lambda\alpha = \left(\sqrt{1-(1-\mu_\alpha^2)^\lambda}, \sqrt{(1-\mu_\alpha^2)^\lambda - (1-(\mu_\alpha^2+\nu_\alpha^2))^\lambda}\right)$, $\lambda > 0$.

(4) $(\alpha)^\lambda = \left(\sqrt{(1-\nu_\alpha^2)^\lambda - (1-(\mu_\alpha^2+\nu_\alpha^2))^\lambda}, \sqrt{1-(1-\nu_\alpha^2)^\lambda}\right), \lambda > 0$.

Equations (1) and (2) can be rewritten as follows:

$$\begin{aligned}\alpha_1 \oplus \alpha_2 &= \Big(\sqrt{1-(1-\mu_{\alpha_1}^2)(1-\mu_{\alpha_2}^2)}, \\ &\quad ((1-\mu_{\alpha_1}^2)(1-\mu_{\alpha_2}^2) - (1-(\mu_{\alpha_1}^2+\nu_{\alpha_1}^2))(1(\mu_{\alpha_2}^2+\nu_{\alpha_2}^2)))^{1/2}\Big) \\ &= \Big(\sqrt{1-\textstyle\prod_{j=1}^2(1-\mu_{\alpha_j}^2)}, \\ &\quad \sqrt{\textstyle\prod_{j=1}^2(1-\mu_{\alpha_j}^2) - \prod_{j=1}^2(1-(\mu_{\alpha_j}^2+\nu_{\alpha_j}^2))}\Big).\end{aligned} \quad (1)$$

$$\begin{aligned}\alpha_1 \otimes \alpha_2 &= \Big(\sqrt{(1-\nu_{\alpha_1}^2)(1-\nu_{\alpha_2}^2) - (1-(\mu_{\alpha_1}^2+\nu_{\alpha_1}^2))(1-(\mu_{\alpha_2}^2+\nu_{\alpha_2}^2))} \\ &\quad \cdot \sqrt{1-(1-\nu_{\alpha_1}^2)(1-\nu_{\alpha_2}^2)}\Big) \\ &\quad \cdot \Big(\sqrt{\textstyle\prod_{j=1}^2 1-\nu_{\alpha_j}^2 - \prod_{j=1}^2(1-(\mu_{\alpha_j}^2+\nu_{\alpha_j}^2))}, \\ &\quad \sqrt{1-\textstyle\prod_{j=1}^2(1-\nu_{\alpha_j}^2)}\Big).\end{aligned} \quad (2)$$

Definition 5. Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be two Pythagorean fuzzy numbers. The Hamming distance between α_1 and α_2 can be defined as follows:

$$d(\alpha_1, \alpha_2) = \frac{1}{2}\left(\left|\mu_{\alpha_1}^2 - \mu_{\alpha_2}^2\right| + \left|\nu_{\alpha_1}^2 - \nu_{\alpha_2}^2\right|\right). \quad (4)$$

3. Pythagorean Fuzzy Weighted Interaction Partitioned Bonferroni Mean Operator

The Bonferroni mean (BM) aggregation operator was defined by Bonferroni [34] in 1950. It was generalized by Yager [35] and others. The BM operator has the following forms.

Definition 6 (see [35]). For any $p, q \geq 0$ with $p + q > 0$, the BM aggregation operator of dimension n is a mapping BM: $(R^+)^n \to R^+$, such that

$$\mathrm{BM}^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n} a_i^p a_j^q\right)^{1/(p+q)}. \quad (5)$$

Definition 7 (see [44]). For any $p, q \geq 0$ with $p + q > 0$ and $T = (a_1, a_2, \ldots, a_n)$ with $a_k \geq 0\,(k = 1, 2, \ldots, n)$, which is partitioned into d distinct sorts $P_1, P_2, \ldots, P_d$, where $\bigcup_{h=1}^d P_h = T$, the partitioned Bonferroni mean aggregation operator of dimension n is a mapping PBM:

$$\mathrm{PBM}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{d}\left(\sum_{h=1}^{d}\left(\frac{1}{|P_h|}\sum_{i\in P_h} a_i^p\left(\frac{1}{|P_h|-1}\sum_{j\in P_h, j\neq i} a_j^q\right)\right)^{1/(p+q)}\right), \quad (6)$$

where $|P_h|$ denotes the cardinality of P_h, d is the number of partitioned sorts, and $\sum_{h=1}^d |P_h| = n$.

Definition 8. Let $T = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a collection of PFNs, which is partitioned into d distinct sorts $P_1, P_2, \ldots, P_d$, where

$\alpha_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ and $\bigcup_{h=1}^{d} P_h = T$. The Pythagorean fuzzy interaction partitioned Bonferroni mean (PFIPBM) operator is defined as follows:

$$\mathrm{PFIPBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|P_h|}\oplus_{i\in P_h}\left(\alpha_i^p \otimes \left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\alpha_j^q\right)\right)\right)^{1/(p+q)}\right), \tag{7}$$

where $p, q \geq 0$, $|P_h|$ denotes the cardinality of P_h, d is the number of the partitioned sorts, and $\sum_{h=1}^{d}|P_h| = n$.

Theorem 1. *Let $\alpha_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ be a collection of PFNs and $p, q \geq 0$. The aggregated result of PFIPBM operator is still of a PFN, which has the following form:*

$$\begin{aligned}
&\mathrm{PFIPBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|P_h|}\oplus_{i\in P_h}\left(\alpha_i^p \otimes \left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\alpha_j^q\right)\right)\right)^{1/(p+q)}\right) \\
&= \left(\left(1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i\in P_h}\left(1-(1-\nu_i^2)^p(1-\xi+\eta)\right.\right.\right.\right.\right. \\
&\quad \left.\left.\left. + \left(1-(\mu_i^2+\nu_i^2)\right)^p\eta\right)^{1/|P_h|} + \prod_{i\in P_h}\left(\left(1-(\mu_i^2+\nu_i^2)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)} \\
&\quad \left.\left. + \left(\prod_{i\in P_h}\left(\left(1-(\mu_i^2+\nu_i^2)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d}\right)^{1/2}, \\
&\quad \left(\prod_{h=1}^{d}\left(1-\left(1-\prod_{i\in P_h}\left(1-(1-\nu_i^2)^p(1-\xi+\eta)\right.\right.\right.\right. \\
&\quad \left.\left.\left. + \left(1-(\mu_i^2+\nu_i^2)\right)^p\eta\right)^{1/|P_h|} + \prod_{i\in P_h}\left(\left(1-(\mu_i^2+\nu_i^2)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)} \\
&\quad \left. + \left(\prod_{i\in P_h}\left(\left(1-(\mu_i^2+\nu_i^2)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d} \\
&\quad \left.\left. - \prod_{h=1}^{d}\left(\left(\prod_{i\in P_h}\left(\left(1-(\mu_i^2+\nu_i^2)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d}\right)^{1/2}\right),
\end{aligned} \tag{8}$$

where $\xi = \prod_{j\in P_h, j\neq i}(1-(1-\nu_j^2)^q + 1-(\mu_j^2+\nu_j^2)^q)^{1/(|P_h|-1)}$ *and* $\eta = (\prod_{j\in P_h, j\neq i}(1-(\mu_j^2+\nu_j^2))^q)^{1/(|P_h|-1)}$.

Proof 1.

$$\alpha_j^q = \left(\left(\left(1-\nu_j^2\right)^q - \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/2}, \sqrt{1-\left(1-\nu_j^2\right)^q}\right),$$

$$\alpha_i^p = \left(\left((1-\nu_i^2)^p - (1-(\mu_i^2+\nu_i^2))^p\right)^{1/2}, \sqrt{1-(1-\nu_i^2)^p}\right),$$

$$\begin{aligned}
&\oplus_{j\in P_h, j\neq i}\alpha_j^q \\
&= \left(\left(1-\prod_{j\in P_h, j\neq i}\left(1-\left(1-\nu_j^2\right)^q + \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)\right)^{1/2},\right. \\
&\quad \left(\prod_{j\in P_h, j\neq i}\left(1-\left(1-\nu_j^2\right)^q + \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)\right. \\
&\quad \left.\left. - \prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/2}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\alpha_j^q \\
&= \left(\left(1-\prod_{j\in P_h, j\neq i}\left(1-\left(1-\nu_j^2\right)^q + \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\right)^{1/2},\right. \\
&\quad \left(\prod_{j\in P_h, j\neq i}\left(1-\left(1-\nu_j^2\right)^q + \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\right. \\
&\quad \left.\left. - \left(\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\right)^{1/2}\right).
\end{aligned} \tag{9}$$

Let

$$\xi = \prod_{j\in P_h, j\neq i}\left(1-\left(1-\nu_j^2\right)^q + \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)},$$

$$\eta = \left(\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)},$$

$$\begin{aligned}
&\alpha_i^p \otimes \left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\alpha_j^q\right) \\
&= \left(\sqrt{(1-\nu_i^2)^p(1-\xi+\eta) - (1-(\mu_i^2+\nu_i^2))^p\eta},\right. \\
&\quad \left.\sqrt{1-(1-\nu_i^2)^p(1-\xi+\eta)}\right),
\end{aligned}$$

$$\begin{aligned}
&\oplus_{i\in P_h}\left(\alpha_i^p \otimes \left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\alpha_j^q\right)\right) \\
&= \left(\left(1-\prod_{i\in P_h}\left(1-(1-\nu_i^2)^p(1-\xi+\eta)\right.\right.\right. \\
&\quad \left.\left. + (1-(\mu_i^2+\nu_i^2))^p\eta\right)\right)^{1/2}, \\
&\quad \left(\prod_{i\in P_h}\left(1-(1-\nu_i^2)^p * (1-\xi+\eta) + (1-(\mu_i^2+\nu_i^2))^p\eta\right)\right. \\
&\quad \left.\left. - \prod_{i\in P_h}(1-(\mu_i^2+\nu_i^2))^p\eta\right)^{1/2}\right),
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{1}{|P_h|}\oplus_{i\in P_h}\left(\alpha_i^p\otimes\left(\frac{1}{|P_h|-1}\oplus_{j\in P_h,j\neq i}\alpha_j^q\right)\right)\right)^{1/(p+q)} \\
&=\left(\left(\left(\left(1-\prod_{i\in P_h}\left(1-\left(1-\nu_i^2\right)^p\left(1-\xi+\eta\right)\right.\right.\right.\right.\right. \\
&\quad\left.+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|} \\
&\quad\left.+\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)} \\
&\quad\left.\left.-\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/2}, \right. \\
&\quad\left(1-\left(1-\prod_{i\in P_h}\left(1-\left(1-\nu_i^2\right)^p\left(1-\xi+\eta\right)\right.\right.\right. \\
&\quad\left.+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|} \\
&\quad\left.\left.\left.+\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/2}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|P_h|}\oplus_{i\in P_h}\left(\alpha_i^p\otimes\left(\frac{1}{|P_h|-1}\oplus_{j\in P_h,j\neq i}\alpha_j^q\right)\right)\right)^{1/(p+q)}\right) \\
&=\left(\left(1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i\in P_h}\left(1-\left(1-\nu_i^2\right)^p\left(1-\xi+\eta\right)\right.\right.\right.\right.\right. \\
&\quad\left.+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|} \\
&\quad\left.\left.+\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)} \\
&\quad\left.\left.+\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d}\right)^{1/2}, \\
&\quad\left(\prod_{h=1}^{d}\left(1-\left(1-\prod_{i\in P_h}\left(1-\left(1-\nu_i^2\right)^p\left(1-\xi+\eta\right)\right.\right.\right.\right. \\
&\quad\left.+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|} \\
&\quad\left.\left.+\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)} \\
&\quad\left.+\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d} \\
&\quad\left.\left.-\prod_{h=1}^{d}\left(\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d}\right)^{1/2}\right).
\end{aligned} \tag{10}$$

Moreover,

$$\begin{aligned}
&\left(\mu_{\mathrm{PFIPBM}^{p,q}}\right)^2+\left(\nu_{\mathrm{PFIPBM}^{p,q}}\right)^2 \\
&=1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i\in P_h}\left(1-\left(1-\nu_i^2\right)^p\left(1-\xi+\eta\right)\right.\right.\right. \\
&\quad\left.+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|} \\
&\quad\left.+\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)} \\
&\quad\left.+\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d} \\
&\quad+\prod_{h=1}^{d}\left(1-\left(1-\prod_{i\in P_h}\left(1-\left(1-\nu_i^2\right)^p\left(1-\xi+\eta\right)\right.\right.\right. \\
&\quad\left.+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|} \\
&\quad\left.+\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)} \\
&\quad\left.+\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d} \\
&\quad-\prod_{h=1}^{d}\left(\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d} \\
&=1-\prod_{h=1}^{d}\left(\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d},
\end{aligned} \tag{11}$$

since

$$\eta=\left(\prod_{j\in P_h,j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}, \tag{12}$$

then $0\le\eta\le1$ and

$$0\le1-\prod_{h=1}^{d}\left(\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d}\le1. \tag{13}$$

Hence, the aggregated result of the $\mathrm{PFIPBM}^{p,q}$ operator is still a PFN.

Theorem 2 (idempotency). *Let $\alpha_i=(\mu_i,\nu_i)$ $(i=1,2,\ldots,n)$ be a collection of PFNs and $p,q\ge0$. If all α_i $(i=1,2,\ldots,n)$ are equal, that is, $\alpha_i=\alpha=(\mu,\nu)$ $(i=1,2,\ldots,n)$, then*

$$\mathrm{PFIPBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\alpha. \tag{14}$$

Proof 2. Let $\mathrm{PFIPBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\sigma, \tau)$. Because $\mu_i = \mu$ and $\nu_i = \nu$, then we get

$$
\begin{aligned}
\xi &= \prod_{j\in P_h, j\neq i} \left(1 - \left(1 - \nu_j^2\right)^q + \left(1 - \left(\mu_j^2 + \nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)} \\
&= \prod_{j\in P_h, j\neq i} \left(1 - \left(1 - \nu^2\right)^q + \left(1 - \left(\mu^2 + \nu^2\right)\right)^q\right)^{1/(|P_h|-1)} \\
&= 1 - \left(1 - \nu^2\right)^q + \left(1 - \left(\mu^2 + \nu^2\right)\right)^q, \\
\eta &= \left(\prod_{j\in P_h, j\neq i} \left(1 - \left(\mu_j^2 + \nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)} \\
&= \left(\prod_{j\in P_h, j\neq i} \left(1 - \left(\mu^2 + \nu^2\right)\right)^q\right)^{1/(|P_h|-1)} \\
&= \left(1 - \left(\mu^2 + \nu^2\right)\right)^q.
\end{aligned}
\tag{15}
$$

Therefore, we have

$$
\begin{aligned}
\sigma &= \Bigg(1 - \prod_{h=1}^{d} \Bigg(1 - \Bigg(1 - \prod_{i\in P_h} \Big(1 - \left(1 - \nu^2\right)^p \left(1 - \nu^2\right)^q \\
&\quad + \left(1 - \left(\mu^2 + \nu^2\right)\right)^p \left(1 - \left(\mu^2 + \nu^2\right)\right)^q\Big)^{1/|P_h|} \\
&\quad + \left(\left(1 - \left(\mu^2 + \nu^2\right)\right)^p \left(1 - \left(\mu^2 + \nu^2\right)\right)^q\right)\Bigg)^{1/(p+q)} \\
&\quad + \Big(\Big(\left(1 - \left(\mu^2 + \nu^2\right)\right)^p \\
&\quad * \left(1 - \left(\mu^2 + \nu^2\right)\right)^q\Big)\Big)^{1/(p+q)}\Bigg)^{1/d}\Bigg)^{1/2} \\
&= \Bigg(1 - \prod_{h=1}^{d} \Bigg(1 - \Big(1 - \Big(1 - \left(1 - \nu^2\right)^{p+q} \\
&\quad + \left(1 - \left(\mu^2 + \nu^2\right)\right)^{p+q}\Big) + \left(1 - \left(\mu^2 + \nu^2\right)\right)^{p+q}\Big)^{1/(p+q)} \\
&\quad + \left(\left(1 - \left(\mu^2 + \nu^2\right)\right)^{p+q}\right)^{1/(p+q)}\Bigg)^{1/d}\Bigg)^{1/2} = \mu, \\
\tau &= \Bigg(\prod_{h=1}^{d} \Bigg(1 - \Bigg(1 - \prod_{i\in P_h} \Big(1 - \left(1 - \nu^2\right)^p \left(1 - \nu^2\right)^q \\
&\quad + \left(1 - \left(\mu^2 + \nu^2\right)\right)^p \left(1 - \left(\mu^2 + \nu^2\right)\right)^q\Big)^{1/|P_h|} \\
&\quad + \prod_{i\in P_h} \left(\left(1 - \left(\mu^2 + \nu^2\right)\right)^p \left(1 - \left(\mu^2 + \nu^2\right)\right)^q\right)^{1/|P_h|}\Bigg)^{1/(p+q)} \\
&\quad + \left(\prod_{i\in P_h} \left(\left(1 - \left(\mu^2 + \nu^2\right)\right)^p \left(1 - \left(\mu^2 + \nu^2\right)\right)^q\right)^{1/|P_h|}\right)^{1/(p+q)}\Bigg)^{1/d} \\
&\quad - \prod_{h=1}^{d} \Bigg(\Bigg(\prod_{i\in P_h} \Big(\left(1 - \left(\mu^2 + \nu^2\right)\right)^p \\
&\quad \cdot \left(1 - \left(\mu^2 + \nu^2\right)\right)^q\Big)^{1/|P_h|}\Bigg)^{1/(p+q)}\Bigg)^{1/d}\Bigg)^{1/2} \\
&= \Bigg(\prod_{h=1}^{d} \Bigg(1 - \Big(1 - \Big(1 - \left(1 - \nu^2\right)^{p+q} + \left(1 - \left(\mu^2 + \nu^2\right)\right)^{p+q}\Big) \\
&\quad + \left(\left(1 - \left(\mu^2 + \nu^2\right)\right)^{p+q}\right)\Big)^{1/(p+q)} \\
&\quad + \left(\left(\left(1 - \left(\mu^2 + \nu^2\right)\right)^{p+q}\right)\right)^{1/(p+q)}\Bigg)^{1/d} \\
&\quad - \prod_{h=1}^{d} \left(\left(\left(\left(1 - \left(\mu^2 + \nu^2\right)\right)^{p+q}\right)\right)^{1/(p+q)}\right)^{1/d}\Bigg)^{1/2} \\
&= \left(1 - \left(1 - \nu^2\right) + \left(1 - \left(\mu^2 + \nu^2\right)\right) - \left(1 - \left(\mu^2 + \nu^2\right)\right)\right)^{1/2} = \nu.
\end{aligned}
\tag{16}
$$

Hence, we get $(\sigma, \tau) = (\mu, \nu)$ and $\mathrm{PFIPBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha$.

Theorem 3 (commutativity). *Let $\alpha_i = (\mu_i, \nu_i)$ $(i = 1, 2, \ldots, n)$ and $\alpha'_i = (\mu'_i, \nu'_i)$ $(i = 1, 2, \ldots, n)$ be two collections of PFNs. If $\alpha'_i = (\mu'_i, \nu'_i)$ is any permutation of $\alpha_i = (\mu_i, \nu_i)$, then*

$$
\mathrm{PFIPBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{PFIPBM}^{p,q}\left(\alpha'_1, \alpha'_2, \ldots, \alpha'_n\right).
\tag{17}
$$

Proof 3. By using (8), we can get

$$
\begin{aligned}
&\mathrm{PFIPBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \Bigg(\Bigg(1 - \prod_{h=1}^{d} \Bigg(1 - \Bigg(1 - \prod_{i\in P_h} \Big(1 - \left(1 - \nu_i^2\right)^p (1 - \xi + \eta) \\
&\quad + \left(1 - \left(\mu_i^2 + \nu_i^2\right)\right)^p \eta\Big)^{1/|P_h|} \\
&\quad + \prod_{i\in P_h} \left(\left(1 - \left(\mu_i^2 + \nu_i^2\right)\right)^p \eta\right)^{1/|P_h|}\Bigg)^{1/(p+q)} \\
&\quad + \left(\prod_{i\in P_h} \left(\left(1 - \left(\mu_i^2 + \nu_i^2\right)\right)^p \eta\right)^{1/|P_h|}\right)^{1/(p+q)}\Bigg)^{1/d}\Bigg)^{1/2} \\
&\quad \cdot \Bigg(\prod_{h=1}^{d} \Bigg(1 - \Bigg(1 - \prod_{i\in P_h} \Big(1 - \left(1 - \nu_i^2\right)^p (1 - \xi + \eta) \\
&\quad + \left(1 - \left(\mu_i^2 + \nu_i^2\right)\right)^p \eta\Big)^{1/|P_h|} \\
&\quad + \prod_{i\in P_h} \left(\left(1 - \left(\mu_i^2 + \nu_i^2\right)\right)^p \eta\right)^{1/|P_h|}\Bigg)^{1/(p+q)}
\end{aligned}
$$

$$+\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\Bigg)^{1/d}$$

$$-\prod_{h=1}^{d}\left(\left(\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\eta\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d}\right)^{1/2}\Bigg), \tag{18}$$

where

$$\xi=\prod_{j\in P_h,j\neq i}\left(1-\left(1-\nu_j^2\right)^q+\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)},$$
$$\eta=\left(\prod_{j\in P_h,j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}.$$

$$\mathrm{PFIPBM}^{p,q}\left(\alpha'_1,\alpha'_2,\ldots,\alpha'_n\right)$$
$$=\Bigg(\Bigg(1-\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i\in P_h}\left(1-\left(1-\nu'^2_i\right)^p\left(1-\xi'+\eta'\right)\right.$$
$$\left.+\left(1-\left(\mu'^2_i+\nu'^2_i\right)\right)^p\eta\right)^{1/|P_h|}$$
$$+\prod_{i\in P_h}\left(\left(1-\left(\mu'^2_i+\nu'^2_i\right)\right)^p\eta'\right)^{1/|P_h|}\Bigg)^{1/(p+q)}$$
$$+\left(\prod_{i\in P_h}\left(\left(1-\left(\mu'^2_i+\nu'^2_i\right)\right)^p\eta'\right)^{1/|P_h|}\right)^{1/(p+q)}\Bigg)^{1/d}\Bigg)^{1/2},$$
$$\Bigg(\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i\in P_h}\left(1-\left(1-\nu'^2_i\right)^p\left(1-\xi'+\eta'\right)\right.$$
$$\left.+\left(1-\left(\mu'^2_i+\nu'^2_i\right)\right)^p\eta'\right)^{1/|P_h|}$$
$$+\prod_{i\in P_h}\left(\left(1-\left(\mu'^2_i+\nu'^2_i\right)\right)^p\eta'\right)^{1/|P_h|}\Bigg)^{1/(p+q)}$$
$$+\left(\prod_{i\in P_h}\left(\left(1-\left(\mu'^2_i+\nu'^2_i\right)\right)^p\eta'\right)^{1/|P_h|}\right)^{1/(p+q)}\Bigg)^{1/d}$$
$$-\prod_{h=1}^{d}\left(\left(\left(\prod_{i\in P_h}\left(\left(1-\left(\mu'^2_i+\nu'^2_i\right)\right)^p\eta'\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d}\right)^{1/2}\Bigg), \tag{19}$$

where

$$\xi'=\prod_{j\in P_h,j\neq i}\left(1-\left(1-\nu'^2_j\right)^q+\left(1-\left(\mu'^2_j+\nu'^2_j\right)\right)^q\right)^{1/(|P_h|-1)},$$
$$\eta'=\left(\prod_{j\in P_h,j\neq i}\left(1-\left(\mu'^2_j+\nu'^2_j\right)\right)^q\right)^{1/(|P_h|-1)}. \tag{20}$$

Since $\alpha'_i=(\mu'_i,\nu'_i)$ is any permutation of $\alpha_i=(\mu_i,\nu_i)$, then we can get

$$\mathrm{PFIPBM}^{p,q}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)=\mathrm{PFIPBM}^{p,q}\left(\alpha'_1,\alpha'_2,\ldots,\alpha'_n\right). \tag{21}$$

Theorem 4 (boundedness). *Let* $\tilde{\alpha}=(1,0)$ *and* $\check{\alpha}=(0,1)$, *then*

$$\check{\alpha}\le\mathrm{PFIPBM}^{p,q}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)\le\tilde{\alpha} \tag{22}$$

Proof 4. The property of boundedness can be proved easily by using Theorem 1.

Some of the special cases of the proposed $\mathrm{PFIPBM}^{p,q}$ operator regarding parameters p and q are as follows:

(i) When $q\to 0$, we can get

$$\xi=\prod_{j\in P_h,j\neq i}\left(1-\left(1-\nu_j^2\right)^q+\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}=1,$$
$$\eta=\left(\prod_{j\in P_h,j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}=1. \tag{23}$$

Thus, we can get

$$\mathrm{PFIPBM}^{p,0}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)$$
$$=\Bigg(\Bigg(1-\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i\in P_h}\left(1-\left(1-\nu_i^2\right)^p\right.$$
$$\left.+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\right)^{1/|P_h|}+\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\right)^{1/|P_h|}\Bigg)^{1/p}$$
$$+\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\right)^{1/|P_h|}\right)^{1/p}\Bigg)^{1/d}\Bigg)^{1/2},$$
$$\Bigg(\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i\in P_h}\left(1-\left(1-\nu_i^2\right)^p\right.$$
$$\left.+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\right)^{1/|P_h|}+\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\right)^{1/|P_h|}\Bigg)^{1/p}$$
$$+\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\right)^{1/|P_h|}\right)^{1/p}\Bigg)^{1/d}$$
$$-\prod_{h=1}^{d}\left(\left(\left(\prod_{i\in P_h}\left(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\right)^{1/|P_h|}\right)^{1/p}\right)^{1/d}\right)^{1/2}\Bigg). \tag{24}$$

(ii) When $q \to 0$ and $p = 1$, we can get

$$\xi = \prod_{j\in P_h, j\neq i}\left(1-\left(1-\nu_j^2\right)^q + \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)} = 1,$$

$$\eta = \left(\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)} = 1. \tag{25}$$

Thus, we can get

$$\begin{aligned}
&\mathrm{PFIPBM}^{1,0}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&= \left(\left(1-\prod_{h=1}^{d}\left(\prod_{i\in P_h}\left(1-\mu_i^2\right)^{1/|P_h|}\right)^{1/d}\right)^{1/2},\right.\\
&\left(\prod_{h=1}^{d}\left(\prod_{i\in P_h}\left(1-\mu_i^2\right)^{1/|P_h|}\right)^{1/d}\right.\\
&\left.\left.-\prod_{h=1}^{d}\left(\prod_{i\in P_h}\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{1/|P_h|}\right)^{1/d}\right)^{1/2}\right).
\end{aligned} \tag{26}$$

(iii) When $p \to 0$, we can get

$$\begin{aligned}
&\mathrm{PFIPBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&= \left(\left(1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{j\in P_h, j\neq i}\left(1-1-\nu_j^{2q}\right.\right.\right.\right.\right.\\
&+\left.\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\\
&+\left.\left(\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\right)^{1/q}\\
&+\left.\left.\left.\left(\left(\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\right)^{1/q}\right)^{1/d}\right)^{1/2},\\
&\left(\prod_{h=1}^{d}\left(1-\left(1-\prod_{j\in P_h, j\neq i}\left(1-\left(1-\nu_j^2\right)^q\right.\right.\right.\right.\\
&+\left.\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\\
&+\left.\left(\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\right)^{1/q}\\
&+\left.\left(\left(\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\right)^{1/q}\right)^{1d}\\
&\left.\left.-\prod_{h=1}^{d}\left(\left(\left(\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(|P_h|-1)}\right)^{1/q}\right)^{1/d}\right)^{1/2}\right).
\end{aligned} \tag{27}$$

(iv) When $p \to 0$ and , $q = 1$, we can get

$$\begin{aligned}
&\mathrm{PFIPBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&= \left(\left(1-\prod_{h=1}^{d}\left(\prod_{j\in P_h, j\neq i}\left(1-\mu_j^2\right)^{1/(|P_h|-1)}\right)^{1/d}\right)^{1/2},\right.\\
&\left(\prod_{h=1}^{d}\prod_{j\in P_h, j\neq i}\left(1-\mu_j^2\right)^{1/(|P_h|-1)}\right)^{1/d}\\
&\left.-\left(\prod_{h=1}^{d}\left(\left(\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)\right)^{1/(|P_h|-1)}\right)^{1/d}\right)^{1/2}\right).
\end{aligned} \tag{28}$$

If all the PFNs are partitioned into one sort, then the PFIPBM operator reduces to the Pythagorean fuzzy interaction Bonferroni mean (PFIBM) operator as follows:

$$\begin{aligned}
&\mathrm{PFIBM}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&= \left(\frac{1}{n(n-1)}\oplus_{i,j=1,i\neq j}^{n}\left(\alpha_i^p\otimes\alpha_j^q\right)\right)^{1/(p+q)}\\
&= \left(\left(\left(1-\left(\sum_{i,j=1,i\neq j}^{n}\left(1-\left(1-\nu_i^2\right)^p\left(1-\nu_j^2\right)^q\right.\right.\right.\right.\right.\\
&+\left.\left.\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)\right)^{1/(p+q)}\\
&+\left(\sum_{i,j=1,i\neq j}^{n}\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\right.\\
&\cdot\left.\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/(n(n-1))}\Bigg)^{1/(p+q)}\\
&-\left(\left(\sum_{i,j=1,i\neq j}^{n}\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\right.\right.\\
&\left.\left.\left.*\left(1-\left(\mu_j^2+\nu_j^2\right)^q\right)^{1/(n(n-1))}\right)^{1/(p+q)}\right)^{1/2},\\
&\left(1-\left(1-\left(\sum_{i,j=1,i\neq j}^{n}\left(1-\left(1-\nu_i^2\right)^p\left(1-\nu_j^2\right)^q\right.\right.\right.\right.\\
&+\left.\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)\Bigg)^{1/(n(n-1))}
\end{aligned}$$

$$+\left(\sum_{i,j=1,i\neq j}^{n}\left(1-\right)\mu_i^2+\nu_i^2\right)\right)^p * \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\Bigg)^{1/(n(n-1))}\Bigg)^{1/(p+q)}\Bigg)^{1/2}\Bigg). \quad (29)$$

Definition 9. Let $T=(\alpha_1,\alpha_2,\dots,\alpha_n)$ be a collection of PFNs, which is partitioned into d distinct sorts $P_1,P_2,\dots,P_d$, where $\alpha_i=(\mu_i,\nu_i)$ $(i=1,2,\dots,n)$ and $\bigcup_{h=1}^{d}P_h=T$. The Pythagorean fuzzy weighted interaction partitioned Bonferroni mean (PFWIPBM) operator is defined as follows:

$$\mathrm{PFWIPBM}^{p,q}(\alpha_1,\alpha_2,\dots,\alpha_n)=\frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|\mathrm{P}_h|(|\mathrm{P}_h|-1)}\oplus_{i,j\in P_h,j\neq i}\left((w_i\alpha_i)^p\otimes\left(w_j\alpha_j\right)^q\right)\right)^{1/(p+q)}\right), \quad (30)$$

where $p,q\geq 0$, $|P_h|$ denotes the cardinality of P_h, d is the number of the partitioned sorts, and $\sum_{h=1}^{d}|P_h|=n$. $\mathbf{W}=(w_1,w_2,\dots,w_n)$ is the weight vector of $(\alpha_1,\alpha_2,\dots,\alpha_n)$ satisfying $w_j\in[0,1]$, $j=1,2,\dots,n$, and $\sum_{j=1}^{n}w_j=1$.

Theorem 5. *Let $\alpha_i=(\mu_i,\nu_i)$ $(i=1,2,\dots,n)$ be a collection of PFNs and $p,q\geq 0$. The aggregated result of PFWIPBM operator is still of a PFN, which has the following form:*

$$\begin{aligned}&\mathrm{PFWIPBM}^{p,q}(\alpha_1,\alpha_2,\dots,\alpha_n)\\&=\frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|P_h|(|P_h|-1)}\oplus_{i,j\in P_h,j\neq i}(w_i\alpha_i)^p\otimes\left(w_j\alpha_j\right)^q\right)^{1/(p+q)}\right)\\&=\left(\left(\left(1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)}+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)\right)^{1/d}\right)^{1/2},\right.\\&\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}+\prod_{i,j\in P_h,j\neq i}\eta^{1/(P_h(|P_h|-1))}\right)^{1/p+q}+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\right.\right.\\&\left.\left.-\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q)d)}\right)^{1/2}\right),\end{aligned} \quad (31)$$

where $\xi=(1-(1-\mu_i^2)^{w_i}+(1-(\mu_i^2+\nu_i^2))^{w_i})^p(1-(1-\mu_j^2)^{w_j}+(1-(\mu_j^2+\nu_j^2))^{w_j})^q$ *and* $\eta=(1-(\mu_i^2+\nu_i^2))^{w_ip}(1-(\mu_j^2+\nu_j^2))^{w_jq}$.

Proof 5.

$$\begin{aligned}w_i\alpha_i&=\left(\sqrt{1-\left(1-\mu_i^2\right)^{w_i}},\sqrt{\left(1-\mu_i^2\right)^{w_i}-\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i}}\right),\\w_j\alpha_j&=\left(\sqrt{1-\left(1-\mu_j^2\right)^{w_j}},\left(\left(1-\mu_j^2\right)^{w_j}-\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j}\right)^{1/2}\right),\\(w_i\alpha_i)^p\otimes(w_j\alpha_j)^q&\Bigg(\Bigg(\Bigg(1-\left(1-\mu_i^2\right)^{w_i}+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i}\Bigg)^p\Bigg(1-\left(1-\mu_j^2\right)^{w_j}+\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j}\Bigg)^q-\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_ip}\\&\cdot\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_jq}\Bigg)^{1/2},\Bigg(1-\Bigg(1-\left(1-\mu_i^2\right)^{w_i}+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i}\Bigg)^p*\Bigg(1-\left(1-\mu_j^2\right)^{w_j}\\&+\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j}\Bigg)^q\Bigg)^{1/2}\Bigg).\end{aligned} \quad (32)$$

Let

$$\xi=\left(1-\left(1-\mu_i^2\right)^{w_i}+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i}\right)^p\cdot\left(1-\left(1-\mu_j^2\right)^{w_j}+\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j}\right)^q,$$

$$\eta=\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_ip}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_jq},$$

$$\oplus_{i,j\in P_h,j\neq i}(w_i\alpha_i)^p\otimes(w_j\alpha_j)^q=\left(\sqrt{1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)},\sqrt{\prod_{i,j\in P_h,j\neq i}1-\xi+\eta-\prod_{i,j\in P_h,j\neq i}\eta}\right),$$

$$\left(\frac{1}{|P_h|(|P_h|-1)} \oplus_{i,j\in P_h, j\neq i}(w_i\alpha_i)^p \otimes (w_j\alpha_j)^q\right)^{1/(p+q)}$$
$$= \left(\left(\left(1 - \prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} - \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/2},\right.$$
$$\left.\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)}\right)^{1/2}\right),$$

$$\frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|P_h|(|P_h|-1)} \oplus_{i,j\in P_h, j\neq i}(w_i\alpha_i)^p \otimes (w_j\alpha_j)^q\right)^{1/(p+q)}\right)$$
$$= \left(\left(1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\right)^{1/2},\right.$$
$$\left(\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}$$
$$\left.\left. - \prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q)d)}\right)^{1/2}\right). \tag{33}$$

Moreover,

$$(\mu_{\mathrm{PFWIPBM}^{p,q}})^2 + (\nu_{\mathrm{PFWIPBM}^{p,q}})^2$$
$$= 1 - \prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}$$
$$+ \prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}$$
$$- \prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q)d)}$$
$$= 1 - \prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q)d)}, \tag{34}$$

since $\eta = (1-(\mu_i^2+\nu_i^2))^{w_ip}(1-(\mu_j^2+\nu_j^2))^{w_jq}$, then $0 \le \eta \le 1$ and we can get $0 \le 1 - \prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q)d)} \le 1$. Hence, the aggregated result of the $\mathrm{PFWIPBM}^{p,q}$ operator is still a PFN.

Some special cases of the PFWIPBM operator are discussed as follows:

(i) If $q \to 0$, we can get $\xi = (1-(1-\mu_i^2)^{w_i} + (1-(\mu_i^2+\nu_i^2))^{w_i})^p$ and $\eta = (1-(\mu_i^2+\nu_i^2))^{w_ip}$. Thus, we can get

$$\mathrm{PFWIPBM}^{p,0}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$
$$= \left(\left(1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/p} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)p)}\right)^{1/d}\right)^{1/2},\right.$$
$$\left(\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/p} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)p)}\right)^{1/d}$$
$$\left.\left. - \prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)pd)}\right)^{1/2}\right). \tag{35}$$

(ii) If $q \to 0$ and $p = 1$, we can get $\xi = 1-(1-\mu_i^2)^{w_i} + (1-(\mu_i^2+\nu_i^2))^{w_i}$ and $\eta = (1-(\mu_i^2+\nu_i^2))^{w_i}$. Thus, we can get

$$\mathrm{PFWIPBM}^{1,0}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$
$$= \left(\left(1-\prod_{h=1}^{d}\left(\prod_{i,j\in P_h, j\neq i}(1-\mu_i^2)^{w_i/(|P_h|(|P_h|-1)d)}\right)\right)^{1/2},\right.$$
$$\left(\prod_{h=1}^{d}\left(\prod_{i,j\in P_h, j\neq i}(1-\mu_i^2)^{w_i/(|P_h|(|P_h|-1))}\right)^{1/d}\right.$$
$$\left.\left. - \prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}(1-(\mu_i^2+\nu_i^2))^{w_i/(|P_h|(|P_h|-1)d)}\right)^{1/2}\right). \tag{36}$$

(iii) If $p \to 0$, we can get $\xi = (1-(1-\mu_j^2)^{w_j} + (1-(\mu_j^2 + \nu_j^2))^{w_j})^q$ and $\eta = (1-(\mu_j^2+\nu_j^2))^{w_j q}$. Thus, we can get

$$\begin{aligned}
&\mathrm{PFWIPBM}^{0,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \Bigg(\Bigg(1 - \prod_{h=1}^{d} \Bigg(1 - \Bigg(1 - \prod_{i,j\in P_h, j\neq i} (1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i} \eta^{1/(P_h(|P_h|-1))} \Bigg)^{1/q} + \prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1)q)} \Bigg)^{1/d} \Bigg)^{1/2}, \\
&\quad \Bigg(\prod_{h=1}^{d} \Bigg(1 - \Bigg(1 - \prod_{i,j\in P_h, j\neq i} (1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i} \eta^{1/(P_h(|P_h|-1))} \Bigg)^{1/q} + \prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1)q)} \Bigg)^{1/d} \\
&\quad - \prod_{h=1}^{d} \prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1)qd)} \Bigg)^{1/2} \Bigg).
\end{aligned} \tag{37}$$

(iv) If $p \to 0$ and $q = 1$, we can get $\xi = 1 - (1-\mu_j^2)^{w_j} + (1-(\mu_j^2+\nu_j^2))^{w_j}$ and $\eta = (1-(\mu_j^2+\nu_j^2))^{w_j}$. Thus, we can get

$$\begin{aligned}
&\mathrm{PFWIPBM}^{0,1}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \Bigg(\Bigg(1 - \prod_{h=1}^{d} \Bigg(\prod_{i,j\in P_h, j\neq i} \left(1-\mu_j^2\right)^{w_j/(|P_h|(|P_h|-1))} \Bigg)^{1/d} \Bigg), \\
&\quad \Bigg(\prod_{h=1}^{d} \Bigg(\prod_{i,j\in P_h, j\neq i} \left(1-\mu_j^2\right)^{w_j/(|P_h|(|P_h|-1))} \Bigg)^{1/d} - \prod_{h=1}^{d} \prod_{i,j\in P_h, j\neq i} \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j/(P_h(|P_h|-1)d)} \Bigg) \Bigg).
\end{aligned} \tag{38}$$

If all the PFNs are partitioned into one sort, the PFWIPBM operator reduces to the Pythagorean fuzzy weighted interaction Bonferroni mean (PFWIBM) operator as follows:

$$\begin{aligned}
&\left(\frac{1}{|P_h|(|P_h|-1)} \oplus_{i,j\in P_h, j\neq i} (w_i\alpha_i)^p \otimes \left(w_j\alpha_j\right)^q \right)^{1/(p+q)} \\
&= \Bigg(\Bigg(\Bigg(1 - \prod_{i,j\in P_h, j\neq i} (1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1))} \Bigg)^{1/(p+q)} - \prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1)(p+q))} \Bigg)^{1/2}, \\
&\quad \Bigg(1 - \Bigg(1 - \prod_{i,j\in P_h, j\neq i} (1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1))} \Bigg)^{1/(p+q)} \Bigg)^{1/2} \Bigg),
\end{aligned} \tag{39}$$

where $\xi = (1-(1-\mu_i^2)^{w_i} + (1-(\mu_i^2+\nu_i^2))^{w_i})^p (1-(1-\mu_j^2)^{w_j} + (1-(\mu_j^2+\nu_j^2))^{w_j})^q$; $\eta = (1-(\mu_i^2+\nu_i^2))^{w_i p}(1-(\mu_j^2+\nu_j^2))^{w_j q}$; $p, q \geq 0$; and $(w_1, w_2, \ldots, w_n)$ is the weight vector of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $w_j \in [0,1]$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$.

Definition 10. Let $T = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a collection of PFNs, which is partitioned into d distinct sorts $P_1, P_2, \ldots, P_d$, where $\alpha_i = (\mu_i, \nu_i)$ $(i = 1, 2, \ldots, n)$ and $\bigcup_{h=1}^{d} P_h = T$. The Pythagorean fuzzy interaction partitioned geometric Bonferroni mean (PFIPGBM) operator is defined as follows:

$$\begin{aligned}
&\mathrm{PFIPGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \left(\otimes_{h=1}^{d} \left(\frac{1}{p+q} \left(\otimes_{i,j\in P_h, j\neq i} \left((p\alpha_i) \oplus \left(q\alpha_j\right) \right) \right)^{1/(|P_h|(|P_h|-1))} \right) \right)^{1/d},
\end{aligned} \tag{40}$$

where $p, q \geq 0$, $|P_h|$ denotes the cardinality of P_h, d is the number of the partitioned sorts, and $\sum_{h=1}^{d} |P_h| = n$.

Theorem 6. *Let $\alpha_i = (\mu_i, \nu_i)$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs and $p, q \geq 0$. The aggregated result of PFIPGBM operator is still of a PFN, which has the following form:*

$$\begin{aligned}
&\mathrm{PFIPGBM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \left(\otimes_{h=1}^{d} \left(\frac{1}{p+q} \left(\otimes_{i,j\in P_h, j\neq i} \left((p\alpha_i) \oplus \left(q\alpha_j\right) \right) \right)^{1/(|P_h|(|P_h|-1))} \right) \right)^{1/d} \\
&= \Bigg(\Bigg(\Bigg(\prod_{h=1}^{d} \Bigg(1 - \Bigg(1 - \prod_{i,j\in P_h, j\neq i} (1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1))} \Bigg)^{1/(p+q)} + \prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1)(p+q))} \Bigg)^{1/d} \\
&\quad - \prod_{h=1}^{d} \Bigg(\prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1)(p+q))} \Bigg)^{1/d} \Bigg)^{1/2}, \\
&\quad \Bigg(1 - \prod_{h=1}^{d} \Bigg(1 - \Bigg(1 - \prod_{i,j\in P_h, j\neq i} (1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i} \eta^{1/(|P_h|(|P_h|-1))} \Bigg)^{1/(p+q)}
\end{aligned}$$

$$+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\Bigg)^{1/d}\Bigg)^{1/2}\Bigg), \tag{41}$$

where $\xi=(1-\mu_i^2)^p(1-\mu_j^2)^q$ *and* $\eta=(1-(\mu_i^2+\nu_i^2))^p(1-(\mu_j^2+\nu_j^2))^q$.

Proof 6.

$$p\alpha_i=\left(\sqrt{1-(1-\mu_i^2)^p},\sqrt{(1-\mu_i^2)^p-(1-(\mu_i^2+\nu_i^2))^p}\right)$$

$$q\alpha_j=\left(\sqrt{1-\left(1-\mu_j^2\right)^q},\left(\left(1-\mu_j^2\right)^q-\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/2}\right),$$

$$\begin{aligned}&(p\alpha_i)\oplus(q\alpha_j)\\&=\left(\sqrt{1-(1-\mu_i^2)^p\left(1-\mu_j^2\right)^q},\left((1-\mu_i^2)^p\left(1-\mu_j^2\right)^q\right.\right.\\&\left.\left.\quad-\left(1-(\mu_i^2+\nu_i^2)\right)^p\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\right)^{1/2}\right).\end{aligned}\tag{42}$$

Let $=(1-\mu_i^2)^p(1-\mu_j^2)^q$ and $\eta=(1-(\mu_i^2+\nu_i^2))^p(1-(\mu_j^2+\nu_j^2))^q$, then

$$\begin{aligned}&\otimes_{i,j\in P_h,j\neq i}\left((p\alpha_i)\oplus(q\alpha_j)\right)\\&=\left(\sqrt{\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)-\prod_{i,j\in P_h,j\neq i}\eta},\sqrt{1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)}\right),\\&\frac{1}{p+q}\left(\otimes_{i,j\in P_h,j\neq i}\left((p\alpha_i)\oplus(q\alpha_j)\right)\right)^{1/(|P_h|(|P_h|-1))}\\&=\left(\left(1-\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)}\right)^{1/2}\right.\\&\quad\left(\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)}\right.\\&\quad\left.\left.-\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/2}\right),\end{aligned}$$

$$\begin{aligned}&\left(\otimes_{h=1}^{d}\left(\frac{1}{p+q}\left(\otimes_{i,j\in P_h,j\neq i}\left((p\alpha_i)\oplus(q\alpha_j)\right)\right)^{1/(|P_h|(|P_h|-1))}\right)\right)^{1/d}\\&=\left(\left(\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}\right.\right.\right.\right.\\&\quad\left.+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)}\left.+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\\&\quad\left.-\prod_{h=1}^{d}\left(\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\right)^{1/2},\\&\quad\left(1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}\right.\right.\right.\\&\quad\left.+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)}\\&\quad\left.\left.\left.+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\right)^{1/2}\right).\end{aligned}\tag{43}$$

Moreover,

$$\begin{aligned}&(\mu_{\text{PFIFGBM}^{p,q}})^2+(\nu_{\text{PFIFGBM}^{p,q}})^2\\&=\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}\right.\right.\\&\quad\left.\left.+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)}+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\\&\quad-\prod_{h=1}^{d}\left(\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\\&\quad+1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}\right.\right.\\&\quad\left.\left.+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)}+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\\&=1-\prod_{h=1}^{d}\left(\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d},\end{aligned}\tag{44}$$

since $\eta=(1-(\mu_i^2+\nu_i^2))^p(1-(\mu_j^2+\nu_j^2))^q$, then $0\le\eta\le1$ and we can get

$$0\le1-\prod_{h=1}^{d}\left(\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\le1. \tag{45}$$

Hence, the aggregated result of the $\mathrm{PFIPGBM}^{p,q}$ operator is still a PFN.

Theorem 7 *(idempotency). Let $T=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ be a collection of PFNs. If all α_k $(k=1,2,\ldots,n)$ are equal, that is, $\alpha_k=\alpha=(\mu,\nu)$ $(k=1,2,\ldots,n)$, then*

$$\mathrm{PFIPGBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\alpha. \quad (46)$$

Theorem 8 *(commutativity). Let $\alpha_k=(\mu_k,\nu_k)$ $(k=1,2,\ldots,n)$ and α'_k $(k=1,2,\ldots,n)$ be two collections of PFNs. If $\alpha'_k=(\mu'_k,\nu'_k)$ $(k=1,2,\ldots,n)$ is any permutation of $\alpha_k=(\mu_k,\nu_k)$, then*

$$\mathrm{PFIPGBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\mathrm{PFIPGBM}^{p,q}\left(\alpha'_1,\alpha'_2,\ldots,\alpha'_n\right). \quad (47)$$

Theorem 9 *(boundedness). Let $\tilde{\alpha}=(1,0)$ and $\breve{\alpha}=(0,1)$, then*

$$\breve{\alpha}\le \mathrm{PFIPGBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\le\tilde{\alpha}. \quad (48)$$

Some special cases of the PFIPGBM operator based on the parameters p and q are discussed as follows:

(i) When $q\to 0$, we can get $\xi=(1-\mu_i^2)^p$, $\eta=(1-(\mu_i^2+\nu_i^2))^p$,

$$\begin{aligned}
&\mathrm{PFIPGBM}^{p,0}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\Bigg(\Bigg(\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}\Big(1-(1-\mu_i^2)^p\\
&+(1-(\mu_i^2+\nu_i^2))^p\Big)^{1/(|P_h|(|P_h|-1))}\\
&+\prod_{i,j\in P_h,j\neq i}(1-(\mu_i^2+\nu_i^2))^{p/(|P_h|(|P_h|-1))}\Bigg)^{1/p}\\
&+\prod_{i,j\in P_h,j\neq i}(1-(\mu_i^2+\nu_i^2))^{p1/(|P_h|(|P_h|-1)p)}\Bigg)^{1/d}\\
&-\prod_{i,j\in P_h,j\neq i}\Big(1-(1-\mu_i^2)^p+(1-(\mu_i^2+\nu_i^2))^p\Big)^{1/(|P_h|(|P_h|-1))}\\
&+\prod_{i,j\in P_h,j\neq i}\Bigg(1-\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}\\
&\cdot(1-(\mu_i^2+\nu_i^2))^{p1/(|P_h|(|P_h|-1)p)}\Bigg)^{1/d}\Bigg)^{1/2},\\
&\Bigg(1-\prod_{h=1}^{d}\Big(1-(1-(\mu_i^2+\nu_i^2))^{p1/(|P_h|(|P_h|-1))}\Big)^{1/p}\\
&+\prod_{i,j\in P_h,j\neq i}(1-(\mu_i^2+\nu_i^2))^{p1/(|P_h|(|P_h|-1)p)}\Bigg)^{1/d}\Bigg)^{1/2}\Bigg).
\end{aligned} \quad (49)$$

(ii) When $q\to 0$ and $p=1$, we can get $\xi=1-\mu_i^2$, $\eta=1-(\mu_i^2+\nu_i^2)$, and

$$\begin{aligned}
&\mathrm{PFIPGBM}^{p,0}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\Bigg(\Bigg(\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}(1-\nu_i^2)^{1/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\\
&-\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}(1-(\mu_i^2+\nu_i^2))^{1/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2},\\
&\Bigg(1-\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}(1-\nu_i^2)^{1/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2}\Bigg).
\end{aligned} \quad (50)$$

(iii) When $p\to 0$, we can get $\xi=(1-\mu_j^2)^q$ and $\eta=(1-(\mu_j^2+\nu_j^2))^q$,

$$\begin{aligned}
&\mathrm{PFIPGBM}^{0,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\Bigg(\Bigg(\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}\Big(1-\left(1-\mu_j^2\right)^q\\
&+\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\Big)^{1/(|P_h|(|P_h|-1))}\\
&+\prod_{i,j\in P_h,j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{q/(|P_h|(|P_h|-1))}\Bigg)^{1/q}\\
&+\prod_{i,j\in P_h,j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{q/(|P_h|(|P_h|-1)(q))}\Bigg)^{1/d}\\
&-\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)q)}\Bigg)^{1/d}\Bigg)^{1/2},\\
&\Bigg(1-\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}\Big(1-\left(1-\mu_j^2\right)^q\\
&+\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\Big)^{1/(|P_h|(|P_h|-1))}\\
&+\prod_{i,j\in P_h,j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{q/(|P_h|(|P_h|-1))}\Bigg)^{1/q}\\
&+\prod_{i,j\in P_h,j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{q/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2}\Bigg).
\end{aligned} \quad (51)$$

(iv) When $p\to 0$ and $q=1$, we can get $\xi=1-\mu_j^2$ and $\eta=1-(\mu_j^2+\nu_j^2)$,

$$\begin{aligned}
&\mathrm{PFIPGBM}^{0,1}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\Bigg(\Bigg(\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}\left(1-\nu_j^2\right)^{1/(|P_h|(|P_h|-1))}\Bigg)^{1/d}
\end{aligned}$$

$$-\prod_{h=1}^{d}\left(\prod_{i,j\in P_h,j\neq i}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{1/(|P_h|(|P_h|-1))}\right)^{1/d}\Bigg)^{1/2},$$

$$\left(1-\prod_{h=1}^{d}\left(\prod_{i,j\in P_h,j\neq i}\left(1-\nu_j^2\right)^{1/(|P_h|(|P_h|-1))}\right)^{1/d}\right)^{1/2}\Bigg). \quad (52)$$

If all PFNs are partitioned into one sort, the PFIPGBM operator reduces to the Pythagorean fuzzy interaction geometric Bonferroni mean (PFIGBM) operator as follows:

$$\begin{aligned}
&\mathrm{PFIGBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\frac{1}{p+q}\left(\otimes_{i,j\in P_h,j\neq i}\left(\left(p\alpha_i\right)\oplus\left(q\alpha_j\right)\right)\right)^{1/(m(m-1))}\\
&=\Bigg(\Bigg(\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}\Big(1-\left(1-\mu_i^2\right)^p\left(1-\mu_j^2\right)^q\\
&\quad+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\Big)^{1/(|P_h|(|P_h|-1))}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\Big(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\\
&\quad\cdot\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\Big)^{1/(|P_h|(|P_h|-1))}\Bigg)^{1/(p+q)}\Bigg)^{1/2},\\
&\quad\Bigg(\Bigg(1-\prod_{i,j\in P_h,j\neq i}\Big(1-\left(1-\mu_i^2\right)^p\left(1-\mu_j^2\right)^q\\
&\quad+\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\Big)^{1/(|P_h|(|P_h|-1))}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\Big(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\\
&\quad\cdot\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\Big)^{1/(|P_h|(|P_h|-1))}\Bigg)^{1/(p+q)}\\
&\quad-\prod_{i,j\in P_h,j\neq i}\Big(\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^p\\
&\quad\cdot\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^q\Big)^{1/(|P_h|(|P_h|-1)(p+q))}\Bigg)^{1/2}\Bigg),
\end{aligned} \quad (53)$$

where $p,q\geq 0$.

Definition 11. Let $T=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ be a collection of PFNs, which is partitioned into d distinct sorts $P_1,P_2,\ldots,P_d$ and $\bigcup_{h=1}^{d}P_h=T$. The Pythagorean fuzzy weighted interaction partitioned geometric Bonferroni mean (PFWIPGBM) operator is defined as follows:

$$\begin{aligned}
&\mathrm{PFWIPGBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left(\otimes_{h=1}^{d}\left(\frac{1}{p+q}\left(\otimes_{i,j\in P_h,j\neq i}\left(p\left(\alpha_i\right)^{w_i}\right.\right.\right.\right.\\
&\quad\left.\left.\left.\left.\oplus\, q\left(\alpha_j\right)^{w_j}\right)\right)^{1/(|P_h|(|P_h|-1))}\right)\right)^{1/d},
\end{aligned} \quad (54)$$

where $\alpha_i=(\mu_i,\nu_i)$ $(i=1,2,\ldots,n)$; $p,q\geq 0$; $|P_h|$ denotes the cardinality of P_h; d is the number of the partitioned sorts; and $\sum_{h=1}^{d}|P_h|=n$. $\mathbf{w}=(w_1,w_2,\ldots,w_n)$ is the weight vector of $(\alpha_1,\alpha_2,\ldots,\alpha_n)$, $w_j\geq 0$, $j=1,2,\ldots,n$, and $\sum_{j=1}^{n}w_j=1$.

Theorem 10. *Let $\alpha_i=(\mu_i,\nu_i)$ $(i=1,2,\ldots,n)$ be a collection of PFNs. For any $p,q\geq 0$, the aggregated result of the PFWIPGBM operator is still a PFN, which has the following forms:*

$$\begin{aligned}
&\mathrm{PFWIPGBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left(\otimes_{h=1}^{d}\left(\frac{1}{p+q}\left(\otimes_{i,j\in P_h,j\neq i}\left(p\left(\alpha_i\right)^{w_i}\oplus q\left(\alpha_j\right)^{w_j}\right)\right)^{1/(|P_h|(|P_h|-1))}\right)\right)^{1/d}\\
&=\Bigg(\Bigg(\Bigg(\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\eta^{1/(P_h(|P_h|-1))}\Bigg)^{1/(p+q)}+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\Bigg)^{1/d}\\
&\quad-\left(\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\Bigg)^{1/2},\\
&\quad\Bigg(1-\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\eta^{1(|P_h|(|P_h|-1))}\Bigg)^{1/(p+q)}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\Bigg)^{1/d}\Bigg)^{1/2}\Bigg),
\end{aligned} \quad (55)$$

where $\xi=(1-(1-\nu_i^2)^{w_i}+(1-(\mu_i^2+\nu_i^2))^{w_i})^p(1-(1-\nu_j^2)^{w_j}+(1-(\mu_j^2+\nu_j^2))^{w_j})^q$ *and* $\eta=(1-(\mu_i^2+\nu_i^2))^{w_ip}(1-(\mu_j^2+\nu_j^2))^{w_jq}$.

Proof 7.

$$\alpha_i^{w_i}=\left(\left(\left(1-\nu_i^2\right)^{w_i}-\left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i}\right)^{1/2},\left(1-\left(1-\nu_i^2\right)^{w_i}\right)^{1/2}\right),$$

$$\alpha_j^{w_j} = \left(\left(\left(1-\nu_j^2\right)^{w_j} - \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j}\right)^{1/2}, \left(1-\left(1-\nu_j^2\right)^{w_j}\right)^{1/2}\right),$$

$$\begin{aligned}\left(p\alpha_i^{w_i}\right) \oplus \left(q\alpha_j^{w_j}\right) = \Bigg(\Bigg(1 - \Big(1 - \left(1-\nu_i^2\right)^{w_i} &+ \left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i}\Big)^p \Big(1 - \left(1-\nu_j^2\right)^{w_j} \\ &+ \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j}\Big)^q\Bigg)^{1/2}, \Bigg(\Big(1-\left(1-\nu_i^2\right)^{w_i} \\ &+ \left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i}\Big)^p \Big(1-\left(1-\nu_j^2\right)^{w_j} \\ &+ \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j}\Big)^q - \left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i p} \\ &\cdot \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j q}\Bigg)^{1/2}\Bigg). \end{aligned} \tag{56}$$

Let

$$\begin{aligned} &\otimes_{i,j\in P_h, j\neq i}\left(\left(p\alpha_i^{w_i}\right) \oplus \left(q\alpha_j^{w_j}\right)\right) \\ &= \xi = \left(1-\left(1-\nu_i^2\right)^{w_i} + \left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i}\right)^p \\ &\quad \cdot \left(1-\left(1-\nu_j^2\right)^{w_j} + \left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j}\right)^q, \\ &\eta = \left(1-\left(\mu_i^2+\nu_i^2\right)\right)^{w_i p}\left(1-\left(\mu_j^2+\nu_j^2\right)\right)^{w_j q}, \\ &\left(\sqrt{\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta) - \prod_{i,j\in P_h, j\neq i}\eta}, \sqrt{1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)}\right), \\ &\frac{1}{p+q}\left(\otimes_{i,j\in P_h, j\neq i}\left(\left(p\alpha_i^{w_i}\right)\oplus\left(q\alpha_j^{w_j}\right)\right)\right)^{1/(|P_h|(|P_h|-1))} \\ &= \left(\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)}\right)^{1/2}, \right. \\ &\quad \left(\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} \right. \\ &\quad \left.\left. - \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/2}\right). \end{aligned}$$

$$\begin{aligned} &\left(\otimes_{h=1}^{d}\left(\frac{1}{p+q}\left(\otimes_{i,j\in P_h, j\neq i}\left(\left(p\alpha_i^{w_i}\right)\oplus\left(q\alpha_j^{w_j}\right)\right)\right)^{1/(|P_h|(|P_h|-1))}\right)\right)^{1/d} \\ &= \left(\left(\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d} \right.\right. \\ &\quad \left. - \left(\prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\right)^{1/2}, \\ &\quad \left(1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} \right.\right. \\ &\quad \left.\left.\left. + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}\right)^{1/2}\right). \end{aligned} \tag{57}$$

Moreover,

$$\begin{aligned} &\left(\mu_{\mathrm{PFWIPGBM}^{p,q}}\right)^2 + \left(\nu_{\mathrm{PFWIPGBM}^{p,q}}\right)^2 \\ &= \prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d} \\ &\quad - \left(\prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d} + 1 \\ &\quad - \prod_{h=1}^{d}\left(1-\left(1-\prod_{i,j\in P_h, j\neq i}(1-\xi+\eta)^{1/(|P_h|(|P_h|-1))} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1))}\right)^{1/(p+q)} + \prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d} \\ &= 1 - \left(\prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d}. \end{aligned} \tag{58}$$

Since $\eta = (1-(\mu_i^2+\nu_i^2))^{w_i p}(1-(\mu_j^2+\nu_j^2))^{w_j q}$, then $0 \leq \eta \leq 1$ and we can get

$$0 \leq 1 - \left(\prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\eta^{1/(|P_h|(|P_h|-1)(p+q))}\right)^{1/d} \leq 1. \tag{59}$$

Hence, the aggregated result of the PFWIPGBM^{p,q} is still a PFN.

Some special cases of the PFWIPGBM operator are discussed as follows:

(i) If $q=0$, we can get $\xi=(1-(1-\nu_i^2)^{w_i}+(1-(\mu_i^2+\nu_i^2))^{w_i})^p$ and $\eta=(1-(\mu_i^2+\nu_i^2))^{w_i p}$.

$$\begin{aligned}
&\mathrm{PFWIPGBM}^{p,0}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\Bigg(\Bigg(\Bigg(\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}\Big(1-\big(1-(1-\nu_i^2)^{w_i}\\
&\quad+\big(1-(\mu_i^2+\nu_i^2)\big)^{w_i}\big)^p+\big(1-(\mu_i^2+\nu_i^2)\big)^{w_i p}\Big)^{1/(|P_h|(|P_h|-1))}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\big(1-(\mu_i^2+\nu_i^2)\big)^{w_i p/(|P_h|(|P_h|-1))}\Bigg)^{1/p}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\big(1-(\mu_i^2+\nu_i^2)\big)^{w_i/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\\
&\quad-\Bigg(\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\big(1-(\mu_i^2+\nu_i^2)\big)^{w_i/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2},\\
&\quad\Bigg(1-\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}\Big(1-\big(1-(1-\nu_i^2)^{w_i}\\
&\quad+\big(1-(\mu_i^2+\nu_i^2)\big)^{w_i}\big)^p+\big(1-(\mu_i^2+\nu_i^2)\big)\Big)^{w_i p/(|P_h|(|P_h|-1))}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\big(1-(\mu_i^2+\nu_i^2)\big)^{w_i p/(|P_h|(|P_h|-1))}\Bigg)^{1/p}\\
&\quad+\prod_{i,j\in P_h,j\neq i}1-\big(\mu_i^2+\nu_i^2\big)^{w_i/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2}\Bigg).
\end{aligned}\tag{60}$$

(ii) If $q=0$ and $p=1$, we can get $\xi=1-(1-\nu_i^2)^{w_i}+(1-(\mu_i^2+\nu_i^2))^{w_i}$ and $\eta=(1-(\mu_i^2+\nu_i^2))^{w_i}$.

$$\begin{aligned}
&\mathrm{PFWIPGBM}^{1,0}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&\cdot\Bigg(\Bigg(\Bigg(\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}(1-\nu_i^2)^{w_i/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\\
&\quad-\Bigg(\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\big(1-(\mu_i^2+\nu_i^2)\big)^{w_i/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2},\\
&\quad\Bigg(1-\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}(1-\nu_i^2)^{w_i/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2}\Bigg).
\end{aligned}\tag{61}$$

(iii) If $p=0$, we can get $\xi=(1-(1-\nu_j^2)^{w_j}+(1-(\mu_j^2+\nu_j^2))^{w_j})^q$ and $\eta=(1-(\mu_j^2+\nu_j^2))^{w_j q}$.

$$\begin{aligned}
&\mathrm{PFWIPGBM}^{0,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\Bigg(\Bigg(\Bigg(\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}\Big(1-\Big(1-\big(1-\nu_j^2\big)^{w_j}\\
&\quad+\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_j}\Big)^q+\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_j q}\Big)^{1/(|P_h|(|P_h|-1))}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_j q/(|P_h|(|P_h|-1))}\Bigg)^{1/q}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_j/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\\
&\quad-\Bigg(\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_j/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2},\\
&\quad\Bigg(1-\prod_{h=1}^{d}\Bigg(1-\Bigg(1-\prod_{i,j\in P_h,j\neq i}\Big(1-\Big(1-\big(1-\nu_j^2\big)^{w_j}\\
&\quad+\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_j}\Big)^q+\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_j q}\Big)^{1/(|P_h|(|P_h|-1))}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_i q/(|P_h|(|P_h|-1))}\Bigg)^{1/q}\\
&\quad+\prod_{i,j\in P_h,j\neq i}\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_j/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2}\Bigg).
\end{aligned}\tag{62}$$

(iv) If $p=0, q=1$, we can get $\xi=(1-(1-\nu_j^2)^{w_j}+(1-(\mu_j^2+\nu_j^2))^{w_j})$ and $\eta=(1-(\mu_j^2+\nu_j^2))^{w_j}$.

$$\begin{aligned}
&\mathrm{PFWIPGBM}^{0,1}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\Bigg(\Bigg(\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}\big(1-\nu_j^2\big)^{w_j/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\\
&\quad-\Bigg(\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\Big(1-\big(\mu_j^2+\nu_j^2\big)\Big)^{w_j/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2},\\
&\quad\Bigg(1-\Bigg(\prod_{h=1}^{d}\Bigg(\prod_{i,j\in P_h,j\neq i}\big(1-\nu_j^2\big)^{w_j/(|P_h|(|P_h|-1))}\Bigg)^{1/d}\Bigg)^{1/2}\Bigg).
\end{aligned}\tag{63}$$

If all PFNs are partitioned into one sort, the PFWIPGBM operator reduces to the Pythagorean fuzzy weighted interaction geometric Bonferroni mean (PFWIGBM) operator as follows:

$$\begin{aligned}&\mathrm{PFWIGBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\&=\frac{1}{p+q}\left(\otimes_{i,j\in P_h,j\neq i}\left(\left(p\alpha_i^{w_i}\right)\oplus\left(q\alpha_j^{w_j}\right)\right)\right)^{1/(m(m-1))}\\&=\left(\left(\left(1-\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1/(m(m-1))}\right.\right.\right.\right.\\&\left.\left.\left.+\prod_{i,j\in P_h,j\neq i}\eta^{1/(m(m-1))}\right)^{1/(p+q)}\right)^{1/2},\\&\left(\left(1-\prod_{i,j\in P_h,j\neq i}(1-\xi+\eta)^{1(m(m-1))}\right.\right.\\&\left.\left.\left.+\prod_{i,j\in P_h,j\neq i}\eta^{1/(m(m-1))}\right)^{1/(p+q)}-\prod_{i,j\in P_h,j\neq i}\eta^{1/(m(m-1)(p+q))}\right)^{1/2}\right).\end{aligned}\tag{64}$$

4. A New Method for Multiple-Attribute Decision-Making Based on the Proposed PFWIPBM and PFWIPGBM Operator

For a MAGDM problem with PFNs, let $\{E_1,E_2,\ldots,E_t\}$ be a collection of decision-makers, let $\{A_1,A_2,\ldots,A_m\}$ be a collection of alternatives, and let $C=\{C_1,C_2,\ldots,C_n\}$ be a collection of attributes. $D^{(k)}=(d_{ij}^{(k)})_{m\times n}$ is a decision matrix given by decision-maker E_k, where $d_{ij}^{(k)}=(\mu_{ij}^{(k)},\nu_{ij}^{(k)})$ is the evaluation value of A_i with respect to attribute C_j given by decision-maker E_k. $(\lambda_1,\lambda_2,\ldots,\lambda_t)$ is the weight vector of decision-makers, where $\lambda_k\geq 0$ and $\sum_{k=1}^{t}\lambda_k=1$. Let $(w_1,w_2,\ldots,w_n)$ be the weight vector of attributes with $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$. The proposed method based on the new operators is presented as follows.

Step 1. Decision-maker E_k gives the evaluation value of alternative A_i with respect to attribute C_j using Pythagorean fuzzy number $d_{ij}^{(k)}$. Then, the decision matrix is formed as $D^{(k)}=(d_{ij}^{(k)})_{m\times n}$, $k=1,2,\ldots,t$.

Step 2. Aggregate different decision matrices into a collective one by using the PFIPBM operator or the PFIPGBM operator.

$$\begin{aligned}r_{ij}&=\left(\mu_{ij},\nu_{ij}\right)=\mathrm{PFIPBM}^{p,q}\left(r_{ij}^{(1)},r_{ij}^{(2)},\ldots,r_{ij}^{(t)}\right)\\&=\frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|P_h|}\oplus_{l\in P_h}\left(\left(\alpha_{ij}^{(l)}\right)^{p}\right.\right.\right.\\&\left.\left.\left.\otimes\left(\frac{1}{|P_h|-1}\oplus_{k\in P_h,k\neq l}\left(\alpha_{ij}^{(k)}\right)^{q}\right)\right)\right)^{1/(p+q)}\right),\end{aligned}$$

$$\begin{aligned}r_{ij}&=\left(\mu_{ij},\nu_{ij}\right)=\mathrm{PFIPGBM}^{p,q}\left(r_{ij}^{(1)},r_{ij}^{(2)},\ldots,r_{ij}^{(t)}\right)\\&=\left(\otimes_{h=1}^{d}\left(\frac{1}{p+q}\left(\otimes_{l,k\in P_h,l\neq k}\left(\left(p\alpha_{ij}^{(l)}\right)\right.\right.\right.\right.\\&\left.\left.\left.\left.\oplus\left(q\alpha_{ij}^{(k)}\right)\right)\right)^{1/(|P_h|(|P_h|-1))}\right)\right)^{1/d},\end{aligned}\tag{65}$$

where $p,q\geq 0$, $|P_h|$ denotes the cardinality of P_h, d is the number of the partitioned sorts and $\sum_{h=1}^{d}|P_h|=t$. By using Eq.(8) and Eq. (41), $r_{ij}=(\mu_{ij},\nu_{ij})$ can be calculated.

Step 3. Calculate the collective evaluation values of each alternatives by using the proposed PFWIPBM operator or PFWIPGBM operator.

$$\begin{aligned}r_i&=(\mu_i,\nu_i)=\mathrm{PFWIPBM}^{p,q}(r_{i1},r_{i2},\ldots,r_{\mathrm{in}})\\&=\frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|P_h|(|P_h|-1)}\right.\right.\\&\left.\left.\oplus_{i,j\in P_h,j\neq i}\left((w_i\alpha_i)^{p}\otimes\left(w_j\alpha_j\right)^{q}\right)\right)^{1/(p+q)}\right),\\r_i&=(\mu_i,\nu_i)=\mathrm{PFWIPGBM}^{p,q}(r_{i1},r_{i2},\ldots,r_{\mathrm{in}})\\&\cdot\left(\otimes_{h=1}^{d}\left(\frac{1}{p+q}\left(\otimes_{i,j\in P_h,j\neq i}\left(p(\alpha_i)^{w_i}\right.\right.\right.\right.\\&\left.\left.\left.\left.\oplus q\left(\alpha_j\right)^{w_j}\right)\right)^{1/(|P_h|(|P_h|-1))}\right)\right)^{1/d},\end{aligned}\tag{66}$$

where $p,q\geq 0$, $|P_h|$ denotes the cardinality of P_h, d is the number of the partitioned sorts, and $\sum_{h=1}^{d}|P_h|=n$. $(w_1,w_2,\ldots,w_n)$ is the weight vector with $w_i\geq 0$ and $\sum_{j=1}^{n}w_j=1$. By using (31) and (55), the collective evaluation values $r_i=(\mu_i,\nu_i)$ $(i=1,2,\ldots,m)$ of alternatives can be calculated.

Step 4. Calculate the score value $S(r_i)$ and accuracy value $A(r_i)$ of the collective evaluation value r_i of alternative A_i by using (2) and (3).

Step 5. Rank alternatives according the method introduced in Definition 3 and select the optimal alternative.

The new method has some desirable advantages as follows: (1) Pythagorean fuzzy numbers are used as the evaluation values, which are more powerful and flexible comparing with other existing tools to model uncertain and fuzzy information; (2) Bonferroni mean has been used to model interrelationship of attributes; (3) partitioned Bonferroni mean operator is used to depict the interrelationship among different sorts, which can lead to more accurate decision results; and (4) the interaction between membership and nonmembership has been considered to avoid unreasonable results caused by extremely small values of membership or nonmembership.

Table 1: Pythagorean fuzzy decision matrix $D^{(1)}$.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.65, 0.30)	(0.60, 0.10)	(0.80, 0.30)	(0.75, 0.40)	(0.80, 0.10)
A_2	(0.70, 0.20)	(0.50, 0.30)	(0.60, 0.40)	(0.80, 0.10)	(0.70, 0.30)
A_3	(0.50, 0.40)	(0.80, 0.20)	(0.40, 0.30)	(0.90, 0.20)	(0.60, 0.30)
A_4	(0.50, 0.60)	(0.40, 0.20)	(0.70, 0.20)	(0.60, 0.40)	(0.85, 0.15)
A_5	(0.80, 0.30)	(0.90, 0.00)	(0.50, 0.10)	(0.40, 0.20)	(0.50, 0.30)

Table 2: Pythagorean fuzzy decision matrix $D^{(2)}$.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.50, 0.40)	(0.60, 0.20)	(0.85, 0.35)	(0.70, 0.30)	(0.80, 0.20)
A_2	(0.75, 0.25)	(0.40, 0.50)	(0.70, 0.30)	(0.85, 0.20)	(0.60, 0.20)
A_3	(0.60, 0.30)	(0.85, 0.10)	(0.50, 0.40)	(0.90, 0.10)	(0.75, 0.25)
A_4	(0.50, 0.65)	(0.30, 0.40)	(0.80, 0.15)	(0.65, 0.30)	(0.80, 0.20)
A_5	(0.85, 0.20)	(0.95, 0.05)	(0.60, 0.20)	(0.40, 0.30)	(0.50, 0.40)

Table 3: Pythagorean fuzzy decision matrix $D^{(3)}$.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.60, 0.35)	(0.50, 0.30)	(0.80, 0.40)	(0.50, 0.25)	(0.85, 0.25)
A_2	(0.80, 0.20)	(0.30, 0.40)	(0.65, 0.20)	(0.90, 0.20)	(0.50, 0.40)
A_3	(0.70, 0.30)	(0.80, 0.25)	(0.60, 0.30)	(0.80, 0.30)	(0.50, 0.30)
A_4	(0.50, 0.50)	(0.40, 0.20)	(0.85, 0.10)	(0.60, 0.20)	(0.90, 0.10)
A_5	(0.75, 0.35)	(0.80, 0.05)	(0.55, 0.25)	(0.30, 0.40)	(0.40, 0.50)

Table 4: Pythagorean fuzzy collective decision matrix D.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.5889, 0.3510)	(0.5706, 0.2119)	(0.8206, 0.3515)	(0.6765, 0.3335)	(0.8210, 0.1927)
A_2	(0.7541, 0.2187)	(0.4115, 0.4089)	(0.6545, 0.3081)	(0.8589, 0.1742)	(0.6122, 0.3090)
A_3	(0.6106, 0.3355)	(0.8186, 0.1889)	(0.5109, 0.3386)	(0.8738, 0.2065)	(0.6350, 0.2869)
A_4	(0.5043, 0.5882)	(0.3712, 0.2815)	(0.7715, 0.1524)	(0.6197, 0.3102)	(0.8555, 0.1521)
A_5	(0.8035, 0.2864)	(0.9010, 0.0448)	(0.5531, 0.1958)	(0.3709, 0.3100)	(0.4711, 0.4070)

5. An Illustrative Example

An investment company wants to invest a large amount of money to the following five possible areas: A_1—real estate, A_2—energy industry, A_3—gold, A_4—stock market, and A_5—artificial intellectual company. The company's board has decided to appoint an expert panel consisting of three decision-makers $E_i (i = 1, 2, 3)$ to evaluate the investment opinions on the basis of following five interrelated attributes: C_1—market potential, C_2—growth potential, C_3—risk of losing capital sum, C_4—the amount of interests received, and C_5—inflation. Based on the interrelationship, the attributes have been partitioned into the following two sets $P_1 = \{C_1, C_2\}$ and $P_2 = \{C_3, C_4, C_5\}$. The proposed multiple-attribute group decision-making method is applied for the selection of the best investment option as follows.

5.1. Decision-Making Steps

Step 1. Decision matrices $D^{(k)}$ $(k = 1, 2, 3)$ are presented by decision-makers when evaluating alternatives with respect to attributes. The results are shown in Tables 1–3.

Step 2. The collective decision matrix is obtained by aggregating different decision-makers' evaluation values. The decision-makers are partitioned into one sort, and (30) is used to calculate the collective decision matrix, and the results are shown in Table 4. Here, $p = 1$ and $q = 2$.

Table 5: Results of different p and q considering interaction.

	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	$S(\alpha_5)$	Ranking of alternatives	OA
$p=1, q=1$	0.0904	0.0950	0.1256	0.0734	0.1091	$A_3 > A_5 > A_2 > A_1 > A_4$	A_3
$p=1, q=2$	0.0904	0.0955	0.1249	0.0742	0.1099	$A_3 > A_5 > A_2 > A_1 > A_4$	A_3
$p=1, q=3$	0.0264	0.0958	0.1163	0.0754	0.2574	$A_5 > A_3 > A_2 > A_4 > A_1$	A_5
$p=2, q=2$	−0.0268	0.0891	0.0630	0.0743	0.2040	$A_5 > A_2 > A_4 > A_3 > A_1$	A_5
$p=1, q=4$	−0.1183	0.2049	0.5881	0.1483	−0.3674	$A_3 > A_2 > A_4 > A_1 > A_5$	A_3
$p=2, q=3$	−0.2481	0.1444	0.4812	0.0797	−0.4889	$A_3 > A_2 > A_4 > A_1 > A_5$	A_3
$p=1, q=5$	−0.0570	0.2625	0.6405	0.2327	−0.3190	$A_3 > A_2 > A_4 > A_1 > A_5$	A_3
$p=2, q=4$	−0.2037	0.1983	0.5309	0.1647	−0.4507	$A_3 > A_2 > A_4 > A_1 > A_5$	A_3
$p=3, q=3$	−0.2567	0.1791	0.4917	0.1474	−0.5051	$A_3 > A_2 > A_4 > A_1 > A_5$	A_3

Step 3. The attribute weight vector is given as (0.10, 0.20, 0.25, 0.30, 0.15). The alternative's collective evaluation values are calculated by using the PFWIPBM operator and $p=1, q=2$. The results are as follows

$$\begin{aligned} r_1 &= (0.3563, 0.3563),\\ r_2 &= (0.3514, 0.1674),\\ r_3 &= (0.3892, 0.1629),\\ r_4 &= (0.3222, 0.1721),\\ r_5 &= (0.3588, 0.1374). \end{aligned} \tag{67}$$

Step 4. The scores of r_i can be calculated as follows:

$$\begin{aligned} S(r_1) &= 0.0904,\\ S(r_2) &= 0.0955,\\ S(r_3) &= 0.1249,\\ S(r_4) &= 0.0742,\\ S(r_5) &= 0.1099. \end{aligned} \tag{68}$$

Step 5. The alternatives can be ranked according to the ranking of scores $S(r_i)$ $(i=1,2,\cdots,5)$ to get

$$A_3 > A_5 > A_2 > A_1 > A_4. \tag{69}$$

The optimal alternative is A_3.

5.2. Comparison Analysis and Discussions

5.2.1. Influence of the Parameters p and q. In order to illustrate influence of parameters p and q on the ranking results, we consider different p and q in Steps 2 and 3. For simplicity, the same p and q are used in Steps 2 and 3. For example, if $p=1$ and $q=2$ are used in Step 2, then $p=1$ and $q=2$ are also used in Step 3. The results are shown in Table 5; here, OA means the optimal alternative. From the results, we can see that A_5 is the optimal alternative in the cases of $p=1$ and $q=3$ and $p=2$ and $q=2$ and A_3 becomes the optimal alternative in other cases. If $p=1$ and $q=1$ and $p=1$ and $q=2$, the optimal alternative is A_3 and the suboptimal alternative is A_5. With the increase of p and q, the suboptimal becomes A_2 and A_5 is ranked last. A_3 has relatively larger memberships and relatively smaller nonmemberships comparing with other alternatives. Though A_5 has the largest membership and the smallest nonmembership among all the evaluation values, it is still ranked last with increasing p and q due to the intersection between membership and nonmembership that is considered. The rankings of alternatives change with different p and q.

The larger the p and q, the more interaction can be emphasized. But in special cases of $p=0$ or $q=0$, there is no interaction between input arguments. From the viewpoint of the risk attitudes, decision-makers are more risk-seeking with the increase of p and q. By taking different p and q in the PFIPBM operator, the PFIPGBM operator, the PFWIPBM, or the PFWIPGBM operator, different risk attitudes of decision-makers can be reflected and different aspects of decision problem can also be reflected, since the arithmetic aggregation operator stresses the impact of the overall input arguments while the geometric aggregation operator emphasizes the balance of the input arguments [47]. In real decision-making, decision-makers can select the corresponding aggregation operator and p and q according to their preferences and real needs. For simplicity, the decision-makers can select $p=1$ and $q=1$ if the decision-maker is risk averse, which is simple and intuitive.

5.2.2. Comparison with Other Methods. If interactions between the memberships and nonmemberships are not considered, the PFIPBM operator and the PFWIPBM operator reduce to the Pythagorean fuzzy partitioned Bonferroni mean (PFPBM) operator and the Pythagorean fuzzy weighted partitioned Bonferroni mean (PFWPBM) operator as follows:

$$\begin{aligned} &\mathrm{PFPBM}^{p,q}(\alpha_1, \alpha_2, \dots, \alpha_n)\\ &= \frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|P_h|}\oplus_{i\in P_h}\left(\alpha_i^p \otimes\left(\frac{1}{|P_h|-1}\oplus_{j\in P_h, j\neq i}\alpha_j^q\right)\right)\right)^{1/(p+q)}\right)\\ &= \left(\left(1-\prod_{h=1}^{d}\left(1-\left(1-\prod_{i\in P_h}\left(1-\left(\mu_i^p \xi\right)^2\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d}\right)^{1/2}\right., \end{aligned}$$

Table 6: Results of different p and q without considering interaction.

	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	$S(\alpha_5)$	Ranking of alternatives	OA
$p=1, q=1$	−0.5132	−0.5198	−0.5082	−0.5213	−0.4771	$A_5 > A_3 > A_1 > A_2 > A_4$	A_5
$p=1, q=2$	−0.5041	−0.5030	−0.4830	−0.5110	−0.4519	$A_5 > A_3 > A_2 > A_1 > A_4$	A_5
$p=1, q=3$	−0.4892	−0.4790	−0.4437	−0.4946	−0.4128	$A_5 > A_3 > A_2 > A_1 > A_4$	A_5
$p=2, q=2$	−0.5103	−0.5073	−0.4989	−0.5185	−0.4678	$A_5 > A_3 > A_2 > A_1 > A_4$	A_5
$p=1, q=4$	−0.4738	−0.4557	−0.4063	−0.4791	−0.3758	$A_5 > A_3 > A_2 > A_1 > A_4$	A_5
$p=2, q=3$	−0.5046	−0.4962	−0.4833	−0.5121	−0.4522	$A_5 > A_3 > A_2 > A_1 > A_4$	A_5
$p=1, q=5$	−0.4599	−0.4349	−0.3743	−0.4656	−0.3444	$A_5 > A_3 > A_2 > A_1 > A_4$	A_5
$p=2, q=4$	−0.4943	−0.4798	−0.4563	−0.5009	−0.4260	$A_5 > A_3 > A_2 > A_1 > A_4$	A_5
$p=3, q=3$	−0.5073	−0.4965	−0.4914	−0.5155	−0.4585	$A_5 > A_3 > A_2 > A_1 > A_4$	A_5

$$\left(\prod_{h=1}^{d}\left(1-\left(1-\prod_{i\in P_h}\left(1-\left(1-\nu_i^2\right)^p \cdot \left(1-\eta^2\right)\right)^{1/|P_h|}\right)^{1/(p+q)}\right)^{1/d}\right)^{1/2}\Bigg), \tag{70}$$

where

$$\xi=\sqrt{1-\prod_{j\in P_h, j\neq i}\left(1-\left(\mu_j^q\right)^2\right)^{1/(|P_h|-1)}},$$

$$\eta=\sqrt{\prod_{j\in P_h, j\neq i}\left(1-\left(1-\nu_j^2\right)^q\right)^{1/(|P_h|-1)}},$$

$$\begin{aligned}
&\mathrm{PFWPBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\frac{1}{d}\left(\oplus_{h=1}^{d}\left(\frac{1}{|P_h|(|P_h|-1)}\oplus_{i,j\in P_h, j\neq i}\left(\left(w_i\alpha_i^p\right)\otimes\left(w_j\alpha_j^q\right)\right)\right)^{1/(p+q)}\right)\\
&=\left(\left(1-\prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\left(1-\left(1-\left(1-\left(\mu_i^p\right)^2\right)^{w_i}\right)\cdot\left(1-\left(1-\left(\mu_j^q\right)^2\right)^{w_j}\right)\right)^{1/(d|P_h|(|P_h|-1))}\right)^{1/2},\right.\\
&\left.\left(\prod_{h=1}^{d}\prod_{i,j\in P_h, j\neq i}\left(1-\left(1-\left(1-\left(1-\nu_i^2\right)^p\right)^{w_i}\right)\cdot\left(1-\left(1-\left(1-\nu_j^2\right)^q\right)^{w_j}\right)\right)^{1/(d|P_h|(|P_h|-1))}\right)^{1/2}\right).
\end{aligned} \tag{71}$$

If the $\mathrm{PFPBM}^{p,q}$ operator is used in Step 2 and the P FWPBM^{p,q} operator is used in Step 3; the interactions between memberships and nonmemberships are not considered and the results are shown in Table 6. From the results, we can see that the ranking of alternatives is different from that of considered interaction between memberships and nonmemberships. A_5 becomes the optimal alternative and A_3 becomes the suboptimal alternative in all cases. In fact, the rankings of alternatives are the same except for the case of $p=1$ and $q=1$. If interactions are not considered, the effect of memberships will reduce if one membership is nearly approaching zero in multiply operation no matter what about the other memberships and the effect of nonmemberships will be reduced if one nonmembership is nearly approaching zero in sum operation no matter what about the other nonmemberships. These shortcomings can be overcome by considering interactions between memberships and nonmemberships.

If the TOPSIS method has been used, the Pythagorean fuzzy interaction averaging (PFIA) operator is used to aggregate different evaluation values given by different decision-makers into collective ones. Then, calculate the weighted decision matrix $\mathbf{D}'$ as in Table 7. The Pythagorean fuzzy positive ideal solution (PFPIS) can be determined as $r^+=(r_1^+, r_2^+, \ldots, r_5^+)=(\max_j r_{1j}, \max_j r_{2j}, \ldots, \max_j r_{5j})=((0.3148, 0.1513), (0.5329, 0.0396), (0.7021, 0.1073), (0.5935, 0.1907), (0.4240, 0.1022))$. The Pythagorean fuzzy negative ideal solution (PFNIS) can be determined as $r^-=(r_1^-, r_2^-, \ldots, r_5^-)=(\min_j r_{1j}, \min_j r_{2j}, \ldots, \min_j r_{5j})=((0.1684, 0.2434), (0.1908, 0.2058), (0.2700, 0.1954), (0.2082, 0.1828), (0.1918, 0.1847))$. The distances of each alternative evaluation values to the PFPIS and the PFNIS can be calculated by using the following equations, respectively, $d_i^+=\sum_{j=1}^5 d(r_{ij}, r_j^+)$, $d_i^-=\sum_{j=1}^5 d(r_{ij}, r_j^-)(\mathrm{i}=1,2,\ldots,5)$. We can get $d_1^+=0.3196$, $d_2^+=0.2887$, $d_3^+=0.2150$, $d_4^+=0.3023$, $d_5^+=0.2978$, $d_1^-=0.3098$, $d_2^-=0.2454$, $d_3^-=0.3047$, $d_4^-=0.2278$, and $d_5^-=0.2175$. The closeness coefficients can be calculated by the equation $CC_i=d_i^-/(d_i^-+d_i^+)$, and we can get $CC_1=0.4923$, $CC_2=0.4594$, $CC_3=0.5863$, $CC_4=0.4297$ and $CC_5=0.4221$. The alternatives can be ranked according to the ranking of CC_i to get $A_3 > A_2 > A_1 > A_4 > A_5$. The optimal alternative is A_3. The optimal alternative is the same as the most case of the proposed method, but the rankings of alternatives are slightly different.

$$\alpha_{ij}=PFIA\left(\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \ldots, \alpha_{ij}^{(t)}\right)=\frac{1}{t}\left(\oplus_{k=1}^{t}\alpha_{ij}^{(k)}\right)$$

Table 7: Pythagorean fuzzy weighted decision matrix $\mathbf{D}'$.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.2046,0.1402)	(0.2751,0.1129)	(0.4920,0.2945)	(0.4046,0.2437)	(0.3914,0.1297)
A_2	(0.2839,0.1038)	(0.1908,0.2058)	(0.5840,0.1981)	(0.5721,0.1664)	(0.2607,0.1515)
A_3	(0.2141,0.1358)	(0.4459,0.1350)	(0.4507,0.1954)	(0.5935,0.1907)	(0.2740,0.1408)
A_4	(0.1684,0.2434)	(0.1706,0.1366)	(0.7021,0.1073)	(0.3664,0.2098)	(0.4240,0.1022)
A_5	(0.3148,0.1513)	(0.5329,0.0396)	(0.4892,0.1139)	(0.2082,0.1828)	(0.1918,0.1847)

Table 8: Pythagorean fuzzy weighted decision matrix $\mathbf{D}''$.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.2046,0.8997)	(0.2751,0.7110)	(0.4920,0.7678)	(0.4046,0.7042)	(0.3914,0.7673)
A_2	(0.2839,0.8577)	(0.3245,0.8290)	(0.3603,0.7329)	(0.5721,0.5757)	(0.2607,0.8299)
A_3	(0.2141,0.8951)	(0.6970,0.7024)	(0.2700,0.7580)	(0.5935,0.5995)	(0.2740,0.8272)
A_4	(0.1684,0.9470)	(0.2912,0.7591)	(0.4501,0.6163)	(0.3664,0.6887)	(0.4240,0.7479)
A_5	(0.3148,0.8792)	(0.7955,0.0000)	(0.2952,0.6431)	(0.2082,0.6887)	(0.1918,0.8688)

Table 9: Results of different p and q considering interaction with one sort.

	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	$S(\alpha_5)$	Ranking of alternatives	OA
$p=1, q=1$	0.1709	0.1893	0.2253	0.1583	0.1706	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3
$p=1, q=2$	0.0979	0.1073	0.1274	0.0903	0.0953	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3
$p=1, q=3$	0.4854	0.1068	0.5529	0.0920	0.5541	$A_5 > A_3 > A_1 > A_2 > A_4$	A_5
$p=2, q=2$	0.4507	0.5012	0.5184	0.0904	0.5241	$A_5 > A_2 > A_3 > A_1 > A_4$	A_5
$p=1, q=4$	0.4996	0.6237	0.5562	0.5878	0.5462	$A_2 > A_4 > A_3 > A_5 > A_1$	A_2
$p=2, q=3$	0.4183	0.5855	0.4759	0.5463	0.4801	$A_2 > A_4 > A_5 > A_3 > A_1$	A_2
$p=1, q=5$	0.5807	0.6883	0.6175	0.6697	0.6104	$A_2 > A_4 > A_3 > A_5 > A_1$	A_2
$p=2, q=4$	0.5076	0.6526	0.5392	0.6334	0.5463	$A_2 > A_4 > A_5 > A_3 > A_1$	A_2
$p=3, q=3$	0.4883	0.6419	0.5146	0.6245	0.5285	$A_2 > A_4 > A_5 > A_3 > A_1$	A_2

$$= \left(\sqrt{1 - \prod_{k=1}^{t} \left(1 - \left(\mu_{ij}^{(k)}\right)^2\right)^{1/t}}, \right.$$
$$\left. \sqrt{\prod_{k=1}^{t} \left(1 - \left(\mu_{ij}^{(k)}\right)^2\right)^{1/t} - \prod_{k=1}^{t} \left(1 - \left(\left(\mu_{ij}^{(k)}\right)^2 + \left(\nu_{ij}^{(k)}\right)^2\right)\right)^{1/t}} \right). \quad (72)$$

If interaction is not considered in TOPSIS as the method in [48], the Pythagorean fuzzy averaging (PFA) operator is first used to aggregate evaluation values given by different decision-makers into collective ones. The PFA operator is defined as

$$\alpha_{ij} = \text{PFA}\left(\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \dots, \alpha_{ij}^{(t)}\right) = \frac{1}{t} \oplus_{k=1}^{t} \alpha_{ij}^{(k)}$$
$$= \left(\sqrt{1 - \prod_{k=1}^{t} \left(1 - \left(\mu_{ij}^{(k)}\right)^2\right)^{1/t}}, \prod_{k=1}^{t} \left(\nu_{ij}^{(k)}\right)^{1/t} \right). \quad (73)$$

The weighted collective decision matrix $\mathbf{D}''$ is calculated as in Table 8, where the weight vector is also taken as $(0.10, 0.20, 0.25, 0.30, 0.15)$. The PFPIS can be determined as $r^+ = ((0.2893, 0.8577), (0.5329, 0.0000), (0.4501, 0.6163), (0.5721, 0.5757), (0.4240, 0.7479))$. The PFNIS can be determined as $r^- = ((0.1684, 0.9470), (0.1908, 0.8290), (0.2700, 0.7580), (0.2082, 0.6887), (0.1918, 0.8688))$. The distances of alternative's weighted evaluation values to the PFPIS and the PFNIS can be calculated as $d_1^+ = 0.7299$, $d_2^+ = 0.7030$, $d_3^+ = 0.6430$, $d_4^+ = 0.6901$, $d_5^+ = 0.4851$, $d_1^- = 0.4651$, $d_2^- = 0.4159$, $d_3^- = 0.5009$, $d_4^- = 0.4361$, $d_5^- = 0.6523$. The closeness coefficients can be calculated as $CC_1 = 0.3892$, $CC_2 = 0.3717$, $CC_3 = 0.4379$, $CC_4 = 0.3872$, and $CC_5 = 0.5735$. The alternatives can be ranked as $A_5 > A_3 > A_1 > A_4 > A_2$ and the optimal alternative is A_5.

If attributes are all are partitioned into one sort, (37) is used to aggregate alternative evaluation values into collective ones in Step 3 and other steps are the same. Then, the results are shown in Table 9. From the results, we can see that A_3 becomes the optimal alternative in the cases of $p=1$, $q=1$, $p=1$, and $q=2$ and A_5 becomes the optimal alternative in

Table 10: Results of different p and q without considering interaction with one sort.

	$S(\alpha_1)$	$S(\alpha_2)$	$S(\alpha_3)$	$S(\alpha_4)$	$S(\alpha_5)$	Ranking of alternatives	OA
$p=1, q=1$	−0.4835	−0.4958	−0.4628	−0.4836	−0.4528	$A_5 > A_3 > A_1 > A_4 > A_2$	A_5
$p=1, q=2$	−0.4628	−0.4627	−0.4270	−0.4496	−0.4147	$A_5 > A_3 > A_4 > A_2 > A_1$	A_5
$p=1, q=3$	−0.4399	−0.4171	−0.3838	−0.4183	−0.3669	$A_5 > A_3 > A_2 > A_4 > A_1$	A_5
$p=2, q=2$	−0.4567	−0.4625	−0.4195	−0.4358	−0.4100	$A_5 > A_3 > A_4 > A_1 > A_2$	A_5
$p=1, q=4$	−0.4199	−0.3720	−0.3441	−0.3920	−0.3225	$A_5 > A_3 > A_2 > A_4 > A_1$	A_5
$p=2, q=3$	−0.4424	−0.4384	−0.3939	−0.4156	−0.3830	$A_5 > A_3 > A_4 > A_2 > A_1$	A_5
$p=1, q=5$	−0.4029	−0.3317	−0.3094	−0.3726	−0.2843	$A_5 > A_3 > A_2 > A_4 > A_1$	A_5
$p=2, q=4$	−0.4272	−0.4057	−0.3649	−0.3968	−0.3507	$A_5 > A_3 > A_2 > A_4 > A_1$	A_5
$p=3, q=3$	−0.4347	−0.4335	−0.3821	−0.4036	−0.3734	$A_5 > A_3 > A_4 > A_2 > A_1$	A_5

the cases of $p=1$ and $q=3$ and $p=2$ and $q=2$. A_2 becomes the optimal alternative in the other cases. The results are different from those of the partitioned one. If attributes can be divided into several classes and there is interaction relationships among the same class and there is no interaction between classes, the PFWIPBM operator can be used to assure accuracy and reasonableness of decision results.

If attributes have been partitioned into one sort and interaction between membership and nonmembership is not considered, the Pythagorean fuzzy Bonferroni mean ($\mathrm{PFBM}^{p,q}$) operator [43] is used in Step 2 and Pythagorean fuzzy weighted Bonferroni mean ($\mathrm{PFWBM}^{p,q}$) operator is used in Step 3; the results are shown in Table 10. A_5 becomes the optimal alternative and A_3 becomes the suboptimal alternative. Though the optimal and suboptimal alternatives are the same as those of partitioned cases, the ranking of alternatives is different.

$$
\begin{aligned}
&\mathrm{PFBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left(\frac{1}{n(n-1)}\oplus_{i,j\in P_h,j\neq i}\left(\alpha_i^p\otimes\alpha_j^q\right)\right)^{1/(p+q)}\\
&=\left(\left(1-\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\left(1-\left(1-\left(1-\left(\mu_i^p\right)^2\right)^{w_i}\right)\right.\right.\right.\\
&\quad\left.\left.\cdot\left(1-\left(1-\left(\mu_j^q\right)^2\right)^{w_j}\right)\right)^{1/(d|P_h|(|P_h|-1))}\right)^{1/2},\\
&\quad\left(\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\left(1-\left(1-\left(1-\left(1-\nu_i^2\right)^p\right)^{w_i}\right)\right.\right.\\
&\quad\left.\left.\cdot\left(1-\left(1-\left(1-\nu_j^2\right)^q\right)^{w_j}\right)\right)^{1/(d|P_h|(|P_h|-1))}\right)^{1/2}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{PFWBM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left(\frac{1}{n(n-1)}\oplus_{i,j\in P_h,j\neq i}\left(\left(w_i\alpha_i\right)^p\otimes\left(w_j\alpha_j\right)^q\right)\right)^{1/(p+q)}\\
&=\left(\left(1-\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\left(1-\left(1-\left(1-\left(\mu_i^p\right)^2\right)^{w_i}\right)\right.\right.\right.\\
&\quad\left.\left.\cdot\left(1-\left(1-\left(\mu_j^q\right)^2\right)^{w_j}\right)\right)^{1/(d|P_h|(|P_h|-1))}\right)^{1/2},\\
&\quad\left(\prod_{h=1}^{d}\prod_{i,j\in P_h,j\neq i}\left(1-\left(1-\left(1-\left(1-\nu_i^2\right)^p\right)^{w_i}\right)\right.\right.\\
&\quad\left.\left.\cdot\left(1-\left(1-\left(1-\nu_j^2\right)^q\right)^{w_j}\right)\right)^{1/(d|P_h|(|P_h|-1))}\right)^{1/2}\Bigg).
\end{aligned} \tag{74}
$$

If the PFIA operator is used in aggregating different decision-makers' evaluation values into collective ones in Step 2 and the Pythagorean fuzzy weighted interaction averaging (PFIWA) operator is used in aggregating alternatives' evaluation values into collective ones in Step 3, the collective evaluation values of alternatives are as $r_1=(0.7243, 0.3242)$, $r_2=(0.7188, 0.2748)$, $r_3=(0.7586, 0.2535)$, $r_4=(0.6825, 0.2825)$, and $r_5=(0.6367, 0.2416)$. The scores of alternatives can be calculated as $S(r_1)=0.4195$, $S(r_2)=0.4411$, $S(r_3)=0.5112$, $S(r_4)=0.3860$, and $S(r_5)=0.3990$, then $S(r_3)>S(r_2)>S(r_1)>S(r_5)>S(r_4)$. The alternatives can be ranked accordingly as $A_3>A_2>A_1>A_5>A_4$. The optimal alternative is A_3.

$$
\begin{aligned}
\alpha_i &= \mathrm{PFIWA}(\alpha_{i1},\alpha_{i2},\ldots,\alpha_{\mathrm{in}})=\oplus_{j=1}^{n}w_j\alpha_{ij}\\
&=\left(\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{ij}^2\right)^{w_j}},\right.\\
&\quad\left.\sqrt{\prod_{j=1}^{n}\left(1-\mu_{ij}^2\right)^{w_j}-\prod_{j=1}^{n}\left(1-\left(\mu_{ij}^2+\nu_{ij}^2\right)\right)^{w_j}}\right).
\end{aligned} \tag{75}
$$

If the PFA operator is used in Step 2 and the PFWA operator is used in the second phase [38], the results are as $r_1 = (0.7243, 0.2654)$, $r_2 = (0.7188, 0.2489)$, $r_3 = (0.7586, 0.2364)$, $r_4 = (0.6825, 0.2282)$, and $r_5 = (0.6367, 0.0000)$, where

$$\alpha_i = \mathrm{PFWA}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) = \oplus_{j=1}^{n} w_j \alpha_{ij} = \left(\sqrt{1 - \prod_{j=1}^{n} \left(1 - \mu_{ij}^2\right)^{w_j}}, \prod_{j=1}^{n} \nu_{ij}^{w_j} \right). \quad (76)$$

The scores can be calculated as $S(r_1) = 0.4542$, $S(r_2) = 0.4547$, $S(r_3) = 0.5195$, $S(r_4) = 0.4138$, and $S(r_5) = 0.4574$. The alternatives can be ranked accordingly to the ranking of scores to get $A_3 > A_5 > A_2 > A_1 > A_4$. The optimal alternative is A_3.

As discussed above, we summarize differences between our proposed method with the existing methods in Table 11. In a word, we can know from Table 11 that our proposed method is based on the Pythagorean fuzzy numbers and partitioned Bonferroni mean operator with parameters p and q. Moreover, interaction between membership and nonmembership has been considered by using the Pythagorean fuzzy interaction operation laws and interaction between attributes have been considered by using the Bonferroni mean. Hence the new method is more general and flexible than the existing methods. Partitioned the input arguments into several sorts can accurately model the interrelationship between attributes.

Table 11: The characteristic comparisons of different methods.

Methods	Information by Pythagorean fuzzy number	Whether to consider the interrelationships between aggregating arguments
Liang et al. [43]	Yes	Yes
Xu and Yager [38]	No	Yes
Zhang and Xu [26]	Yes	No
Our proposed method	Yes	Yes
Methods	**Whether to consider the partition of the input arguments**	**Whether to consider the interactions between membership and nonmembership**
Liang et al. [43]	No	No
Xu and Yager [38]	No	No
Zhang and Xu [26]	No	No
Our proposed method	Yes	Yes

6. Conclusions

Some Pythagorean fuzzy interaction partitioned Bonferroni mean operators have been developed in this paper including the Pythagorean fuzzy interaction partitioned Bonferroni mean operator, the Pythagorean fuzzy weighted interaction partitioned Bonferroni mean operator, the Pythagorean fuzzy interaction partitioned geometric Bonferroni mean operator, and the Pythagorean fuzzy weighted interaction partitioned geometric Bonferroni mean operator. The Bonferroni mean has been used to model interaction between attributes. The attributes have been partitioned into several classes and the attributes in the same class are interrelated, which have been modeled by using Bonferroni mean, while there is no interrelationship between attributes between different classes. Some properties and some special cases of the new aggregation operators have been studied. We have developed new multiple-attribute group decision-making method based on the new aggregation operators. We applied the new method to solve the problem of selecting an investment company. Some comparisons with other existing methods have been made to show its effectiveness and practical advantages.

The proposed method has some desirable advantages: (1) the evaluation values are given as Pythagorean fuzzy numbers, which are more flexible than other tools to model fuzzy and uncertain information; (2) interaction operations between Pythagorean fuzzy numbers can overcome the drawback of the existing methods; (3) interrelationship of attributes have been modeled by using the Bonferroni mean. By using the partitioned structure of attributes considering relationship among attributes, the proposed method can model interrelationship among attributes more meaningfully and accurately; and (4) since attributes in different sorts are not related, the new aggregation operators can avoid the conjunction effect of unrelated attributes during aggregation. The disadvantage of the new method is that the computation amount has increased comparing with the existing methods. But it is still a polynomial time algorithm and can be calculated easily by using software such as MATLAB and Excel.

The proposed method can be used to handle real-life problems involving fuzziness and uncertainty in the decision-making process. In the future, we will apply it in a wide range of practical problems such as supplier selection problems and site selection problems. Although the proposed operators have been developed in the context of decision-making, they can also be applied in the fields of fuzzy clustering, pattern recognition, and so on. It is also meaningful to investigate other characteristics of the proposed operators, such as combing with Choquet integral and Dempster-Shafer belief structure. We will also extend the partitioned Bonferroni mean operators to other uncertain environments [49–54], such as interval neutrosophic sets, linguistic hesitant intuitionistic fuzzy sets, hesitant Pythagorean fuzzy sets, and q-rung fuzzy sets.

Data Availability

All the data used in our paper has been presented in our paper and there in no unavailable data.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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