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Research Article

A Pythagorean Fuzzy Multigranulation Probabilistic Model for Mine Ventilator Fault Diagnosis

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In coal mining industry, the running state of mine ventilators plays an extremely significant role for the safe and reliable operation of various industrial productions. To guarantee the better reliability, safety, and economy of mine ventilators, in view of early detection and effective fault diagnosis of mechanical faults which could prevent unscheduled downtime and minimize maintenance fees, it is imperative to construct some viable mathematical models for mine ventilator fault diagnosis. In this article, we plan to establish a data-based mine ventilator fault diagnosis method to handle situations where engineers are absent or they are incapable of coming to a conclusion from multisource data. In the process of building the mine ventilator fault diagnosis model, considering that probabilistic rough sets (PRSs) could reduce the errors triggered by incompleteness, inconsistency, and inaccuracy without needing any additional assumptions and Pythagorean fuzzy multigranulation rough sets (PF MGRSs) over the two universes' model could effectively handle data representation, fusion, and analysis issues, we generalize the existing PF MGRSs over the two universes' model to the PRS setting, as well as to further establish a novel model named Pythagorean fuzzy multigranulation probabilistic rough sets (PF MG-PRSs) over two universes. In the granular computing paradigm, three types of PF MG-PRSs over two universes based on the risk attitude of engineers are proposed at first. Afterwards, several basic propositions of the newly proposed model are explored. Moreover, a PF multigranulation probabilistic model for mine ventilator fault diagnosis based on PF MG-PRSs over two universes is investigated. At last, a real-world case study of dealing with a mine ventilator fault diagnosis problem is given to illustrate the practicality of the presented model, and a validity test, a sensitivity analysis, and a comparison analysis are further explored to demonstrate the effectiveness of the presented model.

1. Introduction

Nowadays, the development of modern society and economy mainly depends on an adequate and reliable supply of energy to some extent. Among them, as one of the most abundant reserves of fossil fuel in the world, coal resources still play a crucial part in promoting the industrialization and social economic development of some areas; therefore, safe and reasonable exploration of coal resources is the key to most energy-dependent economies. At present, the safety production issue of coal mine is rather complicated, accidents are still in high volume in some regions, and the frequent situation of heavy and serious accidents has not been resolved radically; thus, coal mine is still an industry with high risk and a high accident rate. Specifically, mine ventilator, a rotating electrical machine which ensures the safety of coal mines and underground coal miners through providing fresh air and eliminating harmful gases, acts as an indispensable component of the coal mine safety production, energy conservation, and automated administration. In

consideration of the fault detection and diagnosis of the continuous process which exert a strong influence on the product quality and production safety, it is essential to study on viable mathematical models of fault diagnosis technologies for mine ventilators [1].

In the past decades, a variety of fault diagnosis techniques have been put forward. Among them, artificial intelligence-based fault diagnosis models are recognized as a powerful scheme to enhance the effectiveness of fault diagnosis for electrical machines owing to the advent of big data era, especially in fault detection and maintenance procedures. Moreover, in various artificial intelligence-based fault diagnosis models, data-based methods have received wide attentions from scholars and practitioners since they do not need any prior knowledge about the parameters and models of electrical machines and only an information system of both normal and abnormal situations for locating faults is needed [2, 3]. Usually, neural networks, expert systems, and fuzzy approaches are common data-based fault diagnosis methods. However, the neural network-based fault diagnosis method demands compatible and plenty of training data to guarantee appropriate training, the expert system-based fault diagnosis method is hard to obtain knowledge and maintain a variable database owing to the heuristic nature of the method, and the fuzzy approach-based fault diagnosis method could avoid the above defects to some extent and cope with the complexity and uncertainty existed in mine ventilator fault diagnosis reasonably [4, 5].

During the process of real-world mine ventilator fault diagnosis, we considered that the increased volume of fault diagnosis data available to mine ventilation engineers contains numerous fuzzy, inaccurate, and inconsistent information due to the complexity and severe conditions of the coal mine environment, which is essential information for solving mine ventilator fault diagnosis problems. In addition, a fault type often implies several fuzzy, inaccurate, and inconsistent information for a certain fault characteristic, which features a relationship between fault types and fault characteristics. Hence, we work with the uncertainties to lead mine ventilation engineers to conduct a proper fault diagnosis for faulty mine ventilators. In most of the mine ventilator fault diagnosis problems, there exist some relationships between fault types and fault characteristics provided by multiple engineers, and then, engineers conduct a fault diagnosis on the basis of the similarity between an unknown sample for a faulty mine ventilator and the relationships between fault types and fault characteristics. As stated previously, fuzzy approach-based fault diagnosis methods could be seen as a reasonable model to handle such relationships and unknown samples of faulty mine ventilators.

Ever since the establishment of fuzzy approaches [6], considering that it is essential to express the uncertainty with respect to various real-world applications within a more efficient scheme that is different from classical fuzzy sets, some different types of fuzzy sets have been established one after the other during the past decades [7–10]. Among them, through applying membership degrees and nonmembership degrees simultaneously, the intuitionistic fuzzy set (IFS) theory [11] enables experts to process bipolar notions by virtue of both the epistemic certainty corresponds to membership degrees and the epistemic uncertainty corresponds to nonmembership degrees. Hence, IFSs provide a reasonable scheme to cope with numerous complete and incomplete fuzzy information in society and nature. However, it is worth noting that the sum of the membership degree and the nonmembership degree for a certain element is less than or equal to 1; experts might confront with a situation in which the sum of the membership degree and the nonmembership degree is greater than 1. In order to handle this issue, Yager [12, 13] proposed and developed the notion of PFSs, which is featured by the membership degree and the nonmembership degree as well and whose sum of squares is less than or equal to 1. Afterwards, Yager and Abbasov [14] illustrated the issue by assigning the membership degree and the nonmembership degree as $\sqrt{3}/2$ and 1/2, respectively; it is noted that $(\sqrt{3}/2) + (1/2) > 1$ but $+(1/2)^2 \leq 1$; thus, they are applicable to the context of PFSs, and the authors further pointed out that the PFS is a more reasonable and effective tool to deal with uncertain situations since PFSs could represent much more valid information than IFSs. Currently, theoretical foundations and practical applications for PFSs have become a significant theme in the area of knowledge discovery and data mining [15–24]. Hence, PFSs could better deal with an increasing number of uncertain information in mine ventilator fault diagnosis and offer mine ventilation engineers with a flexible scheme to express relationships and unknown samples of faulty mine ventilators.

Additionally, unlike existing classical information analysis approaches, due to the fact that information systems in rough sets share the identical mathematical structure with various relationships in realistic applications and the rough set theory [25–28] is aimed at handling inconsistent, incomplete, and uncertain information systems by designing lower approximations corresponding to deterministic rules and upper approximations corresponding to possible rules, it is recognized that rough set methods constitute another significant way to aid fault diagnosis technologies [29–33]. More recently, a variety of generalized rough set models have been designed in accordance with practical demands of real-world situations. Among them, the model of MGRSs over two universes [34] acts as a crucial component in solving information fusion and analysis issues for the following reasons. The first one is MGRSs over two universes take advantages of MGRSs [35, 36], which not only accelerate the process of information fusion by parallel computing multiple binary or fuzzy relations but also offer optimistic and pessimistic models corresponding to risk-seeking and risk-averse strategies; hence, MGRSs could be seen as a reasonable information fusion tool. The second advantage is that the two universes' framework [37–45] makes up for deficiencies in representing information systems by providing some inherent relationships between the alternatives set and the attributes set, such as relationships between fault types and fault characteristics in fault diagnosis procedures. In consideration of the above two merits, studies of MGRSs over two universes-based information fusion and analysis problems are being regarded as promising research issues in recent years [19, 46–49].

In view of the importance of MGRSs over two universes in information fusion and analysis, Zhang et al. [19] constructed the notion of PF MGRSs over two universes and further investigated a general risk-based rule within the context of mergers and acquisitions. Furthermore, based on the probability theory and Bayesian procedures, it is noticed that there are families of rough set models which are equipped with the fault tolerance capability and they are robust when processing a number of noisy data [50–58]. Among them, Wong and Ziarko [50] firstly put forward the model of PRSs through combining probabilistic approximation space with rough sets. In PRSs, the corresponding rough approximations could be constructed by virtue of rough membership functions which could be seen as the probability of an arbitrary object being part of a certain set. Moreover, compared with classical rough sets, taking into account the probability of an object in a set to compute rough memberships, PRSs allow the existence of the fault tolerance by setting thresholds α and β, which is characterized by positive rules, negative rules, and boundary rules, respectively. From then on, through allowing a certain acceptable level of error to handle probabilistic practical applications, PRS-based research topics have received increasing attentions [59–64].

In light of the above statements, though the models of PFSs, MGRSs over two universes and PRSs are able to handle mine ventilator fault diagnosis problems by virtue of their respective features; it is noted that the absence of interaction and integrative researches on the three models precludes an effective mathematical formulation that can be utilized for mine ventilator fault diagnosis. The research gaps and motivations are listed as follows.

(1) It is well known that PFSs are able to aid mine ventilation engineers to describe diverse uncertain information existed in mine ventilator fault diagnosis. Since rough set theory is a reasonable and efficient soft computing tool to cope with uncertain information systems, it is necessary to bridge the gap between PFSs and rough sets to better conduct mine ventilator fault diagnosis. Hence, we aim to develop a comprehensive rough set model within the PFS context

(2) In order to discover the fault type of mine ventilators effectively, the invitation of several mine ventilation engineers to conduct a group decision-making outperforms an individual decision-making. Thus, it is necessary to bridge the gap between PF rough sets and group decision-making models. Considering the merits owned by MGRSs over two universes when making group decisions, we aim to investigate a novel MGRSs over the two universes' model within the PFS context

(3) In view of the importance of fault tolerance capability and robustness owned by the mine ventilator fault diagnosis approach, which further avoids the influence of decision errors, according to the advantages of PRSs mentioned above, it is necessary to bridge the gap between PF MGRSs over two universes and PRSs. Thus, in order to let the newly proposed model manifests robustness in mine ventilator fault diagnosis processes, as well as to expand theoretical scopes of PRSs and MGRSs over two universes, we intend to put forward the model of PF MG-PRSs over two universes by integrating PF MGRSs over two universes with PRSs and three types of the proposed model are explored. Moreover, on the basis of PF MG-PRSs over two universes, a general PF multigranulation probabilistic model within the mine ventilator fault diagnosis context is further discussed. Finally, a case study concerning a mine ventilator fault diagnosis issue is explored to show the validation of the proposed method

Comparing with existing models to mine ventilator fault diagnosis, the primary contributions of the article are listed as follows.

(1) In light of the granular computing paradigm in the field of artificial intelligence-based information processing, we put forward the notion of PF MG-PRSs over two universes to handle mine ventilator fault diagnosis, which could be seen as a beneficial supplement for artificial intelligence-based fault diagnosis models. Additionally, we explore several common propositions of the proposed model

(2) The proposed model takes advantages of PF MGRSs over two universes and PRSs simultaneously. The first merit is that the utilization of PF MGRSs over two universes enables engineers to fuse information based on parallel computing and the risk attitude of engineers and the variable type of PF MG-PRSs over two universes could overcome limitations of optimistic and pessimistic counterparts for coping with variable risk attitudes in fault diagnosis. Moreover, the second merit is that the utilization of PRSs makes the proposed model owns the capacity of processing a number of noisy data, which is featured by positive, negative, and boundary fault diagnosis rules

(3) By virtue of variable PF MG-PRSs over two universes, a PF multigranulation probabilistic model for mine ventilator fault diagnosis with PFSs is constructed and a numerical example, a validity test, a sensitivity analysis, and a comparison analysis illustrate that the superiorities of the proposed fault diagnosis rule could cut down uncertainty and improve accuracy of mine ventilator fault diagnosis to a great extent

The rest of this article is set out below. The next section concisely revisits several preliminary backgrounds about PFSs, PF MGRSs over two universes, and PRSs. In Section 3, three types of PF MG-PRSs over two universes are established, i.e., optimistic, pessimistic, and variable PF MG-PRSs over two universes and then, several

fundamental propositions of them are also concluded. Section 4 constructs a general PF multigranulation probabilistic model based on PF MG-PRSs over two universes. In Section 5, an illustrative real-world example concerning a mine ventilator fault diagnosis problem is provided to show the practicability of the proposed model. At last, a few conclusive comments and future study topics are included in Section 6.

2. Preliminaries

For the convenience of the following statements, we present some fundamental knowledge about PFSs, PF MGRSs over two universes, and PRSs.

2.1. PFSs. In [12, 13], Yager proposed the notion of PFSs, which is a generalization of IFSs [11] and the mathematical form of IFSs and PFSs could be presented below.

Definition 1 (see [11]). Suppose U is an arbitrary nonempty finite universe of discourse, an IFS in U, represented by F, is given by

$$F = \{\langle x, \mu_F(x), \nu_F(x)\rangle | x \in U\} = \sum_{x\in U} \frac{\langle \mu_F(x), \nu_F(x)\rangle}{x}, \quad (1)$$

where $\mu_F : U \longrightarrow [0,1]$ and $\nu_F : U \longrightarrow [0,1]$ represent the membership degree and the nonmembership degree of the element $x \in U$ to F, respectively. For all $x \in U$, it holds that $\mu_F(x) + \nu_F(x) \le 1$. Moreover, the degree of indeterminacy is denoted as $\pi_F(x) = 1 - \mu_F(x) - \nu_F(x)$.

Definition 2 (see [12, 13]). Suppose U is an arbitrary nonempty finite universe of discourse, a PFS in U, represented by P, is given by

$$P = \{\langle x, \mu_P(x), \nu_P(x)\rangle | x \in U\} = \sum_{x\in U} \frac{\langle \mu_P(x), \nu_P(x)\rangle}{x}, \quad (2)$$

where $\mu_P : U \longrightarrow [0,1]$ and $\nu_P : U \longrightarrow [0,1]$ represent the membership degree and the nonmembership degree of the element $x \in U$ to P, respectively. For all $x \in U$, it holds that $(\mu_P(x))^2 + (\nu_P(x))^2 \le 1$. Moreover, the degree of indeterminacy is denoted as $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}$. For convenience, $p = (\mu_P, \nu_P)$ is named a Pythagorean fuzzy number (PFN) by Zhang and Xu [16] that satisfies $\mu_P, \nu_P \in [0,1]$ and $(\mu_P)^2 + (\nu_P)^2 \le 1$. In addition, we call the family of all PFSs in U as $\mathrm{PF}(U)$.

Next, given two PFSs denoted by P and Q, then some operations for P and Q were presented in [16, 17].

Definition 3 (see [16, 17]). Suppose $P = \{x, \mu_P(x), \nu_P(x)|_{x\in U}\}$, $Q = \{x, \mu_Q(x), \nu_Q(x)|_{x\in U}\} \in \mathrm{PF}(U)$. For all $x \in U$, we have

$$\begin{aligned}
P = Q &\Leftrightarrow \mu_P(x) = \mu_Q(x), \nu_P(x) = \nu_Q(x),\\
P \subseteq Q &\Leftrightarrow \mu_P(x) \le \mu_Q(x), \nu_P(x) \ge \nu_Q(x),\\
P^c &= \{\langle x, \nu_P(x), \mu_P(x)\rangle|_{x\in U}\},\\
P \cap Q &= \{\langle x, \mu_P(x) \wedge \mu_Q(x), \nu_P(x) \vee \nu_Q(x)\rangle|_{x\in U}\},\\
P \cup Q &= \{\langle x, \mu_P(x) \vee \mu_Q(x), \nu_P(x) \wedge \nu_Q(x)\rangle|_{x\in U}\},\\
P \oplus Q &= \left\{\left\langle x, \sqrt{(\mu_P(x))^2 + (\mu_Q(x))^2 - (\mu_P(x))^2(\mu_Q(x))^2}, \nu_P(x)\nu_Q(x)\right\rangle|_{x\in U}\right\},\\
P \ominus Q &= \left\{\left\langle x, \sqrt{\frac{(\mu_P(x))^2 - (\mu_Q(x))^2}{1 - (\mu_Q(x))^2}}, \frac{\nu_P(x)}{\nu_Q(x)}\right\rangle|_{x\in U}\right\}, \quad \text{if } \mu_P(x) \ge \mu_Q(x)\nu_P(x) \le \min\left\{\nu_Q(x), \frac{\nu_Q(x)\pi_P(x)}{\pi_Q(x)}\right\},\\
P \otimes Q &= \left\{\left\langle x, \mu_P(x)\mu_Q(x), \sqrt{(\nu_P(x))^2 + (\nu_Q(x))^2 - (\nu_P(x))^2(\nu_Q(x))^2}\right\rangle|_{x\in U}\right\},\\
P \oslash Q &= \left\{\left\langle x, \frac{\mu_P(x)}{\mu_Q(x)}, \sqrt{\frac{(\nu_P(x))^2 - (\nu_Q(x))^2}{1 - (\nu_Q(x))^2}}\right\rangle|_{x\in U}\right\}, \quad \text{if } \nu_P(x) \ge \nu_Q(x), \mu_P(x) \le \min\left\{\mu_Q(x), \frac{\mu_Q(x)\pi_P(x)}{\pi_Q(x)}\right\},\\
\lambda P &= \left\{\left\langle x, \sqrt{1 - (1 - (\mu_P(x))^2)^\lambda}\right\rangle, (\nu_P(x))^\lambda | x \in U\right\}, \quad \lambda > 0,\\
P^\lambda &= \left\{\left\langle x, (\mu_P(x))^\lambda, \sqrt{1 - (1 - (\nu_P(x))^2)^\lambda}\right\rangle|_{x\in U}\right\}, \quad \lambda > 0.
\end{aligned} \quad (3)$$

To investigate the magnitude of different PFNs, the comparison law by virtue of score functions and accuracy functions is presented as follows [16, 17].

Definition 4 (see [16, 17]). Suppose $p_1 = (\mu_{P_1}, \nu_{P_1})$ and $p_2 = (\mu_{P_2}, \nu_{P_2})$ are two PFNs, and let $s(p_1) = (\mu_{P_1})^2 - (\nu_{P_1})^2$ and $s(p_2) = (\mu_{P_2})^2 - (\nu_{P_2})^2$ be their score functions and $h(p_1) = (\mu_{P_1})^2 + (\nu_{P_1})^2$ and $h(p_2) = (\mu_{P_2})^2 + (\nu_{P_2})^2$ be their accuracy functions, then

(i) If $s(p_1) < s(p_2)$, then $p_1 < p_2$

(ii) If $s(p_1) = s(p_2)$, then

(1) $h(p_1) < h(p_2)$, then $p_1 < p_2$

(2) $h(\mathrm{p}_1) = h(p_2)$, then $p_1 = p_2$

2.2. PF MGRSs over Two Universes. In view of the superiority of MGRSs over two universes in information fusion and analysis situations, Zhang et al. [19] proposed PF MGRSs over the two universes' model. Before introducing such a model, it is essential to develop the notion of PF relations over two universes.

Definition 5 (see [19]). Suppose U and V are two arbitrary nonempty finite universes of discourse. A PF relation R is a PFS in $U \times V$, i.e., R is given by

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle | (x, y) \in U \times V\}, \quad (4)$$

where $\mu_R : U \times V \longrightarrow [0, 1]$ and $\nu_R : U \times V \longrightarrow [0, 1]$. For all $(x, y) \in U \times V$, it holds that $(\mu_R(x, y))^2 + (\nu_R(x, y))^2 \leq 1$. In addition, we call the family of all PF relations over two universes in $U \times V$ as $\mathrm{PFR}(U \times V)$.

On the basis of PF relations over two universes discussed above, the notion of PF MGRSs over two universes could be established according to the constructive approach.

Definition 6 (see [19]). Suppose U and V are two arbitrary nonempty finite universes of discourse, $R_i \in \mathrm{PFR}(U \times V)$ and $R_i \in \mathscr{R}(i = 1, 2, \ldots, m)$. Then, $(U, V, \mathscr{R})$ is named a PF multigranulation approximation space over two universes. For any $Q \in \mathrm{PF}(V)$, the optimistic PF lower and upper approximations of Q in terms of $(U, V, \mathscr{R})$, represented by $\underline{\sum_{i=1}^m R_i}^O(Q)$ and $\overline{\sum_{i=1}^m R_i}^O(Q)$, are two PFSs given by

$$\begin{aligned}
&\underline{\sum_{i=1}^m R_i}^O(Q) = \left\{x, \mu_{\underline{\sum_{i=1}^m R_i}^O(Q)}(x), \nu_{\underline{\sum_{i=1}^m R_i}^O(Q)}(x)|_{x\in U}\right\},\\
&\overline{\sum_{i=1}^m R_i}^O(Q) = \left\{x, \mu_{\overline{\sum_{i=1}^m R_i}^O(Q)}(x), \nu_{\overline{\sum_{i=1}^m R_i}^O(Q)}(x)|_{x\in U}\right\},
\end{aligned} \quad (5)$$

where

$$\begin{aligned}
&\mu_{\underline{\sum_{i=1}^m R_i}^O(Q)}(x) = \bigvee_{i=1}^m \wedge_{y\in V}\left[\nu_{R_i}(x, y) \vee \mu_Q(y)\right],\\
&\nu_{\underline{\sum_{i=1}^m R_i}^O(Q)}(x) = \bigwedge_{i=1}^m \vee_{y\in V}\left[\mu_{R_i}(x, y) \wedge \nu_Q(y)\right],\\
&\mu_{\overline{\sum_{i=1}^m R_i}^O(Q)}(x) = \bigwedge_{i=1}^m \vee_{y\in V}\left[\mu_{R_i}(x, y) \wedge \mu_Q(y)\right],\\
&\nu_{\overline{\sum_{i=1}^m R_i}^O(Q)}(x) = \bigvee_{i=1}^m \wedge_{y\in V}\left[\nu_{R_i}(x, y) \vee \nu_Q(y)\right].
\end{aligned} \quad (6)$$

In an identical fashion, the pessimistic PF lower and upper approximations of Q in terms of $(U, V, \mathscr{R})$, represented by $\underline{\sum_{i=1}^m R_i}^P(Q)$ and $\overline{\sum_{i=1}^m R_i}^P(Q)$, are also two PFSs given by

$$\begin{aligned}
&\underline{\sum_{i=1}^m R_i}^P(Q) = \left\{\left\langle x, \mu_{\underline{\sum_{i=1}^m R_i}^P(Q)}(x), \nu_{\underline{\sum_{i=1}^m R_i}^P(Q)}(x)\right\rangle|_{x\in U}\right\},\\
&\overline{\sum_{i=1}^m R_i}^P(Q) = \left\{\left\langle x, \mu_{\overline{\sum_{i=1}^m R_i}^P(Q)}(x), \nu_{\overline{\sum_{i=1}^m R_i}^P(Q)}(x)\right\rangle|_{x\in U}\right\},
\end{aligned} \quad (7)$$

where

$$\begin{aligned}
&\mu_{\underline{\sum_{i=1}^m R_i}^P(Q)}(x) = \bigwedge_{i=1}^m \wedge_{y\in V}\left[\nu_{R_i}(x, y) \vee \mu_Q(y)\right],\\
&\nu_{\underline{\sum_{i=1}^m R_i}^P(Q)}(x) = \bigvee_{i=1}^m \vee_{y\in V}\left[\mu_{R_i}(x, y) \wedge \nu_Q(y)\right],\\
&\mu_{\overline{\sum_{i=1}^m R_i}^P(Q)}(x) = \bigvee_{i=1}^m \vee_{y\in V}\left[\mu_{R_i}(x, y) \wedge \mu_Q(y)\right],\\
&\nu_{\overline{\sum_{i=1}^m R_i}^P(Q)}(x) = \bigwedge_{i=1}^m \wedge_{y\in V}\left[\nu_{R_i}(x, y) \vee \nu_Q(y)\right].
\end{aligned} \quad (8)$$

In light of the above statements, we denote $(\underline{\sum_{i=1}^m R_i}^O(Q), \overline{\sum_{i=1}^m R_i}^O(Q))$ and $(\underline{\sum_{i=1}^m R_i}^P(Q), \overline{\sum_{i=1}^m R_i}^P(Q))$ the optimistic and pessimistic PF MGRSs over two universes, respectively.

2.3. PRSs. In classical rough sets, the approximation space is exactly deterministic since rough sets are based on a

deterministic knowledge base; hence, this potential weakness may preclude classical rough sets from including some objects into the approximation region. In order to conquer this limitation, some fundamental notions of PRSs are revisited in a brief way [50].

Suppose U is an arbitrary nonempty finite universes of discourse, R is an equivalence relation in U, $\mathscr{P}$ is a probability measure on the basis of the σ-algebra that is combined by the subset family of U. Then, we denote the triple $(U, R, \mathscr{P})$ as a probabilistic approximation space.

Definition 7 (see [50, 54, 55]). For any $0 \le \beta < \alpha \le 1$ and $X \in U$, the lower and upper approximations of X with respect to $(U, R, \mathscr{P})$, represented by $\underline{R}_\alpha(X)$ and $\bar{R}_\beta(X)$, are given by

$$\begin{aligned} \underline{R}_\alpha(X) &= \{x \in U | P(X|[x]_R) \ge \alpha\}, \\ \bar{R}_\beta(X) &= \{x \in U | P(X|[x]_R) > \beta\}, \end{aligned} \tag{9}$$

where $P(X \mid [x]_R)$ represents the probability of the object in X given that the object is in $[x]_R$. Moreover, the corresponding positive region, negative region, and boundary region in terms of the probabilistic lower and upper approximations are represented below.

$$\begin{aligned} \mathrm{POS}_{\alpha,\beta}(X) &= \underline{R}_\alpha(X), \\ \mathrm{NEG}_{\alpha,\beta}(X) &= U - \bar{R}_\beta(X), \\ \mathrm{BND}_{\alpha,\beta}(X) &= \bar{R}_\beta(X) - \underline{R}_\alpha(X). \end{aligned} \tag{10}$$

It is noted that with the parameter α decreases, the success rate of an object is correctly classified decreases, while if the value of the parameter β is large, the success rate of an object is correctly classified decreases as well. In particular, PRSs reduce to classical rough sets when $\alpha = 1$ and $\beta = 0$.

3. PF MG-PRSs over the Two Universes' Model

In the following section, we generalize the model of PF MGRSs over two universes to the context of PRSs and further develop three types of PF MG-PRSs over two universes, i.e., the model of optimistic, pessimistic, and variable PF MG-PRSs over two universes; then, some of their fundamental propositions will be discussed briefly.

3.1. Optimistic PF MG-PRSs over the Two Universes' Model. According to the original definition of optimistic MGRSs, the reason for using the word "optimistic" lies in at least one granular structure is needed to meet the demand of the inclusion condition between an equivalence class and the approximated target when computing the optimistic multigranulation lower approximation, while the optimistic upper approximation within the multigranulation context is defined based on the complement of the optimistic multigranulation lower approximation.

Following the idea, in the background of PRSs, the corresponding optimistic multigranulation lower approximation should also utilize at least one granular structure to meet the demand of the probability constraint between an equivalence class and the approximated target, while the corresponding optimistic multigranulation upper approximation should be computed in a similar manner.

Next, the concept of PF inclusion degrees, which denotes membership degrees of some objects in corresponding PFSs, is given as follows.

Definition 8. Suppose U and V are two arbitrary nonempty finite universes of discourse, R is a PF relation in $U \times V$. For any $Q \in \mathrm{PF}(V)$, $x \in U$, and $y \in V$, the membership degree of x in Q in terms of R, denoted by $\varpi_Q^R(x)$, is given as

$$\varpi_Q^R(x) = \frac{\sum_{y \in V} R(x, y) Q(y)}{\sum_{y \in V} R(x, y)}. \tag{11}$$

It is worth noting that $\varpi_Q^R(x)$ is constructed based on a single PF relation over two universes. However, the multigranulation context requires experts to utilize multiple PF relations over two universes when computing the corresponding multigranulation lower and upper approximations. Hence, we give the following definition by considering multiple membership degrees.

Definition 9. Suppose U and V are two arbitrary nonempty finite universes of discourse, R_i is a PF relation in $U \times V$. For any $Q \in \mathrm{PF}(V)$, $x \in U$, and $y \in V$, the maximal and minimal membership degrees of x in Q in terms of each R_i, denoted by $\xi_Q^{\sum_{i=1}^m R_i}(x)$ and $\zeta_Q^{\sum_{i=1}^m R_i}(x)$, are given as

$$\begin{aligned} \xi_Q^{\sum_{i=1}^m R_i}(x) &= \max_{i=1}^m \varpi_Q^{R_i}(x), \\ \zeta_Q^{\sum_{i=1}^m R_i}(x) &= \min_{i=1}^m \varpi_Q^{R_i}(x). \end{aligned} \tag{12}$$

By means of the concept of maximal membership degrees $\xi_Q^{\sum_{i=1}^m R_i}(x)$, optimistic PF MG-PRSs over two universes could be designed below.

Definition 10. Suppose U and V are two arbitrary nonempty finite universes of discourse, R_i is a PF relation in $U \times V$. For any $Q \in \mathrm{PF}(V)$, $x \in U$, and $y \in V$, threshold parameters α and β are two PFNs and $\beta < \alpha$; the optimistic PF multigranulation probabilistic lower and upper approximations of Q in terms of $(U, V, \mathscr{R})$, denoted by $\underline{\sum_{i=1}^m R_i}^{\xi,\alpha}(Q)$ and $\overline{\sum_{i=1}^m R_i}^{\xi,\beta}(Q)$, are given as

$$\begin{aligned} \underline{\sum_{i=1}^m Ri}^{\xi,a} &= \left\{ x \in U \middle| \xi_Q^{\sum_{i=1}^m R_i}(x) \ge \alpha \right\}, \\ \overline{\sum_{i=1}^m R_i}^{\xi,\beta}(Q) &= \left\{ x \in U \middle| \xi_Q^{\sum_{i=1}^m R_i}(x) > \beta \right\}. \end{aligned} \tag{13}$$

The pair $(\underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q), \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q))$ is called an optimistic PF MG-PRS over two universes. Moreover, the corresponding positive region, negative region, and boundary region in terms of the optimistic PF multigranulation probabilistic lower and upper approximations are represented below.

$$\begin{aligned}
\mathrm{POS}^{\xi}_{\alpha,\beta}(Q) &= \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q),\\
\mathrm{NEG}^{\xi}_{\alpha,\beta}(Q) &= U - \underline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q),\\
\mathrm{BND}^{\xi}_{\alpha,\beta}(Q) &= \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q) - \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q).
\end{aligned} \tag{14}$$

Example 1. Let $U = \{x_1, x_2, x_3, x_4\}$ and $V = \{y_1, y_2, y_3, y_4, y_5\}$ be two universes, R_1, R_2, and R_3 be three PF relations in $U \times V$, where

$$R_1 = \left\{\begin{array}{cccccc} & y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1 & (0.4, 0.6) & (0.6, 0.4) & (0.9, 0.2) & (0.8, 0.4) & (0.2, 0.6) \\ x_2 & (0.6, 0.5) & (0.2, 0.7) & (0.5, 0.7) & (0.3, 0.5) & (0.9, 0.1) \\ x_3 & (0.8, 0.4) & (0.7, 0.1) & (0.3, 0.8) & (0.7, 0.1) & (0.3, 0.8) \\ x_4 & (0.1, 0.8) & (0.7, 0.4) & (0.4, 0.7) & (0.9, 0.2) & (0.6, 0.3) \end{array}\right\},$$

$$R_2 = \left\{\begin{array}{cccccc} & y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1 & (0.3, 0.7) & (0.6, 0.3) & (0.8, 0.3) & (0.7, 0.4) & (0.1, 0.6) \\ x_2 & (0.7, 0.5) & (0.3, 0.5) & (0.5, 0.6) & (0.3, 0.6) & (0.7, 0.1) \\ x_3 & (0.8, 0.3) & (0.8, 0.2) & (0.4, 0.8) & (0.6, 0.2) & (0.4, 0.7) \\ x_4 & (0.2, 0.8) & (0.8, 0.3) & (0.4, 0.5) & (0.8, 0.2) & (0.5, 0.3) \end{array}\right\},$$

$$R_3 = \left\{\begin{array}{cccccc} & y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1 & (0.3, 0.6) & (0.7, 0.2) & (0.6, 0.2) & (0.8, 0.4) & (0.2, 0.8) \\ x_2 & (0.6, 0.5) & (0.4, 0.6) & (0.6, 0.7) & (0.2, 0.6) & (0.7, 0.2) \\ x_3 & (0.7, 0.3) & (0.7, 0.4) & (0.3, 0.8) & (0.5, 0.1) & (0.3, 0.7) \\ x_4 & (0.4, 0.7) & (0.7, 0.3) & (0.3, 0.5) & (0.7, 0.3) & (0.6, 0.4) \end{array}\right\}. \tag{15}$$

Then, a PFS Q in V is given as follows.

$$\begin{aligned} Q = \{&\langle y_1, 0.7, 0.2\rangle, \langle y_2, 0.3, 0.5\rangle, \langle y_3, 0.6, 0.4\rangle,\\ &\langle y_4, 0.2, 0.6\rangle, \langle y_5, 0.7, 0.4\rangle\}. \end{aligned} \tag{16}$$

According to Definition 8, we obtain

$$\varpi_Q^{R_1}(x_1) = \frac{\sum_{y\in V} R_1(x_1, y)Q(y)}{\sum_{y\in V} R_1(x_1, y)} = (0.64, 0.08). \tag{17}$$

In an identical fashion, we also obtain $\varpi_Q^{R_1}(x_2) = (0.77,0.08)$, $\varpi_Q^{R_1}(x_3) = (0.67,0.10)$, $\varpi_Q^{R_1}(x_4) = (0.54,0.11)$, $\varpi_Q^{R_2}(x_1) = (0.58,0.09)$, $\varpi_Q^{R_2}(x_2) = (0.75,0.07)$, $\varpi_Q^{R_2}(x_3) = (0.69,0.08)$, $\varpi_Q^{R_2}(x_4) = (0.53,0.08)$, $\varpi_Q^{R_3}(x_1) = (0.52,0.08)$, $\varpi_Q^{R_3}(x_2) = (0.75,0.08)$, $\varpi_Q^{R_3}(x_3) = (0.65,0.09)$, and $\varpi_Q^{R_3}(x_4) = (0.60,0.08)$.

Then, according to Definition 9, we obtain $\xi_Q^{\sum_{i=1}^{3} R_i}(x_1) = (0.64,0.08)$, $\xi_Q^{\sum_{i=1}^{3} R_i}(x_2) = (0.77,0.08)$, $\xi_Q^{\sum_{i=1}^{3} R_i}(x_3) = (0.69,0.08)$, and $\xi_Q^{\sum_{i=1}^{3} R_i}(x_4) = (0.60,0.08)$.

If we take $\alpha = (0.66,0.08)$ and $\beta = (0.61,0.08)$, we obtain $\underline{\sum_{i=1}^{3} R_i}^{\xi,(0.66,0.08)}(Q) = \{x_2, x_3\}$ and $\overline{\sum_{i=1}^{3} R_i}^{\xi,(0.61,0.08)}(Q) = \{x_1, x_2, x_3\}$. Then, $\mathrm{POS}^{\xi}_{(0.66,0.08),(0.61,0.08)}(Q) = \{x_2, x_3\}$, $\mathrm{NEG}^{\xi}_{(0.66,0.08),(0.61,0.08)}(Q) = \{x_4\}$, and $\mathrm{BND}^{\xi}_{(0.66,0.08),(0.61,0.08)}(Q) = \{x_1\}$ could be obtained.

Proposition 1. *Given a PF multigranulation approximation space over two universes $(U, V, \mathscr{R})$. For any $Q, Q' \in \mathrm{PF}(V)$, threshold parameters α and β are two PFNs and $\beta < \alpha$; then, the optimistic PF multigranulation probabilistic lower and upper approximations have the following properties.*

$$\begin{aligned}
&\underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q),\\
&\underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}\left(Q \cap Q'\right) \subseteq \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q) \cap \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}\left(Q'\right),\\
&\overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}\left(Q \cup Q'\right) \supseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q) \cup \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}\left(Q'\right),\\
&\underline{\sum_{i=1}^{m} R_i}^{\xi,a}\left(Q \cup Q'\right) \supseteq \underline{\sum_{i=1}^{m} R_i}^{\xi,a}(Q) \cup \underline{\sum_{i=1}^{m} R_i}^{\xi,a}\left(Q'\right),\\
&\overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}\left(Q \cap Q'\right) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q) \cap \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}\left(Q'\right),\\
&Q \subseteq Q' \Rightarrow \underline{\sum_{i=1}^{m} R_i}^{\xi,a}(Q) \subseteq \underline{\sum_{i=1}^{m} R_i}^{\xi,a}\left(Q'\right),\\
&\overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}\left(Q'\right).
\end{aligned} \tag{18}$$

Proof.

(1) According to Definition 10, since $\beta < \alpha$, we have

$$\underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q) = \left\{x \in U \middle| \xi_Q^{\sum_{i=1}^{m} R_i}(x) \geq \alpha\right\} \subseteq \left\{x \in U \middle| \xi_Q^{\sum_{i=1}^{m} R_i}(x) > \beta\right\}. \tag{19}$$

Hence, $\underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q)$ could be obtained easily.

(2) According to Definition 10, we have

$$\begin{aligned}
\underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}\left(Q \cap Q'\right) &= \left\{x \in U \middle| \xi_{Q\cap Q'}^{\sum_{i=1}^{m} R_i}(x) \geq \alpha\right\} \\
&= \left\{x \in U \middle| \max_{i=1}^{m} \omega_{Q\cap Q'}^{R_i}(x) \geq \alpha\right\} \\
&= \left\{x \in U \middle| \max_{i=1}^{m} \frac{\sum_{y\in V} R_i(x,y)\left(Q \cap Q'\right)(y)}{\sum_{y\in V} R_i(x,y)} \geq \alpha\right\} \\
&\leq \left\{x \in U \middle| \max_{i=1}^{m} \frac{\sum_{y\in V} R_i(x,y) Q(y)}{\sum_{y\in V} R_i(x,y)} \geq \alpha\right\}, \\
\underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}\left(Q \cap Q'\right) &= \left\{x \in U \middle| \max_{i=1}^{m} \frac{\sum_{y\in V} R_i(x,y)\left(Q \cap Q'\right)(y)}{\sum_{y\in V} R_i(x,y)} \geq \alpha\right\} \\
&\leq \left\{x \in U \middle| \max_{i=1}^{m} \frac{\sum_{y\in V} R_i(x,y) Q'(y)}{\sum_{y\in V} R_i(x,y)} \geq \alpha\right\}.
\end{aligned} \tag{20}$$

Hence, we obtain $\underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q \cap Q') \subseteq \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q) \cap \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q')$, and $\overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q \cup Q') \supseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q) \cup \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q')$ can be obtained in an identical fashion.

(3) In view of the above conclusions, $\underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q \cup Q') \supseteq \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q) \cup \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q')$ and $\overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q \cap Q') \subseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q) \cap \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q')$ can be proved similarly.

(4) According to Definition 10, since $Q \subseteq Q'$, we have

$$\begin{aligned}
\underline{\sum_{i=1}^{m} R_i}^{\xi,a}(Q) &= \left\{x \in U \middle| \xi_{Q}^{\sum_{i=1}^{m} R_i}(x) \geq \alpha\right\} \\
&= \left\{x \in U \middle| \max_{i=1}^{m} \omega_{Q}^{R_i}(x) \geq \alpha\right\} \\
&= \left\{x \in U \middle| \max_{i=1}^{m} \frac{\sum_{y\in V} R_i(x,y) Q(y)}{\sum_{y\in V} R_i(x,y)} \geq \alpha\right\} \\
&\leq \left\{x \in U \middle| \max_{i=1}^{m} \frac{\sum_{y\in V} R_i(x,y) Q'(y)}{\sum_{y\in V} R_i(x,y)} \geq \alpha\right\} \\
&= \left\{x \in U \middle| \max_{i=1}^{m} \omega_{Q'}^{R_i}(x) \geq \alpha\right\} \\
&= \left\{x \in U \middle| \xi_{Q'}^{\sum_{i=1}^{m} R_i}(x) \geq \alpha\right\} = \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}\left(Q'\right).
\end{aligned} \tag{21}$$

Hence, we obtain $Q \subseteq Q' \Rightarrow \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q) \subseteq \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q')$. Similarly, $Q \subseteq Q' \Rightarrow \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q')$ could be obtained easily.

3.2. Pessimistic PF MG-PRSs over the Two Universes' Model. Contrary to the idea of optimistic MGRSs, according to the original definition of pessimistic MGRSs, the word "pessimistic" indicates that experts utilize all granular structures to meet the demand of the inclusion condition between an equivalence class and the approximated target when computing the pessimistic multigranulation lower approximation, while the pessimistic multigranulation upper approximation is also defined on the basis of the complement of the pessimistic multigranulation lower approximation.

In order to expand the model of pessimistic PF MGRSs over two universes to the PRS setting, the corresponding pessimistic multigranulation lower approximation should be constructed based on all granular structures to meet the demand of the probability constraint between an equivalence class and the approximated target, while the corresponding pessimistic multigranulation upper approximation should also be computed in an identical fashion.

Similar to optimistic PF MG-PRSs over two universes, according to the concept of minimal membership degrees $\zeta_{Q}^{\sum_{i=1}^{m} R_i}(x)$, we present the notion of pessimistic PF MG-PRSs over two universes below.

Definition 11. Suppose U and V are two arbitrary nonempty finite universes of discourse, R_i is a PF relation in $U \times V$. For any $Q \in \mathrm{PF}(V)$, $x \in U$, and $y \in V$, threshold parameters α and β are two PFNs and $\beta < \alpha$; the pessimistic PF multigranulation probabilistic lower and upper approximations of Q in terms of $(U, V, \mathscr{R})$, denoted by $\underline{\sum_{i=1}^{m} R_i}^{\zeta,\alpha}(Q)$ and $\overline{\sum_{i=1}^{m} R_i}^{\zeta,\beta}(Q)$, are given as

$$\begin{aligned}
\underline{\sum_{i=1}^{m} R_i}^{\zeta,\alpha}(Q) &= \left\{x \in U \middle| \zeta_{Q}^{\sum_{i=1}^{m} R_i}(x) \geq \alpha\right\}, \\
\overline{\sum_{i=1}^{m} R_i}^{\zeta,\beta}(Q) &= \left\{x \in U \middle| \zeta_{Q}^{\sum_{i=1}^{m} R_i}(x) > \beta\right\}.
\end{aligned} \tag{22}$$

The pair $(\underline{\sum_{i=1}^{m} R_i}^{\zeta,\alpha}(Q), \overline{\sum_{i=1}^{m} R_i}^{\zeta,\beta}(Q))$ is called a pessimistic PF MG-PRS over two universes. Moreover, the corresponding positive region, negative region, and boundary region in terms of the pessimistic PF multigranulation probabilistic lower and upper approximations are represented below.

$$\begin{aligned}
\mathrm{POS}_{\alpha,\beta}^{\zeta}(Q) &= \underline{\sum_{i=1}^{m} R_i}^{\zeta,\alpha}(Q), \\
\mathrm{NEG}_{\alpha,\beta}^{\zeta}(Q) &= U - \overline{\sum_{i=1}^{m} R_i}^{\zeta,\beta}(Q), \\
\mathrm{BND}_{\alpha,\beta}^{\zeta}(Q) &= \overline{\sum_{i=1}^{m} R_i}^{\zeta,\beta}(Q) - \underline{\sum_{i=1}^{m} R_i}^{\zeta,\alpha}(Q).
\end{aligned} \tag{23}$$

Example 2. Given a PF multigranulation approximation space over two universes $(U, V, \mathscr{R})$ which is defined in Example 1. According to Definition 9, we obtain $\zeta_Q^{\sum_{i=1}^3 R_i}(x_1) = (0.52, 0.08)$, $\zeta_Q^{\sum_{i=1}^3 R_i}(x_2) = (0.75, 0.08)$, $\zeta_Q^{\sum_{i=1}^3 R_i}(x_3) = (0.65, 0.09)$, and $\zeta_Q^{\sum_{i=1}^3 R_i}(x_4) = (0.54, 0.11)$.

If we also take $\alpha = (0.66, 0.08)$ and $\beta = (0.61, 0.08)$, we obtain $\underline{\sum_{i=1}^3 R_i}^{\zeta,(0.66,0.08)}(Q) = \{x_2\}$ and $\overline{\sum_{i=1}^3 R_i}^{\zeta,(0.61,0.08)}(Q) = \{x_2, x_3\}$. Then, $\mathrm{POS}^{\zeta}_{(0.66,0.08),(0.61,0.08)}(Q) = \{x_2\}$, $\mathrm{NEG}^{\zeta}_{(0.66,0.08),(0.61,0.08)}(Q) = \{x_1, x_4\}$, and $\mathrm{BND}^{\zeta}_{(0.66,0.08),(0.61,0.08)}(Q) = \{x_3\}$ could be obtained.

Proposition 2. *Given a PF multigranulation approximation space over two universes $(U, V, \mathscr{R})$. For any $Q, Q' \in \mathrm{PF}(V)$, threshold parameters α and β are two PFNs and $\beta < \alpha$; then, the pessimistic PF multigranulation probabilistic lower and upper approximations have the following properties.*

$$\begin{aligned}
&\underline{\sum_{i=1}^m R_i}^{\zeta,\alpha}(Q) \subseteq \overline{\sum_{i=1}^m R_i}^{\zeta,\beta}(Q),\\
&\underline{\sum_{i=1}^m R_i}^{\zeta,\alpha}\left(Q \cap Q'\right) \subseteq \underline{\sum_{i=1}^m R_i}^{\zeta,\alpha}(Q) \cap \underline{\sum_{i=1}^m R_i}^{\zeta,\alpha}\left(Q'\right),\\
&\overline{\sum_{i=1}^m R_i}^{\zeta,\beta}\left(Q \cup Q'\right) \supseteq \overline{\sum_{i=1}^m R_i}^{\zeta,\beta}(Q) \cup \overline{\sum_{i=1}^m R_i}^{\zeta,\beta}\left(Q'\right),\\
&\underline{\sum_{i=1}^m R_i}^{\zeta,\alpha}\left(Q \cup Q'\right) \supseteq \underline{\sum_{i=1}^m R_i}^{\zeta,\alpha}(Q) \cup \underline{\sum_{i=1}^m R_i}^{\zeta,\alpha}\left(Q'\right),\\
&\overline{\sum_{i=1}^m R_i}^{\zeta,\beta}\left(Q \cap Q'\right) \subseteq \overline{\sum_{i=1}^m R_i}^{\zeta,\beta}(Q) \cap \overline{\sum_{i=1}^m R_i}^{\zeta,\beta}\left(Q'\right),\\
&Q \subseteq Q' \Rightarrow \underline{\sum_{i=1}^m R_i}^{\zeta,\alpha}(Q) \subseteq \underline{\sum_{i=1}^m R_i}^{\zeta,\alpha}\left(Q'\right),\\
&\overline{\sum_{i=1}^m R_i}^{\zeta,\beta}(Q) \subseteq \overline{\sum_{i=1}^m R_i}^{\zeta,\beta}\left(Q'\right).
\end{aligned} \tag{24}$$

3.3. Variable PF MG-PRSs over the Two Universes' Model. From the standpoint of classical optimistic and pessimistic MGRSs, it is worth noting that optimistic and pessimistic MGRSs are established by using at least one granular structure and all granular structures, respectively, which act as two extreme models in the procedure of information fusion and they have no capacity to adjust risks. Furthermore, it is also noted that optimistic and pessimistic PF MG-PRSs over two universes are constructed by virtue of maximal and minimal membership degrees and they lack the adjustable capability as well. Thus, optimistic and pessimistic PF MG-PRSs over two universes are fixed and could not be changed according to the risk preference of experts. In the following, through introducing a risk coefficient $\lambda (\lambda \in [0, 1])$, the concept of variable membership degrees based on maximal and minimal membership degrees could be put forward and variable PF MG-PRSs over two universes could be further developed.

Next, we give the following definition by combining risk coefficients with maximal and minimal membership degrees.

Definition 12. Suppose U and V are two arbitrary nonempty finite universes of discourse, R_i is a PF relation in $U \times V$. For any $Q \in \mathrm{PF}(V)$, $x \in U$, and $y \in V$, the variable membership degree of x in Q in terms of each R_i, denoted by $\psi_Q^{\sum_{i=1}^m R_i}(x)$, is given as

$$\psi_Q^{\sum_{i=1}^m R_i}(x) = \lambda \xi_Q^{\sum_{i=1}^m R_i}(x) \oplus (1 - \lambda) \zeta_Q^{\sum_{i=1}^m R_i}(x). \tag{25}$$

It is noted that the variable membership degree reduces to the maximal membership degree when $\lambda = 1$, while the variable membership degree reduces to the minimal membership degree when $\lambda = 0$. On the basis of variable membership degrees $\psi_Q^{\sum_{i=1}^m R_i}(x)$, we propose the concept of variable PF MG-PRSs over two universes as follows.

Definition 13. Suppose U and V are two arbitrary nonempty finite universes of discourse, R_i is a PF relation in $U \times V$. For any $Q \in \mathrm{PF}(V)$, $x \in U$, and $y \in V$, threshold parameters α and β are two PFNs and $\beta < \alpha$; the variable PF multigranulation probabilistic lower and upper approximations of Q in terms of $(U, V, \mathscr{R})$, denoted by $\underline{\sum_{i=1}^m R_i}^{\psi,\alpha}(Q)$ and $\overline{\sum_{i=1}^m R_i}^{\psi,\beta}(Q)$, are given as

$$\begin{aligned}
&\underline{\sum_{i=1}^m R_i}^{\psi,\alpha}(Q) = \left\{x \in U \middle| \psi_Q^{\sum_{i=1}^m R_i}(x) \geq \alpha\right\},\\
&\overline{\sum_{i=1}^m R_i}^{\psi,\beta}(Q) = \left\{x \in U \middle| \psi_Q^{\sum_{i=1}^m R_i}(x) > \beta\right\}.
\end{aligned} \tag{26}$$

The pair $(\underline{\sum_{i=1}^m R_i}^{\psi,\alpha}(Q), \overline{\sum_{i=1}^m R_i}^{\psi,\beta}(Q))$ is called a variable PF MG-PRS over two universes. It is easy to see variable PF MG-PRSs over two universes reduce to the optimistic counterpart and pessimistic counterpart when $\lambda = 1$ and $\lambda = 0$, respectively. Thus, optimistic and pessimistic PF MG-PRSs over two universes could be regarded as special cases of variable PF MG-PRSs over two universes. Moreover, the corresponding positive region, negative region, and boundary region in terms of the variable PF multigranulation probabilistic lower and upper approximations are represented below.

$$\begin{aligned}
\mathrm{POS}^{\psi}_{\alpha,\beta}(Q) &= \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}(Q),\\
\mathrm{NEG}^{\psi}_{\alpha,\beta}(Q) &= U - \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}(Q),\\
\mathrm{BND}^{\psi}_{\alpha,\beta}(Q) &= \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}(Q) - \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}(Q).
\end{aligned} \tag{27}$$

Example 3. Given a PF multigranulation approximation space over two universes $(U, V, \mathscr{R})$ which is defined in Example 1. According to Definition 12, if we take the risk coefficient $\lambda = 0.6$, it is not difficult to obtain $\psi_Q^{\sum_{i=1}^{3} R_i}(x_1) = (0.59, 0.08)$, $\psi_Q^{\sum_{i=1}^{3} R_i}(x_2) = (0.76, 0.08)$, $\psi_Q^{\sum_{i=1}^{3} R_i}(x_3) = (0.68, 0.08)$, and $\psi_Q^{\sum_{i=1}^{3} R_i}(x_4) = (0.57, 0.09)$.

If we also take $\alpha = (0.66,0.08)$ and $\beta = (0.61,0.08)$, we obtain $\underline{\sum_{i=1}^{3} R_i}^{\psi,(0.66,0.08)}(Q) = \{x_2, x_3\}$ and $\overline{\sum_{i=1}^{3} R_i}^{\psi,(0.61,0.08)}(Q) = \{x_2, x_3\}$. Then, $\mathrm{POS}^{\psi}_{(0.66,0.08),(0.61,0.08)}(Q) = \{x_2, x_3\}$, $\mathrm{NEG}^{\psi}_{(0.66,0.08),(0.61,0.08)}(Q) = \{x_1, x_4\}$, and $\mathrm{BND}_{(0.66,0.08),(0.61,0.08)}{}^{\psi}(Q) = \varnothing$ could be obtained.

Proposition 3. *Given a PF multigranulation approximation space over two universes $(U, V, \mathscr{R})$. For any $Q, Q' \in \mathrm{PF}(V)$, threshold parameters α and β are two PFNs and $\beta < \alpha$; then, the variable PF multigranulation probabilistic lower and upper approximations have the following properties.*

$$\begin{aligned}
&\underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}(Q),\\
&\underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}\left(Q \cap Q'\right) \subseteq \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}(Q) \cap \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}\left(Q'\right),\\
&\overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}\left(Q \cup Q'\right) \supseteq \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}(Q) \cup \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}\left(Q'\right),\\
&\underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}\left(Q \cup Q'\right) \supseteq \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}(Q) \cup \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}\left(Q'\right),\\
&\overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}\left(Q \cap Q'\right) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}(Q) \cap \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}\left(Q'\right),\\
&Q \subseteq Q' \Rightarrow \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}(Q) \subseteq \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}\left(Q'\right),\\
&\overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}\left(Q'\right).
\end{aligned} \tag{28}$$

Proposition 4. *Given a PF multigranulation approximation space over two universes $(U, V, \mathscr{R})$. For any $Q \in \mathrm{PF}(V)$, threshold parameters α and β are two PFNs and $\beta < \alpha$; then, the following properties hold.*

$$\begin{aligned}
&\underline{\sum_{i=1}^{m} R_i}^{\zeta,\alpha}(Q) \subseteq \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}(Q) \subseteq \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q),\\
&\overline{\sum_{i=1}^{m} R_i}^{\zeta,\beta}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q).
\end{aligned} \tag{29}$$

Proof. According to Definitions 10–13, we have

$$\begin{aligned}
\underline{\sum_{i=1}^{m} R_i}^{\xi,a}(Q) &= \left\{x \in U \middle| \xi_Q^{\sum_{i=1}^{m} R_i}(x) \geq \alpha\right\}\\
&= \left\{x \in U \middle| \max_{i=1}^{m} \omega_Q^{R_i}(x) \geq \alpha\right\}\\
&\geq \left\{x \in U \middle| \left(\lambda \max_{i=1}^{m} \omega_Q^{R_i}(x)\right.\right.\\
&\qquad \left.\left. \oplus (1-\lambda) \max_{i=1}^{m} \omega_Q^{R_i}(x)\right) \geq \alpha\right\}\\
&= \left\{x \in U \middle| \psi_Q^{\sum_{i=1}^{m} R_i}(x) \geq \alpha\right\}\\
&= \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}(Q) \geq \left\{x \in U \middle| \max_{i=1}^{m} \omega_Q^{R_i}(x) \geq \alpha\right\}\\
&= \left\{x \in U \middle| \zeta_Q^{\sum_{i=1}^{m} R_i}(x) \geq \alpha\right\} = \underline{\sum_{i=1}^{m} R_i}^{\zeta,\alpha}(Q).
\end{aligned} \tag{30}$$

Hence, we obtain $\underline{\sum_{i=1}^{m} R_i}^{\zeta,\alpha}(Q) \subseteq \underline{\sum_{i=1}^{m} R_i}^{\psi,\alpha}(Q) \subseteq \underline{\sum_{i=1}^{m} R_i}^{\xi,\alpha}(Q)$. Similarly, $\overline{\sum_{i=1}^{m} R_i}^{\zeta,\beta}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\psi,\beta}(Q) \subseteq \overline{\sum_{i=1}^{m} R_i}^{\xi,\beta}(Q)$ could be obtained easily.

4. A PF Multigranulation Probabilistic Model Based on PF MG-PRSs over Two Universes

In the following section, by utilizing the model of PF MG-PRSs over two universes, a general PF multigranulation probabilistic model is constructed. Moreover, we aim to illustrate the proposed PF multigranulation probabilistic model within the context of mine ventilator fault diagnosis, which has become one of the hottest research topics in the field of intelligent fault diagnosis. By developing such a mine ventilator fault diagnosis approach, the corresponding advantages lie in the following aspects. The first one is handling of the mine ventilator fault diagnosis in the PF environment enables mine ventilation engineers to model the fuzziness in available fault diagnosis data from the viewpoint of epistemic certainty and epistemic uncertainty, and the second merit mainly embodies in the PF multigranulation probabilistic model which provides a reasonable risk-based information fusion and analysis rule for mine ventilator fault diagnosis.

4.1. Problem Statement. As a vital natural resource in China nowadays, coal resource owns an indispensable strategic position in national economy construction and development. In specific, Shanxi Province has always been the largest province of China in coal resources: however, its safe and effective utilization is not satisfactory presently. For the sake of handling the safety issue, it is essential to develop several efficient methods to aid the fault diagnosis of machinery in coal mines. Concretely, in coal mine industry, mine ventilators are utilized to provide fresh air and eliminate toxic, harmful gases, which act as a crucial equipment to ensure the safety of coal mines and underground coal miners. Hence, in order to guarantee mine ventilators work in a safe, reliable, and efficient status, mine ventilation engineers are suggested to eliminate the hidden troubles regularly and conduct some necessary fault diagnosis. Generally, the method of vibration analysis, which includes the knowledge-based method, the analytic model-based method, and the signal processing-based method, is often utilized in mine ventilator fault diagnosis.

In addition, owing to the complexity and severe conditions of the coal mine environment, the relationship between fault types and fault characteristics of mine ventilators gradually shows many uncertainties such as fuzziness, inaccuracy, and inconsistency; hence, the fault diagnosis approach based on fuzzy sets could be seen as a reasonable model to cope with such relationships. Among varieties of extended fuzzy set models, PFSs enable mine ventilation engineers to better model fuzziness and inaccuracy in available fault diagnosis information, which offer them an improved scheme to represent fault diagnosis knowledge for mine ventilators. In order to deal with mine ventilator fault diagnosis with PFSs, a specific mine ventilator fault diagnosis approach on the basis of PF MG-PRSs over two universes is proposed in the next section.

4.2. The Model and Methodology. As discussed previously, it is necessary to construct the relationship between fault types and fault characteristics of mine ventilators. Thus, we let the universe of discourse $U = \{x_1, x_2, \dots, x_k\}$ be the set of fault types, such as unbalance fault, misalignment fault, and mechanical looseness fault, and we also let another universe of discourse $V = \{y_1, y_2, \dots, y_n\}$ be the set of frequency ranges, which ranges according to different fault types during the fault diagnosis process.

During the process of mine ventilator fault diagnosis, each fault type of mine ventilators could be determined according to the different amplitude ratios of vibration signals in various frequency ranges. Hence, the relationship between fault types and fault characteristics of mine ventilators, denoted by $R_i \in \mathrm{PFR}(U \times V)$ and $R_i \in \mathscr{R}(i = 1, 2, \dots, m)$, could be obtained on the basis of the universes U and V.

Suppose Q is the information of frequency ranges detected if there is a failure for a mine ventilator, and Q is a PFS in the universe V, i.e., $Q \in \mathrm{PF}(V)$. Moreover, we also suppose $\mathscr{P}$ is the probability measure of a PFS Q in V; then, we denote the triple $(U, V, \mathscr{R}, \mathscr{P})$ as a PF multigranulation probabilistic approximation space over two universes.

In view of $(U, V, \mathscr{R}, \mathscr{P})$, we compute membership degrees of all fault types in Q in terms of each R_i according to Definition 8 at first; then, it is not difficult to obtain maximal membership degrees $\xi_Q^{\sum_{i=1}^m R_i}(x)$ and minimal membership degrees $\zeta_Q^{\sum_{i=1}^m R_i}(x)$, respectively, according to Definition 9. Considering that both optimistic and pessimistic PF MG-PRSs over two universes lack the capacity of processing variable risk attitudes in mine ventilator fault diagnosis processes and could only be seen as two special models, in order to handle the risk attitude of engineers well, we utilize the model of variable PF MG-PRSs over two universes in the process of mine ventilator fault diagnosis.

Next, based on the model of variable PF MG-PRSs over two universes, suppose $\lambda(\lambda \in [0, 1])$ is a risk coefficient provided in advance by mine ventilation engineers, then variable membership degrees $\psi_Q^{\sum_{i=1}^m R_i}(x)$ could be obtained based on $\xi_Q^{\sum_{i=1}^m R_i}(x)$ and $\zeta_Q^{\sum_{i=1}^m R_i}(x)$ according to Definition 12. Then, suppose α and β are the lowest thresholds also provided in advance by mine ventilation engineers on the basis of real-world scenarios or empirical studies of past mine ventilator fault diagnosis events. Next, we can compute the lower approximation $\underline{\sum_{i=1}^m R_i}^{\psi,\alpha}(Q)$ and the upper approximation $\overline{\sum_{i=1}^m R_i}^{\psi,\beta}(Q)$ of Q with respect to $(U, V, \mathscr{R}, \mathscr{P})$ according to Definition 13. At last, the corresponding positive region $\mathrm{POS}_{\alpha,\beta}^{\psi}(Q)$, negative region $\mathrm{NEG}_{\alpha,\beta}^{\psi}(Q)$, and boundary region $\mathrm{BND}_{\alpha,\beta}^{\psi}(Q)$ with respect to $\underline{\sum_{i=1}^m R_i}^{\psi,\alpha}(Q)$ and $\overline{\sum_{i=1}^m R_i}^{\psi,\beta}(Q)$ could be obtained. In light of the above discussions, we sum up fault diagnosis rules for mine ventilators as follows.

(P) If $\psi_Q^{\sum_{i=1}^m R_i}(x) \geq \alpha$, decide $x \in \mathrm{POS}_{\alpha,\beta}^{\psi}(Q)$, then x is a determined fault type

(N) If $\psi_Q^{\sum_{i=1}^m R_i}(x) \leq \beta$, decide $x \in \mathrm{NEG}_{\alpha,\beta}^{\psi}(Q)$, then x is an excluded fault type

(B) If $\beta < \psi_Q^{\sum_{i=1}^m R_i}(x) < \alpha$, decide $x \in \mathrm{BND}_{\alpha,\beta}^{\psi}(Q)$, then mine ventilation engineers are not sure whether x is a determined or excluded fault type, they need more mechanical tests for faulty mine ventilators to come to a conclusion

Remark 1. In this remark, we aim to clarify how to choose the parametric values and what the stopping criteria are. To be specific, according to $\psi_Q^{\sum_{i=1}^m R_i}(x) = \lambda\xi_Q^{\sum_{i=1}^m R_i}(x) \oplus (1-\lambda)\zeta_Q^{\sum_{i=1}^m R_i}(x)$, it is not difficult to obtain $\psi_Q^{\sum_{i=1}^m R_i}(x) = \xi_Q^{\sum_{i=1}^m R_i}(x)$ when $\lambda = 1$, while $\psi_Q^{\sum_{i=1}^m R_i}(x) = \zeta_Q^{\sum_{i=1}^m R_i}(x)$ when $\lambda = 0$. Based on the above results, an interpretation of the mine ventilator fault diagnosis rule could be formulated by virtue of the form of $\xi_Q^{\sum_{i=1}^m R_i}(x)$ and $\zeta_Q^{\sum_{i=1}^m R_i}(x)$ below.

Input A PF multigranulation probabilistic approximation space over two universes $(U, V, \mathscr{R}, \mathscr{P})$, along with the information of frequency ranges Q for a mine ventilator.
Output The determined fault type.
Step 1. Calculate maximal membership degrees $\xi_Q^{\sum_{i=1}^m R_i}(x)$ and minimal membership degrees $\zeta_Q^{\sum_{i=1}^m R_i}(x)$.
Step 2. Determine the risk coefficient λ, and calculate variable membership degrees $\psi_Q^{\sum_{i=1}^m R_i}(x)$.
Step 3. Determine the thresholds α and β, and calculate the lower approximation $\underline{\sum_{i=1}^m R_i}^{\psi,\alpha}(Q)$ and the upper approximation $\overline{\sum_{i=1}^m R_i}^{\psi,\beta}(Q)$ of Q in terms of $(U, V, \mathscr{R}, \mathscr{P})$.
Step 4. Determine the corresponding positive region $POS_{\alpha,\beta}^{\psi}(Q)$, negative region $NEG_{\alpha,\beta}^{\psi}(Q)$ and boundary region $BND_{\alpha,\beta}^{\psi}(Q)$ with respect to $\underline{\sum_{i=1}^m R_i}^{\psi,\alpha}(Q)$ and $\overline{\sum_{i=1}^m R_i}^{\psi,\beta}(Q)$.
Step 5. Confirm the determined fault type for the faulty mine ventilator according to fault diagnosis rules for mine ventilators (P), (N) and (B).

Algorithm 1

According to the viewpoint of risk decision-making with uncertainty from classical operational research [46], $\psi_Q^{\sum_{i=1}^m R_i}(x) = \xi_Q^{\sum_{i=1}^m R_i}(x)$ could be regarded as the "max-min" rule, while $\psi_Q^{\sum_{i=1}^m R_i}(x) = \zeta_Q^{\sum_{i=1}^m R_i}(x)$ could be regarded as the "min-min" rule. Hence, $\psi_Q^{\sum_{i=1}^m R_i}(x) = \lambda\xi_Q^{\sum_{i=1}^m R_i}(x) \oplus (1-\lambda)\zeta_Q^{\sum_{i=1}^m R_i}(x)$ could be regarded as the compromise rule with the risk coefficient λ.

It is noted that the parameter λ reflects the preference of mine ventilator engineers for the risk of mine ventilator fault diagnosis. Generally speaking, the larger the value of parameter λ is when mine ventilator engineers are risk-seeking. The smaller the value of parameter λ is when mine ventilator engineers are risk-averse. Therefore, the value of parameter λ is provided by mine ventilator engineers' risk preference or the empirical studies and inherent knowledge in advance.

4.3. Algorithm for the Model. In what follows, an algorithm for mine ventilator fault diagnosis by using variable PF MG-PRSs over two universes is concluded.

The flow chart of the proposed approach is given in Figure 1.

5. A Real-World Illustrative Example

In the following section, in order to illustrate the practicability of the proposed PF multigranulation probabilistic model in the previous section, we investigate a real-world case study with PFSs in the context of mine ventilator fault diagnosis.

5.1. Case Description. Suppose $U = \{$unbalance fault (x_1) misalignment fault (x_2) mechanical looseness fault (x_3) blade fault (x_4) outer ring of bearing fault (x_5) inner ring of bearing fault (x_6) rolling element fault (x_7) holder fault $(x_8)\}$ are common fault types for mine ventilators and $V = \{f_1(y_1), 2f_1(y_2), f_2(y_3), (2-5)f_1(y_4), f_o(y_5), f_i(y_6), f_r(y_7), f_h(y_8)\}$ are frequency ranges detected from faulty mine ventilators, where f_1 stands for the vibration frequency caused by unbalance fault, f_2 stands for the vibration frequency caused by blade fault, f_o stands for the characteristic frequency of outer rings, f_i stands for the characteristic frequency of inner rings, f_r stands for the characteristic frequency of rolling elements, and f_h stands for the characteristic frequency of holders. Then, by analyzing whether those frequency components are contained in vibration signals, the position of faults for mine ventilators could be found. Thus, several mine ventilator fault diagnosis knowledge bases with PFSs could be constructed by using relationships between fault types and fault characteristics of mine ventilators. Moreover, we denote such relationships as $R_i \in \mathrm{PFR}(U \times V)(i = 1, 2, 3)$ that are listed in the following tables.

For a faulty mine ventilator, the information of frequency ranges Q is given by mine ventilation engineers with PFSs.

$$\begin{aligned} Q = \{&\langle y_1, 0.89, 0.07\rangle, \langle y_2, 0.15, 0.75\rangle, \\ &\langle y_3, 0.33, 0.55\rangle, \langle y_4, 0.10, 0.88\rangle, \langle y_5, 0, 1\rangle, \\ &\langle y_6, 0, 1\rangle, \langle y_7, 0, 1\rangle, \langle y_8, 0, 1\rangle\}. \end{aligned} \tag{31}$$

5.2. Mine Ventilator Fault Diagnosis Process. In what follows, we utilize the algorithm for mine ventilator fault diagnosis presented in Section 4.3 to solve this mine ventilator fault diagnosis problem as stated in Section 5.1.

Step 1. Firstly, we compute membership degrees of all fault types in Q in terms of each R_i according to Definition 8.

For the relationship shown in Table 1, we have

$$\varpi_Q^{R_1}(x_1) = \frac{\sum_{y\in V} R_1(x_1, y)Q(y)}{\sum_{y\in V} R_1(x_1, y)} = (0.89, 0.05). \tag{32}$$

Similarly, we also have $\varpi_Q^{R_1}(x_2) = (0.26, 0.39)$, $\varpi_Q^{R_1}(x_3) = (0.22, 0.46)$, $\varpi_Q^{R_1}(x_4) = (0.90, 0.03)$, $\varpi_Q^{R_1}(x_5) = (0.21, 0.51)$, $\varpi_Q^{R_1}(x_6) = (0.21, 0.51)$, $\varpi_Q^{R_1}(x_7) = (0.21, 0.51)$, and $\varpi_Q^{R_1}(x_8) = (0.21, 0.51)$.

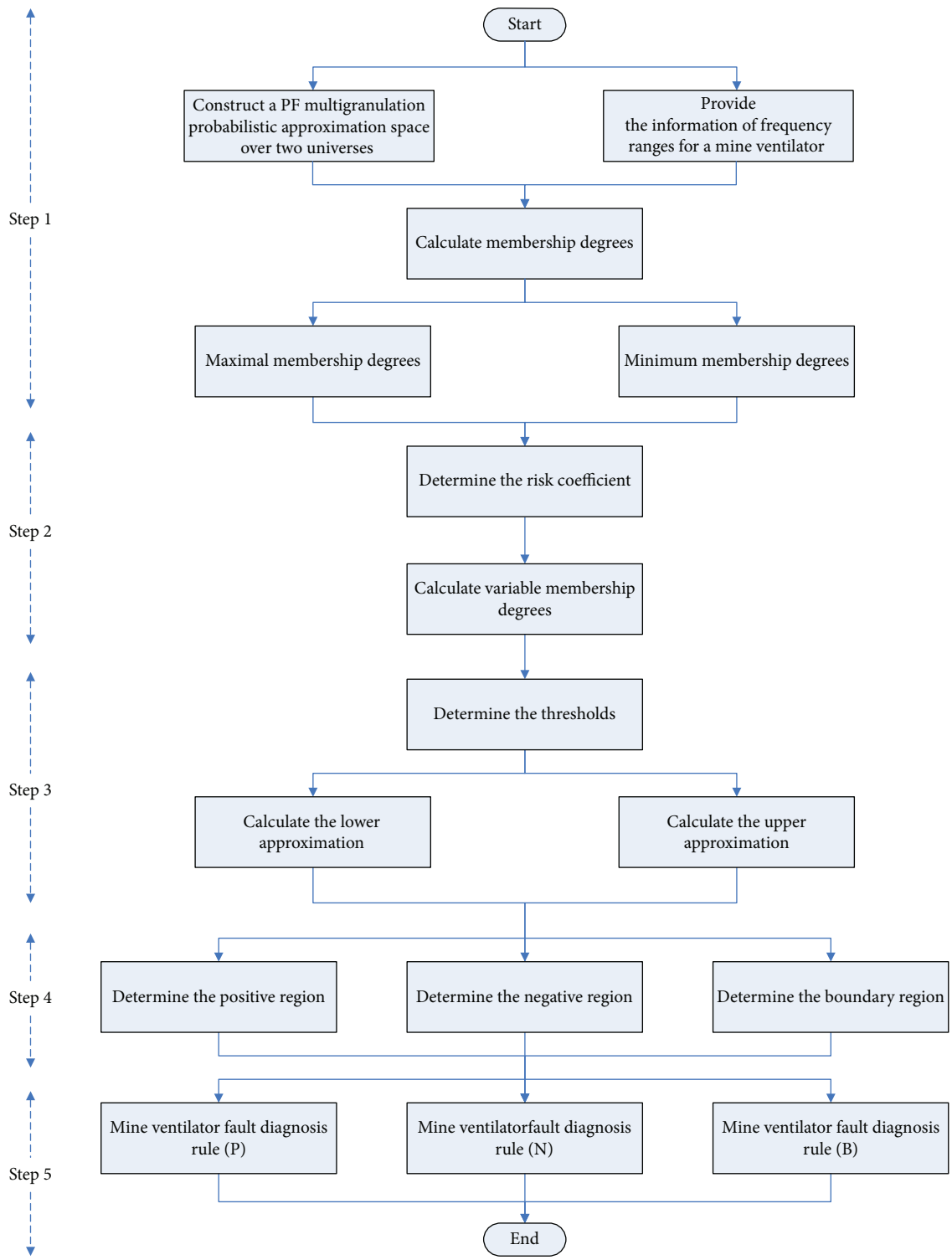

Figure 1: Flow chart of the proposed approach.

For the relationship shown in Table 2, we have $\varpi_Q^{R_2}(x_1) = (0.89,0.06)$, $\varpi_Q^{R_2}(x_2) = (0.24,0.46)$, $\varpi_Q^{R_2}(x_3) = (0.20,0.57)$, $\varpi_Q^{R_2}(x_4) = (0.90,0.04)$, $\varpi_Q^{R_2}(x_5) = (0.18,0.65)$, $\varpi_Q^{R_2}(x_6) = (0.18,0.65)$, $\varpi_Q^{R_2}(x_7) = (0.18,0.65)$, and $\varpi_Q^{R_2}(x_8) = (0.18,0.65)$.

For the relationship shown in Table 3, we have $\varpi_Q^{R_3}(x_1) = (0.89,0.04)$, $\varpi_Q^{R_3}(x_2) = (0.32,0.29)$, $\varpi_Q^{R_3}(x_3) = (0.30,0.36)$, $\varpi_Q^{R_3}(x_4) = (0.90,0.03)$, $\varpi_Q^{R_3}(x_5) = (0.28,0.39)$, $\varpi_Q^{R_3}(x_6) = (0.28,0.39)$, $\varpi_Q^{R_3}(x_7) = (0.28,0.39)$, and $\varpi_Q^{R_3}(x_8) = (0.28,0.39)$.

Table 1: The relationship between fault types and fault characteristics of mine ventilators given by engineer 1.

R_1	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
x_1	(1, 0)	(0.18,0.78)	(0.39,0.59)	(0.09,0.89)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_2	(0.18,0.78)	(1, 0)	(0.39,0.59)	(0.68,0.28)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_3	(0.18,0.78)	(0.18,0.78)	(0.39,0.59)	(0.88,0.08)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_4	(1, 0)	(0.18,0.78)	(0.88,0.08)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_5	(0.18,0.78)	(0.18,0.78)	(0.39,0.59)	(0.09,0.89)	(1, 0)	(0, 1)	(0, 1)	(0, 1)
x_6	(0.18,0.78)	(0.18,0.78)	(0.39,0.59)	(0.09,0.89)	(0, 1)	(1, 0)	(0, 1)	(0, 1)
x_7	(0.18,0.78)	(0.18,0.78)	(0.39,0.59)	(0.09,0.89)	(0, 1)	(0, 1)	(1, 0)	(0, 1)
x_8	(0.18,0.78)	(0.18,0.78)	(0.39,0.59)	(0.09,0.89)	(0, 1)	(0, 1)	(0, 1)	(1, 0)

Table 2: The relationship between fault types and fault characteristics of mine ventilators given by engineer 2.

R_2	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
x_1	(1, 0)	(0.15, 0.88)	(0.36, 0.76)	(0.07, 0.94)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_2	(0.15, 0.88)	(1, 0)	(0.36, 0.66)	(0.63, 0.32)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_3	(0.15, 0.88)	(0.15, 0.88)	(0.36, 0.66)	(0.82, 0.12)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_4	(1, 0)	(0.15, 0.88)	(0.82, 0.28)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_5	(0.15, 0.88)	(0.15, 0.88)	(0.36, 0.66)	(0.07, 0.94)	(1, 0)	(0, 1)	(0, 1)	(0, 1)
x_6	(0.15, 0.88)	(0.15, 0.88)	(0.36, 0.66)	(0.07, 0.94)	(0, 1)	(1, 0)	(0, 1)	(0, 1)
x_7	(0.15, 0.88)	(0.15, 0.88)	(0.36, 0.66)	(0.07, 0.94)	(0, 1)	(0, 1)	(1, 0)	(0, 1)
x_8	(0.15, 0.88)	(0.15, 0.88)	(0.36, 0.66)	(0.07, 0.94)	(0, 1)	(0, 1)	(0, 1)	(1, 0)

Table 3: The relationship between fault types and fault characteristics of mine ventilators given by engineer 3.

R_3	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
x_1	(1, 0)	(0.26,0.68)	(0.48,0.52)	(0.16,0.78)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_2	(0.26,0.68)	(1, 0)	(0.48,0.52)	(0.78,0.19)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_3	(0.26,0.68)	(0.26,0.68)	(0.48,0.52)	(0.96,0.06)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_4	(1, 0)	(0.26,0.68)	(0.96,0.06)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_5	(0.26,0.68)	(0.26,0.68)	(0.48,0.52)	(0.16,0.78)	(1, 0)	(0, 1)	(0, 1)	(0, 1)
x_6	(0.26,0.68)	(0.26,0.68)	(0.48,0.52)	(0.16,0.78)	(0, 1)	(1, 0)	(0, 1)	(0, 1)
x_7	(0.26,0.68)	(0.26,0.68)	(0.48,0.52)	(0.16,0.78)	(0, 1)	(0, 1)	(1, 0)	(0, 1)
x_8	(0.26,0.68)	(0.26,0.68)	(0.48,0.52)	(0.16,0.78)	(0, 1)	(0, 1)	(0, 1)	(1, 0)

Then, on the basis of above results, maximal membership degrees $\xi_Q^{\sum_{i=1}^m R_i}(x)$ and minimal membership degrees $\zeta_Q^{\sum_{i=1}^m R_i}(x)$ could be obtained according to Definition 9.

For maximal membership degrees, $\xi_Q^{\sum_{i=1}^3 R_i}(x_1) = \max_{i=1}^3 \varpi_Q^{R_i}(x_1) = (0.89,0.04)$. Similarly, we also have $\xi_Q^{\sum_{i=1}^3 R_i}(x_2) = (0.32,0.29)$, $\xi_Q^{\sum_{i=1}^3 R_i}(x_3) = (0.30,0.36)$, $\xi_Q^{\sum_{i=1}^3 R_i}(x_4) = (0.90,0.03)$, $\xi_Q^{\sum_{i=1}^3 R_i}(x_5) = (0.28,0.39)$, $\xi_Q^{\sum_{i=1}^3 R_i}(x_6) = (0.28,0.39)$, $\xi_Q^{\sum_{i=1}^3 R_i}(x_7) = (0.28,0.39)$, and $\xi_Q^{\sum_{i=1}^3 R_i}(x_8) = (0.28,0.39)$.

For minimal membership degrees, $\zeta_Q^{\sum_{i=1}^3 R_i}(x_1) = \min_{i=1}^3 \varpi_Q^{R_i}(x_1) = (0.89,0.06)$. Similarly, we also have $\zeta_Q^{\sum_{i=1}^3 R_i}(x_2) = (0.24,0.46)$, $\zeta_Q^{\sum_{i=1}^3 R_i}(x_3) = (0.20,0.57)$, $\zeta_Q^{\sum_{i=1}^3 R_i}(x_4) = (0.90,0.04)$, $\zeta_Q^{\sum_{i=1}^3 R_i}(x_5) = (0.18,0.65)$, $\zeta_Q^{\sum_{i=1}^3 R_i}(x_6) = (0.18,0.65)$, $\zeta_Q^{\sum_{i=1}^3 R_i}(x_7) = (0.18,0.65)$, and $\zeta_Q^{\sum_{i=1}^3 R_i}(x_8) = (0.18,0.65)$.

Step 2. Assume we take the risk coefficient $\lambda = 0.6$, it is not difficult to obtain

$$\begin{aligned}\psi_Q^{\sum_{i=1}^3 R_i}(x_1) &= 0.6\xi_Q^{\sum_{i=1}^3 R_i}(x_1) \oplus (1-0.6)\zeta_Q^{\sum_{i=1}^3 R_i}(x_1) \\ &= (0.89,0.04).\end{aligned} \tag{33}$$

Similarly, we also have $\psi_Q^{\sum_{i=1}^3 R_i}(x_2) = (0.29,0.35)$, $\psi_Q^{\sum_{i=1}^3 R_i}(x_3) = (0.26,0.43)$, $\psi_Q^{\sum_{i=1}^3 R_i}(x_4) = (0.90,0.03)$, $\psi_Q^{\sum_{i=1}^3 R_i}$

$(x_5) = (0.24,0.48)$, $\psi_Q^{\sum_{i=1}^3 R_i}(x_6) = (0.24,0.48)$, $\psi_Q^{\sum_{i=1}^3 R_i}(x_7) = (0.24,0.48)$, and $\psi_Q^{\sum_{i=1}^3 R_i}(x_8) = (0.24,0.48)$.

Step 3. Assume we take the thresholds $\alpha = (0.65,0.15)$ and $\beta = (0.25,0.45)$, it is not difficult to obtain

$$\underline{\sum_{i=1}^3 R_i}^{\psi,(0.65,0.15)}(Q) = \left\{ x \in U \middle| \psi_Q^{\sum_{i=1}^3 R_i}(x) \geq (0.65,0.15) \right\} = \{x_1, x_4\},$$

$$\overline{\sum_{i=1}^3 R_i}^{\psi,(0.25,0.45)}(Q) = \left\{ x \in U \middle| \psi_Q^{\sum_{i=1}^3 R_i}(x) > (0.25,0.45) \right\} = \{x_1, x_2, x_3, x_4\}. \tag{34}$$

Step 4. We determine the positive region $\mathrm{POS}^{\psi}_{(0.65,0.15),(0.25,0.45)}(Q)$, negative region $\mathrm{NEG}^{\psi}_{(0.65,0.15),(0.25,0.45)}(Q)$, and boundary region $\mathrm{BND}^{\psi}_{(0.65,0.15),(0.25,0.45)}(Q)$ with respect to $\underline{\sum_{i=1}^3 R_i}^{\psi,(0.65,0.15)}(Q)$ and $\overline{\sum_{i=1}^3 R_i}^{\psi,(0.25,0.45)}(Q)$ as follows.

$$\begin{aligned}
\mathrm{POS}^{\psi}_{(0.65,0.15),(0.25,0.45)}(Q) &= \underline{\sum_{i=1}^3 R_i}^{\psi,(0.65,0.15)}(Q) = \{x_1, x_4\},\\
\mathrm{NEG}^{\psi}_{(0.65,0.15),(0.25,0.45)}(Q) &= U - \overline{\sum_{i=1}^3 R_i}^{\psi,(0.25,0.45)}(Q) = \{x_5, x_6, x_7, x_8\},\\
\mathrm{BND}^{\psi}_{(0.65,0.15),(0.25,0.45)}(Q) &= \overline{\sum_{i=1}^3 R_i}^{\psi,(0.25,0.45)}(Q) - \underline{\sum_{i=1}^3 R_i}^{\psi,(0.65,0.15)}(Q) = \{x_2, x_3\}.
\end{aligned} \tag{35}$$

Step 5. The conclusions for this mine ventilator fault diagnosis problem could be obtained based on the fault diagnosis rules (P), (N), and (B), where

(P) Unbalance fault and blade fault are determined fault types, which are most likely to occur and mine ventilation engineers shall lay emphasis on the inspection of those two fault types

(N) Outer ring of bearing fault, inner ring of bearing fault, rolling element fault and holder fault are excluded fault types, which are most unlikely to occur and mine ventilation engineers shall rule out the possibility of those four fault types

(B) Mine ventilation engineers are not sure whether misalignment fault and mechanical looseness fault are a determined or excluded fault type, they need more mechanical tests for faulty mine ventilators to come to a conclusion

5.3. Validity Test. In literature [7], the authors developed a novel way to assess whether a group decision-making approach is effective or not. We list the three test criteria below.

In the first test criterion, an effective decision-making approach should not change the selection of the best alternative by replacing a nonoptimal alternative by another worse alternative. According to this test criterion, we change a nonoptimal fault type x_5 which is provided by three mine ventilator engineers. In Tables 1, 2, and 3, we replace x_5 as $(0.22,0.72)$, $(0.22,0.72)$, $(0.42,0.58)$, $(0.12,0.82)$, $(0,1)$, $(0,1)$, $(0,1)$, and $(1,0)$. We next use the proposed approach to deal with the new PF relationships between fault types and fault characteristics of mine ventilators, and the result also shows that unbalance fault and blade fault are determined fault types. Mine ventilation engineers are not sure whether misalignment fault and mechanical looseness fault are a determined or excluded fault type; other fault types are excluded fault types. Thus, the proposed approach passed the first test criterion.

In the second and third test criteria, first, an effective group decision-making approach must be satisfy transitive property. Then, it is necessary to decompose a group decision-making problem into smaller problems, and the same group decision-making approach is applied on smaller problems to rank alternatives; then, the combination of ranking results for alternatives should be identical to the original ranking of the undecomposed problem. Followed by this test criterion, we decompose the original group decision-making problem into two smaller group decision-making problems $\{x_1, x_2, x_3, x_5, x_6, x_7, x_8\}$ and $\{x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$. Through utilizing the proposed approach, we merge the above two results together; the final result is identical with the original mine ventilator fault diagnosis result. Thus, the proposed approach passed the other two test criteria.

5.4. Sensitivity Analysis. In order to explain the above results and the influence of the parameters in a better way, a sensitivity analysis will be conducted by changing the parameter λ. Through setting the same thresholds α and β in the above case study, we utilize the proposed PF multigranulation probabilistic model to obtain the fault diagnosis conclusion for the faulty mine ventilator by increasing or decreasing the value of λ. In specific, we let λ be 0, 0.25, 0.5, 0.75, and 1, respectively; then, the most likely fault diagnosis result could be obtained as described in Table 4.

It is noted that the fault diagnosis result of fault types is not sensitive to different values of λ; the fault diagnosis result remains the same when λ changes. That is to say, the mine ventilator fault diagnosis conclusion is unbalance fault and blade fault in spite of the change of λ. Hence, the mine ventilator fault diagnosis result is reliable and objective.

Table 4: Fault diagnosis results with different values of λ.

Different values of λ	Risk preferences	The fault diagnosis result
$\lambda = 0$	Completely pessimistic (completely risk-averse)	Unbalance fault and blade fault
$\lambda = 0.25$	Somewhat pessimistic (somewhat risk-averse)	Unbalance fault and blade fault
$\lambda = 0.5$	Neutral (risk-neutral)	Unbalance fault and blade fault
$\lambda = 0.75$	Somewhat optimistic (somewhat risk-seeking)	Unbalance fault and blade fault
$\lambda = 1$	Completely optimistic (completely risk-seeking)	Unbalance fault and blade fault

5.5. Comparison Analysis

5.5.1. A Comparison Analysis with the Approach Proposed in Literature [19]. Considering that the concept of variable PF MG-PRSs over two universes is an extended form of PF MGRSs over two universes, in order to show the superiorities of the proposed mine ventilator fault diagnosis rule, a comparison analysis is conducted by means of fault diagnosis rules based on PF MGRSs over two universes. In what follows, we utilize the approach presented in [19] to solve the mine ventilator fault diagnosis problem as stated in Section 5.1.

At first, we compute the optimistic PF lower and upper approximations of Q in terms of $(U, V, \mathscr{R})$.

$$\begin{aligned}
\underline{\sum_{i=1}^{3} R_i}^{O}(Q) &= \{\langle x_1, 0.76,0.36\rangle, \langle x_2, 0.15,0.75\rangle, \\
&\quad \langle x_3, 0.10,0.88\rangle, \langle x_4, 0.76,0.36\rangle, \langle x_5, 0, 1\rangle, \\
&\quad \langle x_6, 0, 1\rangle, \langle x_7, 0, 1\rangle, \langle x_8, 0, 1\rangle\}, \\
\overline{\sum_{i=1}^{3} R_i}^{O}(Q) &= \{\langle x_1, 0.89,0.07\rangle, \langle x_2, 0.33,0.66\rangle, \\
&\quad \langle x_3, 0.33,0.66\rangle, \langle x_4, 0.89,0.07\rangle, \\
&\quad \langle x_5, 0.33,0.66\rangle, \langle x_6, 0.33,0.66\rangle, \\
&\quad \langle x_7, 0.33,0.66\rangle, \langle x_8, 0.33,0.66\rangle\}.
\end{aligned} \tag{36}$$

In an identical fashion, the pessimistic PF lower and upper approximations of Q in terms of $(U, V, \mathscr{R})$ could be obtained as well.

$$\begin{aligned}
\underline{\sum_{i=1}^{3} R_i}^{P}(Q) &= \{\langle x_1, 0.52,0.48\rangle, \langle x_2, 0.15,0.75\rangle, \\
&\quad \langle x_3, 0.10,0.88\rangle, \langle x_4, 0.52,0.48\rangle, \\
&\quad \langle x_5, 0, 1\rangle, \langle x_6, 0, 1\rangle, \\
&\quad \langle x_7, 0, 1\rangle, \langle x_8, 0, 1\rangle\}, \\
\overline{\sum_{i=1}^{3} R_i}^{P}(Q) &= \{\langle x_1, 0.89,0.07\rangle, \langle x_2, 0.33,0.55\rangle, \\
&\quad \langle x_3, 0.33,0.55\rangle, \langle x_4, 0.89,0.07\rangle, \\
&\quad \langle x_5, 0.33,0.55\rangle, \langle x_6, 0.33,0.55\rangle, \\
&\quad \langle x_7, 0.33,0.55\rangle, \langle x_8, 0.33,0.55\rangle\}.
\end{aligned} \tag{37}$$

Then, we further compute $\underline{\sum_{i=1}^{3} R_i}^{O}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q)$ and $\underline{\sum_{i=1}^{3} R_i}^{P}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q)$.

$$\begin{aligned}
\underline{\sum_{i=1}^{3} R_i}^{O}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q) &= \{\langle x_1, 0.96,0.03\rangle, \langle x_2, 0.36,0.50\rangle, \\
&\quad \langle x_3, 0.34,0.58\rangle, \langle x_4, 0.96,0.03\rangle, \\
&\quad \langle x_5, 0.33,0.66\rangle, \langle x_6, 0.33,0.66\rangle, \\
&\quad \langle x_7, 0.33,0.66\rangle, \langle x_8, 0.33,0.66\rangle\}, \\
\underline{\sum_{i=1}^{3} R_i}^{P}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q) &= \{\langle x_1, 0.92,0.03\rangle, \langle x_2, 0.36,0.41\rangle, \\
&\quad \langle x_3, 0.34,0.48\rangle, \langle x_4, 0.92,0.03\rangle, \\
&\quad \langle x_5, 0.33,0.55\rangle, \langle x_6, 0.33,0.55\rangle, \\
&\quad \langle x_7, 0.33,0.55\rangle, \langle x_8, 0.33,0.55\rangle\}.
\end{aligned} \tag{38}$$

At last, it is not difficult to obtain $s(x_1) = s(x_4) > s(x_2) > s(x_3) > s(x_5) = s(x_6) = s(x_7) = s(x_8)$ for both $\underline{\sum_{i=1}^{3} R_i}^{O}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q)$ and $\underline{\sum_{i=1}^{3} R_i}^{P}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q)$, which is consistent with the mine ventilator fault diagnosis result determined by the proposed fault diagnosis rule based on variable PF MG-PRSs over two universes.

5.5.2. A Comparison Analysis with the Approach Proposed in Literature [47]. In literature [47], according to the granular computing paradigm, a steam turbine fault diagnosis approach is explored based on interval-valued hesitant fuzzy (IVHF) MGRSs over two universes. By virtue of the approach in literature [47], the following steps could be conducted to obtain the most likely fault diagnosis result.

At first, according to the above obtained result of $\underline{\sum_{i=1}^{3} R_i}^{O}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q)$ and $\underline{\sum_{i=1}^{3} R_i}^{P}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q)$, we compute the following set $(\underline{\sum_{i=1}^{3} R_i}^{O}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q)) \oplus (\underline{\sum_{i=1}^{3} R_i}^{P}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q))$ by virtue of Definition 3,

where

$$\left(\underline{\sum_{i=1}^{3} R_i}^{O}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q)\right) \oplus \left(\underline{\sum_{i=1}^{3} R_i}^{P}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q)\right)$$
$$= \{\langle x_1, 0.99, 0.0009\rangle, \langle x_2, 0.49, 0.205\rangle, \langle x_3, 0.47, 0.2784\rangle, \langle x_4, 0.99, 0.0009\rangle, \langle x_5, 0.45, 0.363\rangle, \langle x_6, 0.45, 0.363\rangle, \langle x_7, 0.45, 0.363\rangle, \langle x_8, 0.45, 0.363\rangle\}. \tag{39}$$

Then, we calculate the index sets of $\underline{\sum_{i=1}^{3} R_i}^{O}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q)$, $\underline{\sum_{i=1}^{3} R_i}^{P}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q)$, and $(\underline{\sum_{i=1}^{3} R_i}^{O}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q)) \oplus (\underline{\sum_{i=1}^{3} R_i}^{P}(Q) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q))$ by virtue of Definition 4.

$$\begin{aligned}
T_1 &= \left\{ l \middle| \max_{x_l \in U} \left\{ \underline{\sum_{i=1}^{3} R_i}^{O}(Q)(x_l) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q)(x_l) \right\} \right\} \\
&= \{1, 4\}, \\
T_2 &= \left\{ l' \middle| \max_{x_l' \in U} \left\{ \underline{\sum_{i=1}^{3} R_i}^{P}(Q)\left(x_l'\right) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q)\left(x_l'\right) \right\} \right\} \\
&= \{1, 4\}, \\
T_3 &= \left\{ l'' \middle| \max_{x_l'' \in U} \left\{ \left(\underline{\sum_{i=1}^{3} R_i}^{O}(Q)\left(x_l''\right) \oplus \overline{\sum_{i=1}^{3} R_i}^{O}(Q)\left(x_l''\right) \right) \right.\right. \\
&\quad \left.\left. \oplus \left(\underline{\sum_{i=1}^{3} R_i}^{P}(Q)\left(x_l''\right) \oplus \overline{\sum_{i=1}^{3} R_i}^{P}(Q)\left(x_l''\right) \right) \right\} \right\} \\
&= \{1, 4\}.
\end{aligned} \tag{40}$$

Finally, based on the fault diagnosis approaches originated from the risk decision-making guideline in classical operational research, since $T_1 \cap T_2 \cap T_3 = \{1, 4\} \neq \varnothing$, the most likely fault diagnosis result is unbalance fault and blade fault, which is identical with the conclusion obtained from the above case study.

5.5.3. Discussions. Compared with fault diagnosis methods on the basis of PF MGRSs over two universes proposed in literature [19] and IVHF MGRSs over two universes proposed in literature [47], those two group decision-making strategies expose limitations in the following aspects.

(1) Those two models only take into account the construction of reasonable information fusion rules, without developing an effective data-driven error tolerance scheme

(2) Those two models fail to consider the risk preference owned by mine ventilator engineers, if we overlook the necessity of introducing a reasonable risk-based scheme, a more comprehensive mine ventilator fault diagnosis model will be absent

Taking full advantages of PFSs, MGRSs over two universes and PRSs, we summarize the following merits of the proposed mine ventilator fault diagnosis approach.

(1) The proposed PF multigranulation probabilistic method could be seen as a generalization form of the existing works such as PFSs, MGRSs over two universes, and PRSs. For the fault diagnosis information expression, PFSs provide a flexible way to describe uncertain fault diagnosis information from membership degrees and nonmembership degrees. Thus, the proposed model outperforms other similar models within crisp or classical fuzzy information systems

(2) The proposed PF multigranulation probabilistic method not only improves the performance of processing various noisy data through allowing a certain acceptable level of error but also provides a reasonable risk-based scheme for engineers to cope with variable risk attitudes in fault diagnosis; hence, the limitations of optimistic and pessimistic PF MGRSs over two universes could be addressed well

In light of the above discussions, the proposed PF multigranulation probabilistic model could improve the reliability and accuracy of mine ventilator fault diagnosis effectively.

6. Conclusions

In this article, we establish a novel PF multigranulation probabilistic model for mine ventilator fault diagnosis. Concretely, through integrating the superiorities of PRSs with PF MGRSs over two universes, three types of PF MG-PRSs over two universes on the basis of the risk appetite of mine ventilation engineers are developed. Next, for the sake of improving the theoretical framework of the proposed model, some fundamental propositions are explored in detail. In addition, within the context of mine ventilator fault diagnosis, we construct a general PF multigranulation probabilistic model by means of PF MG-PRSs over two universes. Finally, a numerical example, a validity test, a sensitivity analysis, and a comparison analysis concerning mine ventilator fault diagnosis are explored to show the validation and efficiency of the proposed fault diagnosis rule.

In future work, there still remain several thoughtful topics in the theoretical and practical studies of PF MG-PRSs over two universes. For one thing, since plenty of fault diagnosis problems are equipped with distinguishing features of large-scale and dynamic fault types and high-dimensional fault characteristics, it is well worth the effort to put forward efficient fault diagnosis rules for this issue. For another, it is also desirable to apply more soft computing tools such as soft sets [8–10] to fault diagnosis situations for achieving reasonable and convincing results.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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References

[1] E. I. Acuna and I. S. Lowndes, "A review of primary mine ventilation system optimization," *Interfaces*, vol. 44, no. 2, pp. 163–175, 2014.

[2] J. Ma, J. Wu, and X. Wang, "Fault diagnosis method of check valve based on multikernel cost-sensitive extreme learning machine," *Complexity*, vol. 2017, Article ID 8395252, 19 pages, 2017.

[3] X. Shuiqing, Z. Ke, C. Yi, H. Yigang, and F. Li, "Gear fault diagnosis in variable speed condition based on multiscale chirplet path pursuit and linear canonical transform," *Complexity*, vol. 2018, Article ID 3904598, 8 pages, 2018.

[4] D. Song, H. Liu, L. Qi, and B. Zhou, "A general purpose adaptive fault detection and diagnosis scheme for information systems with superheterodyne receivers," *Complexity*, vol. 2018, Article ID 4763612, 9 pages, 2018.

[5] Y. Ding, F. Wang, Z. Wang, and W. Zhang, "Fault diagnosis for hydraulic servo system using compressed random subspace based relieff," *Complexity*, vol. 2018, Article ID 8740989, 14 pages, 2018.

[6] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[7] X. Wang and E. Triantaphyllou, "Ranking irregularities when evaluating alternatives by using some ELECTRE methods," *Omega*, vol. 36, no. 1, pp. 45–63, 2008.

[8] H. Garg and R. Arora, "A nonlinear-programming methodology for multi-attribute decision-making problem with interval-valued intuitionistic fuzzy soft sets information," *Applied Intelligence*, vol. 48, no. 8, pp. 2031–2046, 2018.

[9] H. Garg and R. Arora, "Generalized and group-based generalized intuitionistic fuzzy soft sets with applications in decision-making," *Applied Intelligence*, vol. 48, no. 2, pp. 343–356, 2018.

[10] H. Garg and R. Arora, "Dual hesitant fuzzy soft aggregation operators and their application in decision-making," *Cognitive Computation*, vol. 10, no. 5, pp. 769–789, 2018.

[11] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[12] R. R. Yager, "Pythagorean fuzzy subsets," in *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, pp. 57–61, Edmonton, AB, Canada, June 2013.

[13] R. R. Yager, "Pythagorean membership grades in multicriteria decision making," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 958–965, 2014.

[14] R. R. Yager and A. M. Abbasov, "Pythagorean membership grades, complex numbers, and decision making," *International Journal of Intelligent Systems*, vol. 28, no. 5, pp. 436–452, 2013.

[15] M. Z. Reformat and R. R. Yager, "Suggesting recommendations using pythagorean fuzzy sets illustrated using netflix movie data," *Communications in Computer and Information Science*, vol. 442, pp. 546–556, 2014.

[16] X. Zhang and Z. Xu, "Extension of topsis to multiple criteria decision making with pythagorean fuzzy sets," *International Journal of Intelligent Systems*, vol. 29, no. 12, pp. 1061–1078, 2014.

[17] X. Peng and Y. Yang, "Some results for pythagorean fuzzy sets," *International Journal of Intelligent Systems*, vol. 30, no. 11, pp. 1133–1160, 2015.

[18] R. R. Yager, "Properties and applications of pythagorean fuzzy sets," *Studies in Fuzziness and Soft Computing*, vol. 332, pp. 119–136, 2016.

[19] C. Zhang, D. Li, and R. Ren, "Pythagorean fuzzy multigranulation rough set over two universes and its applications in merger and acquisition," *International Journal of Intelligent Systems*, vol. 31, no. 9, pp. 921–943, 2016.

[20] R. Zhang, J. Wang, X. Zhu, M. Xia, and M. Yu, "Some generalized pythagorean fuzzy bonferroni mean aggregation operators with their application to multiattribute group decision-making," *Complexity*, vol. 2017, Article ID 5937376, 16 pages, 2017.

[21] L. Pérez-Domínguez, L. A. Rodríguez-Picón, A. Alvarado-Iniesta, D. Luviano Cruz, and Z. Xu, "Moora under pythagorean fuzzy set for multiple-criteria decision making," *Complexity*, vol. 2018, Article ID 2602376, 10 pages, 2018.

[22] H. Garg, "Generalised pythagorean fuzzy geometric interactive aggregation operators using einstein operations and their application to decision making," *Journal of Experimental & Theoretical Artificial Intelligence*, pp. 1–32, 2018.

[23] H. Garg, "Some methods for strategic decision-making problems with immediate probabilities in pythagorean fuzzy environment," *International Journal of Intelligent Systems*, vol. 33, no. 4, pp. 687–712, 2018.

[24] H. Garg, "Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process," *International Journal of Intelligent Systems*, vol. 33, no. 6, pp. 1234–1263, 2018.

[25] Z. Pawlak, "Rough sets," *International Journal of Computer and Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.

[26] M. T. Rezvan, A. Z. Hamadani, and S. R. Hejazi, "An exact feature selection algorithm based on rough set theory," *Complexity*, vol. 20, no. 5, 62 pages, 2015.

[27] A. Mardani, M. Nilashi, J. Antucheviciene, M. Tavana, R. Bausys, and O. Ibrahim, "Recent fuzzy generalisations of rough sets theory: a systematic review and methodological critique of the literature," *Complexity*, vol. 2017, Article ID 1608147, 33 pages, 2017.

[28] J. Bai, K. Xia, Y. Lin, and P. Wu, "Attribute reduction based on consistent covering rough set and its application," *Complexity*, vol. 2017, 9 pages, 2017.

[29] Y. Ma, X. Yu, and Y. Niu, "A parallel heuristic reduction based approach for distribution network fault diagnosis,"

International Journal of Electrical Power & Energy Systems, vol. 73, pp. 548–559, 2015.

[30] P. Konar, J. Sil, and P. Chattopadhyay, "Knowledge extraction using data mining for multi-class fault diagnosis of induction motor," *Neurocomputing*, vol. 166, pp. 14–25, 2015.

[31] X.-X. He, L.-t. Chen, and H.-t. Jia, "Building cognizance rule knowledge for fault diagnosis based on fuzzy rough sets1," *Journal of Intelligent & Fuzzy Systems*, vol. 29, no. 6, pp. 2327–2333, 2015.

[32] S. Rawat, A. Patel, J. Celestino, and A. L. M. dos Santos, "A dominance based rough set classification system for fault diagnosis in electrical smart grid environments," *Artificial Intelligence Review*, vol. 46, no. 3, pp. 389–411, 2016.

[33] F. Pacheco, M. Cerrada, R.-V. Sánchez, D. Cabrera, C. Li, and J. Valente de Oliveira, "Attribute clustering using rough set theory for feature selection in fault severity classification of rotating machinery," *Expert Systems with Applications*, vol. 71, pp. 69–86, 2017.

[34] B. Sun and W. Ma, "Multigranulation rough set theory over two universes," *Journal of Intelligent & Fuzzy Systems*, vol. 28, no. 3, pp. 1251–1269, 2015.

[35] Y. Qian, J. Liang, Y. Yao, and C. Dang, "Mgrs: a multi-granulation rough set," *Information Sciences*, vol. 180, no. 6, pp. 949–970, 2010.

[36] Y. Qian, S. Li, J. Liang, Z. Shi, and F. Wang, "Pessimistic rough set based decisions: a multigranulation fusion strategy," *Information Sciences*, vol. 264, no. 6, pp. 196–210, 2014.

[37] Y. Y. Yao, S. K. M. Wong, and L. S. Wang, "A non-numeric approach to uncertain reasoning," *International Journal of General Systems*, vol. 23, no. 4, pp. 343–359, 1995.

[38] W. Xu, W. Sun, Y. Liu, and W. Zhang, "Fuzzy rough set models over two universes," *International Journal of Machine Learning and Cybernetics*, vol. 4, no. 6, pp. 631–645, 2013.

[39] B. Sun and W. Ma, "Rough approximation of a preference relation by multi-decision dominance for a multi-agent conflict analysis problem," *Information Sciences*, vol. 315, pp. 39–53, 2015.

[40] B. Sun, W. Ma, and H. Zhao, "Rough set-based conflict analysis model and method over two universes," *Information Sciences*, vol. 372, pp. 111–125, 2016.

[41] C. Zhang, D. Li, and J. Liang, "Hesitant fuzzy linguistic rough set over two universes model and its applications," *International Journal of Machine Learning and Cybernetics*, vol. 9, no. 4, pp. 577–588, 2018.

[42] B. Sun, W. Ma, X. Chen, and X. Zhang, "Multigranulation vague rough set over two universes and its application to group decision making," *Soft Computing*, pp. 1–30, 2018.

[43] B. Sun, W. Ma, X. Chen, and X. Li, "Heterogeneous multigranulation fuzzy rough set-based multiple attribute group decision making with heterogeneous preference information," *Computers & Industrial Engineering*, vol. 122, pp. 24–38, 2018.

[44] B. Sun, W. Ma, B. Li, and X. Li, "Three-way decisions approach to multiple attribute group decision making with linguistic information-based decision-theoretic rough fuzzy set," *International Journal of Approximate Reasoning*, vol. 93, pp. 424–442, 2018.

[45] B. Sun, W. Ma, and X. Li, "Linguistic value soft set-based approach to multiple criteria group decision-making," *Applied Soft Computing*, vol. 58, pp. 285–296, 2017.

[46] B. Sun, W. Ma, and Y. Qian, "Multigranulation fuzzy rough set over two universes and its application to decision making," *Knowledge-Based Systems*, vol. 123, pp. 61–74, 2017.

[47] C. Zhang, D. Li, Y. Mu, and D. Song, "An interval-valued hesitant fuzzy multigranulation rough set over two universes model for steam turbine fault diagnosis," *Applied Mathematical Modelling*, vol. 42, pp. 693–704, 2017.

[48] B. Sun, W. Ma, and X. Xiao, "Three-way group decision making based on multigranulation fuzzy decision-theoretic rough set over two universes," *International Journal of Approximate Reasoning*, vol. 81, pp. 87–102, 2017.

[49] C. Zhang, D. Li, Y. Zhai, and Y. Yang, "Multigranulation rough set model in hesitant fuzzy information systems and its application in person-job fit," *International Journal of Machine Learning and Cybernetics*, pp. 1–13, 2017.

[50] S. K. M. Wong and W. Ziarko, "Comparison of the probabilistic approximate classification and the fuzzy set model," *Fuzzy Sets and Systems*, vol. 21, no. 3, pp. 357–362, 1987.

[51] Y. Y. Yao and S. K. M. Wong, "A decision theoretic framework for approximating concepts," *International Journal of Man-Machine Studies*, vol. 37, no. 6, pp. 793–809, 1992.

[52] W. Ziarko, "Variable precision rough set model," *Journal of Computer and System Sciences*, vol. 46, no. 1, pp. 39–59, 1993.

[53] Y. Yao, "Decision-theoretic rough set models," in *Rough Sets and Knowledge Technology*, pp. 1–12, Springer, Berlin, Heidelberg, 2007.

[54] Y. Yao, "Probabilistic rough set approximations," *International Journal of Approximate Reasoning*, vol. 49, no. 2, pp. 255–271, 2008.

[55] Y. Yao, "Three-way decisions with probabilistic rough sets," *Information Sciences*, vol. 180, no. 3, pp. 341–353, 2010.

[56] J. P. Herbert and J. Yao, "Game-theoretic rough sets," *Fundamenta Informaticae*, vol. 108, no. 3-4, pp. 267–286, 2011.

[57] X. Deng and Y. Yao, "Decision-theoretic three-way approximations of fuzzy sets," *Information Sciences*, vol. 279, pp. 702–715, 2014.

[58] Y. Yao and B. Zhou, "Two bayesian approaches to rough sets," *European Journal of Operational Research*, vol. 251, no. 3, pp. 904–917, 2016.

[59] H.-L. Yang, X. Liao, S. Wang, and J. Wang, "Fuzzy probabilistic rough set model on two universes and its applications," *International Journal of Approximate Reasoning*, vol. 54, no. 9, pp. 1410–1420, 2013.

[60] B. Sun, W. Ma, and X. Chen, "Fuzzy rough set on probabilistic approximation space over two universes and its application to emergency decision-making," *Expert Systems*, vol. 32, no. 4, pp. 507–521, 2015.

[61] D. Liang and D. Liu, "A novel risk decision making based on decision-theoretic rough sets under hesitant fuzzy information," *IEEE Transactions on Fuzzy Systems*, vol. 23, no. 2, pp. 237–247, 2015.

[62] T. Feng and J.-S. Mi, "Variable precision multigranulation decision-theoretic fuzzy rough sets," *Knowledge-Based Systems*, vol. 91, pp. 93–101, 2016.

[63] G. Lin, J. Liang, Y. Qian, and J. Li, "A fuzzy multigranulation decision-theoretic approach to multi-source fuzzy information systems," *Knowledge-Based Systems*, vol. 91, pp. 102–113, 2016.

[64] B. Q. Hu, H. Wong, and K. F. C. Yiu, "The aggregation of multiple three-way decision spaces," *Knowledge-Based Systems*, vol. 98, pp. 241–249, 2016.