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Research Article

Sum of Squares Approach for Nonlinear H_∞ Control

Ai-ping Pang,[1] **Zhen He**,[2] **Ming-han Zhao**,[2] **Guang-xiong Wang**,[2] **Qin-mu Wu**,[1] **and Ze-tao Li**[1]

[1]*College of Electrical Engineering, Guizhou University, Guizhou 550025, China*
[2]*School of Astronautics, Harbin Institute of Technology, Harbin 150001, China*

Correspondence should be addressed to Zhen He; hezhenhit@163.com

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A proper Hamilton-Jacobi-Isaacs (HJI) inequality must be solved in a nonlinear H_∞ control problem. The sum of squares (SOS) method can now be used to solve an analytically unsolvable nonlinear problem. A HJI inequality suitable for SOS approach is derived in the paper. The SOS algorithm for solving the HJI inequality is also provided. Conservativeness of the SOS method is then discussed in the paper. The conservativeness of the SOS approach is caused by the method itself, because it is really a synthesis method over the entire state space. To reduce the conservativeness, a local H_∞ design on a restricted state-space region is proposed. But the SOS approach for the local H_∞ design also suffers from the conservativeness problem, because the S-procedure for solving the set-containment constraint provides only a sufficient condition. The above-mentioned sources of conservativeness are peculiar for the SOS approaches. So a proper approach must be carefully selected in the design process to get a reasonable result. A design example is also given in the paper.

1. Introduction

The control problems of complicated nonlinear system are always the research hotspots [1–4]. The H_∞ control of nonlinearity system is also called the control problems of the L_2 gain of the system, which can finally come down to the problems of the dissipative system. L_2 gain constraint requires to solve a suitable Hamilton-Jacobi-Isaacs (HJI) partial differential inequality [5, 6]. However, there has never been one valid analytical solution for these inequalities, which is also a project discussed by scholars, as shown in literature [7] and the following literatures. In order to avoid solving this complicated HIJ inequality, researchers get L2 gain controller through constructed the Hamilton function generally [6, 8]. The SOS method [8–10], the abbreviation of the sum of squares, coming out recent years, has opened up a new way to solve the HJI inequality a numerical solution. The SOS polynomial is used in SOS method to study the nonlinear system. Except for the nonlinearity of the object itself, if we want to use Lyapunov function higher than quadratic terms, or design a high order nonlinear control law, a polynomial in general form has to be studied. But if the polynomial can be sorted in SOS modality, it must be nonnegative. Though the method is new, it has been used in some important applications and shows its superiority, such as the estimation of the nonlinear system attraction basins [11, 12], the satellite attitude control under a high maneuver condition [13, 14], the attitude control of the aircraft [15], the predictive control of the nonlinear model [16], and the stability analysis of the time delay system [17, 18]. But there is not so many SOS literatures involving the H_∞ control of the nonlinear system [19, 20]. Though literature [21] has mentioned the H_∞ control of the nonlinear system, there is no positive reference of the HJI inequality. SOS approach in this paper is the method solving HJI inequality directly. This article illustrates how to solve HJI inequality in a SOS way. The SOS method can come down to the solving of the linear matrix inequality virtually. So this article derives a suitable HJI inequality for SOS method and then turns it into a matrix inequality depending on the state variables similar to LMI and uses the functions in SOSTOOLS to solve the linear matrix inequality which is status-dependent before giving the nonlinear H_∞ control law. The article also discusses the conservatism and the treatment countermeasures while using SOS to solve the HJI inequality.

2. HJI Inequalities in SOS Problems

SOS refers to the sum of the polynomial. The polynomial consisted of a finite number of monomials in a linear combination way, such as

$$p(x_1, x_2) = x_1^2 + 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4 \tag{1}$$

Equation (1) is made up of five monomials with two variables. For the polynomial $p(x_1, \cdots, x_n) \triangleq p(\boldsymbol{x})$, if there is a polynomial $f_1(\boldsymbol{x}), \cdots, f_m(\boldsymbol{x})$ which can make $p(\boldsymbol{x})$ written in a way of sum of squares, just as

$$p(\boldsymbol{x}) = \sum_{i=1}^{m} f_i^2(\boldsymbol{x}), \tag{2}$$

the polynomial in this way can be called SOS polynomial, sometimes SOS for short. It is obvious that every SOS polynomial is nonnegative which can be presented as $p(\boldsymbol{x}) \geq 0$. The collection of the SOS polynomial can be presented as $\sum[\boldsymbol{x}]$. So if a polynomial is SOS, it can be written as

$$p(\boldsymbol{x}) \in \sum[\boldsymbol{x}] \tag{3}$$

SOS polynomial can also be presented as the following special quadratic way:

$$p(\boldsymbol{x}) = \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{Q} \boldsymbol{Z}(\boldsymbol{x}) \tag{4}$$

where $\boldsymbol{Q}$ is a positive semidefinite symmetrical matrix and $\boldsymbol{Z}(\boldsymbol{x})$ is a column vector consisting of every monomials whose order numbers are less than d. And the order number of the polynomial $p(\boldsymbol{x})$ is less than or equal to 2d. The polynomial in (1) is

$$\boldsymbol{Z}(\boldsymbol{x}) := \begin{bmatrix} x_1 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & -0.5 & 0 & 5 \end{bmatrix} \tag{5}$$

Suppose the equation of the system is $\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}$. If we want to use SOS to analyze, the equation should be changed into the following class linear form of state dependence [21], such as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x}) \boldsymbol{Z}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x}) \boldsymbol{u} \tag{6}$$

where $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ are the polynomial matrix for x, and $\boldsymbol{Z}(\boldsymbol{x})$ is $N \times 1$ monomial vector which satisfies the following supposes A1:

$$\text{A1: } \boldsymbol{Z}(\boldsymbol{x}) = 0 \quad \text{iff } \boldsymbol{x} = 0 \tag{7}$$

where iff is the symbolic representation of ‘if and only if’. There is a good example for (6) in literature [13].

For the H_∞ control researching, the equation of the system is

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}(\boldsymbol{x}) \boldsymbol{Z}(\boldsymbol{x}) + \boldsymbol{B}_1(\boldsymbol{x}) \boldsymbol{w} + \boldsymbol{B}_2(\boldsymbol{x}) \boldsymbol{u} \\ \boldsymbol{z} &= \begin{bmatrix} \boldsymbol{C}_1(\boldsymbol{x}) & 0 \\ 0 & \boldsymbol{D}_{12}(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} \boldsymbol{Z}(\boldsymbol{x}) \\ \boldsymbol{u} \end{bmatrix} \end{aligned} \tag{8}$$

where $\boldsymbol{u}(t)$ is the control input, $\boldsymbol{w}(t)$ is the disturbance input, and $\boldsymbol{z}(t)$ is the performance output of the system. And the optimal control problem of H_∞ control is to solve the state feedback control law $\boldsymbol{u}(t)$, which makes the gain L_2 the minimum from disturbance $\boldsymbol{w}$ to output $\boldsymbol{z}$.

Definition 1. Supposed a nonnegative function $V(\boldsymbol{x})$ ($V(0) = 0$) (called storage function). If V is differentiable and the Hamilton function H defined by the following equation is nonpositive, then system (1) is dissipative [4, 22].

$$H := \frac{\mathrm{d}V}{\mathrm{d}t}(\boldsymbol{x}(t)) + \|\boldsymbol{z}\|^2 - \gamma^2 \|\boldsymbol{w}\|^2 \tag{9}$$

The gain of this dissipative system L_2 is less than or equal to γ. If $V(\boldsymbol{x})$ is positive definite, then the storage function is Lyapunov function.

The normal method in related literatures now expresses the nonlinear system into an affine system and makes the storage function $V(\boldsymbol{x})$ a positive definite quadratic function. Based on this, the HJI inequality can be deduced and continue related discussion [4, 5, 23]. However, there is no effective analytical solution to get the HJI inequality just like this, while using SOS method, the equation of the system should be presented as (8) of the state dependent class linear systems and define the storage function $V(\boldsymbol{x})$ as

$$V(\boldsymbol{x}) = \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{P}(\widetilde{\boldsymbol{x}}) \boldsymbol{Z}(\boldsymbol{x}) \tag{10}$$

where $\boldsymbol{Z}(\boldsymbol{x})$ is the $N \times 1$ monomial vector shown in (8) and $\boldsymbol{P}(\widetilde{\boldsymbol{x}})$ is a symmetric polynomial matrix. Suppose $\boldsymbol{J} = \{j_1, j_2, \cdots, j_m\}$ showing the label of the line whose value is zero in (8) $[\boldsymbol{B}_1(\boldsymbol{x}) \ \ \boldsymbol{B}_2(\boldsymbol{x})]$. The corresponding items in the corresponding state equation can be presented as follows:

$$\begin{aligned} \dot{x}_{j1} &= \boldsymbol{A}_{j1}(\boldsymbol{x}) \boldsymbol{Z}(\boldsymbol{x}) \\ \dot{x}_{j2} &= \boldsymbol{A}_{j2}(\boldsymbol{x}) \boldsymbol{Z}(\boldsymbol{x}) \\ &\vdots \end{aligned} \tag{11}$$

The corresponding state variables are defined as $\widetilde{\boldsymbol{x}} = \{x_{j1}, x_{j2}, \cdots, x_{jm}\}$. We define $\boldsymbol{P}$ the polynomial matrix of variable $\widetilde{\boldsymbol{x}}$, just $\boldsymbol{P}(\widetilde{\boldsymbol{x}})$. So, in this way, when we calculate the derivation of $V(\boldsymbol{x})$, there will be no derivation of the input signals $\boldsymbol{w}$ and $\boldsymbol{u}$.

Because of the differences between the system equation and $V(\boldsymbol{x})$, the HJI inequality should be derived again in this problem.

Bring (8) and (10) into (9):

$$\begin{aligned} H &= \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})\left[\sum_{j\in J}\frac{\partial \boldsymbol{P}}{\partial x_j}(\widetilde{\boldsymbol{x}})\left(\boldsymbol{A}_j(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x})\right)\right. \\ &\quad \left. + \boldsymbol{P}(\widetilde{\boldsymbol{x}})\boldsymbol{M}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{M}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}(\widetilde{\boldsymbol{x}})\right]\boldsymbol{Z}(\boldsymbol{x}) \\ &\quad + \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}(\widetilde{\boldsymbol{x}})\boldsymbol{M}(\boldsymbol{x})\left[\boldsymbol{B}_1(\boldsymbol{x})\boldsymbol{w} + \boldsymbol{B}_2(\boldsymbol{x})\boldsymbol{u}\right] \\ &\quad + \left[\boldsymbol{w}^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}}(\boldsymbol{x}) + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{B}_2^{\mathrm{T}}(\boldsymbol{x})\right]\boldsymbol{M}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}(\widetilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x}) \\ &\quad + \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{C}_1^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{C}_1(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}) + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{D}_{12}^{\mathrm{T}}\boldsymbol{D}_{12}\boldsymbol{u} \\ &\quad - \gamma^2\boldsymbol{w}^{\mathrm{T}}\boldsymbol{w} \le 0 \end{aligned} \tag{12}$$

where $\boldsymbol{M}(\boldsymbol{x})$ is the transform matrix of the derivation from $\boldsymbol{Z}$ to $\boldsymbol{x}$. And the corresponding elements are

$$\boldsymbol{M}_{ij}(\boldsymbol{x}) = \frac{\partial \boldsymbol{Z}_i}{\partial x_j}(\boldsymbol{x}) \tag{13}$$

To simplify the process, the following process will omit the variable $\boldsymbol{x}$, $\widetilde{\boldsymbol{x}}$ in the symbols. And now we use $H[\boldsymbol{x}, \boldsymbol{P}, \boldsymbol{w}, \boldsymbol{u}]$ to present the Hamilton function in (9). To solve this inequality, the analysis method is to get the saddle singularity in the following Hamilton function [1, 2]:

$$\begin{aligned} H[\boldsymbol{x}, \boldsymbol{P}, \boldsymbol{w}, \breve{\boldsymbol{u}}] &\le H[\boldsymbol{x}, \boldsymbol{P}, \widehat{\boldsymbol{w}}, \breve{\boldsymbol{u}}] \\ &\le H[\boldsymbol{x}, \boldsymbol{P}, \widehat{\boldsymbol{w}}, \boldsymbol{u}] \end{aligned} \tag{14}$$

where $\widehat{\boldsymbol{w}}$ is the worst disturbance making the Hamilton function $H(\bullet)$ maximum. $\breve{\boldsymbol{u}}$ is the control input making $H(\bullet)$ min. This is also the conception of H_∞ optimization solution.

In this way, according to (12),

$$\frac{\partial H}{\partial \boldsymbol{u}} = 2\boldsymbol{Z}^{-1}\boldsymbol{P}\boldsymbol{M}\boldsymbol{B}_2 + 2\boldsymbol{u}^{\mathrm{T}}\boldsymbol{D} = 0, \quad \boldsymbol{D} = \boldsymbol{D}_{12}^{\mathrm{T}}\boldsymbol{D}_{12} \tag{15}$$

and

$$\breve{\boldsymbol{u}} = -\boldsymbol{D}^{-1}\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{Z} \tag{16}$$

In the same way, we can get

$$\widehat{\boldsymbol{w}} = \frac{1}{\gamma^2}\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{Z} \tag{17}$$

Bring (16) and (17) into (12); we can get

$$\begin{aligned} H &= \boldsymbol{Z}^{\mathrm{T}}\left[\sum_{j\in J}\frac{\partial \boldsymbol{P}}{\partial x_j}(\widetilde{\boldsymbol{x}})\left(\boldsymbol{A}_j\boldsymbol{Z}\right) + \boldsymbol{P}\boldsymbol{M}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{P}\right]\boldsymbol{Z} \\ &\quad + \boldsymbol{Z}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{M}\left[\gamma^{-2}\boldsymbol{B}_1\boldsymbol{B}_1^{\mathrm{T}} - \boldsymbol{B}_2\boldsymbol{D}^{-1}\boldsymbol{B}_2^{\mathrm{T}}\right]\boldsymbol{M}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{Z} \\ &\quad + \boldsymbol{Z}^{\mathrm{T}}\boldsymbol{C}_1^{\mathrm{T}}\boldsymbol{C}_1\boldsymbol{Z} \le 0 \end{aligned} \tag{18}$$

Equation (18) is the Hamilton-Jacobi-Isaacs (HJI) inequality in the problem. After getting $\boldsymbol{P}(\widetilde{\boldsymbol{x}})$ from solving the HJI inequality, we can bring it into (16) and get the nonlinear state feedback control law $\boldsymbol{u}(\boldsymbol{x})$.

Notice that each item in (16) has a premultiplication $\boldsymbol{Z}^{\mathrm{T}}$ and a postmultiplication $\boldsymbol{Z}$. So we can get $\boldsymbol{Z}^{\mathrm{T}}[\bullet]\boldsymbol{Z}$. And the inequality requirements for (18) change into the seminegative definite requirements of the synthetic item, just as the seminegative definite requirements of the following equation:

$$\begin{aligned} &\boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{M}\boldsymbol{A} + \sum_{j\in J}\frac{\partial \boldsymbol{P}}{\partial x_j}\left(\boldsymbol{A}_j\boldsymbol{Z}\right) \\ &\quad + \boldsymbol{P}\boldsymbol{M}\left[\gamma^{-2}\boldsymbol{B}_1\boldsymbol{B}_1^{\mathrm{T}} - \boldsymbol{B}_2\boldsymbol{D}^{-1}\boldsymbol{B}_2^{\mathrm{T}}\right]\boldsymbol{M}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{C}_1^{\mathrm{T}}\boldsymbol{C}_1 \\ &\le 0 \end{aligned} \tag{19}$$

The new equation (19) is a kind of Riccati equation. However, Riccati equation is a quadratic equation which is not suitable in SOS. SOS can come down to solve the inequality matrix, exactly the state dependent type LMI. The matrix (nonlinear) is a polynomial matrix, but the decision variables are all linear. We can know from the method of linear matrix inequality (LMI) that the solution $\boldsymbol{P}$ of the Riccati equation does not make up a convex problem, but if we use the inverse of the solution, just $\boldsymbol{P}^{-1}$, to present, we can get LMI [2]. In addition, γ in (19) comes in square form. while we use it to optimize the decision variables of the problem, a linear relationship for γ is also required. Considering the above two points, we can take

$$\boldsymbol{P} = \gamma\boldsymbol{Q}^{-1} \tag{20}$$

Bring $\mathbf{Q}$ into the equation and we can get a matrix inequality of the HJI inequality above, as shown in the following theorem.

Theorem 2. *For system (8), suppose that there is a symmetric polynomial matrix $\mathbf{Q}(\widetilde{\boldsymbol{x}}) > 0$ which satisfies the matrix inequality,*

$$\begin{bmatrix} \boldsymbol{M}\boldsymbol{A}\mathbf{Q} + \mathbf{Q}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}} - \gamma\boldsymbol{M}\boldsymbol{B}_2\boldsymbol{D}^{-1}\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}} - \sum_{j\in J}\frac{\partial \mathbf{Q}}{\partial x_j}\left(\boldsymbol{A}_j\boldsymbol{Z}\right) & \mathbf{Q}\boldsymbol{C}_1^{\mathrm{T}} & \boldsymbol{M}\boldsymbol{B}_1 \\ \boldsymbol{C}_1\mathbf{Q} & -\gamma I & 0 \\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}} & 0 & -\gamma I \end{bmatrix} \le 0 \tag{21}$$

The gain of the system is less than or equal to γ, and the nonlinear H_∞ state feedback controller can be given in the following equation:

$$\boldsymbol{u}(\boldsymbol{x}) = -\gamma \boldsymbol{D}^{-1}(\boldsymbol{x}) \boldsymbol{B}_2^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{Q}^{-1}(\widetilde{\boldsymbol{x}}) \boldsymbol{Z}(\boldsymbol{x}) \quad (22)$$

If $\boldsymbol{Q}(\widetilde{\boldsymbol{x}})$ is a constant matrix, the stability of the system and the boundary of the gain L_2 are both global.

Proof. Bring (20) into HJI inequality equation (19) and then use Schur complement lemma [2]; we can get it proved. Pay attention that we use a relation in literature [21]:

$$\frac{\partial \boldsymbol{P}}{\partial x_j}(\boldsymbol{x}) = -\boldsymbol{P}(\boldsymbol{x}) \frac{\partial \boldsymbol{P}^{-1}}{\partial x_j}(\boldsymbol{x}) \boldsymbol{P}(\boldsymbol{x}) \quad (23)$$

□

3. SOS Algorithm in Solving HJI Inequalities

Equation (21) represents the negative semidefinite requirement of a symmetric polynomial matrix. Because the problems SOS solved are nonnegative, the inequality, just like (21), has to put on a negative sign which makes it become a positive semidefinite problem. Suppose $F(\boldsymbol{x})$ represent such a problem:

$$\boldsymbol{F}(\boldsymbol{x}) \geq 0 \quad (24)$$

Suppose that $\boldsymbol{F}(\boldsymbol{x})$ is a $N \times N$ symmetrical matrix, a $\boldsymbol{x} \in \mathrm{R}^n$ polynomial matrix whose order is equal to $2d$.

The positive semidefinite requirement of $\boldsymbol{F}(\boldsymbol{x})$ is achieved by the quadratic polynomial $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{x})\boldsymbol{v} \in \sum[\boldsymbol{x}]$, $\boldsymbol{v} \in \mathrm{R}^N$ in SOS. Because if

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{x})\boldsymbol{v} \in \sum[\boldsymbol{x}] \quad (25)$$

it presents that there is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{x})\boldsymbol{v} \geq 0$ for every $(\boldsymbol{v}, \boldsymbol{x}) \in \mathrm{R}^{N\times n}$ which is equivalent to $\boldsymbol{F}(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \in \mathrm{R}^n$ [21].

There are two vector variables in (25) of SOS, $\boldsymbol{v}$ and $\boldsymbol{x}$. And we call $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{v}$ a bipartite scalar polynomial. In theory, there are no differences between this polynomial and the normal ones in essence. For example, suppose a bipartite scalar polynomial is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{v}$, $\boldsymbol{v} = [v_1 \ \ v_2]^{\mathrm{T}}$

$$\boldsymbol{F} = \begin{bmatrix} x^2 - 2x + 2 & x \\ x & x^2 \end{bmatrix} \quad (26)$$

After finishing,

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{v} = \begin{bmatrix} v_1 \\ xv_1 \\ v_2 \\ xv_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ xv_1 \\ v_2 \\ xv_2 \end{bmatrix} = (v_1 + xv_2)^2 + (xv_1 - v_1)^2 \quad (27)$$

Equation (27) shows that $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{v}$ is a polynomial in the sum of squares. So, for (26), all x is the positive definite. Though it is the same in theory, if the special sparse multiple structure statements in SOSTOOLS are used, the calculating workload will decrease greatly. Still take (26) for an example, the corresponding command statements are as follows:

```
>> syms x v1 v2 real
>> F = [x^2 – 2*x + 2, x; x, x^2]
>> v = [v1; v2]
>> p = v'*F*v
>> prog = sosprogram([x, v1, v2])
>> prog = sosineq(prog, p, 'sparsemultipartite', {[x], [v1, v2]})
>> prog = sossolve(prog)
```

In this example, prog is the program, and p is the polynomial solution which is solved. sosineq is the SOS inequality constraint [4], while sparsemultipartite refers to the sparse multivariate structure, and the last statement is the solution; the result is given in (27).

Then Theorem 2 and the requirements of (21) can be written SOS constraints as follows.

$$\boldsymbol{v}_1^{\mathrm{T}}(\boldsymbol{Q}(\widetilde{\boldsymbol{x}}) - \varepsilon_1 I)\boldsymbol{v}_1 \in \sum[\boldsymbol{x}] \quad (28)$$

$$-\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}} \left[\begin{array}{c|cc} \boldsymbol{MAQ} + \boldsymbol{QA}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}} - \gamma \boldsymbol{MB}_2\boldsymbol{D}^{-1}\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}} - \sum_{j\in J} \frac{\partial \boldsymbol{Q}}{\partial x_j}(\boldsymbol{A}_j\boldsymbol{Z}) + \varepsilon_2 I & \boldsymbol{QC}_1^{\mathrm{T}} & \boldsymbol{MB}_1 \\ \hline \boldsymbol{C}_1\boldsymbol{Q} & -(\gamma - \varepsilon_2)\boldsymbol{I} & 0 \\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}} & 0 & -(\gamma - \varepsilon_2)\boldsymbol{I} \end{array}\right] \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} \in \sum[\boldsymbol{x}] \quad (29)$$

where (28) corresponds to $Q(\widetilde{\boldsymbol{x}}) > 0$ and (29) corresponds to (21). ε_1, ε_2 are both decimal numbers, such as 0.001, $\boldsymbol{v}_1 \in \mathrm{R}^N$, and the dimension of $\boldsymbol{v}_2$ is determined by the block matrix in (29). The SOS constraints of (28) and (29) are implemented using the sosineq function. The solution of this SOS problem is $\boldsymbol{Q}(\widetilde{\boldsymbol{x}})$, and we can obtain the state feedback control law $\boldsymbol{u}(\boldsymbol{x})$ by substituting (22).

4. Examples

Suppose the parameter array in (8) as

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x}) &= \begin{bmatrix} -1 + x_1 - 1.5x_1^2 - 0.75x_2^2 & 0.25 - x_1^2 - 0.5x_2^2 \\ 0 & 0 \end{bmatrix} \\ \boldsymbol{Z}(\boldsymbol{x}) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \boldsymbol{B}_1(\boldsymbol{x}) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \boldsymbol{B}_2(\boldsymbol{x}) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \boldsymbol{C}_1(\boldsymbol{x}) &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ \boldsymbol{D}_{12}(\boldsymbol{x}) &= 1, \\ \boldsymbol{M} &= \boldsymbol{I} \end{aligned} \quad (30)$$

According to (28) and (29), we calculate the state feedback control law u to minimize the L2 gain from the disturbance to the output system though corresponding SOS program. And let $\varepsilon_1 = \varepsilon_2 = 0.001$ in (29).

In this example, the disturbances w and control inputs u affect the dynamics characteristics of both x_1 and x_2 through $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$. Because $j = 0$ in (11), so $\boldsymbol{P}(\tilde{\boldsymbol{x}})$ is a constant matrix in (10). We use the dichotomy method to find the optimal value, and the minimum value γ obtained is 1.15; that is, the L2 gain from the system is not larger than 1.15. The resulting control law in (22) is as follows:

$$u = -1.1216x_1 - 2.5401x_2 \quad (31)$$

As a comparison, we examine the results of the linearized design of this system, in which the same SOS optimization program is used; just the system matrix $\boldsymbol{A}(\boldsymbol{x})$ in (30) is changed to the state matrix of the linearization system as follows:

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{bmatrix} -1 & 0.25 \\ 0 & 0 \end{bmatrix} \quad (32)$$

The result of the optimization design for this linearization system is $\gamma = 0.83$, and the corresponding control law is

$$u = -116.8346x_1 - 141.2178x_2 \quad (33)$$

Because the system is linear, the L2 gain of the system is the H_∞ norm which can also be obtained from the transfer function T_{zw} from w to z. Figure 1 is the Bode diagram of the singular value $\overline{\sigma}[T_{zw}(j\omega)]$ of this linearized system. It can be read as $\gamma = \|T_{zw}(j\omega)\|_\infty = -1.6$ dB $= 0.83$ from the Figure 1. And this figure also shows Bode diagram of $|T_{x_1w}|$ and $|T_{uw}|$, which are transfer function from input x_1 and u to the output z.

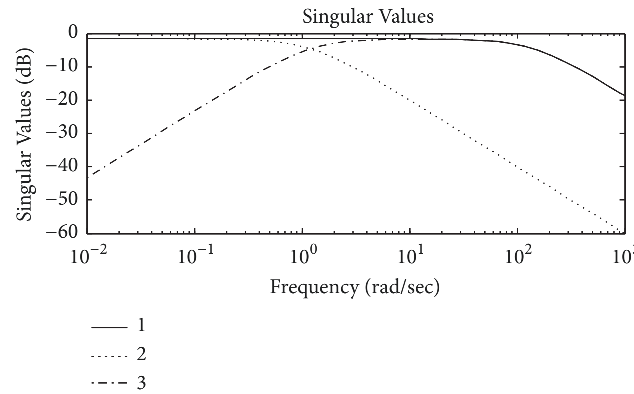

FIGURE 1: Singular value of linearized system. 1—$\overline{\sigma}[T_{zw}]$; 2—$|T_{x_1w}|$; 3—$|T_{uw}|$.

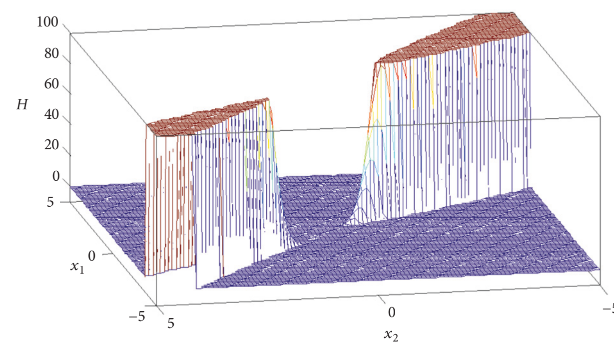

FIGURE 2: Hamiltonian functions of nonlinear systems.

Figure 1 shows that the SOS solution of HJI inequality is consistent with the system's H∞ norm if it is a linear system. However, the actual system is nonlinear. Substituting the matrix P obtained by this linearization design into (18), the Hamilton function H was obtained of the nonlinear system. Figure 2 shows the graph when $H > 0$. Figure 2 shows that in this system the area which satisfies $H \leq 0$, satisfies $\gamma = 0.83$ only near the origin. The area farther away from the origin the graph is upturned and $H > 0$, which does not satisfy the HJI inequality.

The above nonlinear design result in equation (31) shows that, after considering the nonlinearity, the minimum value γ that can be achieved is 1.15, which is greater than the minimum value of the linearized system 0.83. It is necessary to measure the frequency response characteristics from w to z at different frequencies under different input amplitudes for the verification of the gain of the nonlinear system L2. However, the frequency response when the signal amplitude of w is 1, 2, 3, respectively, indicates that the amplitude ratio of the frequency response is only slightly larger than 0.83 and does not exceed 1, let alone 1.15. This conservativeness in solving HJI inequalities is because the HJI inequality contains the entire state space, but some values of the state variables in the actual system may not be so large at all. For example, in (30), plus state feedback, x_2 is just an intermediate variable of the feedback system. As long as the system is stable, x_2 will not be too large. Regardless of these factors, the entire state space is included and the resulting value γ will be greater than the actual value of the actual system. This conservation is due to the positive solution to the HJI inequality (including the entire state space) and is unique to the SOS positive solution method. In order to reduce the conservativeness, it is necessary to limit the state space to the normal working range of the system. This is the local H_∞ control.

5. Local $H\infty$ Control

The local H_∞ control here means the L_2 gain control problem which is related to a state space in a local area of the actual working scope. The local area X in SOS can be described in terms of polynomial inequality $g_l(\boldsymbol{x}) \geq 0$:

$$X = \{\boldsymbol{x} \in R^n : g_l(\boldsymbol{x}) \geq 0,\ l = 1, \ldots, m\} \tag{34}$$

The shape of the local area should be determined by the problem. For an example in this article, x_2 is an intermediate variable of the feedback control system, which will not change a lot. So we can limit this local area in the area of $x_1^2 + 36x_2^2 \leq 9$, which is written in a polynomial inequality as

$$g(\boldsymbol{x}) = 9 - x_1^2 - 36x_2^2 \geq 0 \tag{35}$$

SOS design is designed to solve different inequality constraint originally. While the constraint of the local area needs to be considered, we should satisfy the constraint of (35) with satisfying the original SOS inequality constraint, that is, a set-containment constraint problem. And set-containment constraint can deal with S-procedure problem [24], that is, a constraint item $g(\boldsymbol{x})$ with each one multiplied by $s(\boldsymbol{x}, \boldsymbol{v})$, based on (28) and (29). For this example, they are the two inequality constraints:

$$\boldsymbol{v}_1^T (\boldsymbol{Q}(\tilde{\boldsymbol{x}}) - \varepsilon_1 \boldsymbol{I}) \boldsymbol{v}_1 - s_1(\boldsymbol{x}, \boldsymbol{v}_1) g(\boldsymbol{x}) \in \sum [\boldsymbol{x}] \tag{36}$$

$$-\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}^T W(\boldsymbol{x}) \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} - s_2(\boldsymbol{x}, \boldsymbol{v}_1, \boldsymbol{v}_2) g(\boldsymbol{x}) \in \sum [\boldsymbol{x}] \tag{37}$$

where

$$W(\boldsymbol{x}) = \begin{bmatrix} MAQ + QA^T M^T - \gamma M B_2 D^{-1} B_2^T M^T - \sum_{j \in J} \frac{\partial Q}{\partial x_j} (A_j Z) + \varepsilon_2 I & QC_1^T & MB_1 \\ C_1 Q & -(\gamma - \varepsilon_2) I & 0 \\ B_1^T M^T & 0 & -(\gamma - \varepsilon_2) I \end{bmatrix} \tag{38}$$

where $s_1(\boldsymbol{x}, \boldsymbol{v}_1)$ and $s_2(\boldsymbol{x}, \boldsymbol{v}_1, \boldsymbol{v}_2)$ are the multipliers of the sum of squares of polynomials. In the example, the two multipliers are polynomials with two orders.

Combining with the example equation (30), using SOS to solve (36) and (37), we can get the minimum value of γ as 1.145. But in the local area of (35), making a simulation verify the different amplitudes, the frequency response at different frequencies shows that the amplitudes of frequency response are not more than 1. In other words, local H_∞ control has conservatism. The conservatism is brought by S method. S method makes some collections which contain constrained problems and are not easy to solve, but S method is also a sufficient condition which has some conservatisms.

In a word, not only the design of local area with SOS method but also that of local area with S method contained full state space (in third segment), and they are both conservative. Because the example is easy, the conservatism is not so obvious. But in actual design, we would better make a comparison between the two methods. Especially when the full state space has no solution or has some obvious unreasonable data, local area H_∞ control may give a reasonable result.

6. Conclusion

The method of the sum of squares is a numerical solution method, used to solve the nonlinear problems which are not so easy to get an analytical solution. The article shows an algorithm which is suitable to SOS and the HJI inequality of nonlinear H_∞ control. Solving HJI inequality in a SOS method is different to the existing methods. In practice, the solution thought of the nonlinear H_∞ control problem is based on the design of the linear system first and then attaching the nonlinear control law, making the whole Hamilton function of the system H maintain or less than zero in actual working scope [25, 26]. The method in the article is a direct design method, or is called synthesis, which solve the HJI inequality positively. Because the solution is positive, it contains the whole space of the state. But after designing, because of the negative feedback control, some of the state variables cannot be so large. So there is certain conservatism in this positive direct design. If the design limits the area of the state space, it is a local H_∞ control. But in SOS, the collection of the inequality contains constraints which require the S method to be solve. Though S method is an efficient method, it has a sufficient condition which is also conservative. The two conservations are unique for SOS, so we would better make a comparison to get more reasonable results.

Data Availability

The [DATA TYPE] data used to support the findings of this study are included within the article. If other [DATA TYPE] data or programs used to support the findings of this study are needed, you can obtain them from the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] Y. Wei, J. Qiu, and H.-K. Lam, "A Novel Approach to Reliable Output Feedback Control of Fuzzy-Affine Systems with Time Delays and Sensor Faults," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 6, pp. 1808–1823, 2017.

[2] Y. Wei, J. Qiu, P. Shi, and M. Chadli, "Fixed-Order Piecewise-Affine Output Feedback Controller for Fuzzy-Affine-Model-Based Nonlinear Systems with Time-Varying Delay," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 64, no. 4, pp. 945–958, 2017.

[3] Q. Zhu, L. Liu, W. Zhang, and S. Li, "Control of Complex Nonlinear Dynamic Rational Systems," *Complexity*, vol. 2018, Article ID 8953035, 12 pages, 2018.

[4] A.-P. Pang and Z. He, "H∞ state Feedback design for supercavitating vehicles," *Journal of Control Theory and Applications*, vol. 34, no. 10, pp. 10–17, 2017.

[5] A. Isidori and W. Kang, "H_∞ control via measurement feedback for general nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 40, no. 3, pp. 466–472, 1995.

[6] G. X. Wang and Z. He, *Applied H∞ Control [M]*, Harbin Institute of Technology Press, Harbin, China, 2010, (in Chinese).

[7] T. Ohtsuka, "Solutions to the Hamilton-Jacobi equation with algebraic gradients," *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 56, no. 8, pp. 1874–1885, 2011.

[8] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: A general purpose sum of squares programming solver," in *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 741–746, USA, December 2002.

[9] A. A. Ahmadi and A. Majumdar, "DSOS and SDSOS optimization: LP and SOCP-based alternatives to sum of squares optimization," *Optimization and Control*, 2017.

[10] D. Zhao and J.-L. Wang, "Robust static output feedback design for polynomial nonlinear systems," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 14, pp. 1637–1654, 2010.

[11] W. Tan and A. Packard, "Stability region analysis using polynomial and composite polynomial Lyapunov functions and sum-of-squares programming," *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 565–571, 2008.

[12] U. Topcu, A. Packard, and P. Seiler, "Local stability analysis using simulations and sum-of-squares programming," *Automatica*, vol. 44, no. 10, pp. 2669–2675, 2008.

[13] N. Gollu and L. Rodrigues, "Control of large angle attitude maneuvers for rigid bodies using sum of squares," in *Proceedings of the 2007 American Control Conference, ACC*, pp. 3156–3161, USA, July 2007.

[14] S. Prajna, P. A. Parrilo, and A. Rantzer, "Nonlinear control synthesis by convex optimization," *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 49, no. 2, pp. 310–314, 2004.

[15] K. Krishnaswamy, G. Papageorgiou, S. Glavaski, and A. Papachristodoulou, "Analysis of aircraft pitch axis stability augmentation system using sum of squares optimization," in *Proceedings of the 2005 American Control Conference, ACC*, pp. 1497–1502, USA, June 2005.

[16] G. Franzè, "A nonlinear sum-of-squares model predictive control approach," *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 55, no. 6, pp. 1466–1471, 2010.

[17] A. Papachristodoulou, M. M. Peet, and S. Lall, "Analysis of polynomial systems with time delays via the sum of squares decomposition," *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 1058–1064, 2009.

[18] Y. Zhang, M. Peet, and K. Gu, "Reducing the complexity of the sum-of-squares test for stability of delayed linear systems," *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 56, no. 1, pp. 229–234, 2011.

[19] R. Kamyar and M. M. Peet, "Polynomial optimization with applications to stability analysis and control-alternatives to sum of squares," *Discrete and Continuous Dynamical Systems - Series B*, vol. 20, no. 8, pp. 2383–2417, 2015.

[20] A. Gahlawat and M. M. Peet, "A convex sum-of-squares approach to analysis, state feedback and output feedback control of parabolic PDEs," *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 62, no. 4, pp. 1636–1651, 2017.

[21] S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: A Lyapunov-based approach," in *Proceedings of the 2004 5th Asian Control Conference*, pp. 157–165, Melbourne, Victoria, Australia, July 2004.

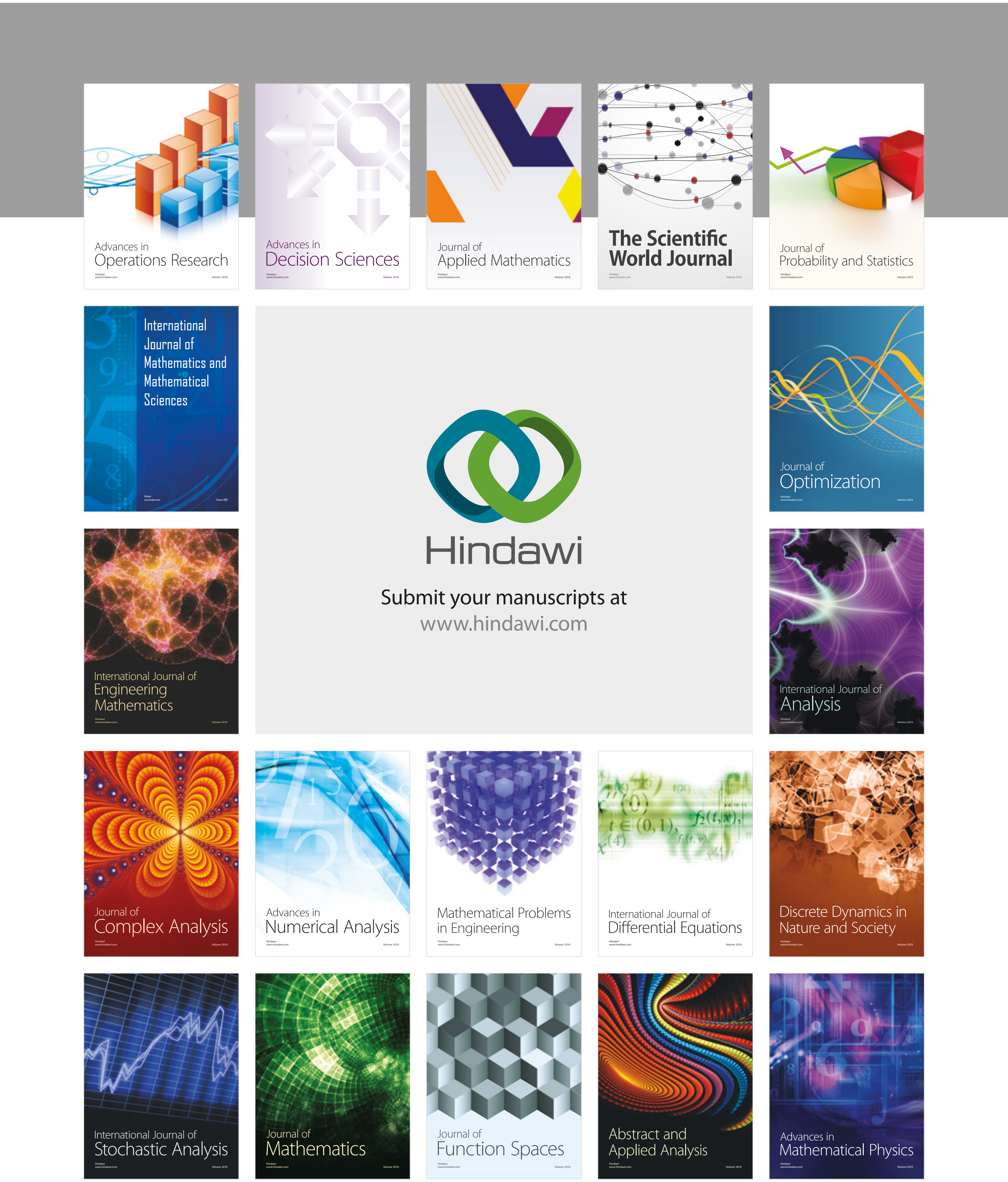
[22] P. Kokotović and M. Arcak, "Constructive nonlinear control: A historical perspective," *Automatica*, vol. 37, no. 5, pp. 637–662, 2001.

[23] A. J. van der Schaft, "L^2-gain analysis of nonlinear system and nonlinear state feedback H_∞ control," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 770–784, 1992.

[24] S. Boyd, L. El Ghaoui, E . Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, Pa, USA, 1994.

[25] P. K. Sinha and A. N. Pechev, "Nonlinear H∞ controllers for electromagnetic suspension systems," *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 49, no. 4, pp. 563–568, 2004.

[26] M. S. Ahmed, "Neural controllers for nonlinear state feedback L2-gain control," *IEE Proceedings Control Theory and Applications*, vol. 147, no. 3, pp. 239–246, 2000.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