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Research Article

Consensus of Delayed Fractional-Order Multiagent Systems Based on State-Derivative Feedback

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A state-derivative feedback (SDF) is added into the designed control protocol in the existing paper to enhance the robustness of a fractional-order multiagent system (FMS) against nonuniform time delays in this paper. By applying the graph theory and the frequency-domain analysis theory, consensus conditions are derived to make the delayed FMS based on state-derivative feedback reach consensus. Compared with the consensus control protocol designed in the existing paper, the proposed SDF control protocol with nonuniform time delays can make the FMS with SDF and nonuniform time delays tolerate longer time delays, which means that the convergence speed of states of the delayed FMS with SDF is accelerated indirectly. Finally, the corresponding results of simulation are given to verify the feasibility of our theoretical results.

1. Introduction

It is well known that the distributed coordination control of multiagent systems has received extensive research attention in various fields including robotics and physics. In the distributed coordination control, it is a critical problem for us to design control laws with the information of states of the agents and their neighbors to insure that multiple agents can agree on certain quantities of interest and this problem is often referred to as the consensus problem [1]. With the development of technologies such as computers, networks, and communications, consensus of multiagent systems has gradually shown enormous potential applications in the field of swarming [2], flocking [3], formation control [4], unmanned air vehicles [5], and distributed sensor networks [6].

With the development of traditional integer-order derivatives and integrals, the concept of fractional calculus has long been proposed. The earliest concept of fractional calculus could be probably traced back to the 17th century [7]. Generally, different from the integer-order derivatives and integrals, the essential characteristic or behavior of an object could be better revealed by the orders of fractional calculus [8]. With the development of fractional-order derivatives and integrals, its applications have been considered by many scholars. The authors in [9] studied the fractional-order derivatives and integrals to establish the stress-strain relationships of viscoelastic materials. The authors in [10] simulated the fractional-order dynamical characteristics of self-similar protein. In [11, 12], the proportional-integral differential (PID) controllers whose dynamics were fractional-order dynamics were proposed and the performance of the fractional-order PID controllers was superior to that of the classical integer-order ones. Moreover, it has been stated in [13] that fractional derivatives were excellent tools for representing the memories and hereditary effects of all manner of materials and processes.

Although fractional-order derivatives and integrals have been studied for a long time, their applications in multiagent systems have just attracted the attention of researchers in recent years. As far as we know, the consensus problem of FMSs was first investigated in [8]. Since then, many research results have been continuously springing up about consensus problems of FMSs [13–17]. The consensus problem of FMSs with a reference state was studied in [13, 14]. The consensus of FMSs about event-triggered control was investigated in [15]. In particular, the authors in [16] studied consensus control protocols for heterogeneous FMSs, which were composed of two different kinds of agents. In [17], the tracking of consensus for the FMSs based on the control method of sliding mode was investigated. Because time delays are ubiquitous in a FMS, some research results begin to take the impact of delays into account. The authors in [18] derived a consensus condition to guarantee the consensus of FMSs whose input delays were identical. In [19, 20], the authors successively studied the FMS with communication delays, whose homogeneous dynamics and heterogeneous dynamics were used to illustrate the agents of a system. In [21], the two types of maximum tolerable delay were obtained to insure reaching consensus for a FMS, whose nonuniform time delays contained up to $n(n-1)$ different values when the FMS consisted of n agents. In [22, 23], a distributed consensus protocol based on the delayed state-derivative feedback (SDF) was designed to improve the robustness against communication delays, which were identical. Hitherto, there are few research works done on the improvement for consensus performance of the FMSs with nonuniform time delays and the consensus of FMSs with SDF and nonuniform time delays.

Hence, we shall study the impact of time delays on the consensus of FMSs based on SDF and how to enhance robustness of the delayed FMSs to nonuniform time delays. First of all, a control protocol based on delayed SDF is designed and the closed-loop dynamics are built by applying graph theory and matrix theory tools. Then, by employing the Laplace transform of the Caputo derivative, the transfer function matrix of the delayed FMS based on SDF is derived and the consensus problems of the delayed FMS based on SDF are transformed into the distribution problems of the eigenvalues of the transfer function matrix in the complex plane, that is, the stability problems of the delayed FMS based on SDF. Finally, the two types of maximum tolerable delay are obtained to guarantee consensus for the delayed FMS based on SDF.

The main contributions of this article are as follows. First, we consider the fractional-order dynamics. The fractional-order dynamics can better reveal the essential characteristic or behavior of an object in a complex environment. Second, we consider the nonuniform time delays which contain symmetric and asymmetric time delays, and obtain the two types of maximum tolerable delay. Third, we can determine the range of fractional-order α to improve the robustness of the FMS with nonuniform time delays by using a graphical method. Finally, we add a SDF into the designed control protocol to enhance the robustness of a FMS against nonuniform time delays.

Compared to the previous research work, the following merits exist in this paper. Firstly, unlike the results in [21], this paper will enhance the consensus performance of the FMS with nonuniform time delays in [21] under the same conditions. Secondly, compared with research on the consensus of integer-order systems based on SDF in [1, 24], this paper mainly studies the consensus of FMSs based on SDF. Finally, although the consensus of delayed FMSs based on SDF in [22, 23] was investigated, all the time delays in [22, 23] were uniform time-delays, which contain the same value. However, this paper considers nonuniform time-delays, which contain up to $n(n-1)$ different values when the FMS consists of n agents.

The main contents of this paper are as follows. Section 2 introduces some basic preliminaries about graph theory, fractional operator, and its Laplace transform. A control protocol based on delayed SDF is designed and the closed-loop dynamics is built in Section 3. The convergence analysis of consensus and the sufficient conditions are obtained in Section 4. In Section 4, we also study the effect of the designed protocol with delayed SDF on the robustness of the FMS against nonuniform time delays. In Section 5, to verify the theoretical results, some examples are simulated. Finally, the conclusions are presented in Section 6.

2. Preliminaries

2.1. Graph Theory. Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ be an interacted graph with the node set $\mathcal{V} = \{v_1, v_2, v_3, \ldots, v_n\}$, the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ik}] \in \mathbb{R}^{n\times n}$. The node indices belong to a finite index set $\mathcal{J} = \{1, 2, \ldots, n\}$. An edge $e_{ik} = (v_k, v_i)$ depicts that node v_i can receive information from node v_k, which means $a_{ik} > 0$, otherwise $a_{ik} = 0$. Besides, we assume $a_{ii} = 0$ for $i \in \mathcal{J}$. Define $N_i = \{k \in \mathcal{J}, k \neq i\}$ as the subscript set of neighbours of node v_i. If a graph describes all the edges $e_{ik} \in \mathcal{E}$ to satisfy $a_{ik} = a_{ki} \geq 0$, then the graph is called an undirected graph; if there exists any $a_{ik} \neq a_{ki}$, then the graph is called a directed graph. A directed path is a sequence of edges in a directed graph with the form $(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots$, where $v_i \in \mathcal{V}$, and if there is a path from every node to every other node, the graph is said to be strongly connected. A spanning tree exists in a directed graph, which means there is a node such that every other node has a directed path to this node. The out-degree of node v_i is defined as $\deg_{\text{out}}(v_i) = \sum_{k=1}^{M} a_{ik}$, and the Laplacian matrix of the interaction graph is $L = \Delta - \mathcal{A} \in \mathbb{R}^{n\times n}$, where $\Delta \triangleq \operatorname{diag}\{\deg_{\text{out}}(v_1), \deg_{\text{out}}(v_2), \ldots, \deg_{\text{out}}(v_i), \ldots, \deg_{\text{out}}(v_n)\}$. For some graphs such as $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_M$, and graph $\mathcal{G}$ composed of the same nodes, the L of graph $\mathcal{G}$ is the sum of the other graphs' Laplacian matrix if the edge set of graph $\mathcal{G}$ is the sum of that of the other graphs $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_M$, that is, $L = \sum_{m=1}^{m} L_m$.

Lemma 1 (see [25]). *If graph $\mathcal{G}$ is an undirected connected graph, then its Laplacian matrix L has a zero eigenvalue and the other eigenvalues are positive real numbers.*

Lemma 2 (see [25]). *If graph $\mathcal{G}$ is a directed graph and has a spanning tree, then its Laplacian matrix L has a zero eigenvalue and the other eigenvalues have a positive real part.*

2.2. Fractional Operator. There are several common fractional operator definitions such as the Caputo operator and Grunwald-Letnikov operator. This paper will use the

Caputo operator to analyze the asymptotic consensus properties because it is widely used in engineering and its physical meaning is easy to understand. The derivative of the Caputo operator of $f(t)$ is defined as follows:

$$ {}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(u-\alpha)} \int_0^t \frac{f^{(u)}(\eta)}{(t-\eta)^{\alpha-u+1}} d\eta, \quad (1)$$

where α is the order of the derivative of Caputo operator, $u-1<\alpha\le u$, $u\in Z^+$, and $\Gamma(\cdot)$ is given by

$$\Gamma(\sigma) = \int_0^{\infty} e^{-t} t^{\sigma-1} dt, \quad (2)$$

where σ is an arbitrary real number.

2.3. Laplace Transform. In order to facilitate the development of the subsequent results, we let $f^{(\alpha)}(t)$ replace ${}_0^C D_t^{\alpha} f(t)$, and let $F(s) = \mathfrak{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt$, then the Laplace transform of the Caputo derivative is obtained:

$$\mathfrak{L}\left\{ \left(f^{(\alpha)}(t) \right\} = \begin{cases} s^{\alpha} F(s) - s^{\alpha-1} f(0^-), & \alpha \in (0,1], \\ s^{\alpha} F(s) - s^{\alpha-1} f(0^-) - s^{\alpha-2} f'(0^-), & \alpha \in (1,2], \end{cases} \quad (3)$$

where $f(0^-) = \lim_{t\to 0^-} f(t)$ and $f'(0^-) = \lim_{t\to 0^-} f'(t)$.

3. Problem Formulation

Assume that a FMS is made up of n agents, each of which is considered as a node in graph $\mathscr{G}$. $\mathscr{G}(\mathscr{V}, \mathscr{E}, \mathscr{A})$ represents the communication topology of the FMS. The dynamic model of agent i is given as follows:

$$x_i^{(\alpha)}(t) = u_i(t), \quad i \in \mathscr{I}, \quad (4)$$

where the ith agent's state is denoted by $x_i(t) \in \mathbb{R}$, the α order Caputo derivative of $x_i(t)$ is denoted by $x_i^{(\alpha)}(t) (\alpha \in (0,1])$, and the control input is denoted by $u_i(t) \in \mathbb{R}$.

Definition 1. The FMS in (4) can achieve consensus when the states of all agents satisfy

$$\lim_{t\to+\infty} (x_i(t) - x_k(t)) = 0, \quad (5)$$

for $\forall i, k \in \mathscr{I}$.

Authors in [21] studied the consensus problems of a FMS with nonuniform time delays, and the distributed control protocol is designed by

$$u_i(t) = \sum_{k\in N_i} a_{ik} [(x_k(t-\tau_{ik}) - x_i(t-\tau_{ik}))], \quad i,k \in \mathscr{I}, \quad (6)$$

where a_{ik} denotes the element of $\mathscr{A}$, N_i denotes the subscript set of neighbours of the agent i, and τ_{ik} is the time delay it takes agent i to receive the information of state of the agent k. If $\tau_{ik} = \tau_{ki}$ holds for all $i, k \in \mathscr{I}$, the time delays are said to be symmetric. Otherwise, the time delays are said to be asymmetric.

In [21], it has been illustrated that the consensus of the FMS in (4) with nonuniform time delays can be achieved by the protocol in (6) when all the $\tau_{ik} < \overline{\tau}$, which is called the maximum tolerable delay. Moreover, the FMS in (4) cannot achieve consensus by the protocol in (6) when all the $\tau_{ik} > \overline{\tau}$. Motivated by the method in [1, 24], we shall use the information $x_k(t-\tau_{ik}) + \gamma x_k^{(\alpha)}(t-\tau_{ik})$ and $x_i(t-\tau_{ik}) + \gamma x_i^{(\alpha)}(t-\tau_{ik})$, respectively, instead of $x_k(t-\tau_{ik})$ and $x_i(t-\tau_{ik})$ to reduce the impact of time delays on consensus, where γ denotes the intensity of the delayed SDF. In addition, we also assume that $\tau_m \in \{\tau_{ik} : i, k \in \mathscr{I}\} (m = 1, 2, \ldots, M)$ denote M different time delays. Finally, the control protocol in (6) can be rewritten to

$$\begin{aligned} u_i(t) = \sum_{k\in N_i} a_{ik} \Big\{ & [x_k(t-\tau_m) - x_i(t-\tau_m)] \\ & + \gamma \left[x_k^{(\alpha)}(t-\tau_m) - x_i^{(\alpha)}(t-\tau_m) \right] \Big\}. \end{aligned} \quad (7)$$

Define $\varphi(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$. By the protocol in (7), the closed-loop dynamics of the FMS in (4) with SDF and nonuniform time delays can be written as

$$\varphi^{(\alpha)}(t) = -\sum_{m=1}^{M} L_m \varphi(t-\tau_m) - \gamma \sum_{m=1}^{M} L_m \varphi^{(\alpha)}(t-\tau_m), \quad (8)$$

where $\varphi^{(\alpha)}(t)$ represents the Caputo derivative of $\varphi(t)$ with α order and L_m represents the Laplacian matrix of a subgraph, which is associated with the time delay τ_m.

4. Consensus Convergence Analysis

4.1. Case 1: Symmetric Time Delays over Undirected Topology

4.1.1. Main Results of Case 1

Theorem 1. *Consider a FMS with SDF and symmetric time-delays over a connected and undirected graph $\mathscr{G}$. By the distributed control protocol in (7), the FMS in (8) with SDF and symmetric time delays can asymptotically achieve consensus if all $\tau_m < \overline{\tau}$, and the FMS in (8) with SDF and symmetric time delays cannot achieve consensus if all $\tau_m > \overline{\tau}$.*

$$\overline{\tau} = \frac{\pi(2-\alpha)}{2\overline{\omega}} + \frac{1}{\overline{\omega}} \arctan \frac{\gamma \overline{\omega}^{\alpha} \sin(\pi\alpha/2)}{1 + \gamma \overline{\omega}^{\alpha} \cos(\pi\alpha/2)}, \quad (9)$$

where

$$\overline{\omega} = \left\{ \frac{\gamma \lambda_n^2 \cos(\pi\alpha/2) + \lambda_n \sqrt{1 + \gamma^2 \cos^2(\pi\alpha/2) \lambda_n^2 - \gamma^2 \lambda_n^2}}{1 - \gamma^2 \lambda_n^2} \right\}^{1/\alpha}, \quad \gamma \in \left[0, \frac{1}{\lambda_n}\right), \quad (10)$$

and λ_n is the maximum eigenvalue of L.

Proof 1. Here, the dynamic performance of the FMS in (8) with SDF and symmetric time delays is studied, so it is not necessary for us to consider the impact of the initial state. Taking the Laplace transform to the FMS in (8) with SDF and symmetric time delays, we have

$$s^{\alpha} I_n \Psi(s) - s^{\alpha-1} I_n \varphi(0^-) = -\left(\sum_{m=1}^{M} L_m e^{-s\tau_m} + \gamma \sum_{m=1}^{M} L_m s^{\alpha} e^{-s\tau_m}\right)\Psi(s), \tag{11}$$

where $\Psi(s)$ is the Laplace transform of $\varphi(t)$, $\varphi(0^-)$ is the initial value of $\varphi(t)$, and $I_n \in \mathbb{R}^{n\times n}$ is the unit matrix.

From (11), we have the characteristic equation of the FMS in (8) with SDF and symmetric time delays:

$$\det\left[s^{\alpha} I_n + \gamma \sum_{m=1}^{M} L_m s^{\alpha} e^{-s\tau_m} + \sum_{m=1}^{M} L_m e^{-s\tau_m}\right] = 0. \tag{12}$$

The roots of (12) are called the eigenvalues of the FMS in (8) with SDF and symmetric time delays. First of all, we assume that the FMS in (8) with SDF and symmetric time delays is stable and can reach consensus when $\tau_m = 0$. Then, it is easy to obtain that as τ_m increases continuously from zero, the eigenvalues of the FMS in (8) with SDF and symmetric time delays in the complex plane will change continuously from the LH (left half-plane) to the RH (right half-plane). Once the trajectories of these eigenvalues reach the RH through the imaginary axis, the FMS in (8) with SDF and symmetric time delays will no longer be stable, which results in the failure of the consensus condition. So, it is essential for us to consider the time delay $\overline{\tau}$ when the nonzero eigenvalues of the FMS in (8) with SDF and symmetric time delays appear on the imaginary axis for the first time, and the time delay $\overline{\tau}$, which is known as maximum tolerable delay, will become the critical point of stability of the FMS in (8) with SDF and symmetric time delays. Now set $s = -j\omega$ and it is the imaginary eigenvalue of the FMS in (8) with SDF and symmetric time delays, $u \in \mathbb{C}^n$ is the corresponding eigenvector, $\|u\| = 1$, and let u^H be the conjugate transpose of u, then the following equation can be obtained:

$$\left[(-j\omega)^{\alpha} I_n + \gamma \sum_{m=1}^{M} L_m (-j\omega)^{\alpha} e^{j\omega\tau_m} + \sum_{m=1}^{M} L_m e^{j\omega\tau_m}\right] u = 0. \tag{13}$$

Since all the roots of (12) appear in the form of conjugate pairs, it is only necessary to study the case $\omega > 0$. Let the left side of (13) be multiplied by u^H, then we have the following series of equations:

$$\begin{aligned}
& u^H\left[(-j\omega)^{\alpha} I_n + \gamma \sum_{m=1}^{M} L_m (-j\omega)^{\alpha} e^{j\omega\tau_m} + \sum_{m=1}^{M} L_m e^{j\omega\tau_m}\right] u = 0,\\
& u^H u(-j\omega)^{\alpha} + u^H\left[\gamma \sum_{m=1}^{M} L_m (-j\omega)^{\alpha} e^{j\omega\tau_m} + \sum_{m=1}^{M} L_m e^{j\omega\tau_m}\right] u = 0,\\
& \sum_{m=1}^{M} u^H L_m u e^{j\omega\tau_m}\left[\gamma(-j\omega)^{\alpha} + 1\right] = -u^H u(-j\omega)^{\alpha},\\
& \sum_{m=1}^{M} \frac{u^H L_m u}{u^H u} e^{j\omega\tau_m} = \frac{-(-j\omega)^{\alpha}}{\left[\gamma(-j\omega)^{\alpha} + 1\right]} = \frac{-\omega^{\alpha}(-j)^{\alpha}}{\left[1 + \gamma(-j)^{\alpha}\omega^{\alpha}\right]}\\
& = \frac{-\omega^{\alpha}\{\cos(-\pi/2) + j\sin(-\pi/2)\}^{\alpha}}{1 + e^{-j(\pi\alpha/2)}\gamma\omega^{\alpha}}\\
& = \frac{-\omega^{\alpha} e^{j(-\pi\alpha/2)}}{1 + \gamma\omega^{\alpha}\cos(\pi\alpha/2) - j\gamma\omega^{\alpha}\sin(\pi\alpha/2)}\\
& = \frac{\omega^{\alpha} e^{j(\pi(2-\alpha)/2)}}{\sqrt{(1 + \gamma\omega^{\alpha}\cos(\pi\alpha/2))^2 + (\gamma\omega^{\alpha}\sin(\pi\alpha/2))^2}\, e^{-j\arctan(\gamma\omega^{\alpha}\sin(\pi\alpha/2)/1+\gamma\omega^{\alpha}\cos(\pi\alpha/2))}}.
\end{aligned} \tag{14}$$

Then, we define

$$F(\omega) \triangleq \sum_{m=1}^{M} a_m e^{j\omega\tau_m} = \frac{\omega^{\alpha} e^{j(\pi(2-\alpha)/2)}}{\sqrt{(1 + \gamma\omega^{\alpha}\cos(\pi\alpha/2))^2 + (\gamma\omega^{\alpha}\sin(\pi\alpha/2))^2}\, e^{-j\arctan(\gamma\omega^{\alpha}\sin(\pi\alpha/2)/1+\gamma\omega^{\alpha}\cos(\pi\alpha/2))}}, \tag{15}$$

where $a_m = u^H L_m u / u^H u$.

According to (15) and Lemma 1, we can get

$$|F(\omega)| = \left|\sum_{m=1}^{M} a_m e^{j\omega\tau_m}\right| \le \sum_{m=1}^{M} a_m = \frac{u^H L u}{u^H u} \le \lambda_n. \qquad (16)$$

Since $|F(\omega)| = \omega^\alpha / \sqrt{(1+\gamma\omega^\alpha \cos(\pi\alpha/2))^2 + (\gamma\omega^\alpha \sin(\pi\alpha/2))^2}$, we have $\omega^\alpha / \sqrt{(1+\gamma\omega^\alpha \cos(\pi\alpha/2))^2 + (\gamma\omega^\alpha \sin(\pi\alpha/2))^2} \le \lambda_n$, which leads to $\omega \le \overline{\omega} = \{\gamma\lambda_n^2 \cos(\pi\alpha/2) + \lambda_n \sqrt{1+\gamma^2\cos^2(\pi\alpha/2)\lambda_n^2 - \gamma^2\lambda_n^2}/1-\gamma^2\lambda_n^2\}^{1/\alpha}$.

Next, we need to discuss the principal value of the argument of $F(\omega)$. Based on (15), we know that

$$\arg[F(\omega)] = \frac{\pi(2-\alpha)}{2} + \arctan\frac{\gamma\omega^\alpha \sin(\pi\alpha/2)}{1+\gamma\omega^\alpha\cos(\pi\alpha/2)} \triangleq \theta(\omega), \qquad (17)$$

where $\theta(\omega) \in [(\pi/2) + \arctan\gamma\omega, \pi]$.

If we suppose that $\delta = \sin(\pi\alpha/2)$ and $\sigma = \cos(\pi\alpha/2)$, then it is easy to arrive at $\delta^2 + \sigma^2 = 1$ and we get

$$\begin{aligned}\tau(\omega) \triangleq \frac{\theta(\omega)}{\omega} &= \underbrace{\frac{\pi(2-\alpha)}{2\omega}}_{\Gamma_1(\omega)} + \underbrace{\frac{1}{\omega}\arctan\frac{\gamma\omega^\alpha\sin(\pi\alpha/2)}{1+\gamma\omega^\alpha\cos(\pi\alpha/2)}}_{\Gamma_2(\omega)} \\ &= \underbrace{\frac{\pi(2-\alpha)}{2\omega}}_{\Gamma_1(\omega)} + \underbrace{\frac{1}{\omega}\arctan\frac{\gamma\omega^\alpha\delta}{1+\gamma\omega^\alpha\sigma}}_{\Gamma_2(\omega)}.\end{aligned} \qquad (18)$$

According to (18), we can calculate the first derivative of $\tau(\omega)$ about ω:

$$\Gamma(\omega) \triangleq \frac{d\tau(\omega)}{d\omega} = \Gamma_1'(\omega) + \Gamma_2'(\omega), \qquad (19)$$

where

$$\Gamma_1'(\omega) = -\frac{\pi(2-\alpha)}{2\omega^2} < 0, \qquad (20)$$

and

$$\begin{aligned}\Gamma_2'(\omega) &= \frac{\left[\alpha\gamma\delta\omega^{\alpha-1}(1+\gamma\sigma\omega^\alpha) - \alpha\gamma^2\delta\sigma\omega^{2\alpha-1}\right]}{\left\{\omega\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]\right\}} - \frac{\left\{\arctan\left[\gamma\delta\omega^\alpha/(1+\gamma\sigma\omega^\alpha)\right]\right\}}{\omega^2} \\ &= \frac{\left[\alpha\gamma\delta\omega^{\alpha}(1+\gamma\sigma\omega^\alpha) - \alpha\gamma^2\delta\sigma\omega^{2\alpha}\right]}{\left\{\omega^2\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]\right\}} - \frac{\left\{\arctan\left[\gamma\delta\omega^\alpha/(1+\gamma\sigma\omega^\alpha)\right]\right\}}{\omega^2} \\ &= \frac{\alpha\gamma\delta\omega^\alpha}{\left\{\omega^2\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]\right\}} - \frac{\left\{\arctan\left[\gamma\delta\omega^\alpha/(1+\gamma\omega^\alpha\sigma)\right]\right\}}{\omega^2} \\ &= \frac{\left\{\alpha\gamma\delta\omega^\alpha/\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right] - \arctan\left[\gamma\delta\omega^\alpha/(1+\gamma\sigma\omega^\alpha)\right]\right\}}{\omega^2}.\end{aligned} \qquad (21)$$

If we assume that there is a function $\mathcal{Z}(\omega)$ established by

$$\mathcal{Z}(\omega) = \arctan\left[\frac{\gamma\delta\omega^\alpha}{(1+\gamma\sigma\omega^\alpha)}\right] - \frac{\alpha\gamma\delta\omega^\alpha}{\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]}, \qquad (22)$$

then the first derivative of $\mathcal{Z}(\omega)$ is as follows:

$$\begin{aligned}\mathcal{Z}'(\omega) &= \frac{\left[\alpha\gamma\delta\omega^{\alpha-1}(1+\gamma\sigma\omega^\alpha) - \alpha\gamma^2\delta\sigma\omega^{2\alpha-1}\right]}{\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]} - \frac{\left\{\alpha^2\gamma\delta\omega^{\alpha-1}\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right] - \left[2(1+\gamma\sigma\omega^\alpha)\alpha\gamma\sigma\omega^{\alpha-1} + 2\alpha\gamma^2\delta^2\omega^{2\alpha-1}\right]\alpha\gamma\delta\omega^\alpha\right\}}{\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]^2} \\ &= \frac{\left\{\alpha\gamma\delta\omega^{\alpha-1}\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right] - \alpha^2\gamma\delta\omega^{\alpha-1}\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right] + \left[2(1+\gamma\sigma\omega^\alpha)\alpha\gamma\sigma\omega^{\alpha-1} + 2\alpha\gamma^2\delta^2\omega^{2\alpha-1}\right]\alpha\gamma\delta\omega^\alpha\right\}}{\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]^2} \\ &= \frac{\left\{\alpha(1-\alpha)\gamma\delta\omega^{\alpha-1}\left[1 + 2\gamma\sigma\omega^\alpha + \gamma^2\omega^{2\alpha}\right] + \left[2\alpha\gamma\sigma\omega^{\alpha-1} + 2\alpha\gamma^2\omega^{2\alpha-1}\right]\alpha\gamma\delta\omega^\alpha\right\}}{\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]^2} \\ &= \frac{\alpha\gamma\delta\omega^{\alpha-1}\left[(1-\alpha) + 2(1-\alpha)\gamma\sigma\omega^\alpha + (1-\alpha)\gamma^2\omega^{2\alpha} + 2\alpha\gamma\sigma\omega^\alpha + 2\alpha\gamma^2\omega^{2\alpha}\right]}{\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]^2} \\ &= \frac{\alpha\gamma\delta\omega^{\alpha-1}\left[(1-\alpha) + 2\gamma\sigma\omega^\alpha + (1+\alpha)\gamma^2\omega^{2\alpha}\right]}{\left[(1+\gamma\sigma\omega^\alpha)^2 + \gamma^2\delta^2\omega^{2\alpha}\right]^2} > 0.\end{aligned} \qquad (23)$$

Since $\mathcal{Z}'(\omega) > 0$, $\mathcal{Z}(\omega)$ is an increasing function. It is very convenient to obtain that $\mathcal{Z}(\omega) > \mathcal{Z}(0) = 0$ when $\omega > 0$. Since $\mathcal{Z}(\omega) > 0$, $\Gamma_2'(\omega) < 0$, which means $\Gamma(\omega) = \Gamma_1'(\omega) + \Gamma_2'(\omega) < 0$. Since $\Gamma(\omega) < 0$, $\tau(\omega)$ is a decreasing function of ω, and when $\omega \le \overline{\omega}$, there is

$$\overline{\tau} = \tau(\overline{\omega}) \le \tau(\omega). \qquad (24)$$

On the other hand, when all $\tau_m < \overline{\tau}$, the following inequality can be obtained:

$$\tau(\omega) = \frac{\theta(\omega)}{\omega} = \frac{\arg\left(\sum_{m=1}^{M} a_m e^{j\omega\tau_m}\right)}{\omega} \le \frac{\max\{\omega\tau_m\}}{\omega} < \frac{\omega\overline{\tau}}{\omega} = \overline{\tau}. \quad (25)$$

The contradiction between the inequality in (25) and the inequality in (24) is obvious. Accordingly, when all τ_m are less than $\overline{\tau}$, we can avoid the eigenvalues of the FMS in (8) with SDF and symmetric time delays crossing the imaginary axis to reach the unstable RH, and the FMS in (8) with SDF and symmetric time delays can reach consensus; when all τ_m are equal to $\overline{\tau}$, $s = -j\overline{\omega}$ is an imaginary eigenvalue of the FMS in (8) with SDF and symmetric time delays, whose corresponding eigenvector $u(\overline{\omega})$ makes $|\sum_{m=1}^{M} a_m| = \lambda_n$ hold; when all τ_m are more than $\overline{\tau}$, there must exist at least one eigenvalue of the FMS in (8) with SDF and symmetric time delays in the RH, and the states of the FMS in (8) with SDF and symmetric time delays will no longer converge and the FMS in (8) with SDF and symmetric time delays cannot reach consensus.

Remark 1. From Theorem 1, we can get $\overline{\tau} > 0$ and $\overline{\omega} > 0$. $\gamma \in [0, 1/\lambda_n)$ is implied in (9). So it is necessary for achieving consensus of the FMS in (8) with symmetric time delays that $\gamma \in [0, 1/\lambda_n)$.

Corollary 1. *Consider a FMS with SDF and symmetric time delays over a connected and undirected graph $\mathcal{G}$. When $\alpha = 1$, by the distributed control protocol in (7), the FMS in (8) with SDF and symmetric time delays can asymptotically achieve consensus if all $\tau_m < \overline{\tau}$, and the FMS in (8) with SDF and symmetric time delays cannot achieve consensus if all $\tau_m > \overline{\tau}$.*

$$\overline{\tau} = \frac{\pi}{2\overline{\omega}} + \frac{1}{\overline{\omega}} \arctan \gamma\overline{\omega}, \quad (26)$$

where $\overline{\omega} = \lambda_n \sqrt{1 - \gamma^2\lambda_n^2 / 1 - \gamma^2\lambda_n^2}$, $\gamma \in [0, 1/\lambda_n)$, *and* λ_n *is the maximum eigenvalue of L.*

4.1.2. Robustness Analysis for Case 1. According to Theorem 1, for the given FMS in (4), if applying the control protocol in (6), that is, the control protocol in (7) with $\gamma = 0$, we obtain

$$\overline{\tau}|_{\gamma=0} = \frac{\pi(2-\alpha)}{2\lambda_n^{1/\alpha}} \triangleq \tau_{up1}. \quad (27)$$

If applying the control protocol in (7), we obtain

$$\overline{\tau}|_{0<\gamma<1/\lambda_n} = \frac{\pi(2-\alpha)}{2\overline{\omega}} + \frac{1}{\overline{\omega}} \arctan \frac{\gamma\overline{\omega}^\alpha \sin(\pi\alpha/2)}{1 + \gamma\overline{\omega}^\alpha \cos(\pi\alpha/2)} \triangleq \tau_{up2}, \quad (28)$$

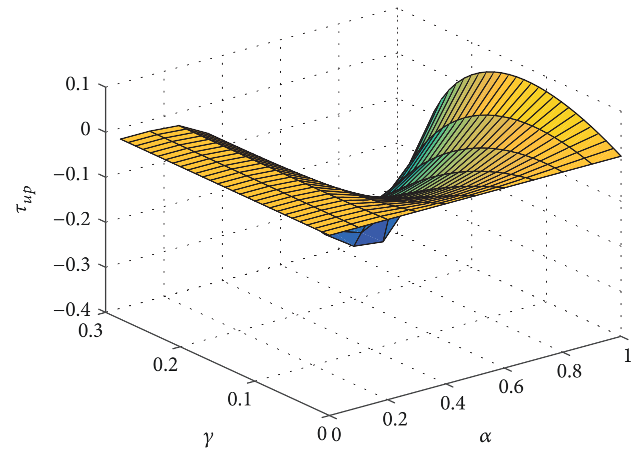

FIGURE 1: The three-dimensional diagram of τ_{up} with respect to parameters α and γ in case 1.

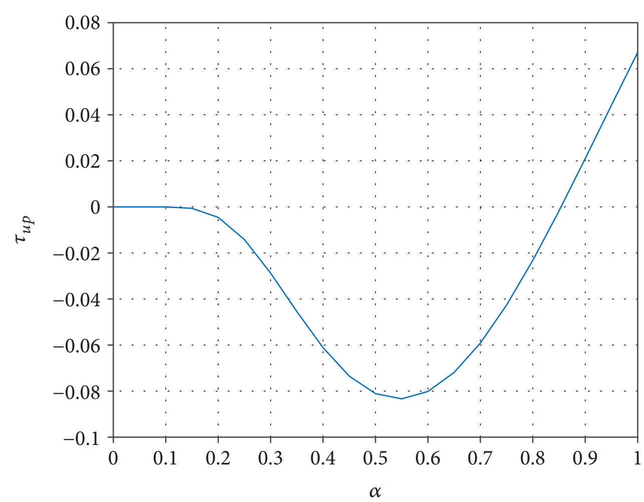

FIGURE 2: The relationship between τ_{up} and α when $\gamma = 0.1$ in case 1.

where $\overline{\omega} = \{\gamma\lambda_n^2 \cos(\pi\alpha/2) + \lambda_n\sqrt{1 + \gamma^2\cos^2(\pi\alpha/2)\lambda_n^2 - \gamma^2\lambda_n^2} / 1 - \gamma^2\lambda_n^2\}^{1/\alpha}$.

In order to show the effect of the protocol in (7) on the robustness against symmetric time delays, we are supposed to further determine the range of α to insure $\tau_{up2} > \tau_{up1}$, which means

$$\tau_{up} \triangleq \tau_{up2} - \tau_{up1} > 0. \quad (29)$$

Obviously, the inequality in (29) contains multiple parameters. However, for a fix undirected interconnection topology, we can determine the range of α and find a proper value of γ to improve the robustness of the FMS with symmetric time delays. Since it is very difficult for us to solve the inequality in (29) by the analytic method, we shall analyze it by the graphical method and let $\lambda_n = 3.5229$. Figure 1 is the three-dimensional diagram of τ_{up} with respect to parameters α and γ, and it is easy for us to find proper parameters α, γ to ensure $\tau_{up} > 0$ by the graphical method. According to Figure 1, we can find that $\alpha \in (\alpha^*, 1]$ when γ changes from 0 to $1/\lambda_n$ and $\tau_{up} > 0$, and α^* which is decided by the inequality in (29) is easy to obtain in Figure 1 when the value of γ is determined.

In particular, if we assume $\gamma = 0.1$, then the relationship between τ_{up} and α can be shown in Figure 2. According to Figure 2, it is obvious that $\alpha \in (0.854, 1](\alpha^* \triangleq 0.854)$ when $\tau_{up} > 0$ and $\gamma = 0.1$.

4.2. Case 2: Asymmetric Time Delays over Directed Topology

4.2.1. Main Results of Case 2

Theorem 2. *Consider a FMS with SDF and asymmetric time delays over a directed interconnection graph $\mathcal{G}$ that has a spanning tree. By the distributed control protocol in (7), the FMS in (8) with SDF and asymmetric time delays can asymptotically achieve consensus if all $\tau_m < \overline{\tau}$, and the FMS in (8) with SDF and asymmetric time delays cannot achieve consensus if all $\tau_m > \overline{\tau}$.*

$$\overline{\tau} = \min_{|\lambda_i|\neq 0}\left\{\frac{\pi(2-\alpha)/2 + \arctan \gamma\overline{\omega}_i^{\alpha} \sin(\pi\alpha/2)/1 + \gamma\overline{\omega}_i^{\alpha}\cos(\pi\alpha/2) - \arg(\lambda_i)}{\overline{\omega}_i}\right\}, \tag{30}$$

where $\overline{\omega}_i = \{\gamma|\overline{\lambda}_i|^2 \cos(\pi\alpha/2) + |\overline{\lambda}_i|\sqrt{1+\gamma^2\cos^2(\pi\alpha/2)|\overline{\lambda}_i|^2 - \gamma^2|\overline{\lambda}_i|^2} / 1 - \gamma^2|\overline{\lambda}_i|^2\}^{1/\alpha}$, $\gamma \in [0, \gamma^*)(\gamma^* = \min_{|\lambda_i|\neq 0}\{1/|\lambda_i|\})$, $\overline{\lambda}_i$ is the λ_i which makes $\overline{\tau}$ minimized, and λ_i is the ith eigenvalue of L.

Proof 2. Let one apply the above frequency-domain proof method, which has been used in proving Theorem 1. Suppose that $s = -j\omega \neq 0$ is the eigenvalue of the FMS in (8) with SDF and asymmetric time delays on the imaginary axis, $u \in \mathbb{C}^n$ is the corresponding eigenvector, and $\|u\| = 1$. According to Lemma 2, one can get

$$\begin{aligned} B_a &\triangleq \sum_{m=1}^{M} a_m e^{j\omega\tau_m} = \frac{-(-j\omega)^{\alpha}}{[\gamma(-j\omega)^{\alpha}+1]} \\ &= \frac{\omega^{\alpha}}{\sqrt{(1+\gamma\omega^{\alpha}\cos(\pi\alpha/2))^2 + (\gamma\omega^{\alpha}\sin(\pi\alpha/2))^2}} e^{j[\pi(2-\alpha)/2 + \arctan(\gamma\omega^{\alpha}\sin(\pi\alpha/2)/1+\gamma\omega^{\alpha}\cos(\pi\alpha/2))]}. \end{aligned} \tag{31}$$

Taking the modulus of both sides of (31) and regarding ω as the function of $|B_a|$, we can get

$$\omega(|B_a|) = \left\{\frac{\gamma|B_a|^2\cos(\pi\alpha/2) + |B_a|\sqrt{1+\gamma^2\cos^2(\pi\alpha/2)|B_a|^2 - \gamma^2|B_a|^2}}{1-\gamma^2|B_a|^2}\right\}^{1/\alpha}, \tag{32}$$

where $\omega(|B_a|)$ is an increasing function of $|B_a|$.

Calculating the principal value of the argument of (31) on both sides separately, and we have

$$\arg(B_a) = \frac{\pi(2-\alpha)}{2} + \arctan\frac{\gamma\omega^{\alpha}\sin(\pi\alpha/2)}{1+\gamma\omega^{\alpha}\cos(\pi\alpha/2)}. \tag{33}$$

According to the definition of B_a in (31), we have

$$\arg(B_a) \leq \arg\left(\sum_{m=1}^{M} a_m\right) + \max(\omega\tau_m), \tag{34}$$

and it yields that

$$\max(\omega\tau_m) \geq \underbrace{\frac{\pi(2-\alpha)}{2} + \arctan\frac{\gamma\omega^{\alpha}\sin(\pi\alpha/2)}{1+\gamma\omega^{\alpha}\cos(\pi\alpha/2)} - \arg\left(\sum_{m=1}^{M} a_m\right)}_{\varepsilon}. \tag{35}$$

Since $\sum_{m=1}^{M} a_m = u^H L u / u^H u$, the possible values of $\sum_{m=1}^{M} a_m$ must be nonzero eigenvalues of L, that is, $\sum_{m=1}^{M} a_m = \lambda_i(\lambda_i \neq 0)$. So when $\gamma < \min_{|\lambda_i|\neq 0}\{1/|\lambda_i|\}$ and $|B_a| \leq |\lambda_i|$, we have $\omega(|B_a|) \leq \omega(|\overline{\lambda}_i|) = \overline{\omega}_i = \{\gamma|\overline{\lambda}_i|^2\cos(\pi\alpha/2) + |\overline{\lambda}_i| \sqrt{1+\gamma^2\cos^2(\pi\alpha/2)|\overline{\lambda}_i|^2 - \gamma^2|\overline{\lambda}_i|^2} / 1 - \gamma^2|\overline{\lambda}_i|^2\}^{1/\alpha}$. If we let all $\tau_m < \overline{\tau}$, we can obtain

$$\begin{aligned} \max(\omega\tau_m) &< \overline{\omega}_i\overline{\tau} = \min_{|\lambda_i|\neq 0}\left\{\frac{[\pi(2-\alpha)/2 + \arctan(\gamma\overline{\omega}_i^{\alpha}\sin(\pi\alpha/2)/1+\gamma\overline{\omega}_i^{\alpha}\cos(\pi\alpha/2)) - \arg(\lambda_i)]}{\overline{\omega}_i}\right\}\overline{\omega}_i \\ &\leq \varepsilon = \frac{\pi(2-\alpha)}{2} + \arctan\frac{\gamma\omega^{\alpha}\sin(\pi\alpha/2)}{1+\gamma\omega^{\alpha}\cos(\pi\alpha/2)} - \arg\left(\sum_{m=1}^{M} a_m\right). \end{aligned} \tag{36}$$

The contradiction between the inequality in (36) and the inequality in (35) is obvious. Accordingly, when all $\tau_m < \overline{\tau}$, the eigenvalues of the FMS in (8) with SDF and asymmetric time delays cannot reach or cross the imaginary axis, then the FMS in (8) with SDF and asymmetric time delays will remain stable and the FMS in (8) with SDF and asymmetric time delays can reach consensus. On the other hand, when all $\tau_m > \overline{\tau}$, there must exist at least one eigenvalue of the FMS in (8) with SDF and asymmetric time delays in the RH, then the states of the FMS in (8) with SDF and asymmetric time delays will no longer converge and the FMS in (8) with SDF and asymmetric time delays cannot reach consensus.

Remark 2. From Theorem 2, let one get $\overline{\tau} > 0$ and $\overline{\omega}_i > 0$. $\gamma \in [0, \gamma^*)$ $(\gamma^* = \min_{|\lambda_i| \neq 0}\{1/|\lambda_i|\})$ is implied in (30). So it is necessary for achieving consensus of the FMS in (8) with SDF and asymmetric time delays that $\gamma \in [0, \gamma^*)(\gamma^* = \min_{|\lambda_i| \neq 0}\{1/|\lambda_i|\})$.

Corollary 2. *Consider a FMS with SDF and asymmetric time delays over a directed interconnection graph $\mathcal{G}$ with a spanning tree. When $\alpha = 1$, by the distributed control protocol in (7), the FMS in (8) with SDF and asymmetric time delays can asymptotically achieve consensus if all $\tau_m < \overline{\tau}$, and the FMS in (8) with SDF and asymmetric time delays cannot achieve consensus if all $\tau_m > \overline{\tau}$.*

$$\overline{\tau} = \min_{|\lambda_i| \neq 0}\left\{\frac{(\pi/2) + \arctan \gamma\overline{\omega}_i - \arg(\lambda_i)}{\overline{\omega}_i}\right\}, \tag{37}$$

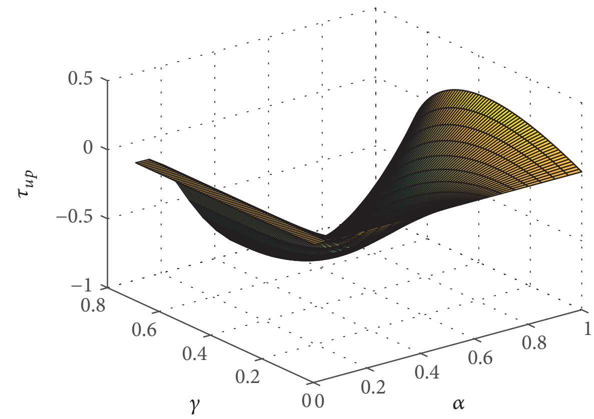

Figure 3: The three-dimensional diagram of τ_{up} with respect to parameters α and γ in case 2.

where $\overline{\omega}_i = |\overline{\lambda}_i|\sqrt{1 - \gamma^2|\overline{\lambda}_i|^2}/1 - \gamma^2|\overline{\lambda}_i|^2$, $\gamma \in [0, \gamma^*)(\gamma^* = \min_{|\lambda_i| \neq 0}\{1/|\lambda_i|\})$, *$\overline{\lambda}_i$ is the λ_i which makes $\overline{\tau}$ minimized, and λ_i is the ith eigenvalue of L.*

4.2.2. Robustness Analysis for Case 2. According to Theorem 2, for the given FMS in (4), if applying the control protocol in (6), that is, the control protocol in (7) with $\gamma = 0$, we obtain

$$\overline{\tau}\big|_{\gamma=0} = \min_{|\lambda_i| \neq 0}\left\{\frac{\pi(2-\alpha)/2 - \arg(\lambda_i)}{|\lambda_i|^{1/\alpha}}\right\} \triangleq \tau_{up1}. \tag{38}$$

When we apply the control protocol in (7), we obtain

$$\overline{\tau}\big|_{0<\gamma<\gamma^*} = \min_{|\lambda_i| \neq 0}\left\{\frac{\pi(2-\alpha)/2 + \arctan\left(\gamma\overline{\omega}_i^{\alpha}\sin(\pi\alpha/2)/1 + \gamma\overline{\omega}_i^{\alpha}\cos(\pi\alpha/2)\right) - \arg(\lambda_i)}{\overline{\omega}_i}\right\} \triangleq \tau_{up2}, \tag{39}$$

where $\overline{\omega}_i = \{\gamma|\overline{\lambda}_i|^2\cos(\pi\alpha/2) + |\overline{\lambda}_i|\sqrt{1 + \gamma^2\cos^2(\pi\alpha/2)|\overline{\lambda}_i|^2 - \gamma^2|\overline{\lambda}_i|^2}/1 - \gamma^2|\overline{\lambda}_i|^2\}^{1/\alpha}$, and $\gamma^* = \min_{|\lambda_i| \neq 0}\{1/|\lambda_i|\}$.

In order to show the effect of the protocol in (7) on the robustness against asymmetric time delays, we are supposed to further determine the range of α to insure $\tau_{up2} > \tau_{up1}$, which means

$$\tau_{up} \triangleq \tau_{up2} - \tau_{up1} > 0. \tag{40}$$

Obviously, the inequality in (40) contains multiple parameters. However, for a fix directed interconnection graph $\mathcal{G}$ that has a spanning tree, we can determine the range of α and find a proper value of γ to improve the robustness of the FMS with asymmetric time delays. Since it is very difficult for us to solve the inequality in (40) by the analytic method, we shall analyze it by the graphical method and let $\gamma^* = 0.6967$. Figure 3 is the three-dimensional diagram of τ_{up} with respect to parameters α and γ, it is easy for us to find proper parameters α, γ to ensure $\tau_{up} > 0$ by the graphical method. According to Figure 3, we can find that $\alpha \in (\alpha^*, 1]$ when γ changes from 0 to γ^* and $\tau_{up} > 0$, and α^* which is decided by the inequality in (40) is easy to obtain in Figure 3 when the value of γ is determined.

In particular, if we assume $\gamma = 0.1$, then the relationship between τ_{up} and α can be shown in Figure 4. According to Figure 4, it is obvious that $\alpha \in (0.7503, 1](\alpha^* \triangleq 0.7503)$ when $\tau_{up} > 0$ and $\gamma = 0.1$.

5. Simulation Results

It is necessary for us to compare the conclusions of this paper with those of [21], so the simulation conditions of this paper must be consistent with those of [21].

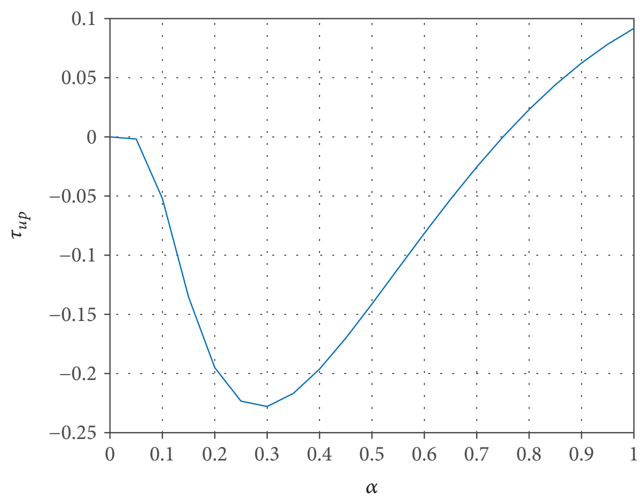

FIGURE 4: The relationship between τ_{up} and α when $\gamma = 0.1$ in case 2.

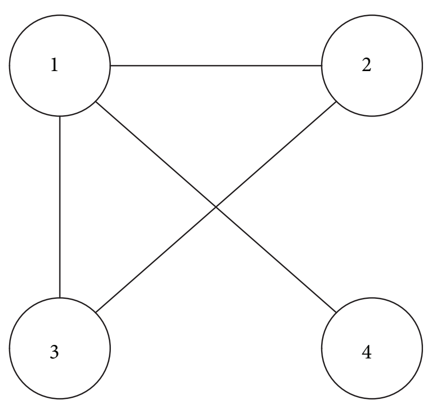

FIGURE 5: The communication topology in example 1.

5.1. Example 1: Simulations for Case 1. Consider a FMS described by (4) with four agents, whose communication topology is given in Figure 5. The L which is associated with this topology graph is

$$L = \begin{bmatrix} 2.6 & -0.7 & -0.9 & -1 \\ -0.7 & 1.5 & 0 & -0.8 \\ -0.9 & 0 & 0.9 & 0 \\ -1 & -0.8 & 0 & 1.8 \end{bmatrix}, \quad (41)$$

and its four eigenvalues are 0, 0.8771, 2.4000, and 3.5229, respectively. From the four eigenvalues of L, we can get $\lambda_n = 3.5229$, which leads to $\gamma < 1/\lambda_n = 0.2838$.

According robustness analysis for case 1, $\alpha \in (0.854, 1]$ ($\alpha^* \triangleq 0.854$) when $\tau_{up} = \tau_{up2} - \tau_{up1} > 0$ and $\gamma = 0.1$. According to Theorem 1 and Remark 1, it is easy to obtain that the maximum tolerable delay of the FMS in (4) by the protocol in (6), that is, the protocol in (7) with $\gamma = 0$ is $\tau_{up1} = \overline{\tau} = 0.4264\,\text{s}$ and the maximum tolerable delay of the FMS in (4) by the protocol in (7) with $\gamma = 0.1$ is $\tau_{up2} = \overline{\tau} = 0.4474\,\text{s}$ when $\alpha = 0.9$. Now let us suppose that the initial states of

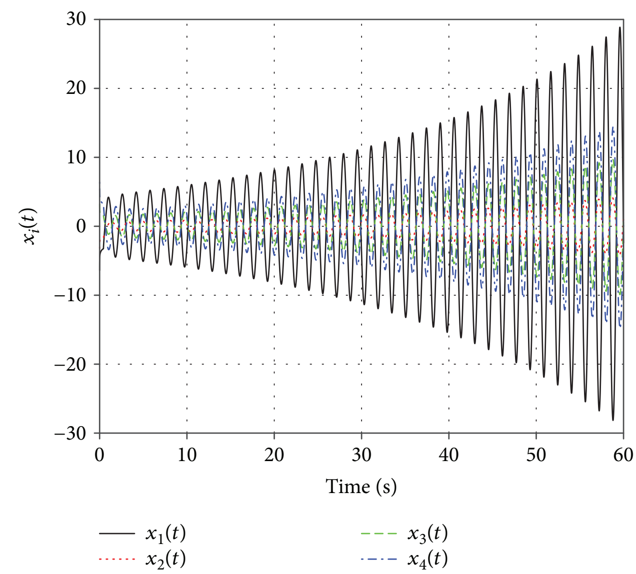

FIGURE 6: The trajectories of $x_i(t)$ applying the protocol in (6) under symmetric time delays when all $\tau_m > \tau_{up1}$.

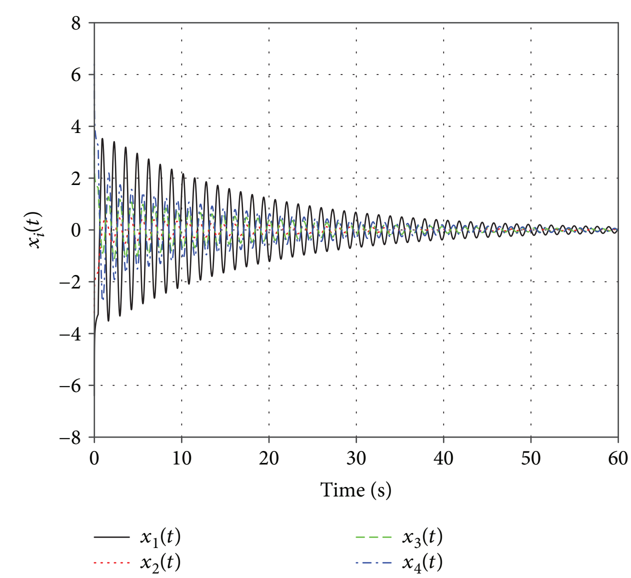

FIGURE 7: The trajectories of $x_i(t)$ applying the protocol in (7) under symmetric time delays when all $\tau_m < \tau_{up2}$.

the FMS (4) are taken as $x_1(t=0) = -6.4$, $x_2(t=0) = -3.2$, $x_3(t=0) = 3.2$, and $x_4(t=0) = 6.4$. Here, a set of symmetric time delays is used to simulate $\tau_{12} = \tau_{21} = 0.430\,\text{s}$, $\tau_{13} = \tau_{31} = 0.435\,\text{s}$, $\tau_{14} = \tau_{41} = 0.440\,\text{s}$, and $\tau_{24} = \tau_{42} = 0.445\,\text{s}$.

It is obvious that all symmetric time delays τ_m are greater than τ_{up1} and less than τ_{up2}.

Figures 6 and 7 show the trajectories of $x_i(t)$ with the symmetric time delays by applying the different control protocols when $\tau_{up1} < \text{all}\,\tau_m < \tau_{up2}$. From these simulation results, it is obvious that the given FMS in (4) by the control protocol in (6) diverges and cannot reach consensus, whereas the FMS in (4) applying the SDF control protocol in (7) converges to the same states and can reach consensus. Hence,

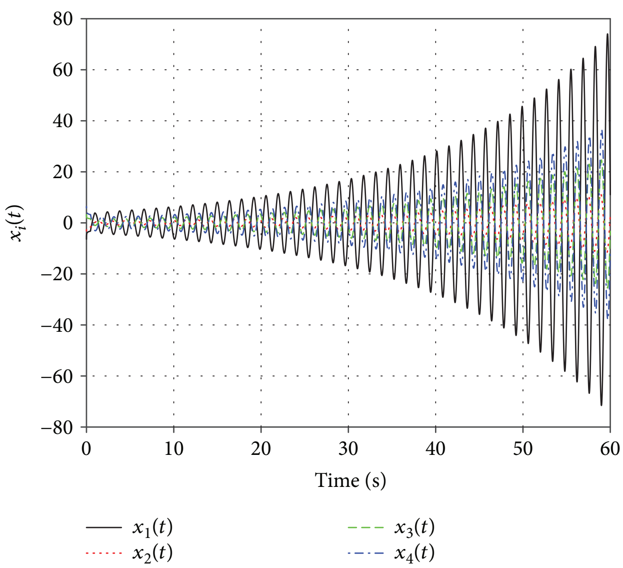

FIGURE 8: The trajectories of $x_i(t)$ applying the protocol in (7) under symmetric time delays when all $\tau_m > \tau_{up2}$.

the introduced SDF control protocol can enhance the robustness of the FMS in (4) to symmetric time delays.

On the other hand, under the same conditions, we suppose that $\tau_{12} = \tau_{21} = 0.45$ s, $\tau_{13} = \tau_{31} = 0.46$ s, $\tau_{14} = \tau_{41} = 0.47$ s, and $\tau_{24} = \tau_{42} = 0.48$ s. Figure 8 shows the trajectories of $x_i(t)$ applying the protocol in (7) under symmetric time delays when all $\tau_m > \tau_{up2}$. It is clear that the FMS in (4) cannot reach consensus.

5.2. Example 2: Simulations for Case 2. Consider a FMS described by (4) with four agents, whose communication topology with a spanning tree is given in Figure 9. The L which is associated with this topology graph is

$$L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -0.7 & 0.7 & 0 & 0 \\ -0.9 & 0 & 0.9 & 0 \\ 0 & -0.8 & 0 & 0.8 \end{bmatrix}, \qquad (42)$$

and its four eigenvalues are 0, 0.9000, $1.2500 + 0.7053i$, and $1.2500 - 0.7053i$, respectively. From the four eigenvalues of L, we can get $\gamma^* = \min_{|\lambda_i| \neq 0}\{1/|\lambda_i|\} = 0.6967$, which leads to $\gamma < \gamma^* = 0.6967$.

According robustness analysis for case 2, $\alpha \in (0.7503, 1]$ $(\alpha^* \triangleq 0.7503)$ when $\tau_{up} = \tau_{up2} - \tau_{up1} > 0$ and $\gamma = 0.1$. According to Theorem 2 and Remark 2, it is also easy to obtain the maximum tolerable delay of the FMS in (4) by the protocol in (6), that is, the protocol in (7) with $\gamma = 0$ is $\tau_{up1} = \overline{\tau} = 0.8126$ s and the maximum delay of the FMS in (4) by the protocol in (7) with $\gamma = 0.1$ is $\tau_{up2} = \overline{\tau} = 0.8751$ s when $\alpha = 0.9$. Now let us suppose that the initial states of the FMS in (4) are taken as $x_1(t=0) = -6.4$, $x_2(t=0) = -3.2$, $x_3(t=0) = 3.2$, and $x_4(t=0) = 6.4$. Here, a set of asymmetric time delays is used to simulate $\tau_{14} = 0.83$ s, $\tau_{21} = 0.84$ s, $\tau_{31} = 0.85$ s, and $\tau_{42} = 0.86$ s.

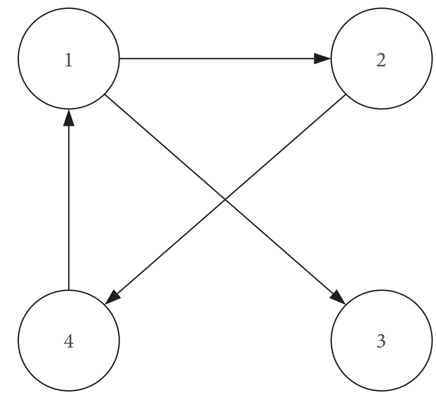

FIGURE 9: The connected interaction topology in example 2.

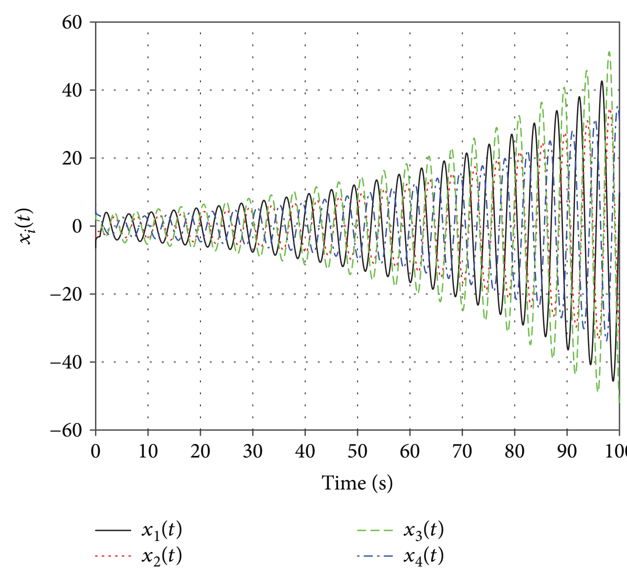

FIGURE 10: The trajectories of $x_i(t)$ applying the protocol in (6) under asymmetric time delays when all $\tau_m > \tau_{up1}$.

It is obvious that all asymmetric time delays τ_m are greater than τ_{up1} and less than τ_{up2}.

Figures 10 and 11 show the trajectories of $x_i(t)$ with the asymmetric time delays by applying the different control protocols when $\tau_{up1} < \text{all}\ \tau_m < \tau_{up2}$. From these simulation results, it is obvious that the given FMS in (4) by the control protocol in (6) diverges and cannot reach consensus, whereas the FMS in (4) applying the SDF control protocol in (7) converges to the same states and can reach consensus. Hence, the introduced SDF control protocol can enhance the robustness of the FMS in (4) to asymmetric time delays.

On the other hand, under the same conditions, we suppose that $\tau_{14} = 0.88$ s, $\tau_{21} = 0.89$ s, $\tau_{31} = 0.90$ s, and $\tau_{42} = 0.91$ s. Figure 12 shows the trajectories of $x_i(t)$ applying the protocol in (7) under asymmetric time delays when all $\tau_m > \tau_{up2}$. It is clear that the FMS in (4) cannot reach consensus.

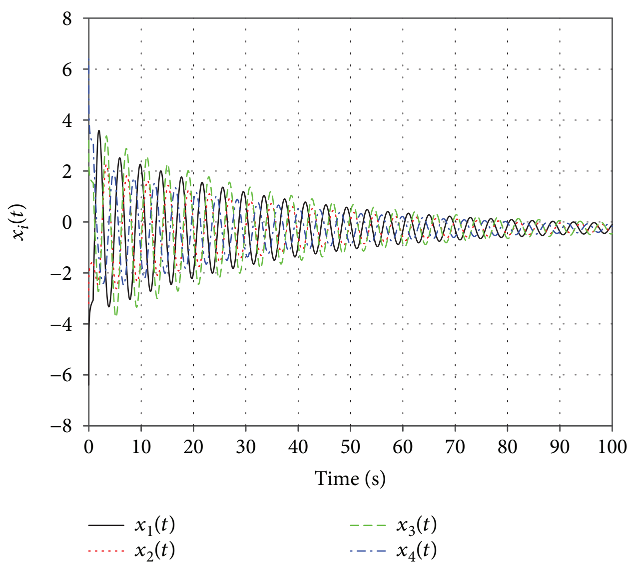

FIGURE 11: The trajectories of $x_i(t)$ applying the protocol in (7) under asymmetric time delays when all $\tau_m < \tau_{up2}$.

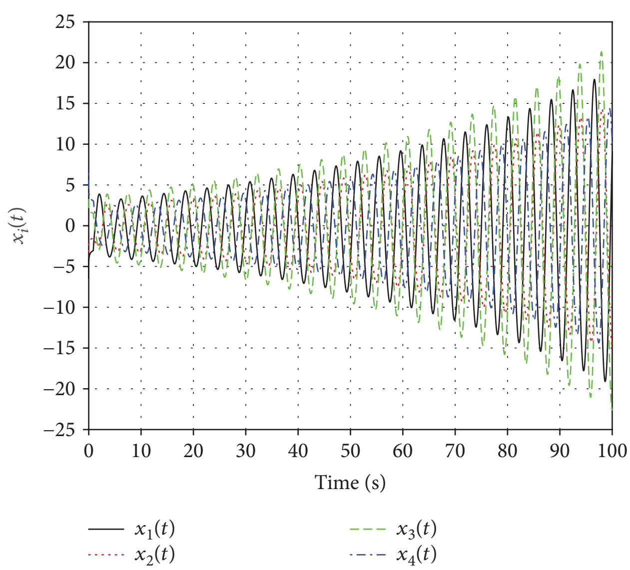

FIGURE 12: The trajectories of $x_i(t)$ applying the protocol in (7) under asymmetric time delays when all $\tau_m > \tau_{up2}$.

6. Conclusion

In order to enhance the robustness of a FMS against nonuniform time delays, a control protocol based on SDF and nonuniform time delays is introduced in this paper. First of all, the consensus problem is investigated for the FMS with SDF and symmetric time delays over undirected topology. Then, the consensus problem is investigated for the FMS with SDF and asymmetric time delays over directed topology. By the robustness analysis, it is obvious that the control protocol-based on SDF with the appropriate intensity can enhance the robustness for the FMS to nonuniform time delays. Finally, the validity of the theoretical analysis is verified by the corresponding simulation results. In addition, inspired by [26, 27], the consensus problems or formation control problems of delayed double-integrator FMSs based on round-robin protocols or attacks will be one of the most interesting topics of our future research work.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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