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Research Article

Optimum Solutions of Fractional Order Zakharov–Kuznetsov Equations

Rashid Nawaz [1], Laiq Zada [1], Abraiz Khattak [2], Muhammad Jibran [3], and Adam Khan [4,5]

[1] *Department of Mathematics, Abdul Wali Khan University Mardan, Mardan, Khaber Pakhtunkhwa, Pakistan*
[2] *Department of Electrical Power Engineering, U.S. Pakistan Center for Advanced Studies in Energy, National University of Sciences and Technology (NUST), Islamabad, Pakistan*
[3] *Department of Computer Science and IT, Sarhad University of Science and Information Technology, Peshawar, Khaber Pakhtunkhwa, Pakistan*
[4] *Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[5] *Faculty of Electrical & Electronics Engineering, Ton Duc Thang University, Ho Chi Minh City, Vietnam*

Correspondence should be addressed to Adam Khan; adamkhan@tdtu.edu.vn

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In this paper, the Optimal Homotopy Asymptotic Method is extended to derive the approximate solutions of fractional order two-dimensional partial differential equations. The fractional order Zakharov–Kuznetsov equation is solved as a test example, while the time fractional derivatives are described in the Caputo sense. The solutions of the problem are computed in the form of rapidly convergent series with easily calculable components using Mathematica. Reliability of the proposed method is given by comparison with other methods in the literature. The obtained results showed that the method is powerful and efficient for determination of solution of higher-dimensional fractional order partial differential equations.

1. Introduction

Fractional calculus is simply an extension of integer order calculus. For many years, it was assumed that fractional calculus is a pure subject of mathematics and having no such applications in real-world phenomena, but this concept is now wrong because of the recent applications of fractional calculus in modeling of the sound waves propagation in rigid porous materials [1], ultrasonic wave propagation in human cancellous bone [2], viscoelastic properties of soft biological tissues [3], the path tracking problem in an autonomous electric vehicles [4], etc. Differential equations of fractional order are the center of attention of many studies due to their frequent applications in the areas of electromagnetic, electrochemistry, acoustics, material science, physics, viscoelasticity, and engineering [5–9]. These kinds of problems are more complex as compared to integer order differential equations. Due to the complexities of fractional calculus, most of the fractional order differential equations do not have the exact solutions, and as an alternative, the approximate methods are extensively used for solution of these types of equations [10–14]. Some of the recent methods for approximate solutions of fractional order differential equations are the Adomian Decomposition Method (ADM), the Homotopy Perturbation Method (HPM), the Variational Iteration Method (VIM), Homotopy Analysis Method (HAM), etc. [15–26].

Marinca and Herisanu introduced the Optimal Homotopy Asymptotic Method (OHAM) for solving nonlinear differential equations which made the perturbation methods independent of the assumption of small parameters and huge computational work [27–31]. The method was recently extended by Sarwar et al. for solution of fractional order differential equations [32–35].

In this paper, OHAM formulation is extended to two-dimensional fractional order partial differential equations. Particularly, the extended formulation is demonstrated by illustrative examples of the following fractional version of the Zakharov–Kuznetsov equations shortly called FZK (p,q,r):

$$D_t^{\alpha}\zeta + a(\zeta^p)_x + b(\zeta^q)_{xxx} + c(\zeta^r)_{yyx} = 0. \quad (1)$$

In above equation, α is a parameter describing theory of the fractional derivative $(0<\alpha\le 1), a, b$ and c are arbitrary constants, and $p,q,r \neq 0$ are integers which govern the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. The FZK equation has been solved by many researchers using different techniques. Some recent well-known techniques are [36–40].

The present paper is divided into six sections. In Section 2, some basic definitions and properties from fractional calculus are given. Section 3 is devoted to analysis of the OHAM for two-dimensional partial differential equations of fractional order. In Section 4, the 1st order approximate solutions of FZK (2, 2, 2) and FZK (3, 3, 3) equations are given, in which the time fractional derivatives are described in the Caputo sense. In Section 5, comparisons of the results of 1st order approximate solution by the proposed method are made with 3rd order variational iteration method (VIM), Perturbation-Iteration Algorithm (PIA), and residual power series method (RPS) solutions [36, 37]. In all cases, the proposed method yields better results.

2. Basic Definitions

In this section, some definitions and results from the literature are stated which are relevant to the current work. Riemann–Liouville, Welyl, Reize, Compos, and Caputo proposed many definitions.

Definition 1. A real function $f(x)$, $x>0$, is said to be in space C_η, $\eta \in \Re$, if there is a real number $p>\eta$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C(0,\infty)$ and it is said to be in the space $C_\eta{}^m$ if only if $f^m \in C_\eta, m \in N$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha \ge 0$ of a function $f\in C_\eta, \eta \ge -1$ is defined as

$$I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\eta)^{\alpha-1} f(\eta)\mathrm{d}\eta, \quad \alpha>0,\ x>0,$$
$$I_a^0 f(x) = f(x). \quad (2)$$

When we formulate the model of real-world problems with fractional calculus, the Riemann–Liouville operator have certain disadvantages. Caputo proposed a modified fractional differential operator D_*^{α} in his work on the theory of viscoelasticity.

Definition 3. The fractional derivative of $f(x)$ in Caputo sense is defined as

$$D_a^{\alpha} f(x) = I_a^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-a)}\int_a^x (x-\zeta)^{m-\alpha-1} f^m(\zeta) d\zeta,$$

for $m-1<\alpha\le m, m\in N, x>0, f\in C_{-1}^m$. (3)

Definition 4. If $m-1<\alpha\le m, m\in N$, and $f\in C_\eta^m, \eta\ge -1$, then

$$D_a^{\alpha} I_a^{\alpha} f(x) = f(x),$$
$$D_a^{\alpha} I_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}\frac{(x-a)}{k!}, \quad x>0. \quad (4)$$

One can found the properties of the operator I^{α} in the literature. We mention the following:

For $f\in C_\eta^m, \alpha,\beta>0, \eta\ge -1$ and $\gamma\ge -1$.

$I_a^{\alpha} f(x)$ exists for almost every $x\in[a,b]$.
$I_a^{\alpha} I_a^{\beta} f(x) = I_a^{\alpha+\beta} f(x)$.
$I_a^{\alpha} I_a^{\beta} f(x) = I_a^{\beta} I_a^{\alpha} f(x)$.
$I_a^{\alpha}(x-a)^{\gamma} = \Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1)(x-a)^{\alpha+\gamma}$.

3. OHAM Analysis for Fractional Order PDEs

In this section, the OHAM for fractional order partial differential equation is introduced. The proposed method is presented in the following steps.

Step 1: write the general fractional order partial differential equation as

$$\frac{\partial^{\alpha}\zeta(r,t)}{\partial t^{\alpha}} = A(\zeta(r,t)) + f(r,t), \quad \alpha>0. \quad (5)$$

Subject to the initial conditions,

$$\begin{aligned} &D_0^{\alpha-k}\zeta(r,0) = h_k(r), \quad k=0,1,2,\ldots,n-1,\\ &D_0^{\alpha-n}\zeta(r,0) = 0, \quad n=[\alpha],\\ &D_0^{k}\zeta(r,0) = g_k(r), \quad k=0,1,2,\ldots,n-1,\\ &D_0^{n}\zeta(r,0) = 0, \quad n=[\alpha]. \end{aligned} \quad (6)$$

In above equations, $\partial^{\alpha}/\partial t^{\alpha}$ is the Caputo or Riemann–Liouville fraction derivative operator, A is the differential operator, $\zeta(r,t)$ is the unknown function and $f(r,t)$ is known analytic function, $r\in\Omega$ is an n-tuple which denotes spatial independent variables, and t represents the temporal independent variable, respectively.

Step 2: construct an optimal homotopy for fractional order partial differential equation, $\phi(r,t;p):\Omega\times[0,1]\longrightarrow R$ which is

$$(1-p)\left(\frac{\partial^\alpha \phi(r,t)}{\partial t^\alpha} - f(r,t)\right) - H(r,p)\left(\frac{\partial^\alpha \phi(r,t)}{\partial t^\alpha} - (A(\phi(r,t)) + f(r,t)\right) = 0. \tag{7}$$

In equation (7) $p \in [0,1]$ is the embedding parameter and $H(r,p)$ is auxiliary function which satisfies the following relation:

$H(r,p) \neq 0$ for $p \neq 0$ and $H(r,0) = 0$.

The solution $\phi(r,t)$ converges rapidly to the exact solution as the value of p increases in the interval $[0,1]$. The efficiency of OHAM depends upon the construction and determination of the auxiliary function which controls the convergence of the solution.

An auxiliary function $H(r,p)$ can be written in the form

$$H(r,p) = pk_1(r,C_i) + p^2k_2(r,C_i) + p^3k_3(r,C_i) + \ldots + p^m k_m(r,C_i). \tag{8}$$

In the above equation, C_i, $i = 1,2,3\ldots$ are the convergence control parameters and $k_i(r)$, $i = 1,2,3,\ldots$ is a function of r.

Step 3: expanding $\phi(r,t;p,C_i)$ in Taylor's series about p, we have

$$\phi(r,t;C_i) = \zeta_0(r,t) + \sum_{k=1}^{m} \zeta_k((r,t;C_i))p^k, \quad i = 1,2,3,\ldots. \tag{9}$$

Remarks: it is clear from equation (9), the convergence of the series depends upon the auxiliary convergence control parameter C_i, $i = 1,2,3,4,\ldots,m$

If it converges at $p = 1$, one has

$$\zeta(r,t;C_i) = \zeta_0(r,t) + \sum_{k=1}^{\infty} \zeta_k(r,t;C_i), \quad i = 1,2,3,\ldots. \tag{10}$$

Step 4: equating the coefficients of like powers of p after substituting equation (10) in equation (7), we get zero order, 1st order, 2nd order, and high-order problems:

$$\begin{aligned}
&p^0: \frac{\partial^\alpha \zeta_0(r,t)}{\partial t^\alpha} - f = 0,\\
&p^1: \frac{\partial^\alpha \zeta_1(r,t,C_1)}{\partial t^\alpha} - (1+C_1)\frac{\partial^\alpha \zeta_0(r,t)}{\partial t^\alpha} + (1+C_1)f\\
&\quad + C_1 A(\zeta_0(r,t)) = 0,\\
&p^2: \frac{\partial^\alpha \zeta_2(r,t,C_1,C_2)}{\partial t^\alpha} - (1+C_1)\frac{\partial^\alpha \zeta_1(r,t,C_1)}{\partial t^\alpha} - C_2\frac{\partial^\alpha \zeta_0(r,t)}{\partial t^\alpha}\\
&\quad + C_1 A(\zeta_1(r,t,C_1)) + C_2(f + A(\zeta_0(r,t))) = 0.
\end{aligned} \tag{11}$$

Step 5: these problems contain the time fractional derivatives. Therefore, we apply the I^α operator on the above problems and obtain a series of solutions as follows:

$$\begin{aligned}
\zeta_0(r,t) &= I^\alpha[f],\\
\zeta_1(r,t;C_1) &= I^\alpha\left[(1+C_1)\frac{\partial^\alpha \zeta_0(r,t)}{\partial t^\alpha} - (1+C_1)f - C_1 A(\zeta_0(r,t))\right],\\
\zeta_2(r,t;C_1,C_2) &= I^\alpha\left[(1+C_1)\frac{\partial^\alpha \zeta_1(r,t;C_1)}{\partial t^\alpha} + C_2\frac{\partial^\alpha \zeta_0(r,t)}{\partial t^\alpha} - C_1 A(\zeta_1(r,t;C_1)) - C_2(f + A(\zeta_0(r,t)))\right]\ldots.
\end{aligned} \tag{12}$$

By putting the above solutions in equation (12), one can get the approximate solution $\zeta(r,t;C_i)$. The residual $R(r,t;C_i)$ is obtained by substituting approximate solution $\zeta(r,t;C_i)$ in equation (5).

Step 6: the convergence control parameters $C_1, C_2, \ldots$ can be found either by the Ritz method, Collocation method, Galerkin's method, or least square method. In this presentation, least square method is used to calculate the convergence control parameters, in which we first construct the functional:

$$\chi(C_i) = \int_0^t \int_0^x R^2(r,t;C_i)\mathrm{d}r\,\mathrm{d}t. \tag{13}$$

And then the convergence control parameters are calculated by solving the following system:

$$\frac{\partial \chi}{\partial C_1} = \frac{\partial \chi}{\partial C_2} = \cdots = \frac{\partial \chi}{\partial C_m} = 0. \tag{14}$$

The approximate solution is obtained by putting the optimum values of the convergence control parameters in equation (10). The method of least squares is a powerful technique and has been used in many other methods such as Optimal Homotopy Perturbation Method (OHPM) and Optimal Auxiliary Functions Method (OAFM) for calculating the optimum values of arbitrary constants [41, 42].

4. OHAM Convergence

If the series (10) converges to $\zeta(x,t)$, where $\zeta_k(x,t) \in L(R^+)$ is produced by zero order problem and the K-order deformation, then $\zeta(x,t)$ is the exact solution of (5).

Proof. since the series

$$\sum_{k=1}^{\infty} \zeta_{i,k}(x,t;C_1,C_2,\ldots,C_k), \tag{15}$$

converges, it can be written as

$$\psi_i(x,t) = \sum_{k=1}^{\infty} \zeta_{i,k}(x,t;C_1,C_2,\ldots,C_k), \tag{16}$$

and its holds that

$$\lim_{k\longrightarrow\infty} \zeta_{i,k}(x,t;C_1,C_2,\ldots,C_k) = 0. \tag{17}$$

In fact, the following equation is satisfied:

$$\begin{aligned}
&\zeta_{i,1}(x,t;C_1) + \sum_{k=2}^{n} \zeta_{i,k}\left(x,t;\overrightarrow{C}_k\right) \\
&\quad - \sum_{k=2}^{n} \zeta_{i,k-1}\left(x,t;\overrightarrow{C}_{k-1}\right) \\
&= \zeta_{i,2}\left(x,t;\overrightarrow{C}_2\right) - \zeta_{i,1}(x,t;C_1) + \cdots + \zeta_{i,n}\left(x,t;\overrightarrow{C}_n\right) \\
&\quad - \zeta_{i,n-1}\left(x,t;\overrightarrow{C}_{n-1}\right) \\
&= \zeta_{i,n}\left(x,t;\overrightarrow{C}_n\right).
\end{aligned} \tag{18}$$

Now, we have

$$\begin{aligned}
&L_{i,1}\left(\zeta_{i,1}(x,t;C_1)\right) + \sum_{k=2}^{\infty} L_1\left(\zeta_{i,k}\left(x,t;\overrightarrow{C}_k\right)\right) \\
&\quad - \sum_{k=2}^{\infty} L_i\left(\zeta_{i,k-1}\left(x,t;\overrightarrow{C}_{k-1}\right)\right) \\
&= L_i\left(\zeta_{i,1}(x,t;C_1)\right) + \sum_{k=2}^{\infty} L_i\left(\zeta_{i,k}\left(x,t;\overrightarrow{C}_k\right)\right) \\
&\quad - \sum_{k=2}^{\infty} L_i\left(\zeta_{i,k-1}\left(x,t;\overrightarrow{C}_{k-1}\right)\right) = 0,
\end{aligned} \tag{19}$$

which satisfies

$$\begin{aligned}
&L_{i,1}\left(\zeta_{i,1}(x,t;C_1)\right) + L_i \sum_{k=2}^{\infty} \left(\zeta_{i,k}\left(x,t;\overrightarrow{C}_k\right)\right) \\
&\quad - L_i \sum_{k=2}^{\infty} \left(\zeta_{i,k-1}\left(x,t;\overrightarrow{C}_{k-1}\right)\right) \\
&= \sum_{k=2}^{\infty} C_m \left[L_i\left(\zeta_{i,k-m}\left(x,t;\overrightarrow{C}_{k-m}\right)\right) + N_{i,k-m}\right. \\
&\quad \left.\cdot \left(\zeta_{i,k-1}(x,t;C_{k-1})\right)\right] + g_i(x,t) = 0.
\end{aligned} \tag{20}$$

Now if C_m, $m = 1,2,3\ldots$, is properly chosen, then the equation leads to

$$L_i(\zeta_i(x,t) + A) = 0, \tag{21}$$

which is the exact solution. □

5. Application of OHAM

5.1. Time Fractional FZK (2, 2, 2). Consider the following Time Fractional FZK (2, 2, 2) equation with initial condition as

$$\begin{gathered}
\frac{\partial^\alpha w}{\partial t^\alpha} + \frac{\partial w^2}{\partial x^2} + \frac{1}{8}\frac{\partial^3 w^2}{\partial x^3} + \frac{1}{8}\frac{\partial^3 w^2}{\partial x\,\partial y^2} = 0, \quad 0 < \alpha \le 1, \\
w(x,y,0) = \frac{4}{3}\lambda \sinh^2(x+y).
\end{gathered} \tag{22}$$

The exact solution of equation (22) for $\alpha = 1$,

$$w(x,y,t) = \frac{4}{3}\lambda \sinh^2(x+y-\lambda t), \tag{23}$$

where λ is an arbitrary constant.

Using the OHAM formulation discussed in Section 3, we have

Zero-order problem:

$$\begin{gathered}
\frac{\partial^\alpha w}{\partial t^\alpha} = 0, \\
w_0(x,y,0) = \frac{4}{3}\lambda \sinh^2(x+y).
\end{gathered} \tag{24}$$

First-order problem:

$$\begin{aligned}
&\frac{\partial^\alpha w_1(x,y,t)}{\partial t^\alpha} = \frac{\partial^\alpha w_0(x,y,t)}{\partial t^\alpha} + C_1 \frac{\partial^\alpha w_0(x,y,t)}{\partial t^\alpha} \\
&\quad + 2C_1 \frac{\partial^\alpha w_0(x,y,t)}{\partial t^\alpha} w_0(x,y,t) + \frac{1}{4}C_1 \frac{\partial^2 w_0(x,y,t)}{\partial y^2}, \\
&\frac{\partial w_0(x,y,t)}{\partial x} + \frac{1}{2}C_1 \frac{\partial w_0(x,y,t)}{\partial y}\frac{\partial^2 w_0(x,y,t)}{\partial x\,\partial y} \\
&\quad + \frac{1}{4}C_1 w_0(x,y,t)\frac{\partial^3 w_0(x,y,t)}{\partial x\,\partial y^2} + \frac{3}{4}C_1 \frac{\partial w_0(x,y,t)}{\partial x}, \\
&\frac{\partial^2 w_0(x,y,t)}{\partial x^2} + \frac{1}{4}C_1 w_0(x,y,t)\frac{\partial^3 w_0(x,y,t)}{\partial x^3}, \quad w_1(x,y,0) = 0.
\end{aligned} \tag{25}$$

The solutions of above problems are as follows:

$$\begin{aligned}
&w_0(x,y,t) = \frac{4}{3}\lambda \sinh^2(x+y), \\
&w_1(x,y,t,C_1) = \frac{-8C_1 t^\alpha \lambda^2 (4\sinh(2(x+y)) - 5\sinh(4(x+y)))}{9\Gamma(1+\alpha)}.
\end{aligned} \tag{26}$$

The 1$^{\text{st}}$ order approximate solution by the OHAM is given by the following expression:

Table 1: Convergence-control parameters for FZK (2, 2, 2) and FZK (3, 3, 3).

FZK (2, 2, 2)		FZK (3, 3, 3)	
α	C_1	α	C_1
1.0	−0.1654570202126229	1.0	−1.0008903783207066
0.75	−0.11770797863128038	0.75	−0.999990856107855
0.67	−0.11303202695535328	0.67	−0.9999963529361133

$$\begin{aligned}\widetilde{w}(x,y,t;C_i) &= w_0(x,y,t) + w_1(x,y,t;C_1),\\ \widetilde{w}(x,y,t;C_1) &= \frac{4}{3}\lambda\sinh^2(x+y) + \frac{1}{9\Gamma(1+a)}\\ &\quad\cdot\left(-8C_1t^{\alpha}\lambda^2(4\sinh(2(x+y))\right.\\ &\quad\left.-5\sinh(4(x+y))\right)).\end{aligned} \tag{27}$$

5.2. Time Fractional FZK (3, 3, 3). Consider the following Time Fractional FZK (2, 2, 2) equation with initial condition as

$$\begin{aligned}&\frac{\partial^{\alpha}w}{\partial t^{\alpha}} + \frac{\partial w^3}{\partial x} + 2\frac{\partial^3 w^3}{\partial x^3} + 2\frac{\partial^3 w^3}{\partial x\, y^2} = 0, \quad 0<\alpha\le 1,\\ &w(x,y,0) = \frac{3}{2}\lambda\sinh\left(\frac{1}{6}(x+y)\right).\end{aligned} \tag{28}$$

The exact solution of equation (22) for $\alpha = 1$,

$$w(x,y,t) = \frac{3}{2}\lambda\sinh\left(\frac{1}{6}(x+y-\lambda t)\right), \tag{29}$$

where λ is an arbitrary constant.

Using the OHAM formulation discussed in Section 3, we have

Zero-order problem:

$$\begin{aligned}\frac{\partial^{\alpha}w(x,y,t)}{\partial t^{\alpha}} &= 0,\\ w_0(x,y,0) &= \frac{3}{2}\lambda\sinh\left(\frac{1}{6}(x+y)\right).\end{aligned} \tag{30}$$

First-order problem:

$$\begin{aligned}\frac{\partial^{\alpha}w_1(x,y,t)}{\partial t^{\alpha}} &= \frac{\partial^{\alpha}w_0(x,y,t)}{\partial t^{\alpha}} + C_1\frac{\partial^{\alpha}w_0(x,y,t)}{\partial t^{\alpha}}\\ &\quad + 3C_1\frac{\partial w_0(x,y,t)}{\partial x}w_0^2(x,y,t)\\ &\quad + 12C_1\frac{\partial^2 w_0^2(x,y,t)}{\partial y^2}\\ &\quad\cdot\frac{\partial w_0(x,y,t)}{\partial x} + 12C_1 w_0(x,y,t)\\ &\quad\frac{\partial^2 w_0(x,y,t)}{\partial y^2}\frac{\partial w_0(x,y,t)}{\partial x}\\ &\quad + 12C_1\frac{\partial w_0^3(x,y,t)}{\partial x^3} + 24C_1 w_0(x,y,t)\\ &\quad\frac{\partial w_0(x,y,t)}{\partial x}\frac{\partial^2 w_0(x,y,t)}{\partial x\,\partial y} + 6C_1 w_0^2(x,y,t)\frac{\partial^3 w_0(x,y,t)}{\partial x\,\partial y^2}\\ &\quad + 36C_1 w_0^2(x,y,t)\frac{\partial w_0(x,y,t)}{\partial x}\frac{\partial^2 w_0(x,y,t)}{\partial x^2}\\ &\quad + 6C_1 w_0^2(x,y,t)\frac{\partial^3 w_0(x,y,t)}{\partial y^3}, \quad w_1(x,y,0) = 0.\end{aligned} \tag{31}$$

The solutions of above problems are as follows:

$$\begin{aligned}w_0(x,y,t) &= \frac{3}{2}\lambda\sinh\left(\frac{1}{6}(x+y)\right),\\ w_1(x,y,t;C_1) &= \frac{1}{32\Gamma(1+\alpha)}\left(3C_1t^{\alpha}\lambda^3\left(-5\cosh\left(\frac{1}{6}(x+y)\right)\right.\right.\\ &\quad\left.\left.+9\cosh\left(\frac{1}{2}(x+y)\right)\right)\right).\end{aligned} \tag{32}$$

The 1st order approximate solution by the OHAM is given by the following expression:

$$\widetilde{w}(x,y,t,C_i) = w_0(x,y,t) + w_1(x,y,t,C_1). \tag{33}$$

6. Results and Discussion

OHAM formulation is tested upon the FZKequation. Mathematica 7 is used for most of computational work.

Table 1 shows the optimum values of the convergence control parameters for FZK (2, 2, 2) and FZK (3, 3, 3) equations at different values of α. In Tables 2 and 3, the results obtained by 1st order approximation of proposed method for the FZK (2, 2, 2) equation are compared with 3rd order approximation of Perturbation-Iteration Algorithm (PIA) and Residual power Series (RPS) method at different values of α. In Tables 4 and 5, the results obtained by 1st order approximation of the proposed method are compared with 3rd order approximation of VIM for FZK (3, 3, 3) equation. Figures 1–4 show the 3D plots of exact versus approximate solution by the proposed method for FZK (2, 2, 2) equation. Figures 1 and 2 show the 3D plots of exact versus approximate solution by the proposed method for FZK (3, 3, 3) equation. Figure 5 shows the 2D plots of approximate solution by the proposed method for FZK (2, 2, 2) equation at different values of α. Figure 6 shows the 2D plots of approximate solution by the proposed method for FZK (3, 3, 3) equation at different values of α.

It is clear from 2D figures that as value of α increases to 1, the approximate solutions tend close to the exact solutions.

Table 2: 1st order approximate solution obtained by the OHAM in comparison with 3 terms approximate solution obtained by PIA and RPS for FZK (2, 2, 2) at $\lambda = 0.001$.

			Solution for $\alpha = 1.0$		Absolute errors for $\alpha = 1.0$		
x	y	t	OHAM solution	Exact solution	PIA [34] error	RPS [34] error	OHAM error
0.1	0.1	0.2	5.3966×10^{-5}	5.39388×10^{-5}	3.85217×10^{-7}	3.85217×10^{-7}	2.71884×10^{-8}
		0.3	5.39248×10^{-5}	5.38841×10^{-5}	5.75911×10^{-7}	5.75912×10^{-7}	4.07394×10^{-8}
		0.4	5.38837×10^{-5}	5.38294×10^{-5}	7.65359×10^{-7}	7.65352×10^{-7}	5.42615×10^{-8}
0.6	0.6	0.2	3.02967×10^{-3}	3.03651×10^{-3}	4.66337×10^{-5}	4.66389×10^{-5}	6.83433×10^{-6}
		0.3	3.02553×10^{-3}	3.03578×10^{-3}	6.86056×10^{-5}	6.86314×10^{-5}	1.02517×10^{-5}
		0.4	3.02138×10^{-3}	3.03505×10^{-3}	8.98263×10^{-5}	8.99046×10^{-5}	1.36692×10^{-5}
0.9	0.9	0.2	1.14455×10^{-2}	1.1537×10^{-2}	5.12131×10^{-4}	5.14241×10^{-4}	9.14704×10^{-5}
		0.3	1.13973×10^{-2}	1.15345×10^{-2}	7.38186×10^{-4}	7.48450×10^{-4}	1.37206×10^{-4}
		0.4	1.13492×10^{-2}	1.15321×10^{-2}	9.57942×10^{-4}	9.89139×10^{-4}	1.82943×10^{-4}

Table 3: 1st order approximate solution obtained by the OHAM in comparison with 3 terms approximate solution obtained by PIA and RPS for FZK (2, 2, 2) at $\lambda = 0.001$.

			$\alpha = 0.67$			$\alpha = 0.75$		
x	y	t	PIA [36] solution	RPS [36] solution	OHAM solution	PIA [36] solution	RPS [36] solution	OHAM solution
0.1	0.1	0.2	5.31854×10^{-5}	5.31244×10^{-5}	5.39424×10^{-5}	5.32747×10^{-5}	5.32479×10^{-5}	5.3953×10^{-5}
		0.3	5.28631×10^{-5}	5.28410×10^{-5}	5.39094×10^{-5}	5.29757×10^{-5}	5.29675×10^{-5}	5.39191×10^{-5}
		0.4	5.25777×10^{-5}	5.25897×10^{-5}	5.38798×10^{-5}	5.27039×10^{-5}	5.27119×10^{-5}	5.38881×10^{-5}
0.6	0.6	0.2	2.95493×10^{-3}	2.95185×10^{-3}	3.0273×10^{-3}	2.96356×10^{-3}	2.96251×10^{-3}	3.02837×10^{-3}
		0.3	2.92662×10^{-3}	2.92709×10^{-3}	3.02397×10^{-3}	2.93717×10^{-3}	2.93780×10^{-3}	3.02496×10^{-3}
		0.4	2.90307×10^{-3}	2.90522×10^{-3}	3.02099×10^{-3}	2.91448×10^{-3}	2.91561×10^{-3}	3.02182×10^{-3}
0.9	0.9	0.2	1.06822×10^{-2}	1.05522×10^{-2}	1.14179×10^{-2}	1.07716×10^{-2}	1.07143×10^{-2}	1.14303×10^{-2}
		0.3	1.04487×10^{-2}	1.01199×10^{-2}	1.13792×10^{-2}	1.05488×10^{-2}	1.03695×10^{-2}	1.13907×10^{-2}
		0.4	9.02777×10^{-2}	9.60606×10^{-2}	1.13447×10^{-2}	1.03736×10^{-2}	9.96743×10^{-2}	1.13543×10^{-2}

Table 4: 1st order solution compared with three terms of VIM at $\lambda = 0.001$, for FZK (3, 3, 3).

			Solution $\alpha = 1$			Absolute errors
x	y	t	VIM [37] solution	OHAM solution	Exact solution	OHAM error
0.1	0.1	0.2	5.00091×10^{-5}	5.00092×10^{-5}	4.99592×10^{-5}	4.99519×10^{-8}
		0.3	5.00091×10^{-5}	5.00091×10^{-5}	4.99342×10^{-5}	7.49278×10^{-8}
		0.4	5.00091×10^{-5}	5.00091×10^{-5}	4.99092×10^{-5}	9.99037×10^{-8}
0.6	0.6	0.2	3.02003×10^{-4}	3.02004×10^{-4}	3.01953×10^{-4}	5.08987×10^{-8}
		0.3	3.02003×10^{-4}	3.02004×10^{-4}	3.01927×10^{-4}	7.63479×10^{-8}
		0.4	3.02003×10^{-4}	3.02004×10^{-4}	3.01902×10^{-4}	1.01797×10^{-7}
0.9	0.9	0.2	4.56780×10^{-4}	4.5678×10^{-4}	4.56728×10^{-4}	5.21227×10^{-8}
		0.3	4.56780×10^{-4}	4.5678×10^{-4}	4.56702×10^{-4}	7.81839×10^{-8}
		0.4	4.56780×10^{-4}	4.5678×10^{-4}	4.56676×10^{-4}	1.04245×10^{-7}

Table 5: 1st order solution compared with 3 terms of VIM for, $\lambda = 0.001$, for FZK (3, 3, 3).

			$\alpha = 0.67$		$\alpha = 0.75$	
x	y	t	VIM [37] solution	OHAM solution	VIM [37] solution	OHAM solution
0.1	0.1	0.2	5.00091×10^{-5}	500091×10^{-5}	5.00091×10^{-5}	5.00091×10^{-5}
		0.3	5.00090×10^{-5}	500091×10^{-5}	5.00090×10^{-5}	5.00091×10^{-5}
		0.4	5.00090×10^{-5}	50009×10^{-5}	5.00090×10^{-5}	5.00091×10^{-5}
0.6	0.6	0.2	3.02003×10^{-4}	302004×10^{-4}	3.02003×10^{-4}	302004×10^{-4}
		0.3	3.02003×10^{-4}	302004×10^{-4}	3.02003×10^{-4}	302004×10^{-4}
		0.4	3.02003×10^{-4}	302004×10^{-4}	3.02003×10^{-4}	302004×10^{-4}
0.9	0.9	0.2	4.56780×10^{-4}	45678×10^{-4}	4.5678×10^{-4}	45678×10^{-4}
		0.3	4.56780×10^{-4}	45678×10^{-4}	4.56780×10^{-4}	45678×10^{-4}
		0.4	4.56780×10^{-4}	45678×10^{-4}	4.56780×10^{-4}	45678×10^{-4}

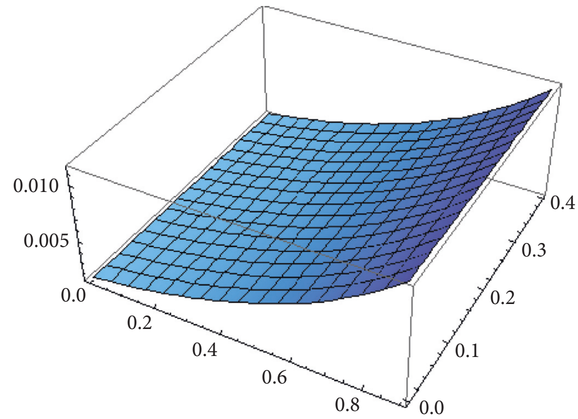

Figure 1: Approximate solution for equation (22) for $\alpha = 1$, ($y = 0.9$).

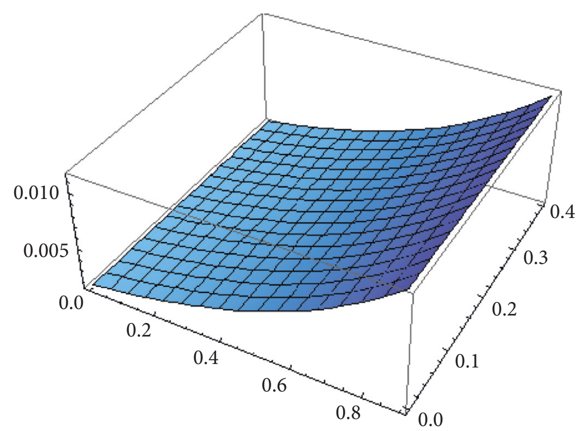

Figure 2: Exact solution for equation (22) for $\alpha = 1$, ($y = 0.9$).

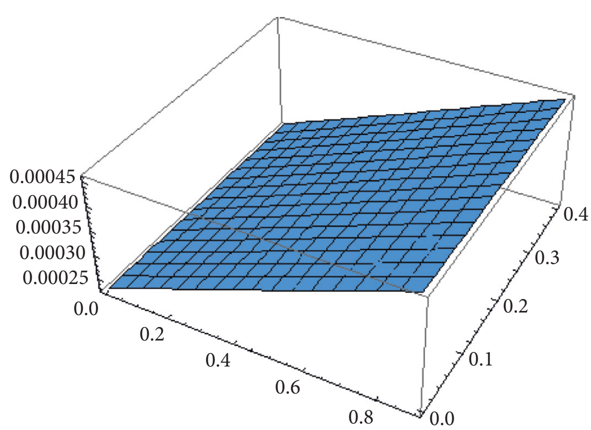

Figure 3: Approximate solution for equation (28) for $\alpha = 1$, ($y = 0.9$).

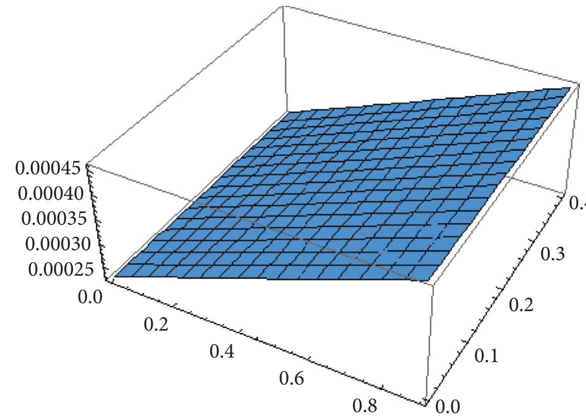

Figure 4: Exact solution for equation (28) for $\alpha = 1$, ($y = 0.9$).

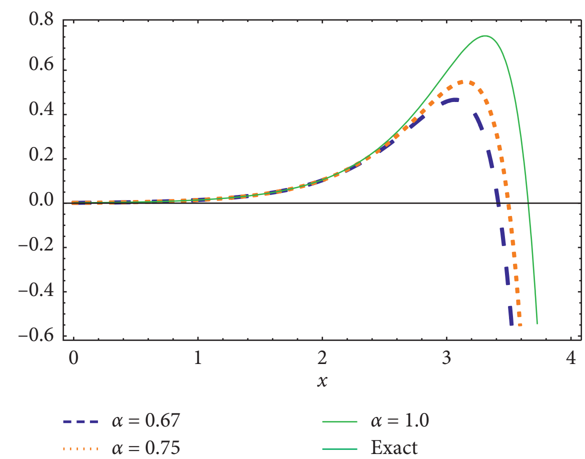

Figure 5: Convergence at different value of alpha for equation (22) at $t = 0.1$, $y = 0.2$.

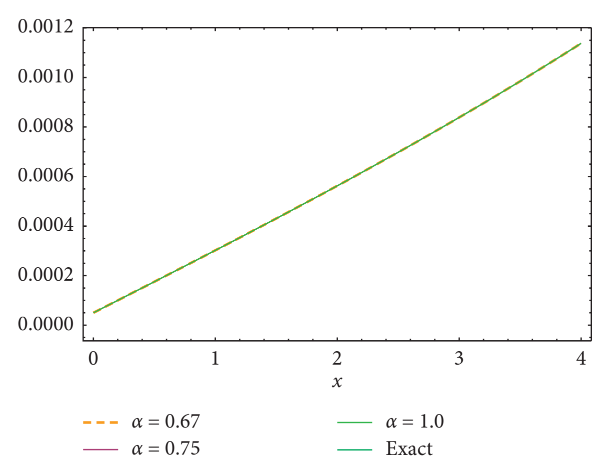

Figure 6: Convergence at different value of alpha for equation (28) at $t = 0.1$, $y = 0.2$.

7. Conclusion

The 1st order OHAM solution gives more encouraging results in comparison to 3rd order approximations of PIA, RPS, and VIM. From obtained results, it is concluded that the proposed method is very effective and convenient for solving higher-dimensional partial differential equations of fractional order. The accuracy of the method can be further improved by taking higher-order approximations.

Data Availability

No data were generated or analyzed during the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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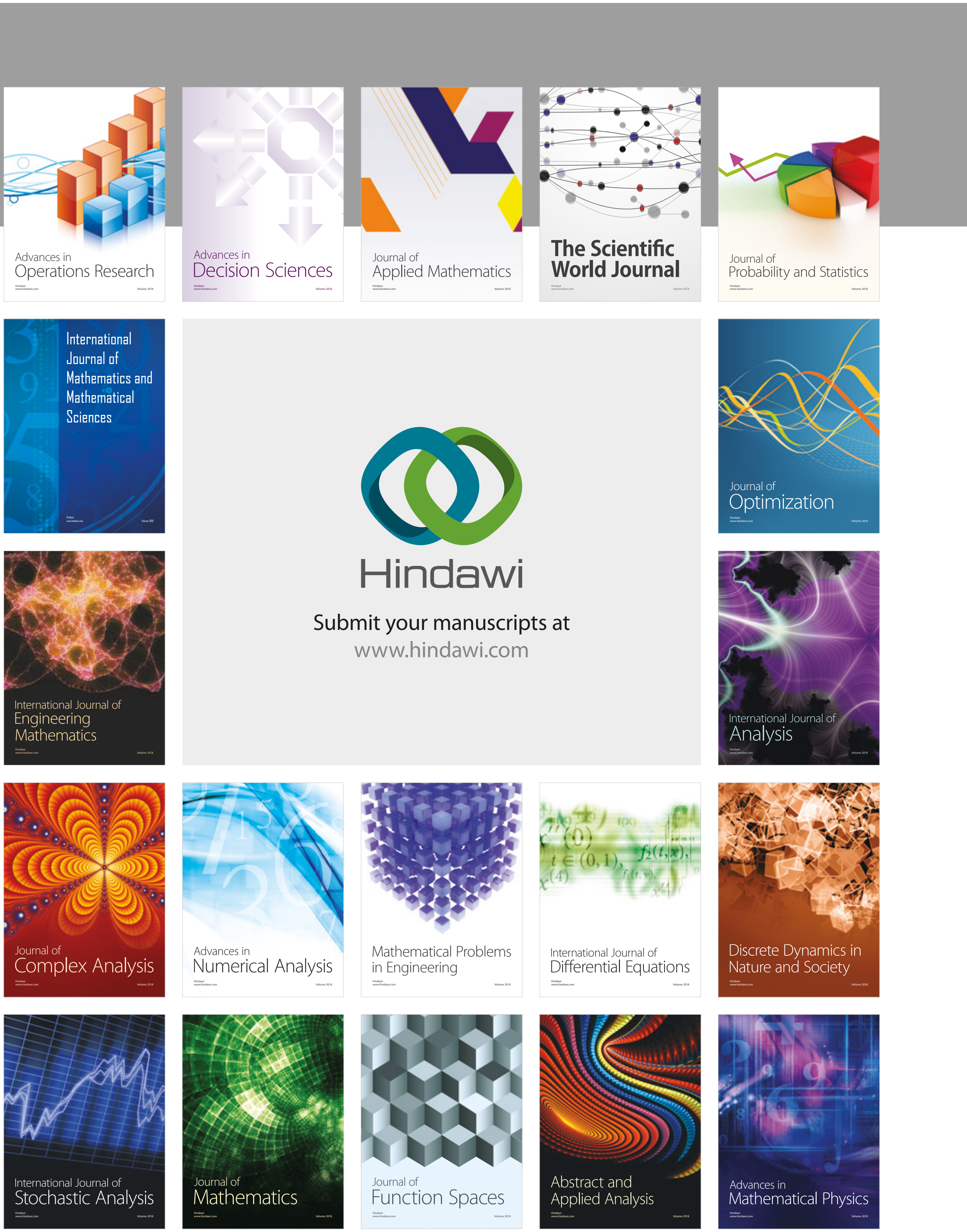
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