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*Research Article*

# Plastic Dynamical Model for Bulk Metallic Glasses

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Based on previous experimental results of the plastic dynamic analysis of metallic glasses upon compressive loading, a dynamical model is proposed. This model includes the sliding speed of shear bands in the plastically strained metallic glasses, the shear resistance of shear bands, the internal friction resulting from plastic deformation, and the influences from the testing machine. This model analysis quantitatively predicts that the loading rate can influence the transition of the plastic dynamics in metallic glasses from chaotic (low loading rate range) to stable behavior (high loading rate range), which is consistent with the previous experimental results on the compression tests of a $Cu_{50}Zr_{45}Ti_5$ metallic glass. Moreover, we investigate the existence of a nonconstant periodic solution for plastic dynamical model of bulk metallic glasses by using Manásevich–Mawhin continuation theorem.

## 1. Introduction

In [1], Cheng et al. investigate a plastic deformation and give the following model

$$\left[\left(\sigma_{\mathrm{p}}-kx\right)-\sigma_{\mathrm{f}}\left(x'\right)\right]\frac{\pi d^2}{4}=Mx'', \quad (1)$$

where $\sigma_{\mathrm{p}}$ is the loading stress of the shear bands, $d$ is the sample diameter, $x$ is the shear sliding displacement, $M$ is the equal effective mass, and it is also the effective inertia of the machine-sample system (MSS) when responding to the stress gradient and is an empirical parameter estimated to be of the order of 10–100 kg for a typical MSS. And, $k=(E/L)(1+S)$, where $L$ is the sample height, $E$ is Young's Modulus, and $S$ is the stiffness ratio of the sample to the testing machine and $S=K_s/K_M=\pi d^2E/4Lk_M$ in [2]. $\sigma_{\mathrm{f}}$ is the shear resistance along the shear plane. As the driving forces exceed the static shear resistance for one block, shear sliding will occur corresponding to the formation of one shear band.

The above model in [1] is considered an ideal situation, i.e., without internal friction. However, when a solid material undergoes plastic deformation, the internal friction reflects the force resisting motion between the elements, which definitely cannot be avoided [3]. In the current study, the internal friction coefficient of the model materials, i.e., metallic glasses [1], was measured by an elastic modulus and internal friction meter (NIHON Techno-Plus Company, Japan), which is 0.0012 at room temperature.

Motivated by this problem, we improve model (1) with internal friction

$$\left[\left(\sigma_{\mathrm{p}}-kx\right)-\sigma_{\mathrm{f}}\left(x'\right)-\gamma x'\right]\frac{\pi d^2}{4}=Mx'', \quad (2)$$

where $\gamma$ is the internal friction coefficient and $\sigma_{\mathrm{f}}$ is a complex function of the loading rate and temperature in the shear bands [1]. Here, we assume $\sigma_{\mathrm{f}}=\sigma_{\mathrm{f}_0}/(1+Ax')$ according to [4], with $\sigma_{\mathrm{f}_0}$ taken as the yielding strength of the sample and $A$ being a constant. So, (2) is translated into

$$\left[\left(\sigma_{\mathrm{p}}-kx\right)-\frac{\sigma_{\mathrm{f}_0}}{1+Ax'}-\gamma x'\right]\frac{\pi d^2}{4}=Mx''. \quad (3)$$

If $\sigma_{\mathrm{p}}=kpt$, where $p$ is the loading rate in [4–7], then (3) can be rewritten into a nonautonomous equation:

$$\left[\left(kpt-kx\right)-\frac{\sigma_{\mathrm{f}_0}}{1+Ax'}-\gamma x'\right]\frac{\pi d^2}{4}=Mx''. \quad (4)$$

Furthermore, we obtain

$$x'' + \gamma B x' + \frac{B\sigma_{f_0}}{1 + Ax'} + Bkx = Bkpt, \tag{5}$$

where $B = \pi d^2/4M$. In Section 2, we conduct a dynamic analysis of model (5). This model analysis quantitatively predicts that the loading rate can influence the transition of the plastic dynamics in metallic glasses from chaotic (low loading rate range) to stable behavior (high loading rate range), which is consistent with the previous experimental results on the compression tests of a $Cu_{50}Zr_{45}Ti_5$ metallic glass.

On the other hand, if $\sigma_p = \sigma(0)$, where $\sigma(0)$ is the initial internal stress, which is equal to the yield stress in [8–10], from (3) and (5), we obtain

$$x'' + \gamma B x' + \frac{B\sigma_{f_0}}{1 + Ax'} + Bkx = B\sigma(0). \tag{6}$$

Using Manásevich–Mawhin continuation theorem, we obtain the existence of a periodic solution for model (6) in Section 3. Moreover, we give the existence of upper and lower bounds of the periodic solution of this equation.

## 2. Dynamical Analysis for Model (5)

Let $x = z + pt$. We have from (5) that

$$z'' + \gamma B z' + \frac{B\sigma_{f_0}}{1 + Ap + Az'} + Bkz + \gamma Bp = 0, \tag{7}$$

and then

$$\begin{cases} z' = y, \\ y' = -\gamma B y - \dfrac{B\sigma_{f_0}}{1 + Ap + Ay} - Bkz - \gamma Bp. \end{cases} \tag{8}$$

Obviously, (8) has one equilibrium point at $E := (-(\gamma p/k) - (\sigma_{f_0}/k + Akp), 0)$. Let $A(z, y)$ be the coefficient matrix of the linearized system of (8) at an equilibrium point $(z, y)$. Then, we have at $E$

$$A\left(-\frac{\gamma p}{k} - \frac{\sigma_{f_0}}{k + Akp}, 0\right) = \begin{pmatrix} 0 & 1 \\ -Bk & \dfrac{AB\sigma_{f_0}}{(1+p)^2} - \gamma B \end{pmatrix}. \tag{9}$$

The characteristic equation of $A(-(\gamma p/k) - (\sigma_{f_0}/k + Akp), 0)$ is

$$f(\lambda) = \lambda^2 - \left(\frac{AB\sigma_{f_0}}{(1+p)^2} - \gamma B\right)\lambda + Bk, \tag{10}$$

which yields

$$\lambda_{1,2} = \frac{\left(\left(AB\sigma_{f_0}/(1+p)^2\right) - \gamma B\right) \pm \sqrt{\left(\left(AB\sigma_{f_0}/(1+p)^2\right) - \gamma B\right)^2 - 4Bk}}{2}. \tag{11}$$

By analyzing, we obtain the following results.

**Theorem 2.1.** *If $\gamma \geq (A\sigma_{f_0}/(1+p)^2) + (4\sqrt{kM\pi}/d\pi)$, then the equilibrium point E is stable node; if $\gamma \leq (A\sigma_{f_0}/(1+p)^2) - (4\sqrt{kM\pi}/d\pi)$, then the equilibrium point E is unstable node.*

**Theorem 2.2.** *If $(A\sigma_{f_0}/(1+p)^2) < \gamma < (A\sigma_{f_0}/(1+p)^2) + (4\sqrt{kM\pi}/d\pi)$, then the equilibrium point E is stable focus; if $(A\sigma_{f_0}/(1+p)^2) - (4\sqrt{kM\pi}/d\pi) < \gamma < (A\sigma_{f_0}/(1+p)^2)$, then the equilibrium point E is unstable focus. Moreover, system (8) has Hopf bifurcation with $\gamma = A\sigma_{f_0}/(1+p)^2$.*

Next, from (8), we denote that

$$\begin{cases} f(z, y) = y, \\ g(z, y) = -\gamma B y - \dfrac{B\sigma_{f_0}}{1 + Ap + Ay} - Bkz - \gamma Bp. \end{cases} \tag{12}$$

Let $\alpha_\gamma = ((AB\sigma_{f_0}/(1+p)^2) - \gamma B)/2$. We know that the bifurcation value is $\alpha_{\gamma_0} = 0$ if $\gamma = A\sigma_{f_0}/(1+p)^2$. And,

$$\left.\frac{d\alpha_\gamma}{d\gamma}\right|_{\gamma = \left(A\sigma_{f_0}/(1+p)^2\right)} = \frac{-B}{2} < 0, \tag{13}$$

i.e., $\alpha_\gamma$ decreases with $\gamma$.

By Andronov–Hopf bifurcation theorem ([11], P. 167), we have $f_z = 0, g_y = 0$, and $f_y g_z = -Bk < 0$. From

$$16K_{r_0} = \frac{8A^3 B^{(3/2)} k^{(1/2)} \sigma_{f_0}}{(1+p)^5} > 0, \tag{14}$$

we have the following.

**Theorem 2.3.** *If $\gamma = (A\sigma_{f_0}/(1+p)^2)$, then the equilibrium point E is weak repell; if $\gamma > (A\sigma_{f_0}/(1+p)^2)$, there exists an unstable periodic orbit of system (11) from the equilibrium point E. This is subcritical bifurcation.*

On the other hand, let $\alpha_{\sigma_{f_0}} = ((AB\sigma_{f_0}/(1+p)^2) - \gamma B)/2$. We know that the bifurcation value is $\alpha_{\sigma_{f_0}} = 0$ if $\sigma_{f_0} = \gamma(1+p)^2/A$. And,

$$\left.\frac{d\alpha_{\sigma_{f_0}}}{d\sigma_{f_0}}\right|_{\sigma_{f_0} = \left(\gamma(1+p)^2/A\right)} = \frac{AB}{2(1+p)^2} > 0, \tag{15}$$

i.e., $\alpha_{\sigma_{f_0}}$ is increasing about $\sigma_{f_0}$.

By Andronov–Hopf bifurcation theorem (see [11], P. 167), we have $f_z = 0, g_y = 0$, and $f_y g_z = -Bk < 0$. From

$$16K_{r_0} = \frac{8A^2\gamma B^{(3/2)}k^{(1/2)}}{(1+p)^3} > 0, \tag{16}$$

we have the following.

**Theorem 2.4.** *If $\sigma_{f_0} = \gamma(1+p)^2/A$, then the equilibrium point E is weak repell; if $\sigma_{f_0} < \gamma(1+p)^2/A$, there exists an unstable periodic orbit of system (11) from the equilibrium point E. This is subcritical bifurcation.*

In the following, we consider Lyapunov exponent of system (5). By substituting $z = t$, nonautonomous system (5) can be rewritten into a three-dimensional autonomous system:

$$\begin{cases} x' = y, \\ y' = -B\gamma y - \dfrac{B\sigma_{f_0}}{1+Ay} - Bkx + Bkpz. \\ z' = 1. \end{cases} \tag{17}$$

From (17), we know that there is a uniform solution (trajectory) in which the shear bands slide at the loading rate:

$$\begin{cases} x = pt - \dfrac{\sigma_{f_0}}{k+Akp} - \dfrac{\gamma p}{k}, \\ y = p, \\ z = t. \end{cases} \tag{18}$$

Let $B(x,y,z)$ be the coefficient matrix of the linearized system of (17) at a trajectory $(x,y,z)$. Then, we have at the trajectory of (18)

$$B\left(pt - \frac{\sigma_{f_0}}{k+Akp} - \frac{\gamma p}{k}, p, t\right) = \begin{pmatrix} 0 & 1 & 0 \\ -Bk & -B\gamma + \dfrac{AB\sigma_{f_0}}{(1+Ap)^2} & Bkp \\ 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

The characteristic equation of $B(pt - (\sigma_{f_0}/k + Akp) - (\gamma p/k), p, t)$ is

$$g(\lambda) = \lambda\left(\lambda^2 - \left(\frac{AB\sigma_{f_0}}{(1+Ap)^2} - B\gamma\right)\lambda + Bk\right), \tag{20}$$

and then

$$\lambda_1 = 0, \lambda_{2,3} = \frac{\left(\left(AB\sigma_{f_0}/(1+Ap)^2\right) - B\gamma\right) \pm \sqrt{\left(\left(AB\sigma_{f_0}/(1+Ap)^2\right) - B\gamma\right)^2 - 4Bk}}{2}. \tag{21}$$

Define $C = 4K_M E/(4LK_M + \pi d^2 E)Ak$. By [12] (P. 727), we obtain the following results.

**Theorem 2.5.** *If $p > C(\sqrt{A\sigma_{f_0}/\gamma} - 1)$, then (5) is the stable closed orbit for trajectory (18). If $p < C(\sqrt{A\sigma_{f_0}/\gamma} - 1)$, then (5) is hyperchaotic for trajectory (18).*

## 3. Periodic Solution for Model (6)

In this section, we prove the existence of a nonconstant $\omega$-periodic solution for model (6) by applying Manásevich–Mawhin continuation theorem. First, we consider the following differential equation with a singularity of derivative:

$$x'' + Cx' + g(x') + Kx = e(t), \tag{22}$$

where $K$ and $C$ are positive constants, $e \in L^2([0,\omega])$ and $e(t) \equiv g(0) - Kc$, for all $c \in \mathbb{R}$, $g\colon (b,+\infty) \longrightarrow \mathbb{R}$ is a continuous function, and $g(0) := \sigma < \|e\| := \max_{t\in[0,\omega]}|e(t)|$, $g$ may have a singularity of derivative at $u = b$, which means that

$$\lim_{u\longrightarrow b^+} \int_1^u g(s)\mathrm{d}s = +\infty, \tag{23}$$

where $b$ is a constant and $b < 0$.

Next, we embed (22) into the following equation family with a parameter $\lambda \in (0,1]$:

$$x'' + \lambda(Cx' + g(x') + Kx) = \lambda e(t). \tag{24}$$

The following lemma is Manásevich–Mawhin continuation theorem ([13], Theorem 3.1).

**Lemma 3.1.** ([13], Theorem 3.1) *Let $\Omega$ be an open bounded set in the space $X := \{\phi \in C^1(\mathbb{R},\mathbb{R})\colon \phi(t+\omega) = \phi(t), \phi'(t+\omega) = \phi'(t), \forall t \in \mathbb{R}\}$. Suppose the following conditions are satisfied:*

*(i) (24) has no solution on $\partial\Omega$.*

*(ii) The following equation*

$$F(a) := \frac{1}{\omega}\int_0^{\omega}(g(0) + Kx(t) - e(t))\mathrm{d}t = 0, \tag{25}$$

*has no solution on* $\partial\Omega \cap \mathbb{R}$.

*(iii) The Brouwer degree of F*

$$\deg\{F, \Omega \cap \mathbb{R}, 0\} \neq 0. \tag{26}$$

*Then, (22) has at least one periodic solution on* $\overline{\Omega}$.

**Theorem 3.1.** *Assume that condition (23) holds. If* $K(\omega/2\pi)^2 < 1$, *then equation (22) has at least one nonconstant periodic solution x with*

$$x \in \left(-\frac{\|e\| - \sigma}{K} - \frac{(\omega/2\pi)\|e\|_2}{1 - K(\omega/2\pi)^2}, \frac{\|e\| - \sigma}{K} + \frac{(\omega/2\pi)\|e\|_2}{1 - K(\omega/2\pi)^2}\right). \tag{27}$$

*Proof.* Suppose that $x$ is a solution of (24) for some $\lambda \in (0, 1]$. Let $t^*, t_*$ be, respectively, the global maximum point and global minimum point of $x(t)$ on $[0, \omega]$, then we obtain $x'(t^*) = 0$, $x'(t_*) = 0$, $x''(t^*) \leq 0$, and $x''(t_*) \geq 0$. Furthermore, we arrive at

$$x''(t^*) + \lambda(g(x'(t^*)) + Kx(t^*)) = \lambda e(t^*). \tag{28}$$

Since $x''(t^*) \leq 0$ and $g(0) = \sigma$, then we obtain

$$x(t^*) \geq \frac{e(t^*) - \sigma}{K} \geq \frac{e_* - \sigma}{K} := D_1, \tag{29}$$

where $e_* := \min_{t\in[0,\omega]} p(t)$.

Similarly, we get

$$x(t_*) \leq \frac{e(t_*) - \sigma}{K} \leq \frac{\|e\| - \sigma}{K} := D_2 > 0, \tag{30}$$

since $\|e\| - \sigma > 0$. From equations (29) and (30), $x$ is a continuous function in $\mathbb{R}$, there exists a point $\tau \in (0, \omega)$ such that

$$D_1 \leq x(\tau) \leq D_2. \tag{31}$$

Multiplying both sides of (24) by $x''(t)$ and integrating over the interval $[0, \omega]$, we have

$$\begin{aligned} &\int_0^{\omega}|x''(t)|^2\mathrm{d}t + \lambda\int_0^{\omega}Cx'(t)x''(t)\mathrm{d}t \\ &\quad + \lambda\int_0^{\omega}g(x'(t))x''(t)\mathrm{d}t + \lambda K\int_0^{\omega}x(t)x''(t)\mathrm{d}t \\ &= \lambda\int_0^{\omega}e(t)x''(t)\mathrm{d}t. \end{aligned} \tag{32}$$

Substituting $\int_0^{\omega}Cx'(t)x''(t)\mathrm{d}t = 0$, $\int_0^{\omega}g(x'(t))x''(t)\mathrm{d}t = 0$, and $\int_0^{\omega}x(t)x''(t)\mathrm{d}t = -\int_0^{\omega}|x'(t)|^2\mathrm{d}t$ into (32) and applying the Hölder inequality, we see that

$$\begin{aligned} \int_0^{\omega}|x''(t)|^2\mathrm{d}t &= \lambda K\int_0^{\omega}|x'(t)|^2\mathrm{d}t + \lambda\int_0^{\omega}e(t)x''(t)\mathrm{d}t \\ &\leq K\int_0^{\omega}|x'(t)|^2 dt + \lambda\int_0^{\omega}|e(t)||x''(t)|\,\mathrm{d}t \\ &\leq K\int_0^{\omega}|x'(t)|^2\mathrm{d}t + \lambda\left(\int_0^{\omega}|e(t)|^2\mathrm{d}t\right)^{(1/2)} \\ &\quad\cdot\left(\int_0^{\omega}|x''(t)|^2\mathrm{d}t\right)^{(1/2)}. \end{aligned} \tag{33}$$

Using the Wirtinger inequality ([14], Lemma 2.4), it is clear that

$$\begin{aligned} \int_0^{\omega}|x''(t)|^2\,\mathrm{d}t &\leq K\left(\frac{\omega}{2\pi}\right)^2\int_0^{\omega}|x''(t)|^2\,\mathrm{d}t \\ &\quad + \lambda\left(\int_0^{\omega}|e(t)|^2\,\mathrm{d}t\right)^{(1/2)}\left(\int_0^{\omega}|x''(t)|^2\,\mathrm{d}t\right)^{(1/2)}. \end{aligned} \tag{34}$$

Since $K(\omega/2\pi)^2 < 1$ and $\int_0^{\omega}|x''(t)|^2\,\mathrm{d}t \neq 0$, it is easy to see that

$$\left(\int_0^{\omega}|x''(t)|^2\,\mathrm{d}t\right)^{(1/2)} \leq \frac{\lambda\|e\|_2}{1 - K(\omega/2\pi)^2} := \lambda M_1', \tag{35}$$

where $e_2 := (\int_0^{\omega}|e(t)|^2\,\mathrm{d}t)^{(1/2)}$. From (31) and the Wirtinger inequality, we have\openup3

$$\begin{aligned} \|x\| &\leq D_2 + \int_0^{\omega}|x'(t)|\mathrm{d}t \\ &\leq D_2 + \sqrt{\omega}\left(\int_0^{\omega}|x'(t)|^2\mathrm{d}t\right)^{(1/2)} \\ &\leq D_2 + \sqrt{\omega}\left(\frac{\omega}{2\pi}\right)\left(\int_0^{\omega}|x''(t)|\mathrm{d}t\right)^{(1/2)} \\ &\leq D_2 + \sqrt{\omega}\left(\frac{\omega}{2\pi}\right)\lambda M_1' \\ &\leq \frac{\|e\| - \sigma}{K} + \frac{(\omega/2\pi)\|e\|_2}{1 - K(\omega/2\pi)^2} := M_1. \end{aligned} \tag{36}$$

Since $x(0) = x(\omega)$, there exists a point $\xi \in (0, \omega)$ such that $x'(\xi) = 0$. Therefore, we have

$$\begin{aligned} x'(t) &= x'(\xi) + \int_{\xi}^{t}x''(s)\mathrm{d}s \\ &\leq \int_0^{\omega}|x''(t)|\mathrm{d}t \\ &\leq \sqrt{\omega}\left(\int_0^{\omega}|x''(t)|^2\mathrm{d}t\right)^{(1/2)} \\ &\leq \sqrt{\omega}\lambda M_1' \\ &\leq \left(\sqrt{\omega}\,e_2/1 - K(\omega/2\pi)^2\right) := M_2. \end{aligned} \tag{37}$$

On the other hand, multiplying both sides of (24) by $x''(t)$, we get

$$(x''(t))^2 + \lambda g(x'(t))x''(t) + \lambda Kx(t)x''(t) = \lambda e(t)x''(t). \tag{38}$$

Let $\xi \in [0,\omega]$ be defined in (37). For any $\xi \le t \le \omega$, integrating (38) on $[\xi, t]$, we obtain

$$\begin{aligned}\lambda \int_{x'(\xi)}^{x'(t)} g(u)\mathrm{d}u &= \lambda \int_{\xi}^{t} g(x'(s))x''(s)\mathrm{d}s \\ &= \int_{\xi}^{t} (x''(s))^2 \mathrm{d}s - \lambda \int_{\xi}^{t} Cx'(s)x''(s)\mathrm{d}s \\ &\quad - \lambda \int_{\xi}^{t} Kx(s)x''(s)\mathrm{d}s + \lambda \int_{\xi}^{t} e(s)x''(s)\mathrm{d}s.\end{aligned} \tag{39}$$

Furthermore, we see that

$$\begin{aligned}\lambda\left|\int_{x'(\xi)}^{x'(t)} g(u)\mathrm{d}u\right| &= \lambda\left|\int_{\xi}^{t} g(x'(s))x''(s)\mathrm{d}s\right| \\ &\le \int_0^{\omega} |x''(s)|^2 \mathrm{d}s \\ &\quad + \lambda C \int_0^{\omega} |x'(s)||x''(s)|\mathrm{d}s \\ &\quad + \lambda K \int_0^{\omega} |x(s)||x''(s)|\mathrm{d}s \\ &\quad + \lambda \int_0^{\omega} |e(s)||x''(s)|\mathrm{d}s.\end{aligned} \tag{40}$$

From (35) and (36), applying the Hölder inequality, we have

$$\begin{aligned}\lambda\left|\int_{x'(\xi)}^{x'(t)} g(u)\mathrm{d}u\right| &\le \int_0^{\omega} |x''(s)|^2 \mathrm{d}s \\ &\quad + \lambda C\|x'\|\sqrt{\omega}\left(\int_0^{\omega} |x''(s)|^2 \mathrm{d}s\right)^{(1/2)} \\ &\quad + \lambda K\|x\|\sqrt{\omega}\left(\int_0^{\omega} |x''(s)|^2 \mathrm{d}s\right)^{(1/2)} \\ &\quad + \lambda\left(\int_0^{\omega} |e(t)|^2 \mathrm{d}t\right)^{(1/2)}\left(\int_0^{\omega} |x''(s)|^2 \mathrm{d}s\right)^{(1/2)} \\ &\le (\lambda M_1')^2 + \lambda^2 CM_2\sqrt{\omega}M_1' \\ &\quad + \lambda^2 KM_1\sqrt{\omega}M_1' + \lambda^2\|e\|_2 M_1'.\end{aligned} \tag{41}$$

The above inequality implies

$$\begin{aligned}\left|\int_{x'(\xi)}^{x'(t)} g(u)\mathrm{d}u\right| &\le M_1'^2 + CM_2\sqrt{\omega}M_1' + KM_1\sqrt{\omega}M_1' \\ &\quad + \|e\|_2 M_1' := M_3'.\end{aligned} \tag{42}$$

From (23), we know that there exists a constant $M_3 > b$ such that

$$x'(t) \ge M_3, \quad \forall t \in [\xi, \omega]. \tag{43}$$

The case $t \in [0, \xi]$ can be treated similarly.

From (36), (37), and (43), we obtain

$$\Omega = \{x \in X\colon \|x\| < M_1 \text{ and } M_3 < x'(t) < M_2\, \forall t \in \mathbb{R}\}. \tag{44}$$

We know that (24) has no solution on $\partial\Omega$ as $\lambda \in (0,1)$, and when $x(t) \in \partial\Omega \cap \mathbb{R}$ and $x(t) = M_1$, from (31), we know that $M_1 > D_2$; So, from (29), we see that

$$\frac{1}{\omega}\int_0^{\omega}\{g(0) + Kx - e(t)\}\mathrm{d}t > 0, \tag{45}$$

since $C\int_0^{\omega} x'(t)\mathrm{d}t = 0$. So, condition (ii) is also satisfied. Set

$$H(x,\mu) = \mu x + (1-\mu)\frac{1}{\omega}\int_0^{\omega}(g(x') + Kx - e(t))\mathrm{d}t, \tag{46}$$

where $x \in \partial\Omega \cap \mathbb{R}$ and $\mu \in [0,1]$, we have

$$xH(x,\mu) = \mu x^2 + (1-\mu)\frac{x}{\omega}\int_0^{\omega}(g(x') + Kx - e(t))\mathrm{d}t > 0, \tag{47}$$

and thus $H(x,\mu)$ is a homotopic transformation and

$$\begin{aligned}\deg\{F, \Omega\cap\mathbb{R}, 0\} &= \deg\left\{\frac{1}{\omega}\int_0^{\omega}(g(x') + Kx - e(t))\mathrm{d}t, \Omega\cap\mathbb{R}, 0\right\} \\ &= \deg\{x, \Omega\cap\mathbb{R}, 0\} \ne 0.\end{aligned} \tag{48}$$

So, condition (iii) is satisfied. By Lemma 3.1, there exists a $\omega$-periodic solution $x$ with

$$x \in \left(\frac{\|e\| - \sigma}{K} - \frac{(\omega/2\pi)\|e\|_2}{1 - K(\omega/2\pi)^2}, \frac{\|e\| - \sigma}{K} + \frac{(\omega/2\pi)\|e\|_2}{1 - K(\omega/2\pi)^2}\right). \tag{49}$$

Finally, observe that $x$ is not a constant. Otherwise, suppose $x \equiv c$ (constant), then we get $x = e(t) - \sigma/K$, which contradicts the assumption $e(t) \equiv \sigma - Kc$; so, the proof is complete.

Next, we apply Theorem 3.1 to the plastic dynamical model of bulk metallic glasses (6). Model (6) is of form (22) with $C = B\gamma$, $g(x') = B\sigma_{f_0}/1 + Ax'$, $e(t) = B\sigma(0)$, and $K = Bk$. It is easy to see that $\lim_{x' \longrightarrow (-1/A)^+} \int_1^u (B\sigma_{f_0}/1 + Ax') = +\infty$. □

**Corollary 3.1.** *Assume that* $(\pi d^2 k/4M)(\omega/2\pi)^2 < 1$, $\sigma(0) > \sigma_{f_0}$, *and* $\sigma(0) \ne \sigma_{f_0} + kc$ *for all* $c \in \mathbb{R}$ *hold. Then, (6) has at least one nonconstant* $\omega$*-periodic solution* $x$ *with*

$$\begin{aligned}x \in &\left(-\frac{\sigma(0) - \sigma_{f_0}}{k} - \frac{\sqrt{\omega}\,\pi d^2\sigma(0)(\omega/2\pi)}{4M - \pi d^2 k(\omega/2\pi)^2}, \frac{\sigma(0) - \sigma_{f_0}}{k}\right. \\ &\left. + \frac{\sqrt{\omega}\,\pi d^2\sigma(0)(\omega/2\pi)}{4M - \pi d^2 k(\omega/2\pi)^2}\right).\end{aligned} \tag{50}$$

## 4. Conclusion and Discussion

In conclusion, we establish a model considering the internal friction during the plastic deformation and investigate how the parameters influence the stability of the system. Meanwhile, we prove the existence of chaotic and periodic solutions by applying mathematical methods. Based on Theorems 2.1 and 2.2, for larger internal friction coefficient, the plastic system manifests a stable state, while for smaller internal friction coefficient, the system becomes unstable. The increasing of the friction coefficient improves the resistance of the motion. As a result, it requires more energy for the plastic deformation, which means the state of the system will not be changed easily, reflecting a stable state.

Theorem 2.5 shows that the plastic dynamics transits from chaotic to stable state as the loading rate increases. For larger loading rate, the system evolves into a stable state. It is corresponding to that the self-organized critical behavior happens at the larger strain rate [15]. While for lower loading rate, the system is chaos, which is corresponding to the chaotic behavior happens at lower strain rate [15]. These results in Theorem 2.5 are consistent with the analysis based on the experimental data considering the loading rate is linearly dependent to the strain rate. Based on the result in Theorem 2.5, we can obtain a critical loading rate, $p = (1/k)(\sqrt{A\sigma_{f_0}/\gamma} - 1)$, and the strain rate can be estimated about $10^{-3}\,s^{-1}$. It is quite accordant with the results that the plastic dynamic behavior changes from chaotic to self-organized critical behavior as the strain rate increases from $4 \times 10^{-3}\,s^{-1}$ to $4 \times 10^{-2}\,s^{-1}$ in [15].

The stick-slip system shows rich dynamic behaviors such as chaos and quasi periodic solution [16, 17]. In this paper, we prove that there is a periodic solution based on mathematical theory, and the periodic solution is accordant with the sinusoidal density variations in shear bands [18]. The chaotic behavior is a result of the shear band instabilities [19]. We illustrate the plastic dynamics transits from chaos to stable state applying nonlinear dynamic theory and demonstrate how the parameters influence the plastic dynamics, which helps us to clarify the internal mechanism of plastic deformation for bulk metallic glasses.

## Data Availability

All data generated or analyzed during this study are included in this article.

## Conflicts of Interest

YSW and CZB declare that they have no competing interests.

## Authors' Contributions

YSW and CZB contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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