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Research Article

Iterative Analysis of the Unique Positive Solution for a Class of Singular Nonlinear Boundary Value Problems Involving Two Types of Fractional Derivatives with *p*-Laplacian Operator

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This article is concerned with a class of singular nonlinear fractional boundary value problems with *p*-Laplacian operator, which contains Riemann–Liouville fractional derivative and Caputo fractional derivative. The boundary conditions are made up of two kinds of Riemann–Stieltjes integral boundary conditions and nonlocal infinite-point boundary conditions, and different fractional orders are involved in the boundary conditions and the nonlinear term, respectively. Based on the method of reducing the order of fractional derivative, some properties of the corresponding Green's function, and the fixed point theorem of mixed monotone operator, an interesting result on the iterative sequence of the uniqueness of positive solutions is obtained under the assumption that the nonlinear term may be singular in regard to both the time variable and space variables. And we obtain the dependence of solution upon parameter. Furthermore, two numerical examples are presented to illustrate the application of our main results.

1. Introduction

In the past decades, fractional differential equations arise in many mathematical disciplines as the analogue modeling of systems and processes in many scientific fields, such as control theory and engineering. In fact, fractional-order models have proved to be valuable tools in modeling many physical phenomena (for details, see [1–3] and the references therein). Accordingly, there has been a substantial development in the research for fractional differential equations, the properties of solutions, especially. We refer the readers to the papers [4–54]. For instance, in [15], Xu and Wei investigated the positive solutions of the following fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}\left(D_{0+}^{\alpha}x(t)\right) = f\left(t, x(t), x'(t), -D_{0+}^{\alpha}x(t)\right), & t \in [0,1], \\ x(0) = x'(0) = x'(1) = 0, \\ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha+1}x(0) = D_{0+}^{\alpha+1}x(1) = 0, \end{cases} \tag{1}$$

where D_{0+}^{α} is the Riemann–Liouville's fractional derivative, $2 < \alpha \le 3$, and $f \in C([0,1] \times \mathbb{R}_+^3, \mathbb{R}_+)$ $(\mathbb{R}_+ = [0, +\infty))$. The existence of positive solutions is obtained by the Krasnoselskii–Zabreiko fixed point theorem. And by using the method of lower and upper solutions, the authors discussed the uniqueness of positive solution of this problem.

On the other hand, the fractional differential equations with the *p*-Laplacian operator can describe various phenomena, such as the flow of some specific fluid. Hence, the study of fractional differential equations with the *p*-Laplacian operator is gaining much significance and attention (see [16, 18–21, 37–39, 48–50]). For example, in [48], in order to research the mechanics phenomenon of turbulent flow in a porous medium, the author introduced the following type equation:

$$\left(\phi_p\left(x'(t)\right)\right)' = f\left(t, x(t), x'(t)\right), \tag{2}$$

where $\phi_p \, (p > 1)$ is the *p*-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s \, (p > 1)$. In the past thirty years, many papers discussed

equation (2) with different boundary value conditions and then drew related conclusion in practice. We refer the reader to [49, 50] and the references cited therein. Recently, Liu et al. in [16] investigated the existence of positive solution of the following fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\alpha}\phi_p\left({}^cD_{0+}^{\beta}x(t)\right) = f\left(t, x(t), {}^cD_{0+}^{\beta}x(t)\right), \\ {}^cD_{0+}^{\beta}x(0) = x'(0) = 0, \\ x(1) = r_1 x(\eta), \quad {}^cD_{0+}^{\beta}x(1) = r_2{}^cD_{0+}^{\beta}x(\xi), \end{cases} \tag{3}$$

where D_{0+}^{α} is the Riemann–Liouville's fractional derivative and ${}^cD_{0+}^{\beta}$ is the Caputo fractional derivative, $1 < \alpha, \beta \leq 2$, r_1, $r_1 \geq 0$, $\phi_p (p > 1)$ is the p-Laplacian operator, and $f \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_-, \mathbb{R}_+)$ ($\mathbb{R}_- = (-\infty, 0]$). The existence of lower and upper solutions is obtained by using the monotone iterative technique.

Based on the above illustration, it is of significance to study the fractional differential equations with p-Laplacian operator. In this article, we investigate the following fractional differential equation, which is a generalized form of the problem (2):

$$^cD_{0+}^{\alpha}\phi_p\left(D_{0+}^{\beta}v(t)\right) + f\left(t, v(t), D_{0+}^{\beta_1}v(t), \ldots, D_{0+}^{\beta_{n-2}}v(t)\right) = 0,$$
$$0 < t < 1, \tag{4}$$

with the boundary conditions

$$v(0) = D_{0+}^{\nu_i}v(0) = D_{0+}^{\beta}v(0) = \left(\phi_p\left(D_{0+}^{\beta}v(0)\right)\right)' = 0,$$
$$i = 1, \ldots, n-2,$$

$$D_{0+}^{r_0}v(1) = \lambda_1 \int_0^1 h_1(s)D_{0+}^{r_1}v(s)dA_1(s)$$
$$+ \lambda_2 \int_0^{\eta} h_2(s)D_{0+}^{r_2}v(s)dA_2(s) + \lambda_3 \sum_{j=1}^{\infty} \rho_j D_{0+}^{r_3}v(\eta_j), \tag{5}$$

where ${}^cD_{0+}^{\alpha}$ denotes the Caputo fractional derivative; D_{0+}^{β} denotes the Riemann–Liouville's fractional derivative; ϕ_p is the p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s (p > 1)$, $1 < \alpha \leq 2$, $n - 1 < \beta \leq n (n \geq 3)$, $1 < \beta - \beta_{n-2} \leq 2$, $i - 1 < \beta_i$, $\nu_i \leq i$, $\beta_{n-2} - \nu_i \leq n - 2 - i (i = 1, 2, \ldots, n-2)$, $\beta_{n-2} \leq r_k \leq r_0 \leq \beta - 1 (k = 1, 2, 3)$, $0 < \eta \leq 1$, $0 < \eta_1 \leq \eta_2 \leq \cdots \leq 1$, $\rho_j \geq 0$ $(j = 1, 2, \ldots)$, $\lambda_k \geq 0 (k = 1, 2, 3)$; $h_1, h_2 \colon (0, 1) \longrightarrow \mathbb{R}_+ = [0, +\infty)$ are continuous with $h_1, h_2 \in L^1(0, 1)$; $\int_0^1 h_1(s)v(s) dA_1(s)$, $\int_0^1 h_2(s)v(s)dA_2(s)$ denote the Riemann–Stieltjes integral, in which $A_1, A_2 \colon [0, 1] \longrightarrow \mathbb{R}$ are functions of bounded variation; and $f \colon (0, 1) \times (0, +\infty)^{n-1} \longrightarrow \mathbb{R}_+$ is continuous. Furthermore, we also consider the following equation with a parameter:

$$^cD_{0+}^{\alpha}\phi_p\left(D_{0+}^{\beta}v(t)\right) + \lambda f\left(t, v(t), D_{0+}^{\beta_1}v(t), \ldots, D_{0+}^{\beta_{n-2}}v(t)\right) = 0,$$
$$0 < t < 1, \tag{6}$$

with the boundary conditions (5), where $\lambda > 0$ is a parameter. In fact, we regard the problem (6)-(5) as a new problem for modeling the problem (4)-(5) involving a parameter. In this paper, a nonnegative function $v \in C[0, 1]$ is called a positive solution of boundary value problem (4)-(5) (resp. (6)-(5)) if it satisfies the problem (4)-(5) (resp. (6)-(5)) for $t \in [0, 1]$ and $v(t) > 0$ for $t \in (0, 1]$.

In this paper, we make the following hypothesis:

$(\mathbf{V}_1)$ $f \colon (0, 1) \times (0, +\infty)^{n-1} \longrightarrow \mathbb{R}_+$ is continuous and satisfies

$$f(t, u_1, \ldots, u_{n-1}) = F(t, u_1, \ldots, u_{n-1}, u_1, \ldots, u_{n-1})$$
$$+ G(t, u_1, \ldots, u_{n-1}, u_1, \ldots, u_{n-1}), \tag{7}$$

for all $(t, u_1, \ldots, u_{n-1}) \in (0, 1) \times (0, +\infty)^{n-1}$, where $F, G \colon (0, 1) \times (0, +\infty)^{2(n-1)} \longrightarrow \mathbb{R}_+$ are continuous.

$(\mathbf{V}_2)$ For any $t \in (0, 1)$, $(v_1, \ldots, v_{n-1}) \in (0, +\infty)^{n-1}$, $F(t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1})$, and $G(t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1})$ are nondecreasing in $(u_1, \ldots, u_{n-1}) \in (0, +\infty)^{n-1}$, and for any $t \in (0, 1)$, $(u_1, \ldots, u_{n-1}) \in (0, +\infty)^{n-1}$, $F(t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1})$, and $G(t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1})$ are nonincreasing in $(v_1, \ldots, v_{n-1}) \in (0, +\infty)^{n-1}$.

$(\mathbf{V}_3)$ For any $l \in (0, 1)$, there exists $\varphi(l) \in (l^{q-1}, 1]$ such that for any $t \in (0, 1)$, $(u_1, \ldots, u_{n-1})$ and $(v_1, \ldots, v_{n-1}) \in (0, +\infty)^{n-1}$:

$$F\left(t, lu_1, \ldots, lu_{n-1}, l^{-1}v_1, \ldots, l^{-1}v_{n-1}\right)$$
$$\geq \varphi^{p-1}(l)F(t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}),$$

$$G\left(t, lu_1, \ldots, lu_{n-1}, l^{-1}v_1, \ldots, l^{-1}v_{n-1}\right)$$
$$\geq l^{p-1}G(t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}). \tag{8}$$

$(\mathbf{V}_4)$ There exists $\varrho > 0$ such that for any $t \in (0, 1)$, $(u_1, \ldots, u_{n-1})$ and $(v_1, \ldots, v_{n-1}) \in (0, +\infty)^{n-1}$:

$$F(t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1})$$
$$\geq \varrho^{p-1}G(t, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-1}). \tag{9}$$

$(\mathbf{V}_5)$

$$0 < \int_0^1 (1-\tau)^{\alpha-1}F\left(\tau, \underbrace{1, 1, \ldots, 1}_{n-1\,\text{times}}, \underbrace{\tau^{\beta-1}, \tau^{\beta-1}, \ldots, \tau^{\beta-1}}_{n-1\,\text{times}}\right)d\tau$$
$$< +\infty,$$

$$0 < \int_0^1 (1-\tau)^{\alpha-1}G\left(\tau, \underbrace{1, 1, \ldots, 1}_{n-1\,\text{times}}, \underbrace{\tau^{\beta-1}, \tau^{\beta-1}, \ldots, \tau^{\beta-1}}_{n-1\,\text{times}}\right)d\tau$$
$$< +\infty. \tag{10}$$

Our main results are as follows.

Theorem 1. *Suppose that* $(V_1) - (V_5)$ *hold, then the problem (4)-(5) has a unique solution* $v^* \in P$, *and there exists* $\delta \in (0, 1)$ *such that*

$$\frac{\Gamma(\beta - \beta_{n-2})\delta}{\Gamma(\beta)}t^{\beta-1} \leq v^*(t) \leq \frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta)\delta}t^{\beta-1}, \quad t \in [0, 1].$$

(11)

And for any initial values $v_0, \widetilde{v}_0 \in P_h$, *by structuring the following sequences:*

$$v_n(t) = I_{0^+}^{\beta_{n-2}} \left\{ \int_0^1 J(t,s)\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}F\left(\tau, I_{0^+}^{\beta_{n-2}}v_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}v_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v_{n-1}(\tau), v_{n-1}(\tau), I_{0^+}^{\beta_{n-2}}\widetilde{v}_{n-1}(\tau), \right. \right.$$

$$\left. I_{0^+}^{\beta_{n-2}-\beta_1}\widetilde{v}_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\widetilde{v}_{n-1}(\tau), \widetilde{v}_{n-1}(\tau)\right)d\tau\right)ds$$

$$+ \int_0^1 J(t,s)\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}G\left(\tau, I_{0^+}^{\beta_{n-2}}v_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}v_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v_{n-1}(\tau), v_{n-1}(\tau), I_{0^+}^{\beta_{n-2}}\widetilde{v}_{n-1}(\tau), \right.\right.$$

$$\left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}\widetilde{v}_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\widetilde{v}_{n-1}(\tau), \widetilde{v}_{n-1}(\tau)\right)d\tau\right)ds\right\},$$

$$\widetilde{v}_n(t) = I_{0^+}^{\beta_{n-2}} \left\{ \int_0^1 J(t,s)\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}F\left(\tau, I_{0^+}^{\beta_{n-2}}\widetilde{v}_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}\widetilde{v}_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\widetilde{v}_{n-1}(\tau), \widetilde{v}_{n-1}(\tau), I_{0^+}^{\beta_{n-2}}v_{n-1}(\tau), \right.\right.$$

$$\left. I_{0^+}^{\beta_{n-2}-\beta_1}v_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v_{n-1}(\tau), v_{n-1}(\tau)d\tau\right)ds$$

$$+ \int_0^1 J(t,s)\phi_q\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}G\left(\tau, I_{0^+}^{\beta_{n-2}}\widetilde{v}_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}\widetilde{v}_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\widetilde{v}_{n-1}(\tau), \widetilde{v}_{n-1}(\tau), I_{0^+}^{\beta_{n-2}}v_{n-1}(\tau), \right.\right.$$

$$\left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}v_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v_{n-1}(\tau), v_{n-1}(\tau)\right)d\tau\right)ds\right\},$$

$$n = 1, 2, \ldots,$$

(12)

we have

$$\sup_{t\in[0,1]} |v_n(t) - v^*(t)| \longrightarrow 0,$$

$$\sup_{t\in[0,1]} |\widetilde{v}_n(t) - v^*(t)| \longrightarrow 0,$$

(13)

$$as\, n \longrightarrow \infty.$$

Theorem 2. *Suppose that* $(V_1) - (V_5)$ *hold, then for any* $\lambda > 0$, *the problem (6)-(5) have a unique solution* $v_\lambda^* \in P$, *and there exists* $\delta_\lambda \in (0, 1)$ *such that*

$$\frac{\Gamma(\beta - \beta_{n-2})\delta_\lambda}{\Gamma(\beta)}t^{\beta-1} \leq v_\lambda^*(t) \leq \frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta)\delta_\lambda}t^{\beta-1}, \quad t \in [0, 1].$$

(14)

And moreover,

(1) If there exists $\gamma \in (0, 1)$ *such that*

$$\varphi(l) \geq \frac{1}{\varrho}(l^\gamma - l) + l^\gamma, \quad \forall l \in (0, 1),$$

(15)

then v_λ^* *is continuous with respect to* $\lambda \in (0, +\infty)$. *That is, for any* $\lambda_0 \in (0, +\infty)$,

$$\sup_{t \in [0,1]} \left| v_\lambda^*(t) - v_{\lambda_0}^*(t) \right| \longrightarrow 0, \quad \text{as } \lambda \longrightarrow \lambda_0. \qquad (16)$$

(2) If

$$\varphi(l) \geq \frac{1}{\varrho} \left(l^{1/2} - l \right) + l^{1/2}, \quad \forall l \in (0,1), \qquad (17)$$

then $0 < \lambda_1 < \lambda_2$ *implies* $v_{\lambda_1}^* < v_{\lambda_2}^*$.

(3) If there exists $\gamma \in (0, 1/2)$ *such that*

$$\varphi(l) \geq \frac{1}{\varrho} \left(l^\gamma - l \right) + l^\gamma, \quad \forall l \in (0,1), \qquad (18)$$

then

$$
\begin{aligned}
\lim_{\lambda \longrightarrow 0^+} \sup_{t \in [0,1]} \left| v_\lambda^*(t) \right| &= 0, \\
\lim_{\lambda \longrightarrow +\infty} \sup_{t \in [0,1]} \left| v_\lambda^*(t) \right| &= +\infty.
\end{aligned}
\qquad (19)
$$

The key argument of the problem (4)-(5) and (6)-(5) is the iterative positive solution by applying the method of reducing the order of fractional derivative and the fixed point theory of mixed monotone operator. The method of reducing the order of fractional derivative is based on certain semigroup properties of the Riemann–Liouville's fractional integral and derivative. We refer the reader to [3, 24, 25, 34, 36, 46]. In this paper, different orders of Riemann–Liouville's fractional derivative are involved in the nonlinearity *f*, which is solved in a more complex space, in most cases. By using the method of reduction, we transform the problem (4)-(5) (resp. (6)-(5)) into an equivalent and low-order problem, in which the nonlinearity *f* contains no fractional derivative. Therefore, the work could proceed in the space $C[0, 1]$, which is more interesting and meaningful. On the other hand, by using the properties of relevant Green's function and cone, the theory of mixed monotone operator could be applied on the research of fractional boundary value problems. We suggest that one refers to [13, 17, 24, 31, 32, 34, 47]. In this paper, by structuring a suitable mixed monotone operator, the problem (4)-(5) (resp. (6)-(5)), which contains the *p*-Laplacian operator, is solved. In [15, 16], the positive solutions are obtained under the assumption that the nonlinear term *f* is continuous. But, we obtain the uniqueness of positive solution with the assumption that the nonlinear term may be singular in regard to both the time variable and space variables. Compared with [15], our equation contains *p*-Laplacian operator, which is more general. Compared with [16], our boundary conditions are of general significance, which would be used to describe more phenomena in practice.

The paper is organized as follows. In Section 2, we present some relevant definitions and lemmas. And we transform the problem (4)-(5) (resp. (6)-(5)) into an equivalent and low-order problem (28)-(29) (resp. (44)-(29)) and obtain some properties of the corresponding Green's function. In Section 3, we give the proof of our main results, as one researches the uniqueness of iterative positive solutions for the problem (4)-(5) and the other one analyses the dependence of solution upon parameter for the problem (6)-(5). In Section 4, we give two examples to illustrate our theoretical results. In Section 5, we give a conclusion of our main results.

2. Preliminaries and Lemmas

Definition 1. Let $\alpha > 0$. The Riemann–Liouville fractional integral of order α of a function $v: (0, \infty) \longrightarrow \mathbb{R}$ is given by

$$I_{0^+}^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds, \qquad (20)$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2. Let $\alpha > 0$. The Riemann–Liouville fractional derivative of order α of a continuous function $v: (0, \infty) \longrightarrow \mathbb{R}$ is given by

$$
\begin{aligned}
D_{0^+}^\alpha v(t) &= \left(\frac{d}{dt} \right)^n I_{0^+}^{n-\alpha} v(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \\
&\quad \cdot \int_0^t \frac{v(s)}{(t-s)^{\alpha-n+1}} ds,
\end{aligned}
\qquad (21)
$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, \infty)$. In particular, if $\alpha = n \in \mathbb{N}_+$, then $D_{0^+}^\alpha v(t) = v^{(n)}(t)$.

Definition 3. Let $\alpha > 0$. The Caputo fractional derivative of order α of a continuous function $v: (0, \infty) \longrightarrow \mathbb{R}$ is given by

$$^c D_{0^+}^\alpha v(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{v^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \qquad (22)$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, \infty)$. In particular, if $\alpha = n \in \mathbb{N}_+$, then $^c D_{0^+}^\alpha v(t) = v^{(n)}(t)$.

Lemma 1 (see [7]). *Let* $\alpha > 0$. *If we assume* $v \in C(0,1) \cap L^1(0,1)$, *then the equation*

$$D_{0^+}^\alpha v(t) = 0, \qquad (23)$$

has

$$v(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R}, \, i = 1, 2, \ldots, n, \qquad (24)$$

as the unique solution, where n is the smallest integer greater than or equal to α.

Lemma 2 (see [7]). *Let $v \in C(0,1) \cap L^1(0,1)$ and $D_{0+}^\alpha v \in C(0,1) \cap L^1(0,1)$. Then,*

$$I_{0+}^\alpha D_{0+}^\alpha v(t) = v(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$
$$c_i \in \mathbb{R}, \ i = 1, 2, \ldots, n, \tag{25}$$

where n is the smallest integer greater than or equal to α.

Lemma 3 (see [4]). *Let $v \in AC^n[0,1]$. Then,*

$$I_{0+}^\alpha \, {}^c D_{0+}^\alpha v(t) = v(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$
$$c_i \in \mathbb{R}, \ i = 1, 2, \ldots, n, \tag{26}$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}_+$ and $n = \alpha$ for $\alpha \in \mathbb{N}_+$.

Lemma 4 (see [4]).

(1) If $u \in L^1(0,1)$, $\alpha \geq \beta > 0$, then

$$D_{0+}^\alpha I_{0+}^\alpha u(t) = u(t),$$
$$D_{0+}^\beta I_{0+}^\alpha u(t) = I_{0+}^{\alpha-\beta} u(t). \tag{27}$$

(2) Let $\alpha \in (n-1, n)$. If $u(0) = u'(0) = \cdots = u^{(n-1)}(0)$, then ${}^c D_{0+}^\alpha v(t) = D_{0+}^\alpha v(t)$.

Let $u = D_{0+}^{\beta_{n-2}} v$, then the problem (4)-(5) could be transformed to the following fractional differential equation:

$${}^c D_{0+}^\alpha \phi_p \left(D_{0+}^{\beta - \beta_{n-2}} u(t) \right) + f\left(t, I_{0+}^{\beta_{n-2}} u(t),\right.$$
$$\left. I_{0+}^{\beta_{n-2} - \beta_1} u(t), \ldots, I_{0+}^{\beta_{n-2} - \beta_{n-3}} u(t), u(t) \right) = 0, \quad 0 < t < 1, \tag{28}$$

with the boundary conditions

$$D_{0+}^{\nu_{n-2} - \beta_{n-2}} u(0) = D_{0+}^{\beta - \beta_{n-2}} u(0) = \left(\phi_p \left(D_{0+}^{\beta - \beta_{n-2}} u(0) \right) \right)' = 0,$$

$$D_{0+}^{r_0 - \beta_{n-2}} u(1) = \lambda_1 \int_0^1 h_1(s) D_{0+}^{r_1 - \beta_{n-2}} u(s) dA_1(s)$$
$$+ \lambda_2 \int_0^\eta h_2(s) D_{0+}^{r_2 - \beta_{n-2}} u(s) dA_2(s)$$
$$+ \lambda_3 \sum_{j=1}^\infty \rho_j D_{0+}^{r_3 - \beta_{n-2}} u(\eta_j), \tag{29}$$

where $1 < \beta - \beta_{n-2} \leq 2$.

Lemma 5. *If the problem (4)-(5) have a solution $v \in C[0,1]$, then $u = D_{0+}^{\beta_{n-2}} v$ is a solution of the problem (28)-(29). And in* turn, if the problem (28)-(29) have a solution $u \in C[0,1]$, then $v = I_{0+}^{\beta_{n-2}} u$ is a solution of the problem (4)-(5).

Proof. Let

$$u(t) = D_{0+}^{\beta_{n-2}} v(t), \quad t \in [0,1], \tag{30}$$

where $v \in C[0,1]$ is a solution of the problem (4)-(5). By using Lemma 2, we have

$$v(t) = I_{0+}^{\beta_{n-2}} u(t) - c_1 t^{\beta_{n-2} - 1} - \cdots - c_{n-2} t^{\beta_{n-2} - (n-2)}, \tag{31}$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n - 2)$. By the boundary condition $v(0) = 0$, we get $c_{n-2} = 0$. So, we have

$$v(t) = I_{0+}^{\beta_{n-2}} u(t) - c_1 t^{\beta_{n-2} - 1} - \cdots - c_{n-3} t^{\beta_{n-2} - (n-3)}. \tag{32}$$

Furthermore, we obtain

$$D_{0+}^{\nu_1} v(t) = I_{0+}^{\beta_{n-2} - \nu_1} u(t) - c_1 \frac{\Gamma(\beta_{n-2})}{\Gamma(\beta_{n-2} - \nu_1)} t^{\beta_{n-2} - \nu_1 - 1}$$

$$- c_2 \frac{\Gamma(\beta_{n-2} - 1)}{\Gamma(\beta_{n-2} - \nu_1 - 1)} t^{\beta_{n-2} - \nu_1 - 2} - \cdots - c_{n-3} \tag{33}$$

$$\cdot \frac{\Gamma(\beta_{n-2} - (n-4))}{\Gamma(\beta_{n-2} - \nu_1 - (n-4))} t^{\beta_{n-2} - \nu_1 - (n-3)},$$

which combines with the boundary condition $D_{0+}^{\nu_1} v(0) = 0$ meaning $c_{n-3} = 0$. In a similar manner, we obtain $c_{n-4} = \cdots = c_1 = 0$ based on the boundary condition $D_{0+}^{\nu_2} v(0) = \cdots = D_{0+}^{\nu_{n-3}} v(0) = 0$. Thus, we have

$$v(t) = I_{0+}^{\beta_{n-2}} u(t), \quad t \in [0,1]. \tag{34}$$

Applying the operator D_{0+}^β to both sides of this equality, we have

$$D_{0+}^\beta v(t) = D_{0+}^\beta \left(I_{0+}^{\beta_{n-2}} u(t) \right) = \frac{d^n}{dt^n} I_{0+}^{n-\beta} \left(I_{0+}^{\beta_{n-2}} u(t) \right)$$

$$= \frac{d^n}{dt^n} I_{0+}^{n - (\beta - \beta_{n-2})} u(t)$$

$$= \frac{d^2}{dt^2} \left(\frac{d^{n-2}}{dt^{n-2}} I_{0+}^{n-2} \right) I_{0+}^{2 - (\beta - \beta_{n-2})} u(t) \tag{35}$$

$$= \frac{d^2}{dt^2} I_{0+}^{2 - (\beta - \beta_{n-2})} u(t)$$

$$= D_{0+}^{\beta - \beta_{n-2}} u(t).$$

Similarly, we know

$$D_{0+}^{\gamma_{n-2}} v(t) = D_{0+}^{\gamma_{n-2}-\beta_{n-2}} u(t), \qquad (36)$$

$$D_{0+}^{r_i} v(t) = D_{0+}^{r_i} I_{0+}^{\beta_{n-2}} u(t) = D_{0+}^{r_i - \beta_{n-2}} u(t), \quad i = 0, 1, 2, 3. \qquad (37)$$

In addition, by means of Lemma 4, we can obtain

$$D_{0+}^{\beta_i} v(t) = D_{0+}^{\beta_i} I_{0+}^{\beta_{n-2}} u(t) = I_{0+}^{\beta_{n-2}-\beta_i} u(t), \quad i = 1, 2, \ldots, n-3. \qquad (38)$$

According to (30), (34), (36), and (38), we have

$$\begin{aligned} {}^c D_{0+}^{\alpha} \phi_p \left(D_{0+}^{\beta-\beta_{n-2}} u(t) \right) &+ f\left(t, I_{0+}^{\beta_{n-2}} u(t), I_{0+}^{\beta_{n-2}-\beta_1} \right.\\ &\left. \cdot u(t), \ldots, I_{0+}^{\beta_{n-2}-\beta_{n-3}} u(t), u(t) \right) \\ =& {}^c D_{0+}^{\alpha} \phi_p \left(D_{0+}^{\beta} v(t) \right) + f\left(t, v(t), D_{0+}^{\beta_1} v(t), \ldots, D_{0+}^{\beta_{n-2}} v(t) \right) \\ =& \, 0. \end{aligned} \qquad (39)$$

It follows from (35)–(37) that

$$D_{0+}^{\beta-\beta_{n-2}} u(0) = D_{0+}^{\beta} v(0) = 0, \qquad (40)$$

$$\left(\phi_p \left(D_{0+}^{\beta-\beta_{n-2}} u(0) \right) \right)' = \left(\phi_p \left(D_{0+}^{\beta} v(0) \right) \right)' = 0, \qquad (41)$$

$$D_{0+}^{\gamma_{n-2}-\beta_{n-2}} u(0) = D_{0+}^{\gamma_{n-2}} v(0) = 0, \qquad (42)$$

$$\begin{aligned} D_{0+}^{r_0-\beta_{n-2}} u(1) = D_{0+}^{r_0} v(1) &= \lambda_1 \int_0^1 h_1(s) D_{0+}^{r_1} v(s) dA_1(s) \\ &+ \lambda_2 \int_0^\eta h_2(s) D_{0+}^{r_2} v(s) dA_2(s) \\ &+ \lambda_3 \sum_{i=1}^\infty \rho_i D_{0+}^{r_3} v(\eta_i) \\ &= \lambda_1 \int_0^1 h_1(s) D_{0+}^{r_1-\beta_{n-2}} u(s) dA_1(s) \\ &+ \lambda_2 \int_0^\eta h_2(s) D_{0+}^{r_2-\beta_{n-2}} u(s) dA_2(s) \\ &+ \lambda_3 \sum_{i=1}^\infty \rho_i D_{0+}^{r_3-\beta_{n-2}} u(\eta_i). \end{aligned} \qquad (43)$$

On the basis of (39)–(43), we know $u = D_{0+}^{\beta_{n-2}} v$ is a solution of the problem (28)-(29).

On the other hand, if the problem (28)-(29) has a solution $u \in C[0,1]$, then $v = I_{0+}^{\beta_{n-2}} u$ is a solution of the problem (4)-(5). Since the proof is similar to Lemma 3 in [44], we omit it here. $\qquad \square$

Remark 5. According to Lemma 5, we conclude that the work on considering the solution of the problem (4)-(5) is equivalent to the search for the solution of (28)-(29). Similarly, we can prove the work on the problem (6)-(5) is equivalent to the following equation:

$$\begin{aligned} {}^c D_{0+}^{\alpha} \phi_p \left(D_{0+}^{\beta-\beta_{n-2}} u(t) \right) &+ \lambda f\left(t, I_{0+}^{\beta_{n-2}} u(t), I_{0+}^{\beta_{n-2}-\beta_1} \right.\\ &\left. \cdot u(t), \ldots, I_{0+}^{\beta_{n-2}-\beta_{n-3}} u(t), u(t) \right) = 0, \quad 0 < t < 1, \end{aligned} \qquad (44)$$

with the boundary conditions (29).

Lemma 6. *Let $x \in C(0,1) \cap L^1(0,1)$. Then, the fractional differential equation*

$$D_{0+}^{\beta-\beta_{n-2}} u(t) + x(t) = 0, \quad 0 < t < 1, \ 1 < \beta - \beta_{n-2} \le 2, \qquad (45)$$

with the boundary conditions

$$\begin{aligned} D_{0+}^{\gamma_{n-2}-\beta_{n-2}} u(0) &= 0, \\ D_{0+}^{r_0-\beta_{n-2}} u(1) &= \lambda_1 \int_0^1 h_1(s) D_{0+}^{r_1-\beta_{n-2}} u(s) dA_1(s) \\ &+ \lambda_2 \int_0^\eta h_2(s) D_{0+}^{r_2-\beta_{n-2}} u(s) dA_2(s) \\ &+ \lambda_3 \sum_{j=1}^\infty \rho_j D_{0+}^{r_3-\beta_{n-2}} u(\eta_j), \end{aligned} \qquad (46)$$

has a unique solution

$$u(t) = \int_0^1 J(t,s) x(s) ds, \qquad (47)$$

where

$$\begin{aligned} J(t,s) = J_0(t,s) &+ t^{\beta-\beta_{n-2}-1} \left(\int_0^1 J_1(\tau,s) h_1(\tau) dA_1(\tau) \right) \\ &+ t^{\beta-\beta_{n-2}-1} \left(\int_0^\eta J_2(\tau,s) h_2(\tau) dA_2(\tau) \right) \\ &+ t^{\beta-\beta_{n-2}-1} \sum_{j=1}^\infty \rho_j J_3(\eta_j, s), \end{aligned} \qquad (48)$$

in which

$$J_0(t,s) = \frac{1}{\Gamma(\beta - \beta_{n-2})} \begin{cases} t^{\beta - \beta_{n-2} - 1}(1-s)^{\beta - r_0 - 1} - (t-s)^{\beta - \beta_{n-2} - 1}, & s \leq t, \\[2ex] t^{\beta - \beta_{n-2} - 1}(1-s)^{\beta - r_0 - 1}, & t \leq s, \end{cases}$$

$$J_1(t,s) = \frac{\lambda_1}{\sigma \Gamma(\beta - \beta_{n-2})\Gamma(\beta - r_1)} \begin{cases} t^{\beta - r_1 - 1}(1-s)^{\beta - r_0 - 1} - (t-s)^{\beta - r_1 - 1}, & s \leq t, \\[2ex] t^{\beta - r_1 - 1}(1-s)^{\beta - r_0 - 1}, & t \leq s, \end{cases}$$

$$J_2(t,s) = \frac{\lambda_2}{\sigma \Gamma(\beta - \beta_{n-2})\Gamma(\beta - r_2)} \begin{cases} t^{\beta - r_2 - 1}(1-s)^{\beta - r_0 - 1} - (t-s)^{\beta - r_2 - 1}, & s \leq t, \\[2ex] t^{\beta - r_2 - 1}(1-s)^{\beta - r_0 - 1}, & t \leq s, \end{cases}$$

$$J_3(t,s) = \frac{\lambda_3}{\sigma \Gamma(\beta - \beta_{n-2})\Gamma(\beta - r_3)} \begin{cases} t^{\beta - r_3 - 1}(1-s)^{\beta - r_0 - 1} - (t-s)^{\beta - r_3 - 1}, & s \leq t, \\[2ex] t^{\beta - r_3 - 1}(1-s)^{\beta - r_0 - 1}, & t \leq s, \end{cases}$$

$$\sigma = \frac{1}{\Gamma(\beta - r_0)} - \frac{\lambda_1}{\Gamma(\beta - r_1)} \int_0^1 s^{\beta - r_1 - 1} h_1(s) dA_1(s) - \frac{\lambda_2}{\Gamma(\beta - r_2)} \int_0^\eta s^{\beta - r_2 - 1} h_2(s) dA_2(s) - \frac{\lambda_3}{\Gamma(\beta - r_3)} \sum_{j=1}^\infty \rho_j \eta_j^{\beta - r_3 - 1} \neq 0.$$

$$(49)$$

Obviously, $J: [0,1] \times [0,1] \longrightarrow \mathbb{R}_+$ is continuous.

Proof. We may apply Lemma 2 to reduce equation (45) to an equivalent integral equation:

$$u(t) = c_1 t^{\beta - \beta_{n-2} - 1} + c_2 t^{\beta - \beta_{n-2} - 2} - I_{0+}^{\beta - \beta_{n-2}} x(t), \quad (50)$$

where $c_1, c_2 \in \mathbb{R}$. Since $D_{0+}^{\gamma_{n-2} - \beta_{n-2}} u(0) = 0$, we have $c_2 = 0$. So

$$u(t) = c_1 t^{\beta - \beta_{n-2} - 1} - I_{0+}^{\beta - \beta_{n-2}} x(t). \quad (51)$$

Applying the operator $D_{0+}^{r_i - \beta_{n-2}}$ ($i = 0, 1, 2, 3$) to both sides of (51), we have

$$D_{0+}^{r_i - \beta_{n-2}} u(t) = c_1 \frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta - r_i)} t^{\beta - r_i - 1} - I_{0+}^{\beta - r_i} x(t), \quad i = 0, 1, 2, 3.$$

$$(52)$$

Hence, according to (52) and the boundary condition

$$D_{0+}^{r_0 - \beta_{n-2}} u(1) = \lambda_1 \int_0^1 h_1(s) D_{0+}^{r_1 - \beta_{n-2}} u(s) dA_1(s)$$

$$+ \lambda_2 \int_0^\eta h_2(s) D_{0+}^{r_2 - \beta_{n-2}} u(s) dA_2(s) \quad (53)$$

$$+ \lambda_3 \sum_{j=1}^\infty \rho_j D_{0+}^{r_3 - \beta_{n-2}} u(\eta_j),$$

we can obtain

$$c_1 = \frac{1}{\sigma \Gamma(\beta - \beta_{n-2})} \left\{ \frac{1}{\Gamma(\beta - r_0)} \int_0^1 (1-s)^{\beta - r_0 - 1} x(s) ds - \frac{\lambda_1}{\Gamma(\beta - r_1)} \right.$$

$$\cdot \int_0^1 \left(\int_0^s (s-\tau)^{\beta - r_1 - 1} x(\tau) d\tau \right) h_1(s) dA_1(s) - \frac{\lambda_2}{\Gamma(\beta - r_2)}$$

$$\cdot \int_0^\eta \left(\int_0^s (s-\tau)^{\beta - r_2 - 1} x(\tau) d\tau \right) h_2(s) dA_2(s)$$

$$\left. - \frac{\lambda_3}{\Gamma(\beta - r_3)} \sum_{j=1}^\infty \rho_j \int_0^{\eta_j} (\eta_j - s)^{\beta - r_3 - 1} x(s) ds \right\},$$

$$(54)$$

where

$$\sigma = \frac{1}{\Gamma(\beta - r_0)} - \frac{\lambda_1}{\Gamma(\beta - r_1)} \int_0^1 s^{\beta - r_1 - 1} h_1(s) dA_1(s)$$

$$- \frac{\lambda_2}{\Gamma(\beta - r_2)} \int_0^\eta s^{\beta - r_2 - 1} h_2(s) dA_2(s) \quad (55)$$

$$- \frac{\lambda_3}{\Gamma(\beta - r_3)} \sum_{j=1}^\infty \rho_j \eta_j^{\beta - r_3 - 1} \neq 0.$$

Then

$$u(t) = -\frac{1}{\Gamma(\beta-\beta_{n-2})}\int_0^t (t-s)^{\beta-\beta_{n-2}-1}x(s)ds + \frac{t^{\beta-\beta_{n-2}-1}}{\sigma\Gamma(\beta-\beta_{n-2})}\left\{\frac{1}{\Gamma(\beta-r_0)}\int_0^1 (1-s)^{\beta-r_0-1}x(s)ds\right.$$

$$-\frac{\lambda_1}{\Gamma(\beta-r_1)}\int_0^1\left(\int_0^s (s-\tau)^{\beta-r_1-1}x(\tau)d\tau\right)h_1(s)dA_1(s) - \frac{\lambda_2}{\Gamma(\beta-r_2)}$$

$$\left.\cdot\int_0^\eta\left(\int_0^s (s-\tau)^{\beta-r_2-1}x(\tau)d\tau\right)h_2(s)dA_2(s) - \frac{\lambda_3}{\Gamma(\beta-r_3)}\sum_{j=1}^\infty \rho_j\int_0^{\eta_j}(\eta_j-s)^{\beta-r_3-1}x(s)ds\right\}$$

$$= -\frac{1}{\Gamma(\beta-\beta_{n-2})}\int_0^t (t-s)^{\beta-\beta_{n-2}-1}x(s)ds$$

$$+\frac{1}{\Gamma(\beta-\beta_{n-2})}\int_0^1 t^{\beta-\beta_{n-2}-1}(1-s)^{\beta-r_0-1}x(s)ds + \frac{t^{\beta-\beta_{n-2}-1}}{\sigma\Gamma(\beta-\beta_{n-2})}\int_0^1 (1-s)^{\beta-r_0-1}x(s)ds$$

$$\times\left\{\frac{\lambda_1}{\Gamma(\beta-r_1)}\int_0^1 s^{\beta-r_1-1}h_1(s)dA_1(s) + \frac{\lambda_2}{\Gamma(\beta-r_2)}\int_0^\eta s^{\beta-r_2-1}h_2(s)dA_2(s) + \frac{\lambda_3}{\Gamma(\beta-r_3)}\sum_{j=1}^\infty \rho_j\eta_j^{\beta-r_3-1}\right\}$$

$$-\left\{\frac{\lambda_1}{\Gamma(\beta-r_1)}\int_0^1\left(\int_0^s (s-\tau)^{\beta-r_1-1}x(\tau)d\tau\right)h_1(s)dA_1(s) + \frac{\lambda_2}{\Gamma(\beta-r_2)}\int_0^\eta\left(\int_0^s (s-\tau)^{\beta-r_2-1}x(\tau)d\tau\right)h_2(s)dA_2(s)\right.$$

$$\left.+\frac{\lambda_3}{\Gamma(\beta-r_3)}\sum_{j=1}^\infty \rho_j\int_0^{\eta_j}(\eta_j-s)^{\beta-r_3-1}x(s)ds\right\}\frac{t^{\beta-\beta_{n-2}-1}}{\sigma\Gamma(\beta-\beta_{n-2})}$$

$$=\frac{1}{\Gamma(\beta-\beta_{n-2})}\left\{\int_0^1 t^{\beta-\beta_{n-2}-1}(1-s)^{\beta-r_0-1}x(s)ds - \int_0^t (t-s)^{\beta-\beta_{n-2}-1}x(s)ds\right\}$$

$$+\frac{\lambda_1 t^{\beta-\beta_{n-2}-1}}{\sigma\Gamma(\beta-\beta_{n-2})\Gamma(\beta-r_1)}\left\{\int_0^1\left(\int_0^1 s^{\beta-r_1-1}(1-\tau)^{\beta-r_0-1}x(\tau)d\tau - \int_0^s (s-\tau)^{\beta-r_1-1}x(\tau)d\tau\right)h_1(s)dA_1(s)\right\}$$

$$+\frac{\lambda_2 t^{\beta-\beta_{n-2}-1}}{\sigma\Gamma(\beta-\beta_{n-2})\Gamma(\beta-r_2)}\left\{\int_0^\eta\left(\int_0^1 s^{\beta-r_2-1}(1-\tau)^{\beta-r_0-1}x(\tau)d\tau - \int_0^s (s-\tau)^{\beta-r_2-1}x(\tau)d\tau\right)h_2(s)dA_2(s)\right\}$$

$$+\frac{\lambda_3 t^{\beta-\beta_{n-2}-1}}{\sigma\Gamma(\beta-\beta_{n-2})\Gamma(\beta-r_3)}\sum_{j=1}^\infty \rho_j\left(\int_0^1 \eta_j^{\beta-r_3-1}(1-s)^{\beta-r_0-1}x(s)ds - \int_0^{\eta_j}(\eta_j-s)^{\beta-r_3-1}x(s)ds\right)$$

$$=\int_0^1 J_0(t,s)x(s)ds + \int_0^1 t^{\beta-\beta_{n-2}-1}\left(\int_0^s J_1(s,\tau)x(\tau)d\tau\right)h_1(s)dA_1(s) + \int_0^\eta t^{\beta-\beta_{n-2}-1}\left(\int_0^s J_2(s,\tau)x(\tau)d\tau\right)h_2(s)dA_2(s)$$

$$+t^{\beta-\beta_{n-2}-1}\int_0^1\sum_{j=1}^\infty \rho_j J_3(\eta_j,s)x(s)ds$$

$$=\int_0^1 J_0(t,s)x(s)ds + \int_0^1 t^{\beta-\beta_{n-2}-1}\left(\int_0^1 J_1(\tau,s)h_1(\tau)dA_1(\tau)\right)x(s)ds$$

$$+\int_0^1 t^{\beta-\beta_{n-2}-1}\left(\int_0^\eta J_2(\tau,s)h_2(\tau)dA_2(\tau)\right)x(s)ds + \int_0^1 t^{\beta-\beta_{n-2}-1}\sum_{j=1}^\infty \rho_j J_3(\eta_j,s)x(s)ds$$

$$=\int_0^1 J(t,s)x(s)ds.$$

$$(56)$$

The proof is complete.   □

Lemma 7. *Let $x \in C(0,1) \cap L^1(0,1)$ be a given function. Then, the following equation*

$$^c D_{0+}^\alpha \phi_p \left(D_{0+}^{\beta-\beta_{n-2}} u(t) \right) + x(t) = 0, \quad 0 < t < 1, \qquad (57)$$

with the boundary conditions

$$D_{0+}^{\gamma_{n-2}-\beta_{n-2}} u(0) = D_{0+}^{\beta-\beta_{n-2}} u(0) = \left(\phi_p \left(D_{0+}^{\beta-\beta_{n-2}} u(0) \right) \right)' = 0,$$

$$D_{0+}^{r_0-\beta_{n-2}} u(1) = \lambda_1 \int_0^1 h_1(s) D_{0+}^{r_1-\beta_{n-2}} u(s) dA_1(s)$$

$$+ \lambda_2 \int_0^\eta h_2(s) D_{0+}^{r_2-\beta_{n-2}} u(s) dA_2(s)$$

$$+ \lambda_3 \sum_{j=1}^\infty \rho_j D_{0+}^{r_3-\beta_{n-2}} u(\eta_j),$$

$$(58)$$

has a unique solution

$$u(t) = \int_0^1 J(t,s) \phi_q \left(\int_0^1 H(s,\tau) x(\tau) d\tau \right) ds, \qquad (59)$$

where $\phi_q = \phi_p^{-1}$, $J(t,s)$ is defined in Lemma 6, and

$$H(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ 0, & 0 \le t \le s \le 1. \end{cases} \qquad (60)$$

Proof. Let $\phi_p(D_{0+}^{\beta-\beta_{n-2}} u(t)) = w(t)$. Then, we can strip down the problem (57)-(58) to the following two problems:

$$\begin{cases} ^c D_{0+}^\alpha w(t) + x(t) = 0, & 0 < t < 1, \\ w(0) = w'(0) = 0, \end{cases} \qquad (61)$$

$$\begin{cases} D_{0+}^{\beta-\beta_{n-2}} u(t) = \phi_q w(t), & 0 < t < 1, \\ D_{0+}^{\gamma_{n-2}-\beta_{n-2}} u(0) = 0, \\ D_{0+}^{r_0-\beta_{n-2}} u(1) = \lambda_1 \int_0^1 h_1(s) D_{0+}^{r_1-\beta_{n-2}} u(s) dA_1(s) \\ + \lambda_2 \int_0^\eta h_2(s) D_{0+}^{r_2-\beta_{n-2}} u(s) dA_2(s) + \lambda_3 \sum_{j=1}^\infty \rho_j D_{0+}^{r_3-\beta_{n-2}} u(\eta_j). \end{cases}$$

$$(62)$$

Obviously, the problem (61) can be written as

$$w(t) = c_0 + c_1 t - I_{0+}^\alpha x(t). \qquad (63)$$

The boundary conditions $w(0) = w'(0) = 0$ imply that $c_0 = c_1 = 0$. Thus,

$$w(t) = -I_{0+}^\alpha x(t) = -\int_0^1 H(t,s) x(s) ds. \qquad (64)$$

On the other hand, by Lemma 6, we know the problem (62) has a unique solution:

$$u(t) = -\int_0^1 J(t,s) \phi_q(w(s)) ds. \qquad (65)$$

Therefore, the problem (57)-(58) has a unique solution:

$$u(t) = \int_0^1 J(t,s) \phi_q \left(\int_0^1 H(s,\tau) x(\tau) d\tau \right) ds. \qquad (66)$$

$\square$

Lemma 8. *Let $\sigma > 0$ (defined in Lemma 6), $\lambda_i \ge 0$ ($i = 1,2,3$), $\int_0^1 s^{\beta-r_1-1} h_1(s) dA_1(s) \ge 0$, $\int_0^\eta s^{\beta-r_2-1} h_2(s) dA_2(s) \ge 0$, and $0 \le \sum_{j=1}^\infty \rho_j \eta_j^{\beta-r_3-1} < +\infty$. Then, the following properties hold:*

(1) $t^{\beta-\beta_{n-2}-1} g_0(s) \le J_0(t,s) \le \Lambda_0 t^{\beta-\beta_{n-2}-1}$, where $\Lambda_0 = 1/(\Gamma(\beta - \beta_{n-2}))$,

$$g_0(s) = \frac{1}{\Gamma(\beta - \beta_{n-2})} (1-s)^{\beta-r_0-1} \left(1 - (1-s)^{r_0-\beta_{n-2}} \right). \qquad (67)$$

(2) $t^{\beta-r_i-1} g_i(s) \le J_i(t,s) \le \Lambda_i t^{\beta-r_i-1}$ ($i = 1,2,3$), where

$$g_i(s) = \frac{1}{\Gamma(\beta - r_i)} (1-s)^{\beta-r_0-1} \left(1 - (1-s)^{r_0-r_i} \right), \quad i = 1,2,3,$$

$$\Lambda_i = \frac{\lambda_i}{\sigma \Gamma(\beta - \beta_{n-2}) \Gamma(\beta - r_i)}, \quad i = 1,2,3.$$

$$(68)$$

(3) $h(t) g(s) \le J(t,s) \le \Lambda h(t)$, where $h(t) = t^{\beta-\beta_{n-2}-1}$,

$$g(s) = g_0(s) + \int_0^1 J_1(\tau,s) h_1(\tau) dA_1(\tau)$$

$$+ \int_0^\eta J_2(\tau,s) h_2(\tau) dA_2(\tau) + \sum_{j=1}^\infty J_3(\eta_j,s),$$

$$\Lambda = \Lambda_0 + \Lambda_1 \left(\int_0^1 \tau^{\beta-r_1-1} h_1(\tau) dA_1(\tau) \right)$$

$$+ \Lambda_2 \left(\int_0^\eta \tau^{\beta-r_2-1} h_2(\tau) dA_2(\tau) \right) + \Lambda_3 \sum_{j=1}^\infty \rho_j \eta_j^{\beta-r_3-1}. \qquad (69)$$

Proof

(1) For $0 \le s \le t \le 1$,

$$J_0(t,s) = \frac{1}{\Gamma(\beta - \beta_{n-2})} \left[t^{\beta-\beta_{n-2}-1} (1-s)^{\beta-r_0-1} - (t-s)^{\beta-\beta_{n-2}-1} \right]$$

$$\ge \frac{1}{\Gamma(\beta - \beta_{n-2})} \left[t^{\beta-\beta_{n-2}-1} (1-s)^{\beta-r_0-1} \right.$$

$$\left. - t^{\beta-\beta_{n-2}-1} (1-s)^{\beta-\beta_{n-2}-1} \right]$$

$$= \frac{1}{\Gamma(\beta - \beta_{n-2})} t^{\beta-\beta_{n-2}-1} (1-s)^{\beta-r_0-1} \left[1 - (1-s)^{r_0-\beta_{n-2}} \right]$$

$$= t^{\beta-\beta_{n-2}-1} g_0(s),$$

$$J_0(t,s) = \frac{1}{\Gamma(\beta - \beta_{n-2})} \left[t^{\beta-\beta_{n-2}-1} (1-s)^{\beta-r_0-1} - (t-s)^{\beta-\beta_{n-2}-1} \right]$$

$$\le \Lambda_0 t^{\beta-\beta_{n-2}-1}.$$

$$(70)$$

For $0 \leq t \leq s \leq 1$,

$$J_0(t,s) = \frac{1}{\Gamma(\beta - \beta_{n-2})} t^{\beta-\beta_{n-2}-1}(1-s)^{\beta-r_0-1}$$

$$\geq \frac{1}{\Gamma(\beta - \beta_{n-2})} t^{\beta-\beta_{n-2}-1}(1-s)^{\beta-r_0-1}\left[1-(1-s)^{r_0-\beta_{n-2}}\right]$$

$$= t^{\beta-\beta_{n-2}-1} g_0(s),$$

$$J_0(t,s) = \frac{1}{\Gamma(\beta - \beta_{n-2})} t^{\beta-\beta_{n-2}-1}(1-s)^{\beta-r_0-1}$$

$$\leq \Lambda_0 t^{\beta-\beta_{n-2}-1}.$$

$$(71)$$

(2) Similarly as (1), we can obtain that

$$t^{\beta-r_i-1} g_i(s) \leq J_i(t,s) \leq \Lambda_i t^{\beta-r_i-1}, \quad i=1,2,3. \quad (72)$$

(3) In view of the conclusions of (1) and (2), we have

$$J(t,s) = J_0(t,s) + t^{\beta-\beta_{n-2}-1}\left(\int_0^1 J_1(\tau,s)h_1(\tau)dA_1(\tau)\right)$$

$$+ t^{\beta-\beta_{n-2}-1}\left(\int_0^\eta J_2(\tau,s)h_2(\tau)dA_2(\tau)\right)$$

$$+ t^{\beta-\beta_{n-2}-1}\sum_{j=1}^\infty \rho_j J_3(\eta_j,s)$$

$$\geq t^{\beta-\beta_{n-2}-1} g_0(s) + t^{\beta-\beta_{n-2}-1}\left(\int_0^1 J_1(\tau,s)h_1(\tau)dA_1(\tau)\right)$$

$$+ t^{\beta-\beta_{n-2}-1}\left(\int_0^\eta J_2(\tau,s)h_2(\tau)dA_2(\tau)\right)$$

$$+ t^{\beta-\beta_{n-2}-1}\sum_{j=1}^\infty \rho_j J_3(\eta_j,s)$$

$$= h(t)g(s),$$

$$J(t,s) = J_0(t,s) + t^{\beta-\beta_{n-2}-1}\left(\int_0^1 J_1(\tau,s)h_1(\tau)dA_1(\tau)\right)$$

$$+ t^{\beta-\beta_{n-2}-1}\left(\int_0^\eta J_2(\tau,s)h_2(\tau)dA_2(\tau)\right)$$

$$+ t^{\beta-\beta_{n-2}-1}\sum_{j=1}^\infty \rho_j J_3(\eta_j,s)$$

$$\leq \Lambda_0 t^{\beta-\beta_{n-2}-1} + \Lambda_1 t^{\beta-\beta_{n-2}-1}\left(\int_0^1 \tau^{\beta-r_1-1}h_1(\tau)dA_1(\tau)\right)$$

$$+ \Lambda_2 t^{\beta-\beta_{n-2}-1}\left(\int_0^\eta \tau^{\beta-r_2-1}h_2(\tau)dA_2(\tau)\right)$$

$$+ \Lambda_3 t^{\beta-\beta_{n-2}-1}\sum_{j=1}^\infty \rho_j \eta_j^{\beta-r_3-1}$$

$$= \Lambda h(t).$$

$$(73)$$

The proof is complete.   □

Let $(E, \|\cdot\|)$ be a Banach space, P be a cone in E, and θ be the zero element of E. P is said to be normal if there exists a constant $N > 0$ such that

$$\theta \leq u \leq v \Longrightarrow \|u\| \leq N\|v\|, \quad \forall u, v \in P. \quad (74)$$

The smallest constant, which satisfies the above inequality, is called the normality constant of P. Then, E is a partially ordered Banach space by P; that is to say,

$$v \leq u \Longleftrightarrow u - v \in P. \quad (75)$$

For $u, v \in E$, the notation $u \sim v$ shows that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda u \leq v \leq \mu u$. Clearly, $\sim$ is an equivalence relation in E. For any $h \in P \setminus \{\theta\}$, let $P_h = \{u \in E \mid u \sim h\} = \{u \in E \mid \mu h \leq u \leq \mu^{-1}h, \ 0 < \mu < 1\}$. Then, $P_h \subset P$ is a component of P. For more details, we suggest readers to refer [10, 11, 13].

Definition 4 (see [11]). Let E be a Banach space and $D \subset E$. The operator $T: D \times D \longrightarrow E$ is called a mixed monotone operator if $T(u,v)$ is increasing in $u \in D$ and decreasing in $v \in D$, i.e.,

$$\begin{aligned} u_1 \leq u_2, \\ v_1 \geq v_2 \Longrightarrow T(u_1,v_1) \leq T(u_2,v_2), \\ \forall u_i, v_i \in D, i=1,2. \end{aligned} \quad (76)$$

Lemma 9 (see [13, 17]). *Let P be a normal cone in the Banach space E and $T, S: P_h \times P_h \longrightarrow P_h$ be mixed monotone operators which satisfy the following:*

(1) For any $l \in (0,1)$, there exists $\varphi(l) \in (l,1]$ such that

$$T(lu, l^{-1}v) \geq \varphi(l)T(u,v), \quad \forall u, v \in P_h. \quad (77)$$

(2) or any $l \in (0,1)$, $u, v \in P_h$,

$$S(lu, l^{-1}v) \geq lS(u,v). \quad (78)$$

(3) There exists a constant $\varrho > 0$ such that for any $u, v \in P_h$, $T(u,v) \geq \varrho S(u,v)$.

Then, the equation $T(u,u) + S(u,u) = u$ has a unique fixed point $u^* \in P_h$. And for any initial values $u_0, v_0 \in P_h$, by structuring the following sequences:

$$\begin{aligned} u_n &= T(u_{n-1}, v_{n-1}) + S(u_{n-1}, v_{n-1}), \\ v_n &= T(v_{n-1}, u_{n-1}) + S(v_{n-1}, u_{n-1}), \\ &n = 1, 2, \ldots, \end{aligned} \quad (79)$$

we have $\|u_n - u^*\| \longrightarrow 0$ and $\|v_n - u^*\| \longrightarrow 0$ in E, as $n \longrightarrow \infty$.

Lemma 10 (see [13, 17]). *Assume T and S satisfy all the conditions of Lemma 9. ien, for any $\lambda > 0$, the equation*

$\lambda T(u, u) + \lambda S(u, u) = u$ has a unique solution $u_\lambda^* \in P_h$, which satisfies the following:

(1) If there exists $\gamma \in (0, 1)$ such that

$$\varphi(l) \geq \frac{1}{\varrho}\left(l^\gamma - l\right) + l^\gamma, \quad \forall l \in (0, 1), \tag{80}$$

then u_λ^* is continuous with respect to $\lambda \in (0, +\infty)$. That is, for any $\lambda_0 \in (0, +\infty)$,

$$\left\| u_\lambda^* - u_{\lambda_0}^* \right\| \longrightarrow 0, \quad as\, \lambda \longrightarrow \lambda_0. \tag{81}$$

(2) If

$$\varphi(l) \geq \frac{1}{\varrho}\left(l^{1/2} - l\right) + l^{1/2}, \quad \forall l \in (0, 1), \tag{82}$$

then $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* < u_{\lambda_2}^*$.

(3) If there exists $\gamma \in (0, 1/2)$ such that

$$\varphi(l) \geq \frac{1}{\varrho}\left(l^\gamma - l\right) + l^\gamma, \quad \forall l \in (0, 1), \tag{83}$$

then

$$\begin{aligned} \lim_{\lambda \longrightarrow 0^+} \left\| u_\lambda^* \right\| &= 0, \\ \lim_{\lambda \longrightarrow +\infty} \left\| u_\lambda^* \right\| &= +\infty. \end{aligned} \tag{84}$$

In this paper, we denote $E = C[0, 1]$ with the norm $\|v\| = \sup_{0 \leq t \leq 1} |v(t)|$. Then $(E, \|\cdot\|)$ is a Banach space. Let $P = \{v \in E : v(t) \geq 0, t \in [0, 1]\}$ be a cone in E. It is easy to check that P is normal in E with the normality constant $N = 1$.

3. Proof of Main Results

In this section, let $P_h = \{u \in E : u \sim h\}$, where $h(t) = t^{\beta - \beta_{n-2} - 1}$, $t \in [0, 1]$ (defined in Lemma 8). Then, P_h is a component of P. Let us define three operators $T : P_h \longrightarrow P$ and $T_1, T_2 : P_h \times P_h \longrightarrow P$ as follows:

$$(Tu)(t) = \int_0^1 J(t, s)\phi_q\left(\int_0^1 H(s, \tau)f\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau)\right)d\tau\right)ds,$$

$$T_1(u, v)(t) = \int_0^1 J(t, s)\phi_q\left(\int_0^1 H(s, \tau)F\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right)d\tau\right)ds,$$

$$T_2(u, v)(t) = \int_0^1 J(t, s)\phi_q\left(\int_0^1 H(s, \tau)G\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right)d\tau\right)ds. \tag{85}$$

It is easy to check that $u \in C[0, 1]$ is a positive solution of the problem (28)-(29) if it is a fixed point of T in P_h.

Proof of Theorem 1. Obviously, by $(\mathbf{V}_1)$, $Tu = T_1(u, u) + T_2(u, u)$. And it is easy to check that $u \in C[0, 1]$ is a solution of the problem (28)-(29) if it satisfies $u = Tu = T_1(u, u) + T_2(u, u)$.

The first work is to prove $T_1, T_2 : P_h \times P_h \longrightarrow P$ are well defined. For any $u, v \in P_h$, there exists a constant $\mu \in (0, 1)$ such that

$$\begin{aligned} \mu h(t) &\leq u(t) \leq \mu^{-1}h(t), \\ \mu h(t) &\leq v(t) \leq \mu^{-1}h(t), \end{aligned} \tag{86}$$

$$t \in [0, 1].$$

On the other hand, it is easy to check that

$$\begin{aligned} I_{0^+}^{\beta_{n-2}}h(t) &= \frac{1}{\Gamma(\beta_{n-2})}\int_0^t (t - s)^{\beta_{n-2}-1}s^{\beta - \beta_{n-2}-1}ds \\ &= \frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta)}t^{\beta-1} \leq 1, \quad t \in [0, 1], \end{aligned} \tag{87}$$

$$\begin{aligned} I_{0^+}^{\beta_{n-2}-\beta_i}h(t) &= \frac{1}{\Gamma(\beta_{n-2} - \beta_i)}\int_0^t (t - s)^{\beta_{n-2}-\beta_i-1}s^{\beta - \beta_{n-2}-1}ds \\ &= \frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta - \beta_i)}t^{\beta-\beta_i-1} \leq 1, \\ &\qquad i = 1, 2, \ldots, n - 3, t \in [0, 1]. \end{aligned} \tag{88}$$

Let

$$r = \min\left\{\frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta)}, \frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta - \beta_1)}, \ldots, \frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta - \beta_{n-3})}, 1\right\}. \tag{89}$$

Then, $0 < r \leq 1$. According to $(\mathbf{V}_1)$–$(\mathbf{V}_3)$ and (87)–(89), we have

$$
\begin{aligned}
&F\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right) \\
&\leq F\left(\tau, I_{0^+}^{\beta_{n-2}}\mu^{-1}h(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}\mu^{-1}h(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\mu^{-1}h(\tau), \mu^{-1}h(\tau), I_{0^+}^{\beta_{n-2}}\mu h(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}\mu h(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\mu h(\tau), \mu h(\tau)\right) \\
&\leq F\left(\tau, \mu^{-1}, \mu^{-1}, \ldots, \mu^{-1}, \mu^{-1}, \mu\frac{\Gamma(\beta-\beta_{n-2})}{\Gamma(\beta)}\tau^{\beta-1}, \mu\frac{\Gamma(\beta-\beta_{n-2})}{\Gamma(\beta-\beta_1)}\tau^{\beta-\beta_1-1}, \ldots, \mu\frac{\Gamma(\beta-\beta_{n-2})}{\Gamma(\beta-\beta_{n-3})}\tau^{\beta-\beta_{n-3}-1}, \tau^{\beta-\beta_{n-2}-1}\right) \\
&\leq F\left(\tau, (\mu r)^{-1}, (\mu r)^{-1}, \ldots, (\mu r)^{-1}, (\mu r)^{-1}, \mu r\tau^{\beta-1}, \mu r\tau^{\beta-\beta_1-1}, \ldots, \mu r\tau^{\beta-\beta_{n-3}-1}, \mu r\tau^{\beta-\beta_{n-2}-1}\right) \\
&\leq \frac{1}{\varphi^{p-1}(\mu r)}F\left(\tau, 1, 1, \ldots, 1, 1, \tau^{\beta-1}, \tau^{\beta-\beta_1-1}, \ldots, \tau^{\beta-\beta_{n-3}-1}, \tau^{\beta-\beta_{n-2}-1}\right) \\
&\leq \frac{1}{\varphi^{p-1}(\mu r)}F\left(\tau, 1, \ldots, 1, \tau^{\beta-1}, \ldots, \tau^{\beta-1}\right),
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
&F\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right) \\
&\geq F\left(\tau, I_{0^+}^{\beta_{n-2}}\mu h(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}\mu h(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\mu h(\tau), \mu h(\tau), I_{0^+}^{\beta_{n-2}}\mu^{-1}h(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}\mu^{-1}h(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\mu^{-1}h(\tau), \mu^{-1}h(\tau)\right) \\
&\geq F\left(\tau, \mu\frac{\Gamma(\beta-\beta_{n-2})}{\Gamma(\beta)}\tau^{\beta-1}, \mu\frac{\Gamma(\beta-\beta_{n-2})}{\Gamma(\beta-\beta_1)}\tau^{\beta-\beta_1-1}, \ldots, \mu\frac{\Gamma(\beta-\beta_{n-2})}{\Gamma(\beta-\beta_{n-3})}\tau^{\beta-\beta_{n-3}-1}, \mu\tau^{\beta-\beta_{n-2}-1}, \mu^{-1}, \mu^{-1}, \ldots, \mu^{-1}, \mu^{-1}\right) \\
&\geq F\left(\tau, \mu r\tau^{\beta-1}, \mu r\tau^{\beta-\beta_1-1}, \ldots, \mu r\tau^{\beta-\beta_{n-3}-1}, \mu r\tau^{\beta-\beta_{n-2}-1}, (\mu r)^{-1}, (\mu r)^{-1}, \ldots, (\mu r)^{-1}, (\mu r)^{-1}\right) \\
&\geq \varphi^{p-1}(\mu r)F\left(\tau, \tau^{\beta-1}, \tau^{\beta-\beta_1-1}, \ldots, \tau^{\beta-\beta_{n-3}-1}, \tau^{\beta-\beta_{n-2}-1}, 1, 1, \ldots, 1, 1\right) \\
&\geq \varphi^{p-1}(\mu r)F\left(\tau, \tau^{\beta-1}, \ldots, \tau^{\beta-1}, 1, \ldots, 1\right).
\end{aligned}
\tag{91}
$$

Similarly, we have

$$
\begin{aligned}
&G\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau),\right. \\
&\left. u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right) \\
&\leq \frac{1}{(\mu r)^{p-1}}G\left(\tau, 1, \ldots, 1, \tau^{\beta-1}, \ldots, \tau^{\beta-1}\right),
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
&G\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau),\right. \\
&\left. I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right) \geq (\mu r)^{p-1} \\
&\cdot G\left(\tau, \tau^{\beta-1}, \ldots, \tau^{\beta-1}, 1, \ldots, 1\right).
\end{aligned}
\tag{93}
$$

On the basis of (90), (92), and $(\mathbf{V}_5)$, we know that for any $t \in [0,1]$,

$$
\begin{aligned}
T_1(u,v)(t) &= \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right)d\tau\right)ds \\
&\leq \frac{\Lambda h(t)}{(\Gamma(\alpha))^{q-1}\varphi(\mu r)}\int_0^1 \phi_q\left(\int_0^1 (1-\tau)^{\alpha-1}F\left(\tau, 1, \ldots, 1, \tau^{\beta-1}, \ldots, \tau^{\beta-1}\right)d\tau\right)ds \\
&< +\infty,
\end{aligned}
\tag{94}
$$

$$
T_2(u,v)(t) \leq \frac{\Lambda h(t)}{(\Gamma(\alpha))^{q-1}(\mu r)}\int_0^1 \phi_q\left(\int_0^1 (1-\tau)^{\alpha-1}G\left(\tau, 1, \ldots, 1, \tau^{\beta-1}, \ldots, \tau^{\beta-1}\right)d\tau\right)ds < +\infty.
\tag{95}
$$

That is, $T_1, T_2 : P_h \times P_h \longrightarrow P$ are well defined. On the other hand, by means of $(\mathbf{V}_2)$, for any (u_1, v_1), (u_2, v_2) satisfying $u_1 \leq u_2$ and $v_1 \geq v_2$, we have

$$
\begin{aligned}
T_1(u_1, v_1)(t) &= \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau, I_{0^+}^{\beta_{n-2}}u_1(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u_1(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u_1(\tau), u_1(\tau), I_{0^+}^{\beta_{n-2}}v_1(\tau),\right.\right.\\
&\qquad\left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}v_1(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v_1(\tau), v_1(\tau)\right)d\tau\right)ds\\
&\leq \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau, I_{0^+}^{\beta_{n-2}}u_2(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u_2(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u_2(\tau), u_2(\tau), I_{0^+}^{\beta_{n-2}}v_2(\tau),\right.\right.\\
&\qquad\left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}v_2(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v_2(\tau), v_2(\tau)\right)d\tau\right)ds\\
&= T_1(u_2, v_2)(t), \quad t \in [0,1],
\end{aligned}
\tag{96}
$$

$$
\begin{aligned}
T_2(u_1, v_1)(t) &= \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)G\tau, I_{0^+}^{\beta_{n-2}}u_1(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u_1(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u_1(\tau), u_1(\tau), I_{0^+}^{\beta_{n-2}}v_1(\tau),\right.\\
&\qquad\left. I_{0^+}^{\beta_{n-2}-\beta_1}v_1(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v_1(\tau), v_1(\tau)d\tau\right)ds\\
&\leq \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)G\left(\tau, I_{0^+}^{\beta_{n-2}}u_2(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u_2(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u_2(\tau), u_2(\tau), I_{0^+}^{\beta_{n-2}}v_2(\tau),\right.\right.\\
&\qquad\left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}v_2(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v_2(\tau), v_2(\tau)\right)d\tau\right)ds\\
&= T_2(u_2, v_2)(t), \quad t \in [0,1].
\end{aligned}
\tag{97}
$$

Hence, $T_1, T_2 : P_h \times P_h \longrightarrow P$ are mixed monotone operators.

In the following, we prove that $T_1, T_2 : P_h \times P_h \longrightarrow P_h$. It follows from (91) and (93) that for any $u, v \in P_h$ and $t \in [0,1]$,

$$
\begin{aligned}
T_1(u, v)(t) &= \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau),\right.\right.\\
&\qquad\left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \dots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right)d\tau\right)ds\\
&\geq \int_0^1 J(t,s)\phi_q\left(\varphi^{p-1}(\mu r)\int_0^1 H(s,\tau)F\left(\tau, \tau^{\beta-1}, \dots, \tau^{\beta-1}, 1, \dots, 1\right)d\tau\right)ds\\
&\geq \varphi(\mu r)h(t)\int_0^1 g(s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau, \tau^{\beta-1}, \dots, \tau^{\beta-1}, 1, \dots, 1\right)d\tau\right)ds\\
&\geq M^{-1}h(t),
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
T_2(u, v)(t) &\geq \mu r h(t)\int_0^1 g(s)\phi_q\left(\int_0^1 H(s,\tau)G\left(\tau, \tau^{\beta-1}, \dots, \tau^{\beta-1}, 1, \dots, 1\right)d\tau\right)ds\\
&\geq M^{-1}h(t),
\end{aligned}
\tag{99}
$$

where M is a constant, which satisfies

$$\max\left\{\frac{\Lambda}{(\Gamma(\alpha))^{q-1}\varphi(\mu r)}\int_0^1\phi_q\left(\int_0^1(1-\tau)^{\alpha-1}F\left(\tau,1,\ldots,1,\tau^{\beta-1},\ldots,\tau^{\beta-1}\right)d\tau\right)ds,\right.$$

$$\cdot\frac{\Lambda}{(\Gamma(\alpha))^{q-1}(\mu r)}\int_0^1\phi_q\left(\int_0^1(1-\tau)^{\alpha-1}G\left(\tau,1,\ldots,1,\tau^{\beta-1},\ldots,\tau^{\beta-1}\right)d\tau\right)ds,$$

$$\cdot\left(\varphi(\mu r)\int_0^1 g(s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau,\tau^{\beta-1},\ldots,\tau^{\beta-1},1,\ldots,1\right)d\tau\right)ds\right)^{-1},\qquad(100)$$

$$\left.\cdot\left(\mu r\int_0^1 g(s)\phi_q\left(\int_0^1 H(s,\tau)G\left(\tau,\tau^{\beta-1},\ldots,\tau^{\beta-1},1,\ldots,1\right)d\tau\right)ds\right)^{-1}\right\}<M.$$

According to (94) and (95), we have

$$T_1(u,v)(t)\le\frac{\Lambda h(t)}{(\Gamma(\alpha))^{q-1}\varphi(\mu r)}\int_0^1\phi_q\left(\int_0^1(1-\tau)^{\alpha-1}F\left(\tau,1,\ldots,1,\tau^{\beta-1},\ldots,\tau^{\beta-1}\right)d\tau\right)ds$$
$$\le Mh(t),\qquad(101)$$

$$T_2(u,v)(t)\le\frac{\Lambda h(t)}{(\Gamma(\alpha))^{q-1}(\mu r)}\int_0^1\phi_q\left(\int_0^1(1-\tau)^{\alpha-1}G\left(\tau,1,\ldots,1,\tau^{\beta-1},\ldots,\tau^{\beta-1}\right)d\tau\right)ds$$
$$\le Mh(t).\qquad(102)$$

On the basis of (98), (99), (101), and (102) we infer that $T_1,T_2:P_h\times P_h\longrightarrow P_h$.

Moreover, it follows from $(\mathbf{V}_3)$, for any $l\in(0,1)$ and u, $v\in P_h$ and $t\in[0,1]$, we have

$$T_1\left(lu,l^{-1}v\right)(t)=\int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau,I_{0^+}^{\beta_{n-2}}lu(\tau),I_{0^+}^{\beta_{n-2}-\beta_1}lu(\tau),\ldots,I_{0^+}^{\beta_{n-2}-\beta_{n-3}}lu(\tau),lu(\tau),I_{0^+}^{\beta_{n-2}}l^{-1}v(\tau),\right.\right.$$
$$I_{0^+}^{\beta_{n-2}-\beta_1}l^{-1}v(\tau),\ldots,I_{0^+}^{\beta_{n-2}-\beta_{n-3}}l^{-1}v(\tau),l^{-1}v(\tau))d\tau\Big)ds$$

$$=\int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau,lI_{0^+}^{\beta_{n-2}}u(\tau),lI_{0^+}^{\beta_{n-2}-\beta_1}u(\tau),\ldots,lI_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau),lu(\tau),l^{-1}I_{0^+}^{\beta_{n-2}}v(\tau),\right.\right.$$
$$l^{-1}I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau),\ldots,l^{-1}I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau),l^{-1}v(\tau))d\tau\Big)ds$$

$$\ge\int_0^1 J(t,s)\phi_q\varphi^{p-1}(l)\left(\int_0^1 H(s,\tau)F\left(\tau,I_{0^+}^{\beta_{n-2}}u(\tau),I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau),\ldots,I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau),u(\tau),I_{0^+}^{\beta_{n-2}}v(\tau),\right.\right.$$
$$I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau),\ldots,I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau),v(\tau))d\tau\Big)ds$$

$$=\varphi(l)\int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau,I_{0^+}^{\beta_{n-2}}u(\tau),I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau),\ldots,I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau),u(\tau),I_{0^+}^{\beta_{n-2}}v(\tau),\right.\right.$$
$$I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau),\ldots,I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau),v(\tau))d\tau\Big)ds$$

$$=\varphi(l)T_1(u,v)(t),$$

$$(103)$$

$$
\begin{aligned}
T_2\left(lu, l^{-1}v\right)(t) &= \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)G\left(\tau, I_{0^+}^{\beta_{n-2}}lu(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}lu(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}lu(\tau), lu(\tau), I_{0^+}^{\beta_{n-2}}l^{-1}v(\tau),\right.\right. \\
&\qquad \left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}l^{-1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}l^{-1}v(\tau), l^{-1}v(\tau)\right)d\tau\right)ds \\
&\geq l\int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)G\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau),\right.\right. \\
&\qquad \left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right)d\tau\right)ds \\
&= lT_2(u,v)(t).
\end{aligned}
\tag{104}
$$

From $(\mathbf{V}_4)$, for any $u, v \in P_h$ and $t \in [0,1]$, we can obtain

$$
\begin{aligned}
T_1(u,v)(t) &= \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau),\right.\right. \\
&\qquad \left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right)d\tau\right)ds \\
&\geq \int_0^1 J(t,s)\phi_q\left(\varrho^{p-1}\int_0^1 H(s,\tau)G\left(\tau, I_{0^+}^{\beta_{n-2}}u(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u(\tau), u(\tau), I_{0^+}^{\beta_{n-2}}v(\tau),\right.\right. \\
&\qquad \left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}v(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}v(\tau), v(\tau)\right)\right)d\tau\right)ds \\
&= \varrho T_2(u,v)(t).
\end{aligned}
\tag{105}
$$

So, based on (103)–(105) and Lemma 9, we know the equation $T_1(u,u) + T_2(u,u) = u$ has a unique fixed point $u^* \in P_h$, which means u^* is the unique positive solution of the problem (28)-(29), and there exists $\delta \in (0,1)$ such that

$$
\delta t^{\beta-\beta_{n-2}-1} = \delta h(t) \leq u^*(t) \leq \delta^{-1}h(t) = \delta^{-1}t^{\beta-\beta_{n-2}-1}, \quad t \in [0,1].
\tag{106}
$$

And for any initial values $u_0, \tilde{u}_0 \in P_h$, we can construct the following sequences:

$$
\begin{aligned}
u_n(t) &= \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau, I_{0^+}^{\beta_{n-2}}u_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u_{n-1}(\tau), u_{n-1}(\tau), I_{0^+}^{\beta_{n-2}}\tilde{u}_{n-1}(\tau),\right.\right. \\
&\qquad \left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}\tilde{u}_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\tilde{u}_{n-1}(\tau), \tilde{u}_{n-1}(\tau)\right)d\tau\right)ds \\
&\quad + \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)G\left(\tau, I_{0^+}^{\beta_{n-2}}u_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u_{n-1}(\tau), u_{n-1}(\tau), I_{0^+}^{\beta_{n-2}}\tilde{u}_{n-1}(\tau),\right.\right. \\
&\qquad \left.\left. I_{0^+}^{\beta_{n-2}-\beta_1}\tilde{u}_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\tilde{u}_{n-1}(\tau), \tilde{u}_{n-1}(\tau)\right)d\tau\right)ds, \\
\tilde{u}_n(t) &= \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)F\left(\tau, I_{0^+}^{\beta_{n-2}}\tilde{u}_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}\tilde{u}_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\tilde{u}_{n-1}(\tau), \tilde{u}_{n-1}(\tau),\right.\right. \\
&\qquad \left.\left. I_{0^+}^{\beta_{n-2}}u_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u_{n-1}(\tau), u_{n-1}(\tau)\right)d\tau\right)ds \\
&\quad + \int_0^1 J(t,s)\phi_q\left(\int_0^1 H(s,\tau)G\left(\tau, I_{0^+}^{\beta_{n-2}}\tilde{u}_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}\tilde{u}_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}\tilde{u}_{n-1}(\tau), \tilde{u}_{n-1}(\tau),\right.\right. \\
&\qquad \left.\left. I_{0^+}^{\beta_{n-2}}u_{n-1}(\tau), I_{0^+}^{\beta_{n-2}-\beta_1}u_{n-1}(\tau), \ldots, I_{0^+}^{\beta_{n-2}-\beta_{n-3}}u_{n-1}(\tau), u_{n-1}(\tau)\right)d\tau\right)ds,
\end{aligned}
\tag{107}
$$

$$
t \in [0,1], \quad n = 1, 2, \ldots,
$$

such that

$$\sup_{t \in [0,1]} \left| u_n(t) - u^*(t) \right| = \left\| u_n - u^* \right\| \longrightarrow 0,$$

$$\sup_{t \in [0,1]} \left| \tilde{u}_n(t) - u^*(t) \right| = \left\| \tilde{u}_n - u^* \right\| \longrightarrow 0, \qquad (108)$$

$$\text{as } n \longrightarrow \infty.$$

Finally, on the basis of Lemma 5, we know $v^* = I^{\beta_{n-2}} u^*$ is the unique positive solution of problem (4)-(5). By (106), we have

$$\frac{\Gamma(\beta - \beta_{n-2})\delta}{\Gamma(\beta)} t^{\beta-1} \le v^*(t) \le \frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta)\delta} t^{\beta-1}, \quad t \in [0,1]. \tag{109}$$

Let $v_n = I_{0^+}^{\beta_{n-2}} u_n$ and $\tilde{v}_n = I_{0^+}^{\beta_{n-2}} \tilde{u}_n$; by using the monotonicity and continuity of fractional integral, we have

$$\sup_{t \in [0,1]} \left| v_n(t) - v^*(t) \right| = \left\| v_n - v^* \right\| \longrightarrow 0,$$

$$\sup_{t \in [0,1]} \left| \tilde{v}_n(t) - v^*(t) \right| = \left\| \tilde{v}_n - v^* \right\| \longrightarrow 0, \qquad (110)$$

$$\text{as } n \longrightarrow \infty. \qquad \square$$

Proof of Theorem 2. For any $\lambda > 0$, let us define three operators $\tilde{T}, \tilde{T}_1, \tilde{T}_2 : P_h \times P_h \longrightarrow P$ by $\tilde{T} = (\lambda^{q-1} T)$, $\tilde{T}_1 = (\lambda^{q-1} T_1)$, and $\tilde{T}_2 = (\lambda^{q-1} T_2)$, respectively. It is easy to check that $u \in C[0,1]$ is a positive solution of the problem (44)–(29) if it is a fixed point of $\tilde{T}$ in P_h, i.e., $u = \tilde{T}u = \tilde{T}_1(u,u) + \tilde{T}_2(u,u)$. Considering the results of Theorem 1 and Lemma 10 together, we know the following equation

$$\tilde{T}_1(u,u) + \tilde{T}_2(u,u) = \left(\lambda^{q-1} T_1\right)(u,u) + \left(\lambda^{q-1} T_2\right)(u,u) = u, \tag{111}$$

has a unique solution $u_\lambda^* \in P_h$, which implies u_λ^* is the unique positive solution of the problem (44)–(29), and there exists $\delta_\lambda \in (0,1)$ such that

$$\delta_\lambda t^{\beta - \beta_{n-2} - 1} = \delta_\lambda h(t) \le u_\lambda^*(t) \le \delta_\lambda^{-1} h(t) = \delta_\lambda^{-1} t^{\beta - \beta_{n-2} - 1}, \quad t \in [0,1]. \tag{112}$$

Furthermore, u_λ^* satisfies the following conclusion:

(1) If there exists $\gamma \in (0,1)$ such that
$$\varphi(l) \ge \frac{1}{\varrho} \left(l^\gamma - l \right) + l^\gamma, \quad \forall l \in (0,1), \tag{113}$$

then u_λ^* is continuous with respect to $\lambda \in (0, +\infty)$. That is, for any $\lambda_0 \in (0, +\infty)$,

$$\sup_{t \in [0,1]} \left| u_\lambda^*(t) - u_{\lambda_0}^*(t) \right| = \left\| u_\lambda^* - u_{\lambda_0}^* \right\| \longrightarrow 0, \quad \text{as } \lambda \longrightarrow \lambda_0. \tag{114}$$

(2) If
$$\varphi(l) \ge \frac{1}{\varrho} \left(l^{1/2} - l \right) + l^{1/2}, \quad \forall l \in (0,1), \tag{115}$$

then $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* < u_{\lambda_2}^*$.

(3) If there exists $\gamma \in (0, 1/2)$ such that

$$\varphi(l) \ge \frac{1}{\varrho} \left(l^\gamma - l \right) + l^\gamma, \quad \forall l \in (0,1), \tag{116}$$

then

$$\lim_{\lambda \longrightarrow 0^+} \sup_{t \in [0,1]} \left| u_\lambda^*(t) \right| = \lim_{\lambda \longrightarrow 0^+} \left\| u_\lambda^* \right\| = 0,$$

$$\lim_{\lambda \longrightarrow +\infty} \sup_{t \in [0,1]} \left| u_\lambda^*(t) \right| = \lim_{\lambda \longrightarrow +\infty} \left\| u_\lambda^* \right\| = +\infty. \tag{117}$$

Finally, by Lemma 5, we deduce $v_\lambda^* = I_{0^+}^{\beta_{n-2}} u_\lambda^*$ is a unique positive solution of the problem (6)-(5), which satisfies

$$\frac{\Gamma(\beta - \beta_{n-2})\delta_\lambda}{\Gamma(\beta)} t^{\beta-1} \le v_\lambda^*(t) \le \frac{\Gamma(\beta - \beta_{n-2})}{\Gamma(\beta)\delta_\lambda} t^{\beta-1}, \quad t \in [0,1]. \tag{118}$$

And by using the monotonicity and continuity of fractional integral, we have the following:

(1) If there exists $\gamma \in (0,1)$ such that

$$\varphi(l) \ge \frac{1}{\varrho} \left(l^\gamma - l \right) + l^\gamma, \quad \forall l \in (0,1), \tag{119}$$

then v_λ^* is continuous with respect to $\lambda \in (0, +\infty)$. That is, for any $\lambda_0 \in (0, +\infty)$,

$$\sup_{t \in [0,1]} \left| v_\lambda^*(t) - v_{\lambda_0}^*(t) \right| = \left\| v_\lambda^* - v_{\lambda_0}^* \right\| \longrightarrow 0, \quad \text{as } \lambda \longrightarrow \lambda_0. \tag{120}$$

(2) If

$$\varphi(l) \ge \frac{1}{\varrho} \left(l^{1/2} - l \right) + l^{1/2}, \quad \forall l \in (0,1), \tag{121}$$

then $0 < \lambda_1 < \lambda_2$ implies $v_{\lambda_1}^* < v_{\lambda_2}^*$.

(3) If there exists $\gamma \in (0, 1/2)$ such that

$$\varphi(l) \ge \frac{1}{\varrho} \left(l^\gamma - l \right) + l^\gamma, \quad \forall l \in (0,1), \tag{122}$$

then

$$\lim_{\lambda \longrightarrow 0^+} \sup_{t \in [0,1]} \left| v_\lambda^*(t) \right| = \lim_{\lambda \longrightarrow 0^+} \left\| v_\lambda^* \right\| = 0,$$

$$\lim_{\lambda \longrightarrow +\infty} \sup_{t \in [0,1]} \left| v_\lambda^*(t) \right| = \lim_{\lambda \longrightarrow +\infty} \left\| v_\lambda^* \right\| = +\infty. \tag{123}$$

$\square$

4. Numerical Examples

In this section, we give two simple theoretical numerical examples which justify Theorems 1 and 2.

Example 1. We consider the following equation:

$$^c D_{0^+}^{3/2} \phi_{3/2} \left(D_{0^+}^{29/10} v(t) \right) + t^{-(19/20)} v^{1/2}(t) + t^{-(9/20)} v^{1/2}(t)$$

$$+ 2t^{-(29/40)} v^{1/4}(t) \left(D_{0^+}^{9/10} v(t) \right)^{1/4} + 2t^{19/20} v^{-(1/2)}(t)$$

$$+ 2t^{1/2} \left(D_{0^+}^{9/10} v(t) \right)^{-(1/2)} = 0, \quad 0 < t < 1, \tag{124}$$

with the boundary conditions

$$v(0) = D_{0+}^{10/11} v(0) = D_{0+}^{29/10} v(0) = \left(\phi_{3/2} \left(D_{0+}^{29/10} v(0) \right) \right)' = 0,$$

$$D_{0+}^{19/10} v(1) = \frac{1}{100} \Gamma\left(\frac{3}{2}\right) \int_0^1 s^{-(1/2)} D_{0+}^{7/5} v(s) dA_1(s)$$

$$+ \frac{1}{100} \int_0^1 s^{-1} D_{0+}^{9/10} v(s) dA_2(s)$$

$$+ \frac{1}{100} \sum_{i=1}^{\infty} \frac{1}{i(i+1)} D_{0+}^{9/10} v(1),$$

(125)

where $\phi_{3/2}(s) = |s|^{-(1/2)} s$ and

$$A_1(t) = \begin{cases} \dfrac{2}{11}, & t \in [0, 1) \\[2mm] \dfrac{13}{11}, & t = 1, \end{cases}$$

$$A_2(t) = \begin{cases} \dfrac{1}{13}, & t \in [0, 1), \\[2mm] \dfrac{14}{13}, & t = 1. \end{cases}$$

(126)

Proof. Let

$$f(t, u_1, u_2) = t^{-(19/20)} u_1^{1/2} + t^{-(9/20)} u_1^{1/2} + 2t^{-(29/40)} u_1^{1/4} u_2^{1/4}$$

$$+ 2t^{19/20} u_1^{-(1/2)} + 2t^{1/2} u_2^{-(1/2)},$$

(127)

$\phi_p(s) = \phi_{3/2}(s)$, $\alpha = 3/2$, $\beta = 29/10$, $\beta_1 = 9/10$, $\nu_1 = 10/11$, $r_0 = 19/10$, $r_1 = 7/5$, $r_2 = 9/10$, $r_3 = 9/10$, $\rho_i = 1/i(i+1)$, $\eta_i = 1$, $(i = 1, 2, \ldots)$, $\eta = 1$, $\lambda_1 = (1/100)\Gamma(3/2)$, $\lambda_2 = 1/100$, $\lambda_3 = 1/100$, $h_1(t) = t^{-(1/2)}$, and $h_2(t) = t^{-1}$. Then, the problem (124)-(125) can be expressed as the problem (4)-(5). Notice that

$$\int_0^1 s^{\beta - r_1 - 1} h_1(s) dA_1(s) = 1 > 0,$$

$$\int_0^\eta s^{\beta - r_2 - 1} h_2(s) dA_2(s) = 1 > 0,$$

$$\sum_{i=1}^{\infty} \rho_i \eta_i^{\beta - r_3 - 1} = 1,$$

$$\sigma = \frac{1}{\Gamma(\beta - r_0)} - \frac{\lambda_1}{\Gamma(\beta - r_1)}$$

$$\cdot \int_0^1 s^{\beta - r_1 - 1} h_1(s) dA_1(s) - \frac{\lambda_2}{\Gamma(\beta - r_2)}$$

$$\cdot \int_0^\eta s^{\beta - r_2 - 1} h_2(s) dA_2(s) - \frac{\lambda_3}{\Gamma(\beta - r_3)}$$

$$\cdot \sum_{i=1}^{\infty} \rho_i \eta_i^{\beta - r_3 - 1} = 0.97 > 0.$$

(128)

We infer that the properties of Green's function in Lemma 8 are achieved. Let

$$F(t, u_1, u_2, v_1, v_2) = t^{-(19/20)} u_1^{1/2} + t^{-(29/40)} u_1^{1/4} u_2^{1/4}$$

$$+ t^{19/20} v_1^{-(1/2)} + t^{1/2} v_2^{-(1/2)},$$

$$G(t, u_1, u_2, v_1, v_2) = t^{-(9/20)} u_1^{1/2} + t^{-(29/40)} u_1^{1/4} u_2^{1/4}$$

$$+ t^{19/20} v_1^{-(1/2)} + t^{1/2} v_2^{-(1/2)}.$$

(129)

It is easy to check the following conditions:

(V_1) $f(t, u_1, u_2) = F(t, u_1, u_2, u_1, u_2) + G(t, u_1, u_2, u_1, u_2)$.

(V_2) For fixed $t \in (0, 1)$, and $(v_1, v_2) \in (0, +\infty)^2$, $F(t, u_1, u_2, v_1, v_2)$, $G(t, u_1, u_2, v_1, v_2)$ are increasing in $(u_1, u_2) \in (0, +\infty)^2$; for fixed $t \in (0, 1)$ and $(u_1, u_2) \in (0, +\infty)^2$, $F(t, u_1, u_2, v_1, v_2)$ and $G(t, u_1, u_2, v_1, v_2)$ are decreasing in $(v_1, v_2) \in (0, +\infty)^2$.

(V_3) Let $\varphi(l) = l$. Then, for $l \in (0, 1)$, $t \in (0, 1)$, and $(u_1, u_2, v_1, v_2) \in (0, +\infty)^4$,

$$F\left(t, lu_1, lu_2, l^{-1} v_1, l^{-1} v_2\right)$$

$$= t^{-(19/20)} \left(lu_1\right)^{1/2} + t^{-(29/40)} \left(lu_1\right)^{1/4} \left(lu_2\right)^{1/4}$$

$$+ t^{19/20} \left(l^{-1} v_1\right)^{-(1/2)} + t^{1/2} \left(l^{-1} v_2\right)^{-(1/2)}$$

$$\geq l^{1/2} F(t, u_1, u_2, v_1, v_2) = \varphi^{1/2}(l) F(t, u_1, u_2, v_1, v_2)$$

$$= \varphi^{p-1}(l) F(t, u_1, u_2, v_1, v_2),$$

$$G\left(t, lu_1, lu_2, l^{-1} v_1, l^{-1} v_2\right)$$

$$= t^{-(9/20)} \left(lu_1\right)^{1/2} + t^{-(29/40)} \left(lu_1\right)^{1/4} \left(lu_2\right)^{1/4}$$

$$+ t^{19/20} \left(l^{-1} v_1\right)^{-(1/2)} + t^{1/2} \left(l^{-1} v_2\right)^{-(1/2)}$$

$$\geq l^{1/2} G(t, u_1, u_2, v_1, v_2) = l^{p-1} G(t, u_1, u_2, v_1, v_2).$$

(130)

(V_4) Let $\varrho = 1$. Then, for all $(u_1, u_2, v_1, v_2) \in (0, +\infty)^4$,

$$F(t, u_1, u_2, v_1, v_2) \geq \varrho G(t, u_1, u_2, v_1, v_2).$$

(131)

(V_5) The functions F and G satisfy

$$0 < \int_0^1 (1 - \tau)^{\alpha - 1} F\left(\tau, 1, 1, \tau^{\beta - 1}, \tau^{\beta - 1}\right) d\tau$$

$$= B(1.5, 0.05) + B(1.5, 0.275) + B(1.5, 1)$$

$$+ B(1.5, 0.55) < +\infty,$$

$$0 < \int_0^1 (1 - \tau)^{\alpha - 1} G\left(\tau, 1, 1, \tau^{\beta - 1}, \tau^{\beta - 1}\right) d\tau$$

$$= B(1.5, 0.55) + B(1.5, 0.275) + B(1.5, 1)$$

$$+ B(1.5, 0.55) < +\infty.$$

(132)
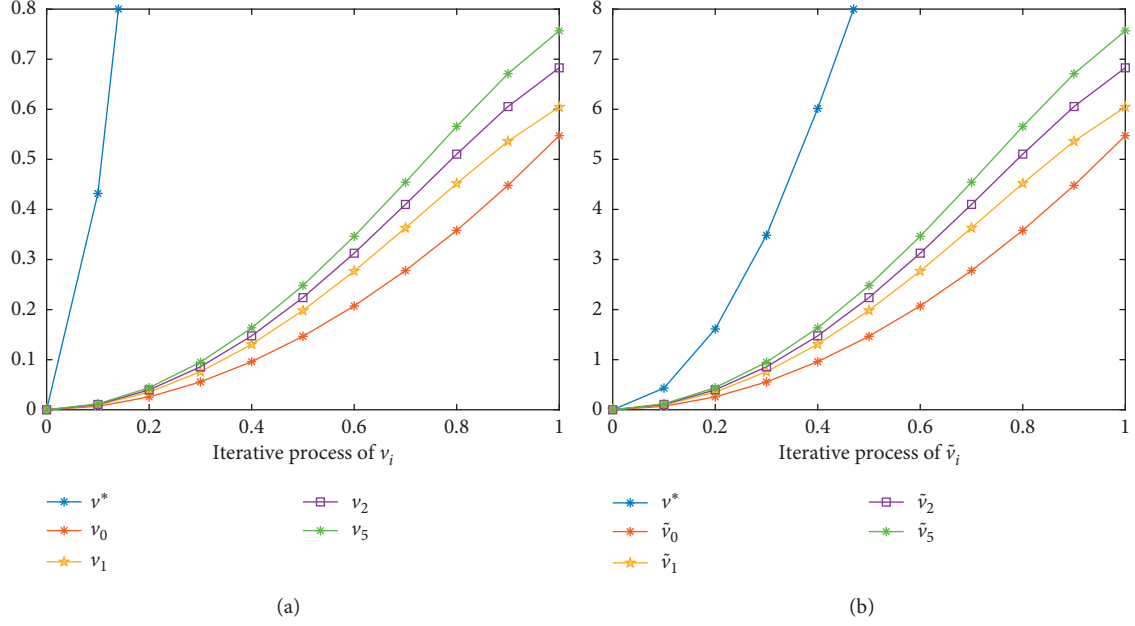
FIGURE 1: Iterative process of v_i and $\widetilde{v}_i$.

Thus, the assumptions of Theorem 1 are satisfied. By calculation, we obtain the approximate solution of the problem (124)-(125) is $v^*(t) \approx 34.315t^{1.9}$. On the other hand, let $u_0(t) = t$ and $\widetilde{u}_0(t) = 10t$. We take $v_0(t) = I^{\beta_1}u_0(t)$ and $\widetilde{v}_0 = I^{\beta_1}\widetilde{u}_0(t)$. By iterating the sequences, the numerical results of the iterative process are shown as follows (Figure 1 and Tables 1 and 2). □

Example 2. We consider the equation

$$^cD_{0+}^{3/2}\phi_{3/2}\left(D_{0+}^{35/12}v(t)\right) + \lambda\left(2t^{-(35/144)}v^{1/12}(t)\left(D_{0+}^{11/12}v(t)\right)^{1/12}\right.$$

$$\left. + 2t^{23/72}v^{-(1/6)}(t) + 2t^{1/6}\left(D_{0+}^{11/12}v(t)\right)^{-(1/6)} + 1\right) = 0,$$

$$0 < t < 1,$$

$$(133)$$

with the boundary conditions

$$v(0) = D_{0+}^{12/13}v(0) = D_{0+}^{35/12}v(0)$$

$$= \left(\phi_{3/2}\left(D_{0+}^{35/12}v(0)\right)\right)' = 0,$$

$$D_{0+}^{23/12}v(1) = \frac{1}{100}\Gamma\left(\frac{3}{2}\right)\int_0^1 s^{-(1/2)}D_{0+}^{17/12}v(s)dA_1(s)$$

$$(134)$$

$$+ \frac{1}{100}\int_0^1 s^{-1}D_{0+}^{11/12}v(s)dA_2(s)$$

$$+ \frac{1}{100}\sum_{i=1}^{\infty}\frac{1}{i(i+1)}D_{0+}^{11/12}v(1),$$

TABLE 1: Iterative process of v_i.

t	v_0	v_1	v_2	v_3	v_4	v_5	v_*
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0069	0.0094	0.0106	0.0113	0.0116	0.0118	0.4320
0.2	0.0257	0.0352	0.0397	0.0421	0.0433	0.0440	1.6123
0.3	0.0556	0.0759	0.0857	0.0908	0.0935	0.0950	3.4835
0.4	0.0960	0.1307	0.1476	0.1563	0.1610	0.1635	6.0172
0.5	0.1466	0.1983	0.2239	0.2373	0.2443	0.2481	9.1945
0.6	0.2073	0.2769	0.3126	0.3312	0.3411	0.3464	13.0008
0.7	0.2779	0.3631	0.4100	0.4344	0.4474	0.4543	17.4249
0.8	0.3581	0.4520	0.5104	0.5408	0.5570	0.5657	22.4572
0.9	0.4480	0.5361	0.6054	0.6415	0.6606	0.6710	28.0895
1.0	0.5472	0.6045	0.6828	0.7234	0.7450	0.7567	34.3150

where $\phi_{3/2}(s) = |s|^{-(1/2)}s$ and

$$A_1(t) = \begin{cases} \dfrac{2}{11}, & t \in [0,1), \\ \dfrac{13}{11}, & t = 1, \end{cases}$$

$$(135)$$

$$A_2(t) = \begin{cases} \dfrac{1}{13}, & t \in [0,1), \\ \dfrac{14}{13}, & t = 1. \end{cases}$$

Proof. Let

$$f(t, u_1, u_2) = \lambda\left(2t^{-(35/144)}u_1^{1/12}u_2^{1/12} + 2t^{23/72}u_1^{-(1/6)}\right.$$

$$\left. + 2t^{1/6}u_2^{-(1/6)} + 1\right),$$

$$(136)$$

$\phi_p(s) = \phi_{3/2}(s)$, $\alpha = 3/2$, $\beta = 35/12$, $\beta_1 = 11/12$, $\nu_1 = 12/13$, $r_0 = 23/12$, $r_1 = 17/12$, $r_2 = 11/12$, $r_3 = 11/12$,

TABLE 2: Iterative process of $\tilde{v}_i$.

t	$\tilde{v}_0$	$\tilde{v}_1$	$\tilde{v}_2$	$\tilde{v}_3$	$\tilde{v}_4$	$\tilde{v}_5$	v_*
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0689	0.0943	0.1064	0.1128	0.1161	0.1179	0.4320
0.2	0.2571	0.3518	0.3971	0.4207	0.4333	0.4400	1.6123
0.3	0.5555	0.7591	0.8570	0.9079	0.9350	0.9496	3.4835
0.4	0.9596	1.3071	1.4757	1.5633	1.6100	1.6351	6.0172
0.5	1.4663	1.9835	2.2394	2.3724	2.4433	2.4814	9.1945
0.6	2.0733	2.7688	3.1262	3.3120	3.4109	3.4641	13.0008
0.7	2.7788	3.6311	4.1000	4.3438	4.4735	4.5433	17.4249
0.8	3.5814	4.5203	5.1043	5.4079	5.5695	5.6565	22.4572
0.9	4.4796	5.3612	6.0542	6.4144	6.6062	6.7093	28.0895
1.0	5.4724	6.0459	6.8277	7.2340	7.4502	7.5666	34.3150

$\rho_i = 1/i(i+1)$, $\eta_i = 1$, $(i = 1, 2, \ldots)$, $\eta = 1$, $\lambda_1 = (1/100)\Gamma(3/2)$, $\lambda_2 = 1/100$, $\lambda_3 = 1/100$, $h_1(t) = t^{-(1/2)}$, and $h_2(t) = t^{-1}$. Then, the problems (133)-(134) can be expressed as the problems (6)-(5). Notice that

$$
\begin{aligned}
\sigma = {} & \frac{1}{\Gamma(\beta - r_0)} - \frac{\lambda_1}{\Gamma(\beta - r_1)} \int_0^1 s^{\beta - r_1 - 1} h_1(s) dA_1(s) \\
& - \frac{\lambda_2}{\Gamma(\beta - r_2)} \int_0^\eta s^{\beta - r_2 - 1} h_2(s) dA_2(s) \\
& - \frac{\lambda_3}{\Gamma(\beta - r_3)} \sum_{i=1}^\infty \rho_i \eta_i^{\beta - r_3 - 1}
\end{aligned}
\tag{137}
$$

$$
= 0.97 > 0.
$$

We infer that the properties of Green's function in Lemma 8 are achieved. Let

$$
\begin{aligned}
F(t, u_1, u_2, v_1, v_2) = {} & \lambda \big(t^{-(35/144)} u_1^{1/12} u_2^{1/12} + t^{23/72} v_1^{-(1/6)} \\
& + t^{1/6} v_2^{-(1/6)} + 1 \big), \\
G(t, u_1, u_2, v_1, v_2) = {} & \lambda \big(t^{-(35/144)} u_1^{1/12} u_2^{1/12} + t^{23/72} v_1^{-(1/6)} \\
& + t^{1/6} v_2^{-(1/6)} \big).
\end{aligned}
\tag{138}
$$

It is easy to check the conditions $(\mathbf{V}_1)$–$(\mathbf{V}_5)$ are all satisfied with $\varphi(l) = l^{1/3}$ and $\varrho = 1$. On the other hand, let $\gamma = (4/9) < (1/2)$. Then,

(1) $\varphi(l) = l^{1/3} \geq (1/\varrho)(l^\gamma - l) + l^\gamma = (6/5)l^{4/9} - (29/40)l$, $\forall l \in (0, 1)$

(2) $\varphi(l) = l^{1/3} \geq (1/\varrho)(l^{1/2} - l) + l^{1/2} = (6/5)l^{1/2} - (29/40)l$, $\forall l \in (0, 1)$

So, the assumptions of Theorem 2 are satisfied. In the following, we give the graphical simulations and table of the solution v_λ^* with respect to $\lambda = 0.5, 1, 1.5, 2, 3, 10$, respectively (Figure 2 and Table 3). □

Based on the graphical simulations and table, we obtain the following conclusions:
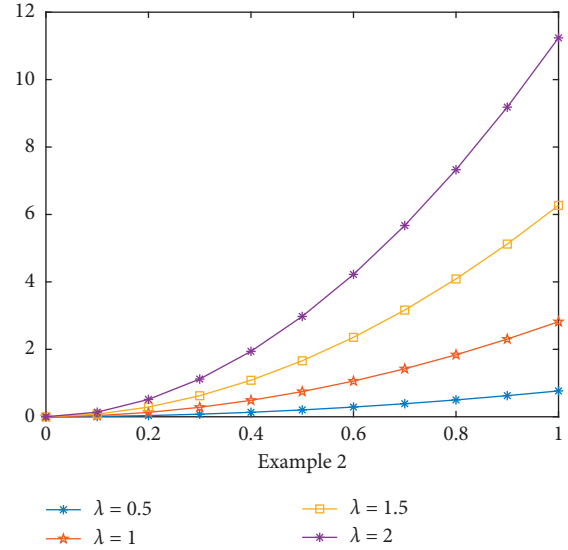
FIGURE 2: Dependence of the solution v_λ^* upon λ.

TABLE 3: Numerical value of the solution v_λ^*.

t	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 3$	$\lambda = 10$
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0093	0.0342	0.0759	0.1362	0.3198	5.0454
0.2	0.0351	0.1289	0.2867	0.5140	1.2073	19.0488
0.3	0.0763	0.2805	0.6237	1.1182	2.6262	41.4358
0.4	0.1324	0.4868	1.0825	1.9407	4.5582	71.9187
0.5	0.2030	0.7466	1.6603	2.9765	6.9909	110.3027
0.6	0.2880	1.0589	2.3548	4.2216	9.9151	156.4408
0.7	0.3870	1.4229	3.1642	5.6727	13.3233	210.2155
0.8	0.4998	1.8379	4.0871	7.3272	17.2093	271.5289
0.9	0.6264	2.3034	5.1222	9.1830	21.5678	340.2972
1.0	0.7666	2.8188	6.2684	11.2379	26.3942	416.4475

(1) v_λ^* is continuous with respect to $\lambda \in (0, +\infty)$. That is, for any $\lambda_0 \in (0, +\infty)$,

$$
\sup_{t \in [0,1]} \left| v_\lambda^*(t) - v_{\lambda_0}^*(t) \right| = \left\| v_\lambda^* - v_{\lambda_0}^* \right\| \longrightarrow 0, \quad \text{as } \lambda \longrightarrow \lambda_0.
\tag{139}
$$

(2) $0 < \lambda_1 < \lambda_2$ implies $v_{\lambda_1}^* < v_{\lambda_2}^*$.

(3) $\lim_{\lambda \to 0^+} \sup_{t \in [0,1]} |v_\lambda^*(t)| = \lim_{\lambda \to 0^+} \|v_\lambda^*\| = 0$, $\lim_{\lambda \to +\infty}$

$$
\sup_{t \in [0,1]} |v_\lambda^*(t)| = \lim_{\lambda \to +\infty} \|v_\lambda^*\| = +\infty.
$$

5. Conclusion

In this paper, we introduce the fixed point theorem of mixed monotone operator for finding the uniqueness of positive solution of a class of fractional boundary value problems, which is a generalized form of turbulent flow problem in a porous medium. Two theoretical numerical examples are given to illustrate Theorems 1 and 2; the results then bring us a step closer to research the characters of solutions.

Furthermore, as the application of mixed monotone operator operator, further research and discussion are required in practice.

Data Availability

The datasets used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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