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Research Article

Sliding Dynamics of a Filippov Forest-Pest Model with Threshold Policy Control

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A novel Filippov forest-pest system with threshold policy control (TPC) is established while an economic threshold (ET) is used to guide switching. The aim of our work is to address how to reasonably and successfully control pests by means of sliding dynamics for the Filippov system. On the basis of the above considerations, conditions for the existence and stability of equilibria of subsystems are addressed, and the sliding segments and several types of equilibria of the proposed system are defined. These equilibria include the regular/virtual equilibrium, pseudoequilibrium, boundary equilibrium, and tangent point. Further, not only are the relations between nullclines and equilibria of the Filippov system discussed, but the relations between pseudoequilibrium, nullclines, and the sliding segment are discussed. More importantly, four cases of sliding bifurcations of the Filippov system with respect to different types of equilibria of subsystems are investigated, and the corresponding biological implications concerning integrated pest management (IPM) are analyzed. Our results show that the points of intersection between nullclines are equilibria of the system, and the two endpoints of the sliding segment are on the nullclines. It is also verified that the pseudoequilibrium is the point of intersection of the sliding segment and nullclines of the Filippov system, and the pseudoequilibrium exists on the sliding segment. More interestingly, sliding dynamics analysis reveals that the Filippov system has sliding limit cycles, a bistable state and a stable refuge equilibrium point, and the optimal time and strategy for controlling pests are provided.

1. Introduction

Because insect infestations increase forest mortality, which in turn affects the carbon cycle and causes air pollution [1], many scholars have paid much attention to the effects of disturbances on forests [2–7]. Disturbances are classified as natural and anthropogenic disturbances. Natural disturbances include wildfires, insect infestations, floods, droughts, and bad weather while anthropogenic disturbances include deforestation, the logging of timber, and the spraying of pesticides [4]. Hence, it is vital that the effects of disturbances on the forest are solved. To do this, the age structures of trees in forests where pests exist have been investigated, and strategies for forest management and fire protection have been developed by considering the size of fires, fire control methods, and the behaviors of planted trees [6–10]. However, few researchers have addressed the insect infestations from a biological point of view.

The dynamics of numerous real-world systems can be modeled using discontinuous differential equations [11–15]. As an example, Filippov pest control models have been proposed to investigate sliding bifurcations [16]. Yang and Liao studied the Filippov Hindmarsh–Rose neuron model for several sliding bifurcation phenomena, and the threshold policy control (TPC) was considered [17]. Wang et al. proposed a Filippov epidemic model in discussing the effects of factors on controlling epidemic diseases [18]. Moreover, research on the related Filippov system has focused on bifurcation analysis [16,19–22]. Tan et al. discussed the sliding bifurcation analysis for a Filippov predator-prey system [23]. Qu and Li investigated the sliding phenomenon in Filippov dynamical systems based on Chua's circuit [24]. Therefore, combining integrated pest management (IPM), a Filippov system with TPC (i.e., a discontinuous piecewise smooth system [25]) is established to reasonably and successfully control pests. It is a challenging but important work to determine the functional response of pests [26–28].

IPM is a well-known strategy of controlling pests effectively [29–35], which includes biological strategies (i.e., releasing natural enemies), cultural strategies (i.e., acquiring or capturing pests artificially), and chemical strategies (i.e., spraying a certain dosage of pesticides) that are used for pest control to avoid exceeding an economic threshold (ET) [36]. In other words, the three types of strategies aim to reduce the number of pests below the economic injury level (EIL), i.e., the environment is not destroyed [37]. TPC, a control strategy, is implemented once the number of pests reaches and exceeds the ET [38–42].

Recently, Xiao and Bosch have built a model with respect to the effects of pest on biologically based technologies for controlling pest (BBTs), which revealed that biological control has two effects including stabilization and destabilization [43]. Stern et al. have verified that integrated control was the most effective where chemical treatment was carried out for successful pest control and pest eradication was not necessary to be considered, and an ET was applied to determine the use of pesticides [36]. Liang and Tang have analyzed the key factors (i.e., optimum timing, the dosage of pesticide, and ET) which affect pest management by employing different impulsive differential equations [44]. A pest-natural enemy system has been proposed by Liang et al., and pesticide resistance was taken into account, which aimed to determine the number of releasing natural enemies to ensure pest eradication [45]. The present paper focuses on the control of pests (beetles), and the sliding dynamics of a Filippov system based on the tree-beetle model is discussed.

First, a novel Filippov forest-pest model is proposed by combining IPM and TPC. Then, conditions for the existence and stability of equilibria of subsystems are addressed, which is helpful to study the dynamics of the proposed system. Further, the sliding segments and several types of equilibria of the system are defined, and these equilibria include the regular/virtual equilibrium, pseudoequilibrium, boundary equilibrium, and tangent point. The relations between nullclines and equilibria of the Filippov system are discussed, which reveal that the points of intersection between nullclines are these equilibrium points of the system. Moreover, the relations between pseudoequilibrium, nullclines and the sliding segment are also discussed. The results verify that pseudoequilibrium is the point of intersection of the sliding segment and nullclines of the Filippov system, and the pseudoequilibrium exists on the sliding segment.

The sliding bifurcations of the Filippov system are addressed for different values of ET. Results show that the value of ET affects the control range and also the time that we decide to take action to prevent the density of the pest population from reaching the value of ET. Therefore, it is important to choose the optimal strategy when the pest density reaches the ET. More importantly, the Filippov system has sliding limit cycles, a bistable state, and a stable refuge equilibrium point (i.e., when the pest population with relatively small intrinsic growth rate, it will be stable at the small density).

The remainder of the paper is organized as follows. In Section 2, the model description and preliminaries of the Filippov system are introduced. In Section 3, equilibria and sliding dynamics including the conditions for the existence and stability of equilibria of subsystems are addressed. Moreover, the sliding regions and equilibria of the Filippov system are discussed. In Section 4, the sliding bifurcations of the Filippov system with respect to different types of equilibria of subsystems are presented, and the corresponding biological implications related to pest control are analyzed. Conclusions drawn from the results of our work are presented in Section 5.

2. Model Description and Preliminaries

2.1. Model Description. To explore the effects of insect infestations on the forest, the classical tree-beetle model was considered [4]:

$$\begin{cases} \dot{V}(t) = r_v V(t)\left(1 - \dfrac{V(t)}{k_v} - m(B)\right), \\ \dot{B}(t) = r_b B(t)\left(1 - \dfrac{B(t)}{k_b}\right) - \dfrac{\alpha B^2(t)}{1 + \beta B^2(t)}, \end{cases} \tag{1}$$

where the number of susceptible trees is denoted by $V(t)$, $B(t)$ represents the population density of mountain pine beetles per tree, r_v and r_b are the intrinsic growth rates of $V(t)$ and $B(t)$, respectively, and their carrying capacity is denoted by k_v and k_b, respectively, α is a positive constant that depends upon trees, and β is also a positive constant that determines the scale at which beetle density generates saturation.

Note that the beetles can be treated as herbivore type insects, which consume trees [46–48]. The density growth of trees is subject to a logistic self-interaction in the absence of beetles where the forest does not have any defense against insect predation. In particular, a functional response (i.e., the Holling type-III functional response) is assumed [26]. The functional response of the predator is an important part of the prey-predator relationship, which is the impact of the predator (pest) per unit time on the change in prey (forest) density. Types of functional response include Holling types I-III [26, 49], the ratio-dependence type [27, 50], and the Hassell-Varley type [28].

When the beetles are introduced in the first equation of model (1), a linear function is assumed. It means that the number of trees decreases because the beetles feed on trees. So a novel tree-beetle model is obtained as

$$\begin{cases} \dot{V}(t) = r_v V(t)\left(1 - \dfrac{V(t)}{k_v}\right) - \delta B(t)V(t), \\ \dot{B}(t) = r_b B(t)\left(1 - \dfrac{B(t)}{k_b}\right) + \eta\delta B(t)V(t) - \dfrac{\alpha B^2(t)}{1 + \beta B^2(t)}, \end{cases} \tag{2}$$

where δ represents the rate that every beetle eats tree; η is the conversion coefficient of the beetles; $\eta\delta B(t)V(t)$ denotes the number of the beetles increases after they eat trees; and $V(t)$ and $B(t)$ can be treated as the forest (tree) and pest (beetle). Thus, model (2) is called the forest-pest model. To a certain extent, they can also be seen as prey and predator.

Furthermore, we let q_1 represent the proportion of pests (prey) that is captured, transferred, or killed by using the cultural and chemical strategies. Therefore, the control model for $B(t) > \mathrm{ET}$ is written as

$$\begin{cases} \dot{V}(t) = r_v V(t)\left(1 - \dfrac{V(t)}{k_v}\right) - \delta B(t)V(t), \\ \dot{B}(t) = r_b B(t)\left(1 - \dfrac{B(t)}{k_b}\right) + \eta\delta B(t)V(t) \\ -\dfrac{\alpha B^2(t)}{1+\beta B^2(t)} - q_1 B(t), \end{cases} \tag{3}$$

where the ET is a switching threshold, i.e., system (2) is satisfied if $B(t) < \mathrm{ET}$ while system (3) is satisfied if $B(t) > \mathrm{ET}$.

In the next subsection, some preliminaries of the Filippov system are provided to understand this paper easily, which prepares for the later sections.

2.2. Preliminaries. On the basis of IPM strategies and TPC, models (2) and (3) are combined and rewritten as [11, 51]

$$\begin{cases} \dot{V}(t) = r_v V(t)\left(1 - \dfrac{V(t)}{k_v}\right) - \delta B(t)V(t), \\ \dot{B}(t) = r_b B(t)\left(1 - \dfrac{B(t)}{k_b}\right) + \eta\delta B(t)V(t) \\ -\dfrac{\alpha B^2(t)}{1+\beta B^2(t)} - \varepsilon q_1 B(t), \end{cases} \tag{4}$$

with

$$\varepsilon = \begin{cases} 0, & B(t) < \mathrm{ET}, \\ 1, & B(t) > \mathrm{ET}. \end{cases} \tag{5}$$

Models (4) and (5) are related to TPC. More details on the Filippov system have been given in the literature [52, 53].

Let $H(Z) = B(t) - \mathrm{ET}$ with column vector $Z = (V, B)^T$ and

$$\begin{aligned} F_{S_1}(Z) &= \left(r_v V\left(1 - \frac{V}{k_v}\right) - \delta BV, r_b B\left(1 - \frac{B}{k_b}\right) + \eta\delta BV - \frac{\alpha B^2}{1+\beta B^2}\right)^T, \\ F_{S_2}(Z) &= \left(r_v V\left(1 - \frac{V}{k_v}\right) - \delta BV, r_b B\left(1 - \frac{B}{k_b}\right) + \eta\delta BV - \frac{\alpha B^2}{1+\beta B^2} - q_1 B\right)^T. \end{aligned} \tag{6}$$

Then, system (6) is rewritten as a Filippov system [52, 53]:

$$\dot{Z}(t) = \begin{cases} F_{S_1}(Z), & Z \in S_1, \\ F_{S_2}(Z), & Z \in S_2. \end{cases} \tag{7}$$

Moreover, the discontinuity boundary set is defined; that is, $\Sigma = \{Z \in R_+^2 \mid H(Z) = 0\}$, which divides R_+^2 into two regions, i.e.,

$$\begin{aligned} S_1 &= \left\{Z \in R_+^2 \mid H(Z) < 0\right\}, \\ S_2 &= \left\{Z \in R_+^2 \mid H(Z) > 0\right\}. \end{aligned} \tag{8}$$

In this paper, Filippov system (7) in region S_1 or S_2 is, respectively, referred to as subsystem S_1 or subsystem S_2.

Let

$$\sigma(Z) = \langle H_Z(Z), F_{S_1}(Z)\rangle\langle H_Z(Z), F_{S_2}(Z)\rangle, \tag{9}$$

where H_Z is a nonvanishing gradient of the smooth scale function on Σ and $\langle\cdot\rangle$ denotes the standard scalar product. Then, sliding regions are defined as

$$\Sigma_S = \{Z \in \Sigma \mid \sigma(Z) \le 0\}. \tag{10}$$

Meanwhile, the following regions on Σ are distinguished:

(i) Escaping region: if $\langle H_Z(Z), F_{S_1}(Z)\rangle < 0$ and $\langle H_Z(Z), F_{S_2}(Z)\rangle > 0$

(ii) Sliding region: if $\langle H_Z(Z), F_{S_1}(Z)\rangle > 0$ and $\langle H_Z(Z), F_{S_2}(Z)\rangle < 0$

(iii) Sewing region: if $\langle H_Z(Z), F_{S_1}(Z)\rangle\langle H_Z(Z), F_{S_2}(Z)\rangle > 0$

The essential definitions for different types of equilibria of Filippov system (7) are taken from the literature [54, 55].

Definition 1. Z^* is called a regular equilibrium of Filippov system (7) if $F_{S_1}(Z^*) = 0$ and $H(Z^*) < 0$, or $F_{S_2}(Z^*) = 0$ and $H(Z^*) > 0$. Z^* is called a virtual equilibrium of Filippov system (7) if $F_{S_1}(Z^*) = 0$ and $H(Z^*) > 0$, or $F_{S_2}(Z^*) = 0$ and $H(Z^*) < 0$.

Definition 2. Z^* is called a pseudoequilibrium if it is an equilibrium on the sliding segment of Filippov system (7), i.e., $(1 - \lambda(Z))F_{S_1}(Z^*) + \lambda(Z)F_{S_2}(Z^*) = 0$, $H(Z^*) = 0$, and $0 < \lambda(Z) < 1$, where

$$\lambda(Z) = \frac{\langle H_Z(Z), F_{S_1}(Z)\rangle}{\langle H_Z(Z), F_{S_1}(Z) - F_{S_2}(Z)\rangle}. \tag{11}$$

Definition 3. Z^* is called a boundary equilibrium of Filippov system (7) if $F_{S_1}(Z^*) = 0$ with $H(Z^*) = 0$ or $F_{S_2}(Z^*) = 0$ with $H(Z^*) = 0$.

Definition 4. Z^* is called a tangent point of Filippov system (7) if $Z^* \in \Sigma_S$ and $\langle H_Z(Z^*), F_{S_1}(Z^*)\rangle = 0$ or $\langle H_Z(Z^*), F_{S_2}(Z^*)\rangle = 0$.

3. Equilibria and Sliding Dynamics

This section discusses conditions for the existence and stability of equilibria of subsystems S_1 and S_2. The sliding segment/region and equilibria of Filippov system (7) are addressed, which prepares for Section 4.

3.1. Existence of Equilibria for Subsystems S_1 and S_2. To discuss the types of equilibria and sliding bifurcations of Filippov system (7), the existence of equilibria for subsystems S_1 and S_2 needs to be solved. For subsystem S_1, let $\dot{V}(t) = 0$ and $\dot{B}(t) = 0$. Then, the equilibrium $E_{1*}(V_{1*}, B_{1*})$ satisfies

$$B^3(t) + a_1 B^2(t) + b_1 B(t) + c_1 = 0, \tag{12}$$

where $a_1 = ((-k_b r_b r_v - \eta\delta k_b k_v r_v)/(r_b r_v + \delta^2 \eta k_b k_v))$, $b_1 = ((r_b r_v + \delta^2 \eta k_b k_v + \alpha k_b r_v)/(r_b r_v \beta + \delta^2 \eta k_b k_v \beta))$, and $c_1 = ((-k_b r_b r_v - \eta \delta k_b k_v r_v)/(r_b r_v \beta + \delta^2 \eta k_b k_v \beta))$. Let $Q_1 = ((a_1^2 - 3b_1)/9)$, $R_1 = -((2a_1^3 - 9a_1 b_1 + 27c_1)/54)$, $M_1 = R_1^2 - Q_1^3$, and $\theta_1 = \arccos(R_1/Q_1^{3/2})$.

(i) If $M_1 < 0$, then (12) has three fixed solutions given by

$$\begin{cases} B_{1i} = -2\sqrt{Q_1}\cos\left(\dfrac{\theta_1 + 2k\pi}{3}\right) - \dfrac{a_1}{3}, & k = -1, 0, 1; \\ & i = 1, 2, 3, \\ V_{1i} = k_v - \dfrac{\delta B_{1i} k_v}{r_v}. \end{cases} \tag{13}$$

(ii) If $M_1 > 0$, then (12) has a unique equilibrium point given by

$$\begin{cases} B_1 = \left(-R_1 + \sqrt{M_1}\right)^{1/3} + \left(-R_1 - \sqrt{M_1}\right)^{1/3} - \dfrac{a_1}{3}, \\ V_1 = k_v - \dfrac{\delta B_1 k_v}{r_v}. \end{cases} \tag{14}$$

(iii) If $M_1 = 0$, then (12) has two equilibrium points. We do not discuss this case in the present paper.

In particular, the equilibrium E_1^* must be positive, i.e., $V_{1i} > 0, B_{1i} > 0$ $(i = 1, 2, 3)$. Analogously, the existence conditions of equilibria for subsystem S_2 can be obtained.

3.2. Stability of Equilibria for Subsystems S_1 and S_2. In general, the Jacobian matrix is employed to explore the stability of equilibrium $E_{1*}(V_{1*}, B_{1*})$ for subsystem S_1, i.e.,

$$J_{1*} = \begin{pmatrix} r_v - \dfrac{2r_v V_{1*}}{k_v} - \delta B_{1*} & -\delta V_{1*} \\ \eta\delta B_{1*} & r_b - \dfrac{2r_b B_{1*}}{k_b} + \eta\delta V_{1*} - \dfrac{2\alpha B_{1*}}{(1 + \beta B_{1*}^2)^2} \end{pmatrix}. \tag{15}$$

Then, the characteristic equation is $\lambda^2 + p_1\lambda + q_1 = 0$, where

$$p_1 = -r_v + \frac{2r_v V_{1*}}{k_v} + \delta B_{1*} - r_b + \frac{2r_b B_{1*}}{k_b} - \eta\delta V_{1*} + \frac{2\alpha B_{1*}}{(1 + \beta B_{1*}^2)^2}, \tag{16}$$

$$q_1 = \left(r_v - \frac{2r_v V_{1*}}{k_v} - \delta B_{1*}\right)\left(r_b - \frac{2r_b B_{1*}}{k_b} + \eta\delta V_{1*} - \frac{2\alpha B_{1*}}{(1 + \beta B_{1*}^2)^2}\right) + \eta\delta^2 B_{1*} V_{1*}. \tag{17}$$

If inequalities $p_1 > 0$ and $q_1 > 0$ are satisfied, then E_{1*} is asymptotically stable. Similarly, the conditions for the stability of equilibria of subsystem S_2 are addressed.

3.3. Sliding Segment and Region. From Definition 2, the equation is obtained:

$$\lambda(Z) = \frac{r_b(1 - (B(t)/k_b)) + \eta\delta V(t) - (\alpha B(t)/(1 + \beta B^2(t)))}{q_1}. \tag{18}$$

The sliding regions are determined by solving $0 \le \lambda(Z) \le 1$ with respect to $V(t)$. Thus, the algebraic equations need to be considered, i.e.,

$$\begin{cases} r_b\left(1 - \dfrac{\mathrm{ET}}{k_b}\right) + \eta\delta V(t) - \dfrac{\alpha \mathrm{ET}}{1 + \beta \mathrm{ET}^2} = 0, \\ r_b\left(1 - \dfrac{\mathrm{ET}}{k_b}\right) + \eta\delta V(t) - \dfrac{\alpha \mathrm{ET}}{1 + \beta \mathrm{ET}^2} - q_1 = 0. \end{cases} \tag{19}$$

Then, solving equation (19) with respect to $V(t)$ yields two real roots:

$$V_1 = \frac{(\alpha \mathrm{ET}/(1 + \beta \mathrm{ET}^2)) - r_b(1 - (\mathrm{ET}/k_b))}{\eta\delta},$$
$$V_2 = \frac{q_1 + (\alpha \mathrm{ET}/(1 + \beta \mathrm{ET}^2)) - r_b(1 - (\mathrm{ET}/k_b))}{\eta\delta}. \tag{20}$$

Because of the relation between V_1 and V_2 $(V_1 < V_2)$, the sliding segment of Filippov system (7) can be defined as

$$\Sigma_S = \{(V, B) \mid V_1 \le V \le V_2, B = \mathrm{ET}\}. \tag{21}$$

3.4. Equilibria of the Filippov System. Nullclines of both subsystems S_1 and S_2 are related to the existence of equilibria, which is conducive to sliding bifurcation analysis and to estimating the existence of equilibria of Filippov system (7). For the two subsystems, nullclines are determined by using equations $\dot{V}(t) = 0$ and $\dot{B}(t) = 0$ as

$$f_{S_1} = f_{S_2} = \frac{k_v}{r_v}\left(r_v - \delta B(t)\right),$$

$$g_{S_1} = \frac{1}{\eta\delta}\left(-r_b\left(1 - \frac{B(t)}{k_b}\right) + \frac{\alpha B(t)}{1 + \beta B^2(t)}\right), \tag{22}$$

$$g_{S_2} = \frac{1}{\eta\delta}\left(q_1 - r_b\left(1 - \frac{B(t)}{k_b}\right) + \frac{\alpha B(t)}{1 + \beta B^2(t)}\right).$$

Different types of equilibria of Filippov system (7) were defined in Section 2. Several types of equilibria for Filippov system (7) are defined, which include the regular equilibrium, virtual equilibrium, pseudoequilibrium, boundary equilibrium, and a special point called the tangent point. These equilibria are denoted by E^R, E^V, E^P, E^B, and E^T. For Filippov system (7), more detailed definitions of the equilibria mentioned are in the following section.

3.4.1. Regular/Virtual Equilibrium. According to Definition 1, $Z_{ji} = (V_{ji}, B_{ji})$ $(j = 1, 2; i = 1, 2, 3)$ is a regular equilibrium of Filippov system (7) if $F_{S_j}(Z_{ji}) = 0$, for subsystem S_1 (i.e., $j = 1$), $B_{1i} < \mathrm{ET}$, or for subsystem S_2 (i.e., $j = 2$), $B_{2i} > \mathrm{ET}$. These equilibria are denoted by E^R_{1i} and E^R_{2i}, respectively. $Z_{ji} = (V_{ji}, B_{ji})$ $(j = 1, 2; i = 1, 2, 3)$ is a virtual equilibrium of Filippov system (7) if $F_{S_j}(Z_{ji}) = 0$, for subsystem S_1 (i.e., $j = 1$), $B_{1i} > \mathrm{ET}$, or for subsystem S_2 (i.e., $j = 2$), $B_{2i} < \mathrm{ET}$. These equilibria are denoted by E^V_{1i} and E^V_{2i}, respectively.

3.4.2. Pseudoequilibrium. According to Definition 2, by employing the Utkin equivalent control method [51] and letting $H_Z = \dot{B}(t) = 0$, equations with respect to ε are obtained:

$$\varepsilon = \frac{r_b\left(1 - (\mathrm{ET}/k_b)\right) + \eta\delta V - \left(\alpha\mathrm{ET}/\left(1 + \beta\mathrm{ET}^2\right)\right)}{q_1}. \tag{23}$$

Thus, the equation $\dot{V}(t) = r_v V(t)(1 - (V(t)/k_v)) - \delta B(t)V(t)$ determines the dynamics on Σ_S. If pseudoequilibrium $E^P(V_P, \mathrm{ET})$ satisfies $\dot{V}(t) = 0$ (i.e., $V_P = ((k_v(r_v - \delta B(t)))/r_v)$ and $B(t) = \mathrm{ET}$), then $E^P((k_v(r_v - \delta\mathrm{ET})/r_v), \mathrm{ET}) \in \Sigma_S$. Therefore, V_P must satisfy the inequality $V_1 \le V_P \le V_2$, i.e., if the inequalities

$$\frac{k_b k_v \delta^2 \eta \mathrm{ET}}{k_b k_v \delta\eta + r_b\left(k_b - \mathrm{ET}\right) - \left(\alpha\mathrm{ET}k_b/\left(1 + \beta\mathrm{ET}^2\right)\right)} < r_v < \frac{k_b k_v \delta^2 \eta \mathrm{ET}}{k_b k_v \delta\eta - q_1 + r_b\left(k_b - \mathrm{ET}\right) - \left(\alpha\mathrm{ET}k_b/1 + \beta\mathrm{ET}^2\right)}, \tag{24}$$

hold, then Filippov system (7) has a pseudoequilibrium E^P.

3.4.3. Boundary Equilibrium. According to Definition 3, the boundary equilibrium of Filippov system (7) satisfies

$$\begin{cases} r_v V(t)\left(1 - \dfrac{V(t)}{k_v}\right) - \delta B(t)V(t) = 0, \\ r_b\left(1 - \dfrac{B(t)}{k_b}\right) + \eta\delta V(t) - \dfrac{\alpha B(t)}{1 + \beta B^2(t)} - q_1 = 0, \\ B(t) = \mathrm{ET}, \end{cases} \tag{25}$$

with ε = 0 or 1. If ε = 0, the boundary equilibrium

$$E^B_1 = \left(\frac{\left(\alpha\mathrm{ET}/\left(1 + \beta\mathrm{ET}^2\right)\right) - r_b\left(1 - (\mathrm{ET}/k_b)\right)}{\eta\delta}, \mathrm{ET}\right), \tag{26}$$

is obtained. If ε = 1, we have the boundary equilibrium

$$E^B_2 = \left(\frac{q_1 + \left(\alpha\mathrm{ET}/\left(1 + \beta\mathrm{ET}^2\right)\right) - r_b\left(1 - (\mathrm{ET}/k_b)\right)}{\eta\delta}, \mathrm{ET}\right). \tag{27}$$

3.4.4. Tangent Point. According to Definition 4, the tangent point $E^T(V_T, \mathrm{ET})$ on Σ_S satisfies

$$\begin{cases} r_b\left(1 - \dfrac{B(t)}{k_b}\right) + \eta\delta V(t) - \dfrac{\alpha B(t)}{1 + \beta B^2(t)} - q_1 = 0, \\ B(t) = \mathrm{ET}. \end{cases} \tag{28}$$

Solving equations (28) with respect to $V(t)$ and $B(t)$ yields

$$E^T_1 = \left(\frac{\left(\alpha\mathrm{ET}/\left(1 + \beta\mathrm{ET}^2\right)\right) - r_b\left(1 - (\mathrm{ET}/k_b)\right)}{\eta\delta}, \mathrm{ET}\right),$$

$$E^T_2 = \left(\frac{q_1 + \left(\alpha\mathrm{ET}/\left(1 + \beta\mathrm{ET}^2\right)\right) - r_b\left(1 - (\mathrm{ET}/k_b)\right)}{\eta\delta}, \mathrm{ET}\right). \tag{29}$$

4. Numerical Simulation

In this section, four cases of sliding bifurcations for Filippov system (7) with different equilibria of subsystems S_1 and S_2 are investigated, which are summarized in Table 1.

4.1. Case A. Subsystem S_1 has an unstable equilibrium point, and there exist stable limit cycles. There are three cases of subsystem S_2:

(i) Subsystem S_2 has only an unstable equilibrium point and stable limit cycles
(ii) Subsystem S_2 is a bistable state
(iii) Subsystem S_2 has only one stable equilibrium point

Figure 1(a) shows that there is only one unstable equilibrium E_{11} and stable limit cycles in subsystem S_1. Figure 1(b) shows that subsystem S_2 is similar to subsystem S_1 in that there is only one unstable equilibrium E_{21} and stable limit cycles. Figure 1(c) shows that there are three

Table 1: The summary of numerical simulation.

Case	Subsystem	Description of the equilibria
A	S_1	An unstable equilibrium point, stable limit cycles An unstable equilibrium point, stable limit cycles
	& S_2	A bistable state Only one stable refuge equilibrium point
B	S_1	Only one stable equilibrium point Only one stable equilibrium point
	& S_2	A bistable state A stable refuge equilibrium point
C	S_1	Three equilibrium points (two stable equilibrium points)
	& S_2	A bistable state A stable refuge equilibrium point
D	S_1 & S_2	Three equilibrium points (only a stable refuge equilibrium point)

equilibria (i.e., E_{21}, E_{22} and E_{23}) in subsystem S_2. E_{21} and E_{23} are stable, i.e., subsystem S_2 is a bistable state. Figure 1(d) shows that there is only one stable equilibrium E_{23}, a stable refuge equilibrium point, in subsystem S_2.

The sliding mode phenomena in the three subcases of Case A are discussed as follows.

Case A.1: subsystem S_2 has only one unstable equilibrium point and stable limit cycles.

Figure 2(a) shows the sliding bifurcation of the Filippov system when the value of ET varies in the range [3.8, 5.8]. V_1 and V_2 are two endpoints of the sliding segment. Moreover, sliding limit cycles appear. Figures 2(b)–2(d) present the three cases of Figure 2(a). In Figure 2(b), if $B_{11} > B_{21} > \mathrm{ET}$ (ET = 3.9), Filippov system (7) has limit cycles and stabilizes to the larger limit cycle and virtual equilibria E_{11}^V of subsystem S_1. In Figure 2(c), if $B_{11} > \mathrm{ET} > B_{21}$ (ET = 4.8), the Filippov system has a pseudoequilibrium E^P and stabilizes on the sliding segment with E^P. In Figure 2(d), if $\mathrm{ET} > B_{11} > B_{21}$ (ET = 5.9), there are limit cycles. Filippov system (7) stabilizes to the larger limit cycle and virtual equilibria E_{21}^V of subsystem S_2.

The results show that the Filippov system has limit cycles regardless of the initial values ($V(t)$ and $B(t)$) and stabilizes to the larger limit cycle. In other words, when $q_1 = 0.2$, the pest population density stabilizes in a relatively small range, i.e., no pest outbreaks will occur. Other interesting problems can be considered, for example, how to decide the time and choose strategies for controlling pests, how to determine the equilibria of the Filippov system, and how to find regions of pseudoequilibrium when there is sliding bifurcation. To solve these problems, sliding bifurcation diagrams with nullclines of the Filippov system for different situations are investigated.

Case A.2: subsystem S_2 is a bistable state.

In Figure 3, V_1V_2 is the sliding segment. In this section, for the Filippov system, red, black and magenta dashed lines, respectively, indicate nullclines f_{S_1}/f_{S_2}, g_{S_1}, and g_{S_2}. The points of intersection between the nullclines are equilibria of the Filippov system. In Figure 3(a), when the ET is set as different values in the range [2, 9], the Filippov system shows sliding bifurcation and limit cycles. The difference compared with Figure 2(a) is that there is only one larger limit cycle. Subsystem S_1 has the equilibrium E_{11} while subsystem S_2 has the three equilibria E_{21}, E_{22}, and E_{23}. In particular, E_{21} and E_{23} are stable. Therefore, subsystem S_2 is a bistable state. Figures 3(b)–3(d) present the three cases of Figure 3(a), i.e., if $B_{11} > B_{21} > \mathrm{ET} > B_{22} > B_{23}$ (ET = 3), regardless of the initial values, the Filippov system is stable at E_{21}. E_{21} is regular and denoted by E_{21}^R (see Figure 3(b)). If $B_{11} > \mathrm{ET} > B_{21} > B_{22} > B_{23}$ (ET = 4.7), the Filippov system has a pseudoequilibrium E^P but no sliding limit cycle, and it stabilizes on the V_1V_2 with E^P (see Figure 3(c)). If $\mathrm{ET} > B_{11} > B_{21} > B_{22} > B_{23}$ (ET = 6.45), the Filippov system has only one sliding limit cycle and stabilizes to this limit cycle (see Figure 3(d)). Therefore, when $q_1 = 0.45$, the pest population density will be stable in a larger range than in the case of $q_1 = 0.2$ as shown in Figure 1.

We conclude that the equilibria (E_{11}, E_{21}, E_{22}, and E_{23}) are the points of intersection of nullclines (i.e., f_{S_1}/f_{S_2}, g_{S_1}, and g_{S_2}) for the Filippov system. The endpoints (V_1 and V_2) of the sliding segment are located on two nullclines (i.e., g_{S_1} and g_{S_2}). The pseudoequilibrium is the point of intersection of the sliding segment and nullclines of the Filippov system. In addition, the pseudoequilibrium exists on the sliding segment ($E_{11}E_{21}$). These findings are also confirmed in the following discussions.

Case A.3: subsystem S_2 has only one stable refuge equilibrium point.

Figure 4(a) presents the sliding bifurcation of the Filippov system for the different values of ET in the range [0.5, 7.8]. Subsystem S_2 has only one stable equilibrium E_{21}. Figures 4(b)–4(d) present the three cases of Figure 4(a). If $B_{11} > \mathrm{ET} > B_{21}$ (ET = 1.54), the Filippov system has a pseudoequilibrium E^P and stabilizes on the sliding segment V_1V_2 with E^P. There exists E^P on the $E_{11}E_{21}$ (see Figure 4(b)). If $B_{11} > B_{21} > \mathrm{ET}$ (ET = 0.5), the ET is sufficiently small for the Filippov system to be stable at a refuge equilibrium point (see Figure 4(c)). If $\mathrm{ET} > B_{11} > B_{21}$ (ET = 6.76), regardless of the initial values, there is a sliding limit cycle and the Filippov system stabilizes to the limit cycle as shown in Figure 4(d). Hence, when $q_1 = 0.6$, the pest population density will be stable in a smaller range than in the case of $q_1 = 0.45$ as shown in Figure 3.

Therefore, when $q_1 = 0.45$, the range of pests that can be controlled is largest. At the same time, the strategy is the most suitable for controlling pests.

4.2. Case B. Subsystem S_1 has a stable equilibrium point while there are three cases of subsystem S_2.

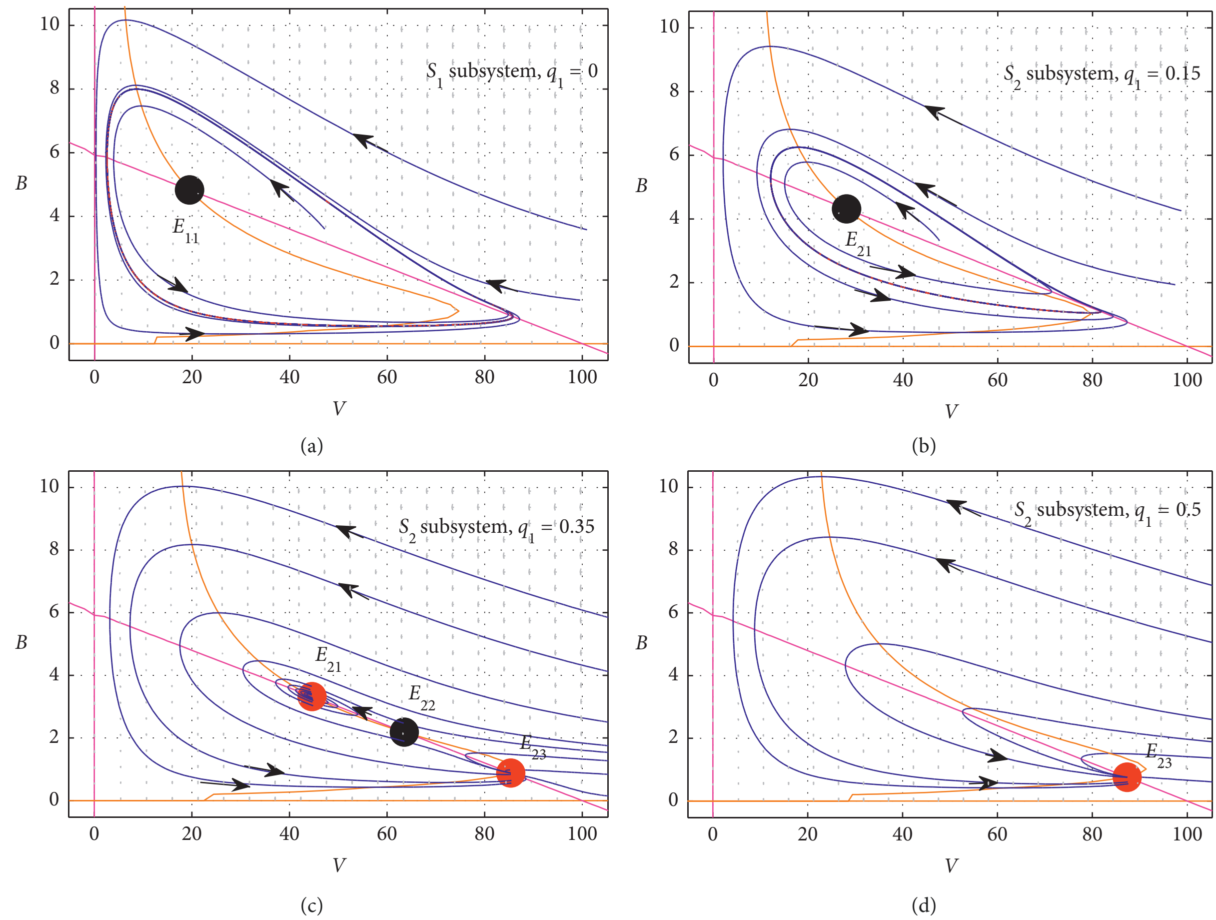

FIGURE 1: Existence of equilibria for Case A of Filippov system (7). Parameters are $r_b = 1$, $k_b = 20$, $\eta = 0.06$, $\delta = 0.5$, $\alpha = 6$, $\beta = 1$, $r_v = 3$, and $k_v = 100$. (a) $q_1 = 0$; (b) $q_1 = 0.15$; (c) $q_1 = 0.35$; (d) $q_1 = 0.5$.

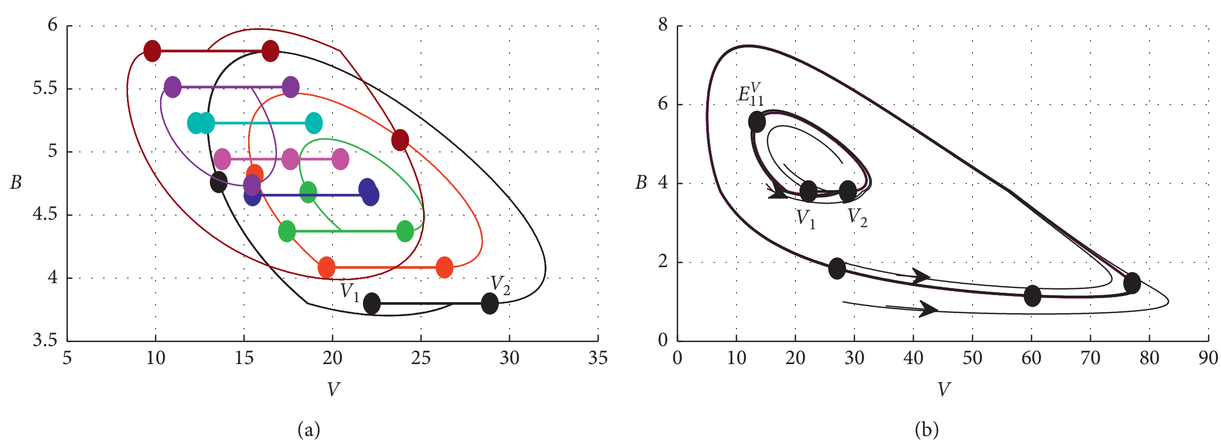

FIGURE 2: Continued.

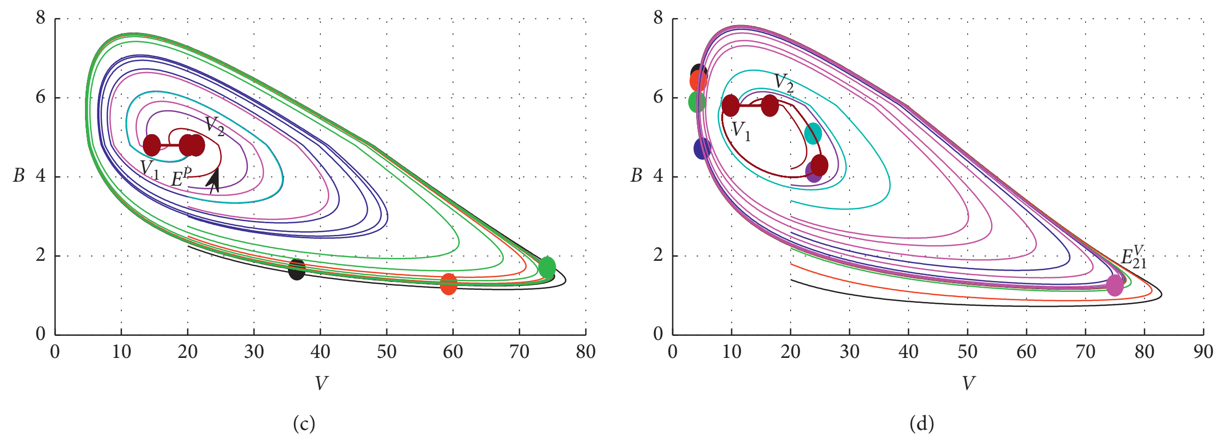

FIGURE 2: Sliding bifurcation for Case A.1 of Filippov system (7). Subsystem S_2 has only one unstable equilibrium point, and stable limit cycles exist. $q_1 = 0.2$, the other parameters are the same as those in Figure 1. (a) ET $\in [3.8, 5.8]$; (b) ET $= 3.9$; (c) ET $= 4.8$; (d) ET $= 5.9$.

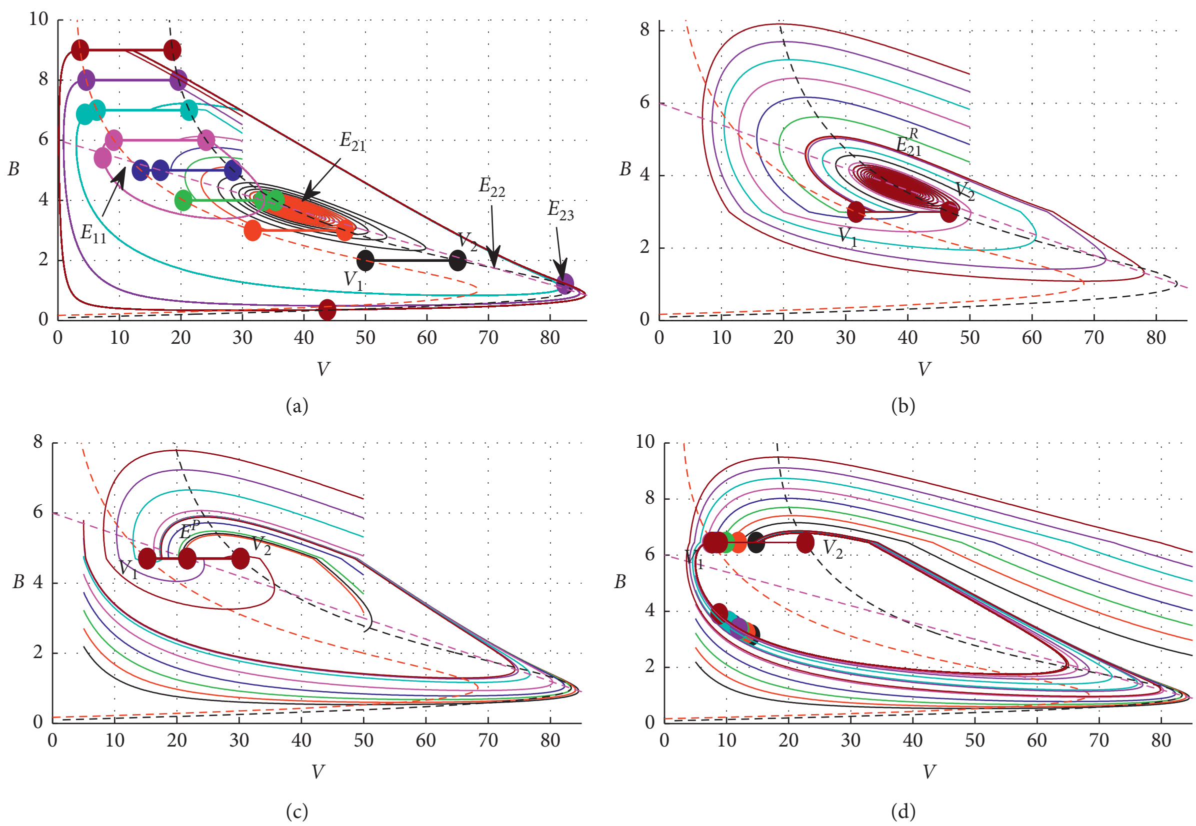

FIGURE 3: Sliding bifurcation for Case A.2 of Filippov system (7). Subsystem S_2 is a bistable state. $q_1 = 0.45$, the other parameters are the same as those in Figure 1. (a) ET $\in [2, 9]$; (b) ET $= 3$; (c) ET $= 4.7$; (d) ET $= 6.45$.

(i) Subsystem S_2 has only one stable equilibrium point

(ii) Subsystem S_2 has two stable equilibrium points (i.e., Subsystem S_2 is a bistable state)

(iii) Subsystem S_2 has only one stable refuge equilibrium point

In Figure 5(a), only one stable equilibrium E_{11} emerges in subsystem S_1. In Figure 5(b), subsystem S_2 has only one stable equilibrium E_{21}. In Figure 5(c), subsystem S_2 has two stable equilibria E_{21} and E_{23}, which is a bistable state. In Figure 5(d), subsystem S_2 has a stable equilibrium E_{23} that becomes a stable refuge equilibrium point with relatively small number of pests.

The sliding mode phenomena in the three subcases of Case B are discussed as follows:

Case B.1: subsystem S_2 has a stable equilibrium point.

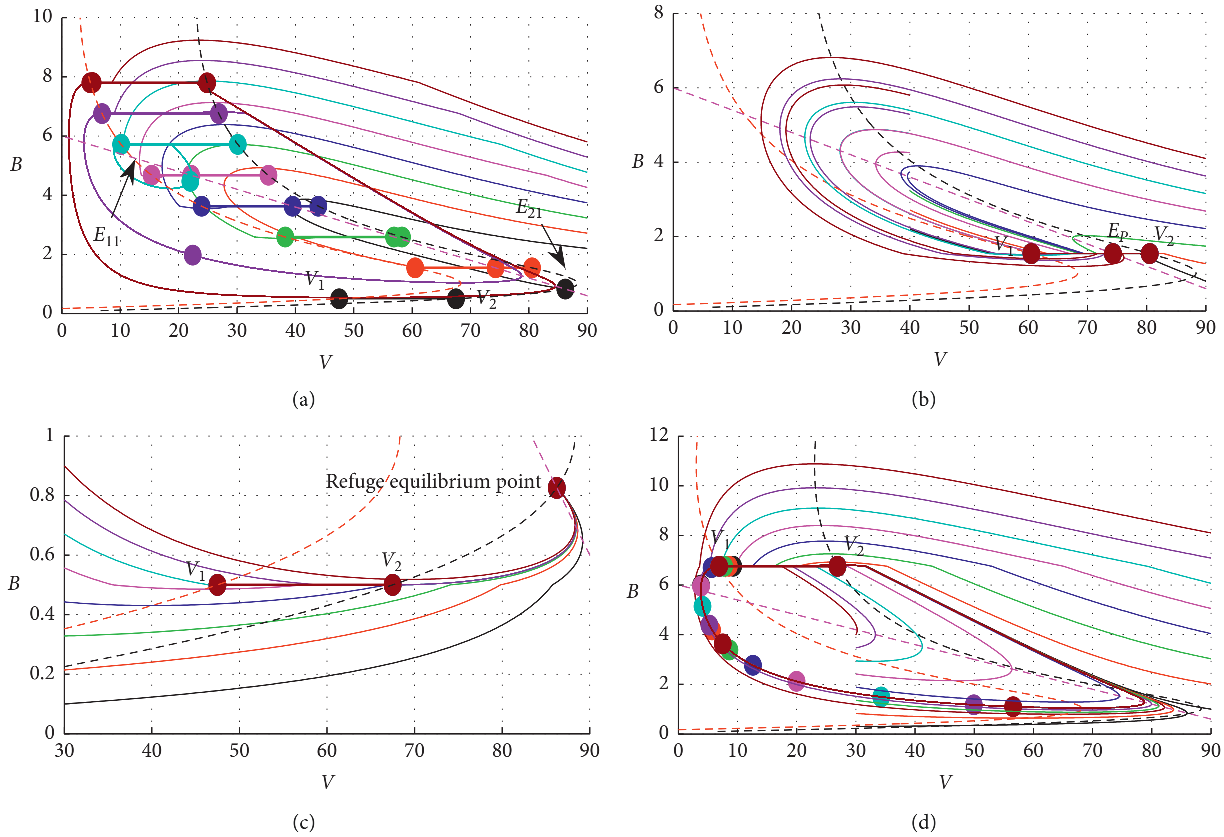

Figure 4: Sliding bifurcation for Case A.3 of Filippov system (7). Subsystem S_2 has only one stable refuge equilibrium point. $q_1 = 0.6$, the other parameters are the same as those in Figure 1. (a) ET $\in [0.5, 7.8]$; (b) ET $= 1.54$; (c) ET $= 0.5$; (d) ET $= 6.76$.

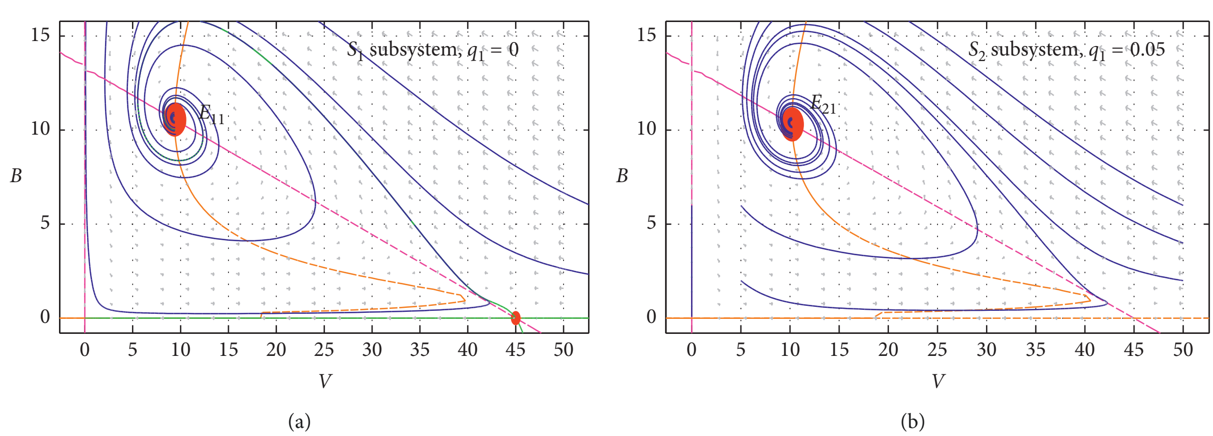

Figure 5: Continued.

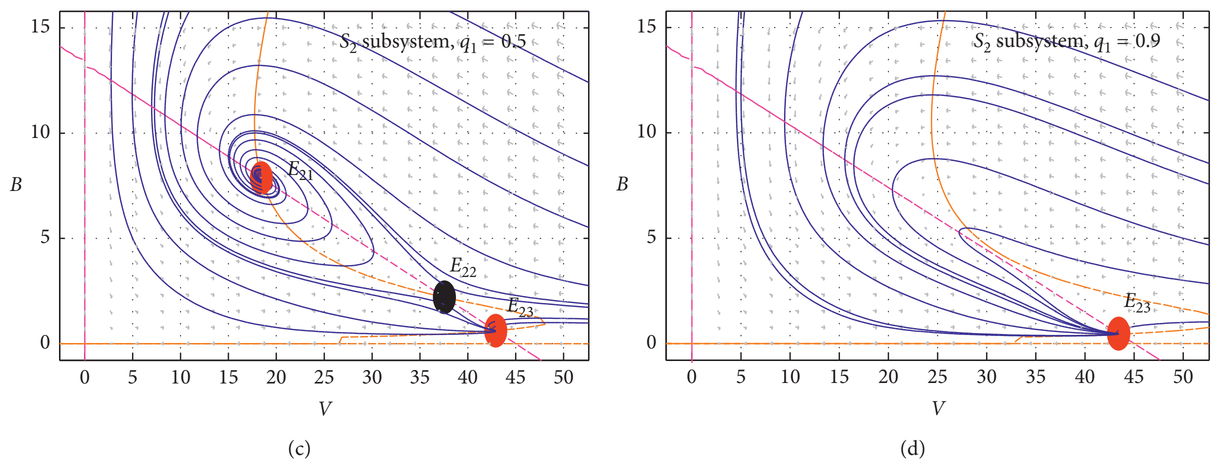

FIGURE 5: Existence of equilibria for Case B of Filippov system (7). Parameters are fixed as $r_b = 0.5$, $k_b = 10$, $\eta = 0.4$, $\delta = 0.15$, $\alpha = 5.7$, $\beta = 1$, $r_v = 2$, and $k_v = 45$. (a) $q_1 = 0$; (b) $q_1 = 0.05$; (c) $q_1 = 0.5$; (d) $q_1 = 0.9$.

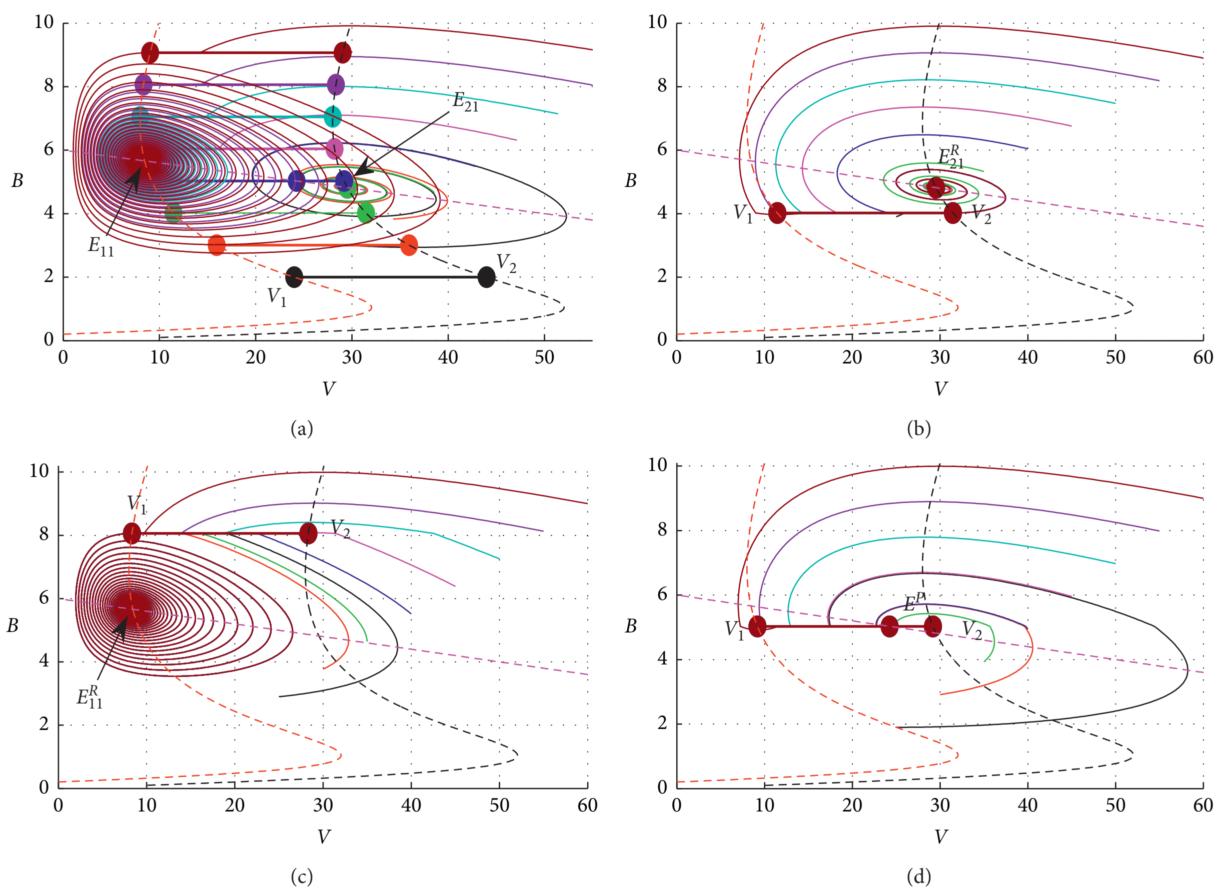

FIGURE 6: Sliding bifurcation for Case B.1 of Filippov system (7). Both subsystem S_1 and subsystem S_2 have one stable equilibrium point. $q_1 = 0.6$, the other parameters are $r_b = 0.6$, $k_b = 10$, $\eta = 0.06$, $\delta = 0.5$, $\alpha = 3$, $\beta = 1$, $r_v = 3$, and $k_v = 150$. (a) ET $\in [2, 9.07]$; (b) ET = 4; (c) ET = 8; (d) ET = 5.

Figure 6(a) shows that there is sliding bifurcation of Filippov system (7) when the ET is set as the different values in $[2, 9.07]$. There is only one stable equilibrium E_{11} of subsystem S_1 and a stable equilibrium E_{21} of subsystem S_2. The three cases of Figure 6(a) are presented in Figures 6(b)–6(d). If $B_{11} > B_{21} > \mathrm{ET}$ (ET = 4), the Filippov system stabilizes at the equilibrium E_{21}, which is regular and denoted by E_{21}^R (see Figure 6(b)). If $\mathrm{ET} > B_{11} > B_{21}$ (ET = 8), the Filippov system stabilizes at E_{11}. E_{11} is virtual and defined by E_{11}^R (see Figure 6(c)). If

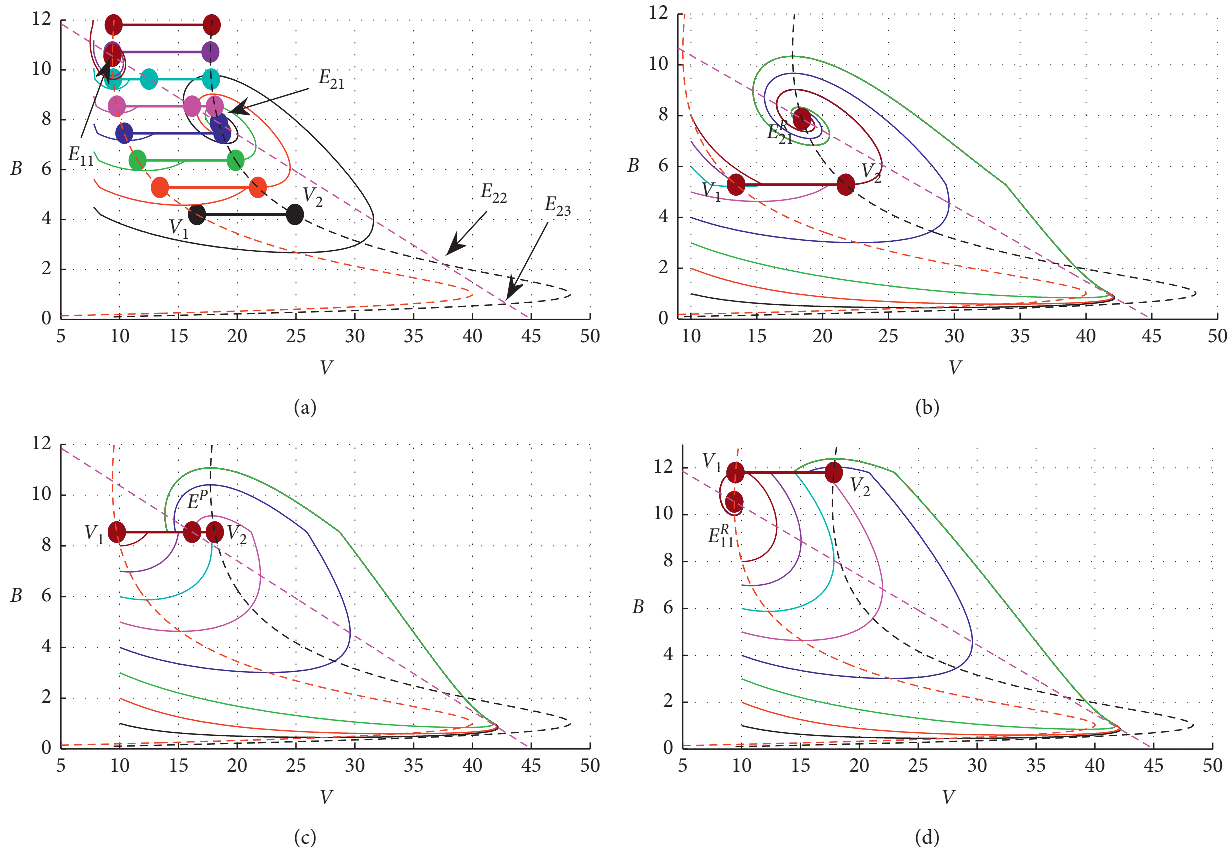

FIGURE 7: Sliding bifurcation for Case B.2 of Filippov system (7). Subsystem S_1 has one stable equilibrium point and subsystem S_2 is a bistable state. $q_1 = 0.5$ and the other parameters are fixed as $r_b = 0.5$, $k_b = 10$, $\eta = 0.4$, $\delta = 0.15$, $\alpha = 5.7$, $\beta = 1$, $r_v = 2$, and $k_v = 45$. (a) $\mathrm{ET} \in [4.2, 11.8]$; (b) ET = 5.29; (c) ET = 8.54; (d) ET = 11.8.

$B_{11} > \mathrm{ET} > B_{21}$ (ET = 5), the Filippov system has a pseudoequilibrium E^P and stabilizes on the sliding segment V_1V_2 with E^P (see Figure 6(d)). In particular, there exists a pseudoequilibrium on the $E_{11}E_{21}$. While there is no limit cycle in this case, there are no pest outbreaks and the system stabilizes within a relatively small range.

Case B.2: subsystem S_2 is a bistable state.

Figure 7(a) shows the sliding bifurcation of the Filippov system for the value of ET varying in the range [4.2, 11.8]. Subsystem S_1 has only one stable equilibrium E_{11}, subsystem S_2 is a bistable state where E_{21} and E_{23} are stable. Figures 7(b)–7(d) presents the three cases of Figure 7(a). In Figure 7(b), if $B_{11} > B_{21} > \mathrm{ET} > B_{22} > B_{23}$ (ET = 5.29), the Filippov system always stabilizes at E_{21}, which is real and denoted by E^R_{21}. In Figure 7(c), if $B_{11} > \mathrm{ET} > B_{21} > B_{22} > B_{23}$ (ET = 8.54), the Filippov system has a pseudoequilibrium E^P and stabilizes on the sliding segment V_1V_2 with E^P. There is a pseudoequilibrium on the $E_{11}E_{21}$. In Figure 7(d), if $\mathrm{ET} > B_{11} > B_{21} > B_{22} > B_{23}$ (ET = 11.8), Filippov system (7) is stable at E_{11} with different initial values. E_{11} is regular and denoted by E^R_{11}. In this case, when $q_1 = 0.5$, the pest population density will stabilize within a relatively large range.

Case B.3: subsystem S_2 has only one stable refuge equilibrium point.

Figure 8(a) presents the sliding mode phenomena for the different values of ET ($\mathrm{ET} \in [0.1, 12]$). Subsystem S_1 has only one stable equilibrium E_{11} while subsystem S_2 has only one stable equilibrium E_{21}, which is a stable refuge equilibrium point. Figures 8(b)–8(d) present the three cases of Figure 8(a). In Figure 8(b), if $B_{11} > B_{21} > \mathrm{ET}$ (ET = 0.1), no matter how large the initial values are, the Filippov system stabilizes at E_{21} (i.e., a stable refuge equilibrium point). In Figure 8(c), if $B_{11} > \mathrm{ET} > B_{21}$ (ET = 6.9), the Filippov system has a pseudoequilibrium E^P and stabilizes on the V_1V_2 (i.e., the sliding segment) with E^P. Moreover, E^P emerges on the $E_{11}E_{21}$. In Figure 8(d), if $\mathrm{ET} = B_{11} > B_{21}$ (ET = 10.6), the equilibrium E_{11} becomes a stable refuge equilibrium point. The Filippov system stabilizes on the V_1V_2 with E_{11}. E_{11} is virtual and denoted by E^V_{11}. When $q_1 = 0.9$, the pest population density stabilizes within a smaller range than in the case of $q_1 = 0.5$ as shown in Figure 7. This means that the proper strategy can better control pests.

4.3. *Case C.*

Subsystem S_1 has three equilibrium points, two of which are stable. There are two subcases for subsystem S_2.

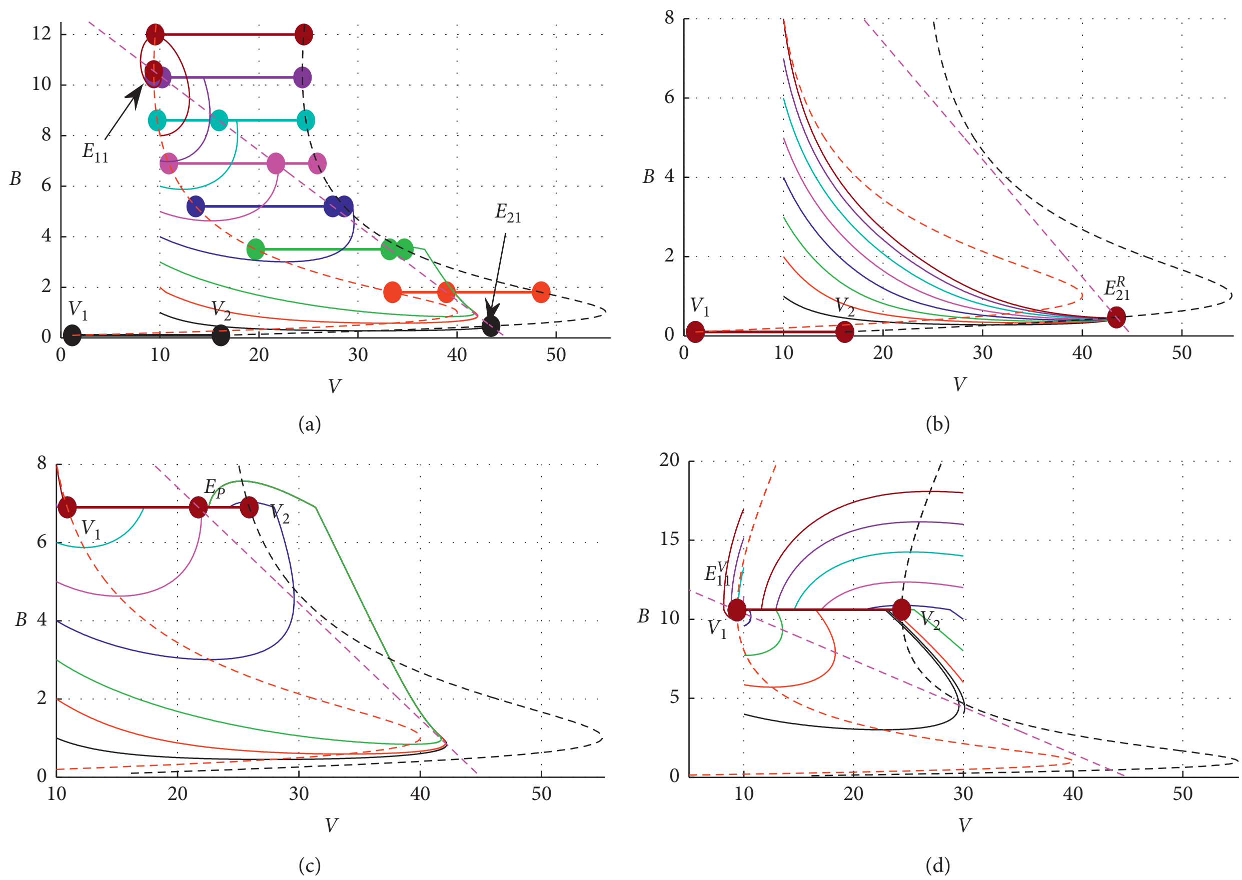

Figure 8: Sliding bifurcation for Case B.3 of Filippov system (7). Subsystem S_1 has a stable equilibrium point. Subsystem S_2 has only one stable refuge equilibrium point. $q_1 = 0.9$, the other parameters are the same as those in Figure 7. (a) ET $\in [0.1, 12]$; (b) ET = 0.1; (c) ET = 6.9; (d) ET = 10.6.

(i) Subsystem S_2 is a bistable state

(ii) Subsystem S_2 has a stable refuge equilibrium point

The sliding mode phenomena in the two subcases of Case *C* are discussed as follows:

Case C.1: Subsystem S_2 is bistable.

Figure 9(a) presents the sliding bifurcation of the Filippov system for the different values of ET in the range [6.4, 13.6]. Both subsystem S_1 and subsystem S_2 have three equilibrium points, namely E_{11}, E_{12}, E_{13}, E_{21}, E_{22} and E_{23}. E_{11}, E_{13}, E_{21} and E_{23} are stable. Subsystems S_1 and S_2 are bistable states. The three cases of Figure 9(a) are presented in Figures 9(b)–9(d). If $\mathrm{ET} > B_{11} > B_{21} > B_{22} > B_{12} > B_{13} > B_{23}$ (ET = 15), the Filippov system stabilizes at E_{11}, which is called a regular equilibrium of subsystem S_1 and denoted by E_{11}^R (see Figure 9(b)). If $B_{11} > \mathrm{ET} > B_{21} > B_{22} > B_{12} > B_{13} > B_{23}$ (ET = 10.5), the Filippov system has a pseudoequilibrium E^P and stabilizes on the sliding segment V_1V_2 with E^P (see Figure 9(c)). If $B_{11} > B_{21} > \mathrm{ET} > B_{22} > B_{13} > B_{23}$ (ET = 6.4), the Filippov system stabilizes at E_{21}, which is called a regular equilibrium of subsystem S_2 and denoted by E_{21}^R (see Figure 9(d)). Therefore, in this case, if the density of the pest population is high, the pest population can be well controlled; if the initial values is relatively small, the Filippov system stabilize at a refuge equilibrium point.

Case C.2: Subsystem S_2 has only one stable refuge equilibrium point.

Figure 10(a) shows the sliding bifurcation of the Filippov system under the different values of ET in the range [5.3, 13.8]. Subsystem S_1 has three equilibrium points, namely E_{11}, E_{12} and E_{13}. Subsystem S_2 has an equilibrium E_{21}. There is a pseudoequilibrium on the $E_{11}E_{12}$. Figure 10(b) shows that a pseudoequilibrium E^P emerges if $B_{11} > \mathrm{ET} > B_{12} > B_{13} > B_{21}$ (ET = 6.5). When the value of B is large, the Filippov system is stable on the V_1V_2 with E^P. When the value of B is small, the Filippov system stabilizes at a refuge equilibrium point. Therefore, when $q_1 = 0.6$, the pest population density can be controlled in a larger range than in the case of $q_1 = 0.4$ as shown in Figure 9.

4.4. Case D. Subsystem S_1 and subsystem S_2 have three equilibrium points. Only one is a stable refuge equilibrium point.

In Figure 11(a), subsystem S_1 has the three equilibria E_{11}, E_{12}, and E_{13}. E_{11} and E_{12} are unstable and colored black. Only the red E_{13} is stable. In Figure 11(b), subsystem S_1 has

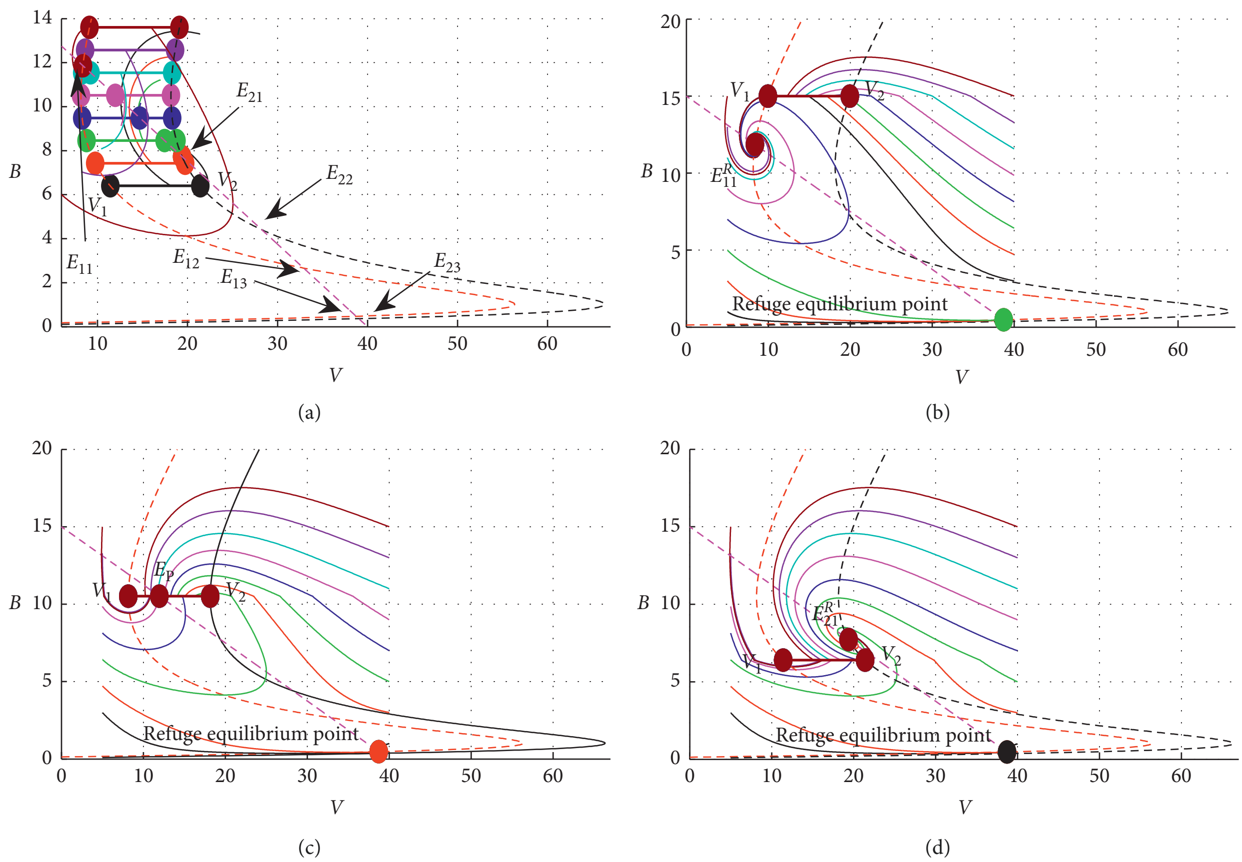

Figure 9: Sliding bifurcation for Case C.1 of Filippov system (7). Both subsystem S_1 and subsystem S_2 are bistable states. $q1 = 0.4$, parameters are $r_b = 0.8$, $k_b = 15$, $\eta = 0.4$, $\delta = 0.1$, $\alpha = 6$, $\beta = 1$, $r_v = 1.5$, and $k_v = 40$. (a) $\mathrm{ET} \in [6.4, 11.6]$; (b) $\mathrm{ET} = 15$; (c) $\mathrm{ET} = 10.5$; (d) $\mathrm{ET} = 6.4$.

the three equilibria E_{21}, E_{22} and E_{23}. The black E_{21} and E_{22} are unstable, and only the smallest red dot E_{23} is stable.

The sliding mode phenomena of Case D are discussed as follows.

Figure 12(a) shows the sliding bifurcation of the Filippov system for the ET in the range [3.6, 16.4]. Both subsystem S_1 and subsystem S_2 have three equilibrium points. Only one is a stable refuge equilibrium point. Figures 12(b)–12(d) present the three cases of Figure 12(a). If $B_{11} > \mathrm{ET} > B_{21} > B_{22} > B_{12} > B_{13} > B_{23}$ ($\mathrm{ET} = 9.1$), the Filippov system has a pseudoequilibrium E^P and stabilizes on the sliding segment V_1V_2 with E^P (see Figure 12(b)). If $\mathrm{ET} > B_{11} > B_{21} > B_{22} > B_{12} > B_{13} > B_{23}$ ($\mathrm{ET} = 12.7$), Filippov system (7) has a stable sliding limit cycle and stabilizes on it (see Figure 12(c)). Figure 12(d) has no limit cycle in contrast with Figure 12(c). Therefore, regardless of their initial values, the Filippov system stabilizes at a stable refuge equilibrium point as shown in Figures 12(b)–12(d). In this case, when $q_1 = 0.4$, the pest population remains stable within a relatively large range.

Overall, in Cases A–D, the value of q_1 is large and the number of beetles is controlled within a certain range that is better. Taking spraying pesticides as an example, from the perspective of economic costs, there is no desire to use more pesticides. In addition, from a biomathematical point of view, the spraying of pesticides (i.e., a chemical measure) is performed frequently to kill beetles, which creates pollution. Therefore, the results show that it is crucial to choose a optimal strategy (or the dosage of pesticide) if the pest density reaches the ET.

5. Conclusion

A novel Filippov forest-pest system with TPC concerning IPM was proposed. The system comprises two subsystems S_1 and S_2. Our aim is to find the best time and strategy for successful pest control. The conditions for the existence and stability of equilibria of subsystems were addressed. Moreover, the sliding segment was determined. Several types of equilibria of the proposed system were defined, i.e., the regular equilibrium, virtual equilibrium, pseudoequilibrium, boundary equilibrium, and a special point called the tangent point. Meanwhile, nullclines of the Filippov system were also defined, which were related to the equilibria of these two susystems and used to determine their existence. That is, the points of intersection between nullclines were these equilibrium points of the system, and the two endpoints of the sliding segment (V_1V_2) were on the nullclines.

Four cases of the sliding bifurcations of the Filippov system with respect to different types of equilibrium points of subsystems were discussed. Details are given in Table 1. In Case A.1, both subsystem S_1 and subsystem S_2 had only

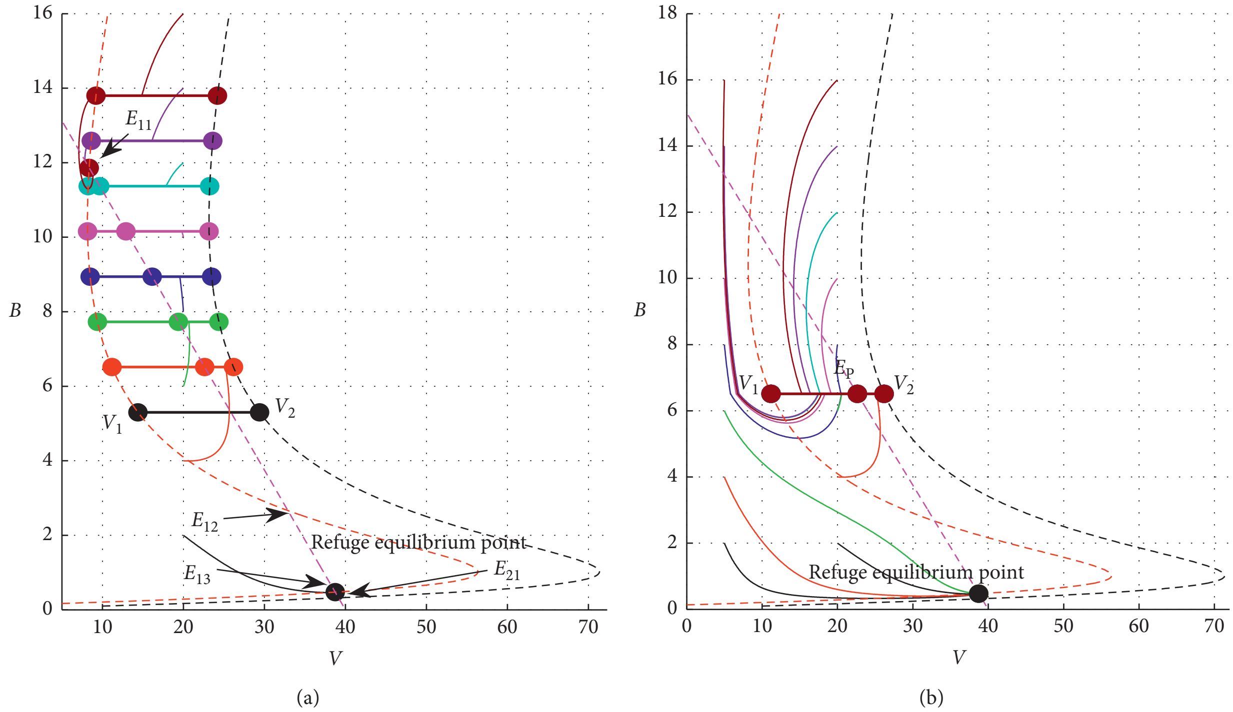

FIGURE 10: Sliding bifurcation for Case C.2 of Filippov system (7). Subsystem S_2 has only one refuge equilibrium point. $q_1 = 0.6$, the other parameters are the same as those in Figure 9. (a) ET $\in [5.3, 13.8]$; (b) ET $= 6.5$.

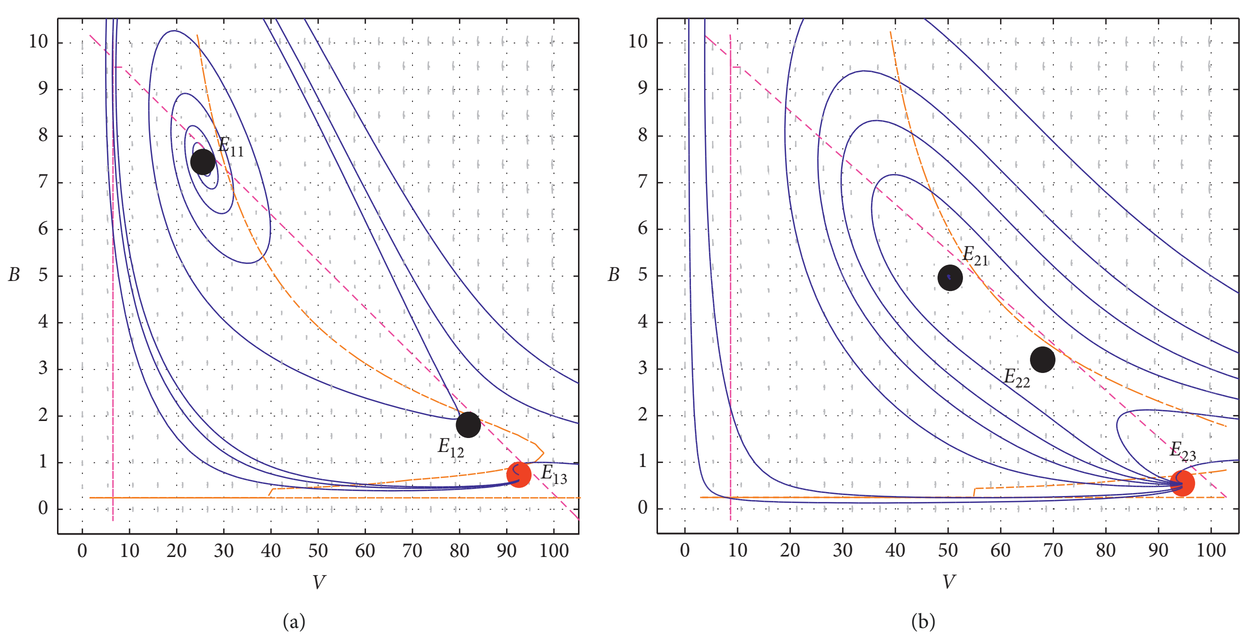

FIGURE 11: Existence of equilibria for Case D of Filippov system (7). Parameters are fixed as $r_b = 0.1$, $k_b = 10$, $\eta = 0.15$, $\sigma = 0.2$, $\alpha = 6$, $\beta = 1$, $r_v = 2$, and $k_v = 100$. (a) $q_1 = 0$; (b) $q_1 = 0.4$.

an unstable equilibrium point (see Figures 1(a) and 1(b)), the system had limit cycles and stabilized to the larger limit cycle regardless of the initial values (V and B) (see Figure 2).

Subsystem S_2 was a bistable state in Cases A.2, B.2, and C.1 (see Figures 3, 7, and 9). For instance, in Cases A.2 and B.2, subsystem S_1 had an unstable equilibrium point (see Figure 1(a)) and subsystem S_1 had a stable equilibrium point (see Figure 5(a)), subsystem S_2 had three equilibria (i.e., E_{21}, E_{22}, and E_{23}) and equilibria (i.e., E_{21} and E_{23}) were stable (see Figures 1(c) and 5(c)). In Case C.1, both subsystem S_1 and subsystem S_2 are bistable states.

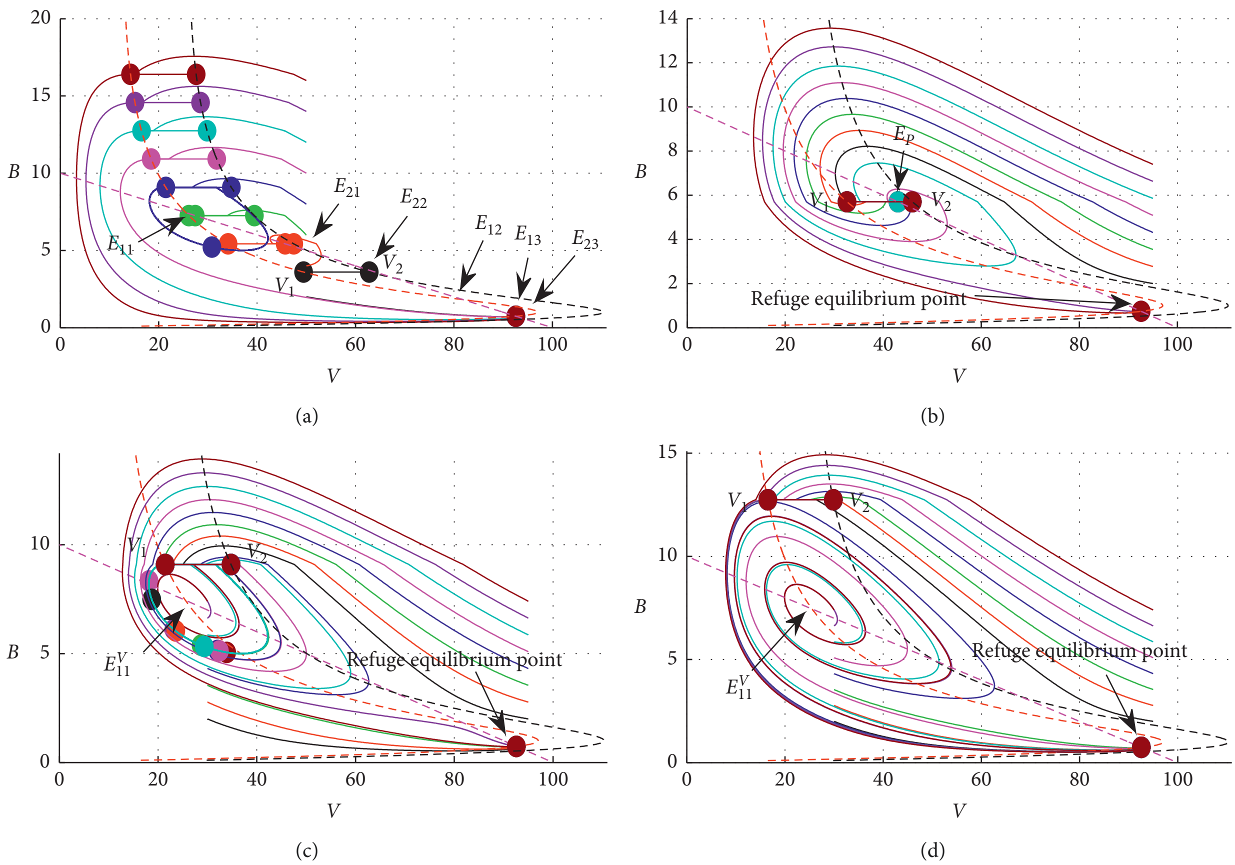

Figure 12: Sliding bifurcation for Case D of Filippov system (7). $q1 = 0.4$, the other parameters are the same as those in Figure 11. (a) ET $\in$ [3.6, 16.4]; (b) ET = 5.7; (c) ET = 9.1; (d) ET = 12.7.

Subsystem S_2 had only one stable refuge equilibrium point in Cases A.3, B.3, C.2, and D. For example, in Case A.3, subsystem S_1 had only an unstable equilibrium point. In Case B.3, subsystem S_1 had only a stable equilibrium point. When the number of pests and forests was relatively small, subsystem S_2 had only one stable refuge equilibrium point (see Figures 1(d) and 5(d)), the system was stable at it (see Figures 4 and 8). In Case C.2, subsystem S_1 was the bistable state and only one stable refuge equilibrium point existed in subsystem S_2. Moreover, the system stabilized on the V_1V_2 with pseudoequilibrium E^{P} (see Figure 10). In Case D, both subsystems S_1 and S_2 had three equilibrium points (see Figure 11) and only one was a stable refuge equilibrium point. The Filippov system stabilized at a stable refuge equilibrium point with ET in the range [3.6, 16.4] (see Figure 12). A stable equilibrium point appeared in subsystem S_1 and subsystem S_2 (see Figures 5(a) and 5(b)). Especially, in Case A, the stable limit cycles existed in subsystem S_1.

In addition, it was verified that the pseudoequilibrium was the point of intersection of the sliding segment and nullclines of the Filippov system, and the pseudoequilibrium existed on the V_1V_2. In Case A, when $q_1 = 0.45$, it was the most suitable for controlling pests. In Case B, the results revealed that the proper use of pesticides can effectively control pests. In Case C, when $q_1 = 0.6$, the effects of pest control were the best. Therefore, to some extent, the more the pesticides were used, the better the number of beetles will be controlled within a certain range so that the aim of IPM strategies was achieved.

In summary, the Filippov system had sliding limit cycles, a bistable state, and a stable refuge equilibrium point by means of sliding bifurcation analysis. From the perspective of practical biological significance, when the pest density reached the ET, the optimal strategy should be used, namely, the proper number of pests was captured, transferred, or killed by using the cultural and chemical strategies (i.e., the right amount of pesticide needed to be sprayed). From an economic point of view, it was beneficial to reduce the use of manpower, material resources, and financial resources and protected the environment. In the future, different control policies in which the threshold is also a linear or nonlinear function about forests and pests deserve investigation.

Data Availability

No data were used to support this study. In our study, there are only some numerical simulations to support our main result, and some parameter values to support the result of this paper are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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