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Research Article

Positive Solutions for a System of Fractional Integral Boundary Value Problems Involving Hadamard-Type Fractional Derivatives

Haiyan Zhang,[1] **Yaohong Li**,[1] **and Jiafa Xu**[2]

[1]*School of Mathematics and Statistics, Suzhou University, Suzhou, Anhui 234000, China*
[2]*School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China*

Correspondence should be addressed to Jiafa Xu; xujiafa292@sina.com

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In this paper, we use fixed-point index to study the existence of positive solutions for a system of Hadamard fractional integral boundary value problems involving nonnegative nonlinearities. By virtue of integral-type Jensen inequalities, some appropriate concave and convex functions are used to depict the coupling behaviors for our nonlinearities $f_i (i = 1, 2)$.

1. Introduction

In this paper, we study the existence of positive solutions for the system of Hadamard fractional integral boundary value problems:

$$\begin{cases} -{}^H D^\alpha u(t) = f_1(t, u(t), v(t)), & t \in (1, e), \\ -{}^H D^\alpha v(t) = f_2(t, u(t), v(t)), & t \in (1, e), \\ u^{(j)}(1) = v^{(j)}(1) = 0, \\ u(e) = \int_1^e h(t) u(t) \frac{dt}{t}, \\ v(e) = \int_1^e h(t) v(t) \frac{dt}{t}, \end{cases} \tag{1}$$

where $\alpha \in (n-1, n]$ is a real number with $n \geq 3$, $j = 0, 1, 2, \dots, n-2$, and ${}^H D^\alpha$ is the Hadamard fractional derivative. The nonlinearities $f_i \in C([1, e] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$. Moreover, the function h on $[1, e]$ satisfies the condition:

(H0) $h \geq 0$ with $\int_1^e h(t) (\log t)^{\alpha - 1} (dt/t) \in [0, 1)$.

In recent years, the fractional calculus and fractional differential equations are of importance in mathematics, physics, electroanalytical chemistry, capacitor theory, electrical circuits, biology, control theory, and fluid dynamics [1–20]. For example, in [1], the author considered the fractional $(n-1, 1)$-type conjugate boundary value problems:

$$\begin{cases} D_{0+}^\alpha u(t) + \lambda f(t, u(t)) = 0, & 0 < t < 1, \\ u^{(j)}(0) = 0, \ u(1) = 0, & 0 \leq j \leq n-2, \end{cases} \tag{2}$$

where $\alpha \in (n-1, n]$, $n \geq 3$, and D_{0+}^α is the Riemann–Liouville's fractional derivative. By means of Leray–Schauder type and Krasnosel'skii's fixed-point theorems, the author derived an interval of parameter λ such that (2) has multiple positive solutions when any λ lies in the interval.

On the other hand, we note that coupled systems of fractional differential equations have also been investigated by many authors, see [21–32]. For example, in [21], the authors used a fixed-point theorem of increasing φ-(h, r)-concave operators to establish the existence and uniqueness of solutions for a system of four-point boundary value problems involving Hadamard fractional derivatives:

$$\begin{cases} {}^H D^\alpha u(t) + f(t, v(t)) = l_f, & t \in (1, e), \\ {}^H D^\beta v(t) + g(t, u(t)) = l_g, & t \in (1, e), \\ u^{(j)}(1) = v^{(j)}(1) = 0, & 0 \leq j \leq n-2, \\ u(e) = a v(\xi), \\ v(e) = b u(\eta), \ \xi, & \eta \in (1, e), \end{cases} \tag{3}$$

where $f, g \in C([1, e] \times \mathbb{R}, \mathbb{R})$ and l_f and l_g are two positive parameters. In [22], the authors established positive solutions for the coupled Hadamard fractional integral boundary value problems:

$$\begin{cases} {}^{H}D^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0, & t \in (1, e), \lambda > 0, \\ {}^{H}D^{\beta}v(t) + \lambda g(t, u(t), v(t)) = 0, & t \in (1, e), \lambda > 0, \\ u^{(j)}(1) = v^{(j)}(1) = 0, & 0 \le j \le n-2, \\ u(e) = \mu \int_1^e v(s)\frac{ds}{s}, \\ v(e) = \nu \int_1^e u(s)\frac{ds}{s}, \end{cases} \tag{4}$$

where the nonlinearities f and g satisfy either of the following conditions:

$(\mathrm{H})_{\mathrm{Yang1}}$: there exists $[\theta_1, \theta_2] \subset (1, e)$ such that $\liminf_{u\longrightarrow+\infty} \min_{t\in[\theta_1,\theta_2]}(f(t,u,v)/u) = +\infty$ and $\liminf_{v\longrightarrow+\infty} \min_{t\in[\theta_1,\theta_2]}(g(t,u,v)/v) = +\infty$.

$(\mathrm{H})_{\mathrm{Yang2}}$: there exists $[\theta_1, \theta_2] \subset (1, e)$ such that $\liminf_{v\longrightarrow+\infty} \min_{t\in[\theta_1,\theta_2]}(f(t,u,v)/v) = +\infty$ and $\liminf_{u\longrightarrow+\infty} \min_{t\in[\theta_1,\theta_2]}(g(t,u,v)/u) = +\infty$.

Inspired by the aforementioned works, in this paper, we use the fixed-point index to consider the existence of positive solutions for system (1) of fractional integral boundary value problems involving Hadamard-type fractional derivatives. Based on integral-type Jensen inequalities, some appropriate concave and convex functions are used to depict the coupling behaviors for the nonlinearities $f_i (i = 1, 2)$. Moreover, our a priori estimates for positive solutions are derived by developing some appropriate nonnegative matrices when $f_i (i = 1, 2)$ grow sublinearly at ∞. These conditions here are different from that in $(\mathrm{H})_{\mathrm{Yang1}}$ and $(\mathrm{H})_{\mathrm{Yang2}}$.

2. Preliminaries

In this paper, we only provide some necessary definitions and lemmas for the Hadamard fractional derivative. For more details about Hadamard fractional calculus, see the book [33].

Definition 1. The Hadamard derivative of fractional order q for a function $g: [1,\infty) \longrightarrow \mathbb{R}$ is defined as

$$ {}^{H}D^{q}g(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^{n}\int_1^t (\log t - \log s)^{n-q-1} g(s)\frac{ds}{s}, \quad n-1 < q < n, \tag{5}$$

where $n = [q] + 1$, $[q]$ denotes the integer part of the real number q, and $\log(\cdot) = \log_e(\cdot)$.

Definition 2. The Hadamard fractional integral of order q for a function g is defined as

$$ {}^{H}I^{q}g(t) = \frac{1}{\Gamma(q)}\int_1^t (\log t - \log s)^{q-1} g(s)\frac{ds}{s}, \quad q > 0. \tag{6}$$

Lemma 1. *Let $q > 0$ and $u \in C[1,\infty) \cap L^1[1,\infty)$. Then, the Hadamard fractional differential equation ${}^{H}D^{q}u(t) = 0$ has the solution*

$$u(t) = c_1(\log t)^{q-1} + c_2(\log t)^{q-2} + \cdots + c_n(\log t)^{q-n}, \tag{7}$$

where $c_i \in \mathbb{R}$, $n-1 < q < n, n = [q] + 1$, and $i = 1, 2, \ldots, n$.

Lemma 2. *Let $q > 0$ and $u \in C[1,\infty) \cap L^1[1,\infty)$. Then, we have the following formula:*

$$ {}^{H}I^{q}\,{}^{H}D^{q}u(t) = u(t) + c_1(\log t)^{q-1} + c_2(\log t)^{q-2} + \cdots + c_n(\log t)^{q-n}, \tag{8}$$

where c_i and n are as in Lemma 1 and $i = 1, 2, \ldots, n$.

Lemma 3. *Suppose that (H0) holds. Let $f \in C[1, e]$. Then, the boundary value problems*

$$\begin{cases} -{}^{H}D^{\alpha}u(t) = f(t), & t \in (1, e), \\ u^{(j)}(1) = 0, \\ u(e) = \int_1^e h(t)u(t)\frac{dt}{t}, \end{cases} \tag{9}$$

has a unique solution

$$u(t) = \int_1^e G(t,s) f(s)\frac{ds}{s}, \tag{10}$$

where

$$G(t,s) = G_1(t,s) + \frac{(\log t)^{\alpha-1}}{1-\int_1^e h(t)(\log t)^{\alpha-1}dt/t}\int_1^e h(t)G_1(t,s)\frac{dt}{t},$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (\log t)^{\alpha-1}(1-\log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1}, & 1 \le s \le t \le e, \\ (\log t)^{\alpha-1}(1-\log s)^{\alpha-1}, & 1 \le t \le s \le e. \end{cases} \tag{11}$$

Proof. Using Lemma 2, we have

$$u(t) = c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} + \cdots + c_n(\log t)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}\int_1^t (\log t - \log s)^{\alpha-1} f(s)\frac{ds}{s}, \tag{12}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. By $u^{(j)}(1) = 0, j = 0, 1, \ldots, n-2$, we have $c_i = 0$, $i = 2, 3, \ldots, n$. Hence,

$$u(t) = c_1(\log t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_1^t (\log t - \log s)^{\alpha-1} f(s)\frac{ds}{s}. \tag{13}$$

Then, we know $u(e) = c_1 - (1/\Gamma(\alpha))\int_1^e (1-\log s)^{\alpha-1} f(s)(ds/s)$. Using the condition $u(e) = \int_1^e h(t)u(t)(dt/t)$, we have

$$\begin{aligned} c_1 - \frac{1}{\Gamma(\alpha)}\int_1^e (1-\log s)^{\alpha-1} f(s)\frac{ds}{s} &= c_1\int_1^e h(t)(\log t)^{\alpha-1}\frac{dt}{t} \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_1^e h(t)\int_1^t (\log t - \log s)^{\alpha-1} f(s)\frac{ds}{s}\frac{dt}{t}. \end{aligned} \tag{14}$$

Then, (H0) implies that

$$
\begin{aligned}
c_1 &= \frac{1}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e (1-\log s)^{\alpha-1} f(s)\frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e h(t)\int_1^t (\log t-\log s)^{\alpha-1} f(s)\frac{ds}{s}\frac{dt}{t} \\
&= \frac{1}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e (1-\log s)^{\alpha-1} f(s)\frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e (1-\log s)^{\alpha-1} f(s)\frac{ds}{s}.
\end{aligned}
\tag{15}
$$

As a result, we have

$$
\begin{aligned}
u(t) &= \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e (1-\log s)^{\alpha-1} f(s)\frac{ds}{s} \\
&\quad - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e h(t)\int_s^e (\log t-\log s)^{\alpha-1} f(s)\frac{dt}{t}\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_1^t (\log t-\log s)^{\alpha-1} f(s)\frac{ds}{s} \\
&= \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e (1-\log s)^{\alpha-1} f(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^e (\log t)^{\alpha-1}(1-\log s)^{\alpha-1} f(s)\frac{ds}{s} \\
&\quad - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e h(t)\int_1^e (\log t-\log s)^{\alpha-1} f(s)\frac{dt}{t}\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_1^t (\log t-\log s)^{\alpha-1} f(s)\frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_1^e (\log t)^{\alpha-1}(1-\log s)^{\alpha-1} f(s)\frac{ds}{s} \\
&= \int_1^e G_1(t,s) f(s)\frac{ds}{s} + \frac{\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e (\log t)^{\alpha-1}(1-\log s)^{\alpha-1} f(s)\frac{ds}{s} \\
&\quad - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\int_1^e h(t)\int_s^e (\log t-\log s)^{\alpha-1} f(s)\frac{dt}{t}\frac{ds}{s} \\
&= \int_1^e G_1(t,s) f(s)\frac{ds}{s} + \frac{(\log t)^{\alpha-1}\left[\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\int_1^e (1-\log s)^{\alpha-1} f(s)(ds/s) - \int_1^e h(t)\int_s^e (\log t-\log s)^{\alpha-1} f(s)(dt/t)(ds/s)\right]}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)} \\
&= \int_1^e G_1(t,s) f(s)\frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\int_1^e\int_1^e h(t)G_1(t,s)\frac{dt}{t} f(s)\frac{ds}{s} \\
&= \int_1^e G(t,s) f(s)\frac{ds}{s}.
\end{aligned}
\tag{16}
$$

This completes the proof. □

In what follows, we study some useful inequalities for Green's functions in (11). We first provide a result in [1].

Let $h(t) \in C[0,1]$, and then the Riemann–Liouville boundary-value problem

$$
\begin{cases}
D_{0+}^{\alpha} u(t) + h(t) = 0, & 0 < t < 1, 2 \le n-1 < \alpha \le n, \\
u^{(j)}(0) = u(1) = 0, & 0 \le j \le n-2,
\end{cases}
\tag{17}
$$

has a unique solution $u(t) = \int_0^1 H(t,s)h(s)ds$, where

$$H(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0\le t\le s\le 1.\end{cases} \quad (18)$$

Moreover, Green's function H satisfies the inequalities:

$$\Gamma(\alpha)k(t)q(s)\le H(t,s)\le(\alpha-1)q(s), \quad \text{for } t,s\in[0,1], \quad (19)$$

where $k(t)=(t^{\alpha-1}(1-t)/\Gamma(\alpha))$ and $q(s)=(s(1-s)^{\alpha-1}/\Gamma(\alpha))$.

Comparing G_1 with H, using $\log t$ and $\log s$ to replace t and s, from (11) and (19), we obtain the function G_1 satisfies the inequalities:

$$\frac{(\log t)^{\alpha-1}(1-\log t)(\log s)(1-\log s)^{\alpha-1}}{\Gamma(\alpha)}\le G_1(t,s)\le\frac{(\alpha-1)(\log s)(1-\log s)^{\alpha-1}}{\Gamma(\alpha)}, \quad t,s\in[1,e]. \quad (20)$$

This, for all $t,s\in[1,e]$, implies that

$$\frac{(\log t)^{\alpha-1}}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\int_1^e h(t)G_1(t,s)\frac{dt}{t}\le\frac{(\alpha-1)\int_1^e h(t)(dt/t)}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}(\log s)(1-\log s)^{\alpha-1}, \quad (21)$$

and

$$\frac{(\log t)^{\alpha-1}}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\int_1^e h(t)G_1(t,s)\frac{dt}{t}\ge\frac{(\log t)^{\alpha-1}\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\cdot(\log s)(1-\log s)^{\alpha-1}. \quad (22)$$

Lemma 4. *Let $\phi(t)=(\log t)(1-\log t)^{\alpha-1}$, where $t\in[1,e]$. Then there exist*

$$\kappa_1=\frac{\alpha^2\Gamma(\alpha)}{\Gamma(2\alpha+2)}+\frac{\Gamma(\alpha)}{2\Gamma(2\alpha)}\frac{\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)},$$

$$\kappa_2=\frac{\alpha-1}{\Gamma(\alpha+2)}\left[1+\frac{\int_1^e h(t)(dt/t)}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right], \quad (23)$$

such that

$$\kappa_1\phi(s)\le\int_1^e G(t,s)\phi(t)\frac{dt}{t}\le\kappa_2\phi(s), \quad \text{for } s\in[1,e]. \quad (24)$$

Proof. Using (20)–(22), for all $s\in[1,e]$, we have

$$\begin{aligned}\int_1^e G(t,s)\phi(t)\frac{dt}{t}&\le\int_1^e\frac{(\alpha-1)\phi(s)}{\Gamma(\alpha)}\phi(t)\frac{dt}{t}+\int_1^e\frac{(\alpha-1)\int_1^e h(t)(dt/t)}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\cdot\phi(s)\phi(t)\frac{dt}{t}=\kappa_2\phi(s),\\ \int_1^e G(t,s)\phi(t)\frac{dt}{t}&\ge\int_1^e\frac{(\log t)^{\alpha-1}(1-\log t)\phi(s)}{\Gamma(\alpha)}\phi(t)\frac{dt}{t}+\int_1^e\frac{(\log t)^{\alpha-1}\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{\Gamma(\alpha)\left(1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\cdot\phi(s)\phi(t)\frac{dt}{t}=\kappa_1\phi(s).\end{aligned} \quad (25)$$

This completes the proof. □

From Lemma 3, we know (1) is equivalent to the following Hammerstein-type integral equations:

$$\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}=\begin{pmatrix}\int_1^e G(t,s)f_1(s,u(s),v(s))\frac{ds}{s}\\ \int_1^e G(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\end{pmatrix}. \quad (26)$$

Let $E:=C[1,e]$, $|u|:=\max_{t\in[1,e]}|u(t)|$, and $P:=\{u\in E:u(t)\ge 0,\ \forall t\in[1,e]\}$. Then $(E,\|\cdot\|)$ becomes a real Banach space and P a cone on E. Moreover, $E\times E$ is a Banach space with the norm $(x,y)=\|x\|+\|y\|$, and $P\times P$ is a cone on $E\times E$. Therefore, we define operators $A_i\,(i=1,2)$ and A as follows:

$$\begin{aligned}A_1(u,v)(t)&=\int_1^e G(t,s)f_1(s,u(s),v(s))\frac{ds}{s},\\ A_2(u,v)(t)&=\int_1^e G(t,s)f_2(s,u(s),v(s))\frac{ds}{s},\\ A(u,v)(t)&=(A_1,A_2)(u,v)(t), \quad \text{for } u,v\in P,\ t\in[1,e].\end{aligned} \quad (27)$$

Note that G and $f_i\,(i=1,2)$ are nonnegative continuous functions, so the operators $A_i:P\times P\longrightarrow P\,(i=1,2)$ and $A:P\times P\longrightarrow P\times P$ are three completely continuous operators. Moreover, if $(u,v)\in(P\times P)\backslash\{0\}$ is a fixed point of A, then (u,v) is a positive solution for (1). Therefore, in what follows, we turn to study the existence of fixed points of the operator A.

Lemma 5. *Let p be a continuous concave function. Then, if φ is an integrable function on $[0,1]$, we have*

$$p\left(\int_0^1\varphi(t)dt\right)\ge\int_0^1 p(\varphi(t))dt. \quad (28)$$

Proof. Let $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$, for all $n \in \mathbb{N}_+$, and $\Delta t_i = t_i - t_{i-1}, d = \max\{\Delta t_i\}, i = 1, 2, \ldots, n$. Then, note that $\sum_{i=1}^{n} \Delta t_i = 1$, for all $\xi_i \in [t_{i-1}, t_i]$, where $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} p\left(\int_0^1 \varphi(t)dt\right) &= p\left(\lim_{d\longrightarrow 0}\sum_{i=1}^{n}\varphi(\xi_i)\Delta t_i\right) = \lim_{d\longrightarrow 0} p\left(\sum_{i=1}^{n}\varphi(\xi_i)\Delta t_i\right) \\ &\geq \lim_{d\longrightarrow 0}\sum_{i=1}^{n} p(\varphi(\xi_i))\Delta t_i = \int_0^1 p(\varphi(t))dt. \end{aligned} \tag{29}$$

This completes the proof. □

Remark 1. If p is a continuous convex function in Lemma 5, then (28) can be changed into the inverse inequality:

$$p\left(\int_0^1 \varphi(t)dt\right) \leq \int_0^1 p(\varphi(t))dt. \tag{30}$$

Lemma 6 (see [34]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If there exists a $\omega_0 \in P\backslash\{0\}$ such that*

$$\omega - A\omega \neq \lambda\omega_0, \quad \forall\lambda \geq 0, \omega \in \partial\Omega \cap P, \tag{31}$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed-point index on P.

Lemma 7 (see [34]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \neq 0, \quad \forall\lambda \in [0, 1], \omega \in \partial\Omega \cap P, \tag{32}$$

then $i(A, \Omega \cap P, P) = 1$.

3. Main Results

Lemma 8. *Let $P_0 = \left\{u \in P : \int_1^e u(t)\phi(t)(dt/t) \geq \omega_0\|u\|\right\}$. Then $Bu \in P_0$, where*

$$(Bu)(t) = \int_1^e G(t, s)u(s)\frac{ds}{s}, \quad u \in P, \tag{33}$$

where

$$\begin{aligned} \omega_0 = {} & \frac{\alpha^2\Gamma^2(\alpha)}{(\alpha-1)\Gamma(2\alpha+2)}\left[1 + \frac{\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right] \\ & \cdot \left[1 + \frac{\int_1^e h(t)(dt/t)}{1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]^{-1}. \end{aligned} \tag{34}$$

Proof. From the definition of G, for all $t, \tau, s \in [1, e]$, we have

$$\begin{aligned} G(t,s) &\geq \frac{(\log t)^{\alpha-1}(1-\log t)\phi(s)}{\Gamma(\alpha)} + \frac{(\log t)^{\alpha-1}\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{\Gamma(\alpha)\left(1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t)\right)}\phi(s) \\ &\geq \frac{(\log t)^{\alpha-1}(1-\log t)}{\Gamma(\alpha)}\left[1 + \frac{\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]\phi(s) \\ &= \frac{(\log t)^{\alpha-1}(1-\log t)}{\Gamma(\alpha)}\left[1 + \frac{\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right] \\ &\quad \cdot \frac{\alpha-1}{\Gamma(\alpha)}\left[1 + \frac{\int_1^e h(t)(dt/t)}{1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]\phi(s) \cdot \frac{\Gamma(\alpha)}{\alpha-1}\left[1 + \frac{\int_1^e h(t)(dt/t)}{1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]^{-1} \\ &\geq \frac{(\log t)^{\alpha-1}(1-\log t)}{\alpha-1}\left[1 + \frac{\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]\left[1 + \frac{\int_1^e h(t)(dt/t)}{1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]^{-1} G(\tau, s). \end{aligned} \tag{35}$$

Then if $u \in P$, we have

$$\int_1^e (Bu)(t)\phi(t)\frac{dt}{t} = \int_1^e \phi(t)\int_1^e G(t,s)u(s)\frac{ds}{s}\frac{dt}{t}$$
$$\geq \int_1^e \phi(t)\int_1^e \frac{(\log t)^{\alpha-1}(1-\log t)}{\alpha-1}\left[1+\frac{\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]$$
$$\cdot\left[1+\frac{\int_1^e h(t)(dt/t)}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]^{-1} G(\tau,s)u(s)\frac{ds}{s}\frac{dt}{t}$$
$$= \frac{\alpha^2\Gamma^2(\alpha)}{(\alpha-1)\Gamma(2\alpha+2)}\left[1+\frac{\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]\left[1+\frac{\int_1^e h(t)(dt/t)}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right]^{-1}(Bu)(\tau). \tag{36}$$

Note that the arbitrariness of $\tau \in [0,1]$, we have

$$\int_1^e (Bu)(t)\phi(t)\frac{dt}{t} \geq \omega_0\|Bu\|. \tag{37}$$

This completes the proof. □

Let $\mathscr{K} = (\alpha - 1/\Gamma(\alpha))[1 + (\int_1^e h(t)(dt/t)/1 - \int_1^e h(t)(\log t)^{\alpha-1}(dt/t))]$. Then, $\max_{t,s\in[1,e]} G(t,s) \leq \mathscr{K}$. Now, we list our assumptions for $f_i\,(i=1,2)$:

(H1) $f_i \in C([1,e]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+), i=1,2$.

(H2) There exist $p_1, q_1 \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $c_1 > 0$ such that

(i) p_1 is a strictly increasing concave function on $\mathbb{R}^+$ and $\lim_{z\longrightarrow+\infty} p_1(z) = +\infty$

(ii) $\begin{pmatrix} f_1(t,u,v) \\ f_2(t,u,v)\end{pmatrix} \geq \begin{pmatrix} p_1(v)-c_1 \\ q_1(u)-c_1\end{pmatrix}, \forall (t,u,v)\in[1,e]\times\mathbb{R}^+\times\mathbb{R}^+$

(iii) $\exists \gamma_1 \in (\kappa_1^{-2},+\infty)$ such that $p_1(\mathscr{K}q_1(z)) \geq \gamma_1\mathscr{K}z - c_1$, $\forall z\in\mathbb{R}^+$

(H3) There exist $p_2, q_2 \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $r_1 > 0$ such that

(i) p_2 is a strictly increasing convex function on $\mathbb{R}^+$ and $p_2(0) = 0$

(ii) $\begin{pmatrix} f_1(t,u,v) \\ f_2(t,u,v)\end{pmatrix} \leq \begin{pmatrix} p_2(v) \\ q_2(u)\end{pmatrix}, \forall (t,u,v)\in[1,e]\times[0,r_1]\times[0,r_1]$

(iii) $\exists \gamma_2 \in (0,\kappa_2^{-2})$ such that $p_2(\mathscr{K}q_2(z)) \leq \gamma_2\mathscr{K}z$, $\forall z\in[0,r_1]$

(H4) There exist $p_3, q_3 \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $r_2 > 0$ such that

(i) p_3 is a strictly increasing concave function on $\mathbb{R}^+$

(ii) $\begin{pmatrix} f_1(t,u,v) \\ f_2(t,u,v)\end{pmatrix} \geq \begin{pmatrix} p_3(v) \\ q_3(u)\end{pmatrix}, \forall (t,u,v)\in[1,e]\times[0,r_2]\times[0,r_2]$

(iii) $\exists \gamma_3 \in (\kappa_1^{-2},+\infty)$ such that $p_3(\mathscr{K}q_3(z)) \geq \gamma_3\mathscr{K}z$, $\forall z\in[0,r_2]$

(H5) There exist $a_{11}, b_{11}, a_{12}, b_{12} \geq 0$ and $l_1, l_2 > 0$ such that

$$a_{11}\kappa_2 < 1, b_{12}\kappa_2 < 1, \det\begin{pmatrix} 1-a_{11}\kappa_2 & -a_{12}\kappa_2 \\ -b_{11}\kappa_2 & 1-b_{12}\kappa_2\end{pmatrix} > 0,$$
$$\begin{pmatrix} f_1(t,u,v) \\ f_2(t,u,v)\end{pmatrix} \leq \begin{pmatrix} a_{11}u + a_{12}v + l_1 \\ b_{11}u + b_{12}v + l_2\end{pmatrix}, \quad \forall (t,u,v)\in[1,e]\times\mathbb{R}^+\times\mathbb{R}^+. \tag{38}$$

Define $B_\rho = \{z \in E : \|z\| < \rho\}$ for $\rho > 0$. We adopt the convention in the sequel that $c_1, c_2, \ldots$ stand for different positive constants.

Theorem 1. *Suppose that (H1)–(H3) hold. Then, (1) has at least one positive solution.*

Proof. Let $M_1 = \{(u,v)\in P\times P : (u,v) = A(u,v) + \mu(u^*,v^*), \mu \geq 0\}$, where $u^*, v^* \in P_0$ are two given elements. Then, we claim that M_1 is a bounded set in $P\times P$. We define operators $\mathbf{f}_i : P\times P \longrightarrow P\,(i=1,2)$ as follows:

$$\mathbf{f}_i(u,v)(t) = f_i(t,u(t),v(t)), \quad \text{for } u,v\in P, t\in[1,e], i=1,2. \tag{39}$$

Now, if there exists $(u,v)\in M_1$, then we have $u = A_1(u,v) + \mu u^* = B\mathbf{f}_1(u,v) + \mu u^*$ and $v = A_2(u,v) + \mu v^* = B\mathbf{f}_2(u,v) + \mu v^*$. From Lemma 8, we have

$$u, v \in P_0. \tag{40}$$

Moreover, together with (H2) (ii), we can obtain that

$$u(t) \geq A_1(u,v)(t) \geq \int_1^e G(t,s)(p_1(v(s)) - c_1)\frac{ds}{s},$$
$$v(t) \geq A_2(u,v)(t) \geq \int_1^e G(t,s)(q_1(u(s)) - c_1)\frac{ds}{s}, \tag{41}$$
$$\geq \int_1^e G(t,s)q_1(u(s))\frac{ds}{s} - c_2.$$

Using (H2) (i) and (iii), we have

$$\begin{aligned} p_1(v(t)) &\geq p_1(v(t)+c_2)-p_1(c_2) \\ &\geq p_1\left(\int_1^e G(t,s)q_1(u(s))\frac{ds}{s}\right)-p_1(c_2) \\ &= p_1\left(\int_0^1 G(t,e^x)q_1(u(e^x))dx\right)-p_1(c_2) \\ &\geq \int_0^1 p_1(G(t,e^x)q_1(u(e^x)))dx-p_1(c_2) \\ &= \int_0^1 p_1\left(\frac{G(t,e^x)}{\mathscr{K}}\mathscr{K}q_1(u(e^x))+\left(1-\frac{G(t,e^x)}{\mathscr{K}}\right)\cdot 0\right)dx-p_1(c_2) \\ &\geq \int_0^1 \frac{G(t,e^x)}{\mathscr{K}}p_1(\mathscr{K}q_1(u(e^x)))dx-p_1(c_2) \\ &\geq \int_0^1 \frac{G(t,e^x)}{\mathscr{K}}(\gamma_1\mathscr{K}u(e^x)-c_1)dx-p_1(c_2) \\ &\geq \gamma_1\int_1^e G(t,s)u(s)\frac{ds}{s}-c_3. \end{aligned} \tag{42}$$

Therefore, we have

$$\begin{aligned} u(t) &\geq \int_1^e G(t,s)\left(\gamma_1\int_1^e G(s,\tau)u(\tau)\frac{d\tau}{\tau}-c_3\right)\frac{ds}{s}-c_2 \\ &\geq \gamma_1\int_1^e\int_1^e G(t,s)G(s,\tau)u(\tau)\frac{ds}{s}\frac{d\tau}{\tau}-c_4. \end{aligned} \tag{43}$$

Recall that $\phi(t)=(\log t)(1-\log t)^{\alpha-1}$, where $t\in[1,e]$. Therefore, we multiply both sides of the above by $\phi(t)$, integrate over $[1,e]$, and use Lemma 4 to obtain

$$\begin{aligned} \int_1^e u(t)\phi(t)\frac{dt}{t} &\geq \gamma_1\int_1^e\phi(t)\int_1^e\int_1^e G(t,s)G(s,\tau)u(\tau)\frac{ds}{s}\frac{d\tau}{\tau}\frac{dt}{t} \\ &\quad - c_4\int_1^e\phi(t)\frac{dt}{t}\geq\gamma_1\kappa_1^2\int_1^e u(t)\phi(t)\frac{dt}{t}-\frac{c_4\Gamma(\alpha)}{\Gamma(\alpha+2)}. \end{aligned} \tag{44}$$

Solving this inequality, from (40), we have

$$\|u\|\leq\omega_0^{-1}\int_1^e u(t)\phi(t)\frac{dt}{t}\leq\frac{\omega_0^{-1}c_4\Gamma(\alpha)}{(\gamma_1\kappa_1^2-1)\Gamma(\alpha+2)}. \tag{45}$$

On the other hand, we estimate the norm of v. Multiplying both sides of the first inequality of (41) by $\phi(t)$, integrating over $[1,e]$, and using Lemma 4, we obtain

$$\int_1^e u(t)\phi(t)\frac{dt}{t}\geq\kappa_1\int_1^e\phi(t)(p_1(v(t))-c_1)\frac{dt}{t}. \tag{46}$$

This implies that

$$\int_1^e\phi(t)p_1(v(t))\frac{dt}{t}\leq\frac{\kappa_1^{-1}c_4\Gamma(\alpha)}{(\gamma_1\kappa_1^2-1)\Gamma(\alpha+2)}+\frac{c_1\Gamma(\alpha)}{\Gamma(\alpha+2)}. \tag{47}$$

Without loss of generality, we may assume $v(t)\not\equiv 0$, then $v>0$. Note that $v\in P_0$, we have

$$\begin{aligned} \|v\| &\leq\frac{1}{\omega_0}\int_1^e v(t)\phi(t)\frac{dt}{t}=\frac{\|v\|}{\omega_0 p_1(\|v\|)}\int_0^1\frac{v(e^x)}{\|v\|}p_1(\|v\|)\phi(e^x)dx \\ &\leq\frac{\|v\|}{\omega_0 p_1(\|v\|)}\int_0^1 p_1(v(e^x))\phi(e^x)dx, \\ p_1(\|v\|) &\leq\frac{1}{\omega_0}\int_0^1 p_1(v(e^x))\phi(e^x)dx=\frac{1}{\omega_0}\int_1^e p_1(v(t))\phi(t)\frac{dt}{t} \\ &\leq\frac{1}{\omega_0}\left[\frac{\kappa_1^{-1}c_4\Gamma(\alpha)}{(\gamma_1\kappa_1^2-1)\Gamma(\alpha+2)}+\frac{c_1\Gamma(\alpha)}{\Gamma(\alpha+2)}\right]. \end{aligned} \tag{48}$$

Combining (H2) (i) ($\lim_{z\longrightarrow+\infty}p_1(z)=+\infty$), there exists $\mathscr{N}_1$ such that $\|v\|\leq\mathscr{N}_1$.

Up to now, we have proved the boundedness of M_1. Taking $R_1>\mathscr{N}_1+(\omega_0^{-1}c_4\Gamma(\alpha)/(\gamma_1\kappa_1^2-1)\Gamma(\alpha+2))$ and $R_1>r_1$ (r_1 is defined by (H3)), we have

$$(u,v)\neq A(u,v)+\mu(u^*,v^*),\ \text{for}\ (u,v)\in\partial B_{R_1}\cap(P\times P),\mu\geq 0. \tag{49}$$

Then, Lemma 6 enables us to obtain

$$i\left(A,B_{R_1}\cap(P\times P),P\times P\right)=0. \tag{50}$$

Next, we show that

$$(u,v)\neq\mu A(u,v),\ \text{for}\ (u,v)\in\partial B_{r_1}\cap(P\times P),\mu\in[0,1]. \tag{51}$$

If this claim is not true, then there exist $(u,v)\in\partial B_{r_1}\cap(P\times P)$, $\mu\in[0,1]$ such that

$$(u,v)=\mu A(u,v). \tag{52}$$

Combining (H3) (ii), we obtain

$$\begin{aligned} u(t) &\leq A_1(u,v)(t)\leq\int_1^e G(t,s)p_2(v(s))\frac{ds}{s}, \\ v(t) &\leq A_2(u,v)(t)\leq\int_1^e G(t,s)q_2(u(s))\frac{ds}{s}. \end{aligned} \tag{53}$$

From (H3) (i) and (iii), we have

$$\begin{aligned} p_2(v(t)) &\leq p_2\left(\int_1^e G(t,s)q_2(u(s))\frac{ds}{s}\right) \\ &\leq\int_0^1 p_2(G(t,e^x)q_2(u(e^x)))dx \\ &=\int_0^1 p_2\left(\frac{G(t,e^x)}{\mathscr{K}}\mathscr{K}q_2(u(e^x))+\left(1-\frac{G(t,e^x)}{\mathscr{K}}\right)\cdot 0\right)dx \\ &\leq\int_0^1\frac{G(t,e^x)}{\mathscr{K}}p_2(\mathscr{K}q_2(u(e^x)))dx \\ &\leq\gamma_2\int_1^e G(t,s)u(s)\frac{ds}{s}. \end{aligned} \tag{54}$$

Consequently, we have

$$u(t) \le \gamma_2 \int_1^e \int_1^e G(t,s)G(s,\tau)u(\tau)\frac{d\tau}{\tau}\frac{ds}{s}. \quad (55)$$

Multiplying both sides of the above by $\phi(t)$, integrating over $[1,e]$, and using Lemma 4, we obtain

$$\int_1^e u(t)\phi(t)\frac{dt}{t} \le \gamma_2\kappa_2^2 \int_1^e u(t)\phi(t)\frac{dt}{t}. \quad (56)$$

Note that $\gamma_2 \in (0, \kappa_2^{-2})$, we have $\int_1^e u(t)\phi(t)(dt/t) = 0$ and $u(t) \equiv 0$ for $t \in [1,e]$. Moreover, using (54), we have $p_2(v(t)) \equiv 0$ for $t \in [1,e]$. From (H3) (i), we have $v(t) \equiv 0$ for $t \in [1,e]$. Therefore, this contradicts to $(u,v) \in \partial B_{r_1} \cap (P \times P)$, $r_1 > 0$. This also implies that (51) holds. Then, Lemma 7 enables us to obtain

$$i\left(A, B_{r_1} \cap (P\times P), P\times P\right) = 1. \quad (57)$$

From (50) and (57), we have

$$\begin{aligned} i\left(A, \left(B_{R_1}\backslash \overline{B}_{r_1}\right) \cap (P\times P), P\times P\right) &= i\left(A, B_{R_1} \cap (P\times P), P\times P\right) \\ &\quad - i\left(A, B_{r_1} \cap (P\times P), P\times P\right) = 0 - 1 = -1. \end{aligned} \quad (58)$$

Therefore, the operator A has at least one fixed point on $(B_{R_1}\backslash \overline{B}_{r_1}) \cap (P\times P)$. Equivalently, (1) has at least one positive solution. This completes the proof. □

Theorem 2. *Suppose that (H1) and (H4)-(H5) hold. Then, (1) has at least one positive solution.*

Proof. For r_2 in (H4), we first show that

$$(u,v) \ne A(u,v) + \mu(u^*, v^*), \quad \text{for } (u,v) \in \partial B_{r_2} \cap (P\times P), \mu \ge 0, \quad (59)$$

where $u^*, v^* \in P$ are two given elements. Indeed, if this claim is false, there exist $(u,v) \in \partial B_{r_2} \cap (P\times P), \mu \ge 0$ such that

$$(u,v) = A(u,v) + \mu(u^*, v^*). \quad (60)$$

This, together with (H4) (ii), implies that

$$\begin{aligned} u(t) &\ge A_1(u,v)(t) \ge \int_1^e G(t,s)p_3(v(s))\frac{ds}{s}, v(t) \ge A_2(u,v)(t) \\ &\ge \int_1^e G(t,s)q_3(u(s))\frac{ds}{s}. \end{aligned} \quad (61)$$

Similar to (42), we have

$$\begin{aligned} p_3(v(t)) &\ge p_3\left(\int_1^e G(t,s)q_3(u(s))\frac{ds}{s}\right) \\ &\ge \int_1^e \frac{G(t,s)}{\mathcal{K}} p_3(\mathcal{K} q_3(u(s)))\frac{ds}{s}. \end{aligned} \quad (62)$$

From (H4) (iii), we have

$$\begin{aligned} u(t) &\ge \int_1^e G(t,s)p_3(v(s))\frac{ds}{s} \\ &\ge \gamma_3 \int_1^e \int_1^e G(t,s)G(s,\tau)u(\tau)\frac{ds}{s}\frac{d\tau}{\tau}. \end{aligned} \quad (63)$$

Multiplying both sides of the above by $\phi(t)$, integrating over $[1,e]$, and using Lemma 4, we obtain

$$\int_1^e u(t)\phi(t)\frac{dt}{t} \ge \gamma_3\kappa_1^2 \int_1^e u(t)\phi(t)\frac{dt}{t}, \quad (64)$$

where $\phi(t) = (\log t)(1-\log t)^{\alpha-1}, t \in [1,e]$. Consequently, $\gamma_3\kappa_1^2 > 1$ implies that $\int_1^e u(t)\phi(t)(dt/t) = 0$ and $u(t) \equiv 0$ for $t \in [1,e]$. Note that (65), should be

$$\int_1^e G(t,s)p_3(v(s))\frac{ds}{s} \le u(t) \equiv 0, \forall t \in [1,e]. \quad (65)$$

From (H4) (i), this indicates that $p_3(v(s)) \equiv 0$ and $v(s) \equiv 0$ for $s \in [1,e]$. Therefore, $\|u\| = \|v\| = 0$ contradicts to $(u,v) \in \partial B_{r_2} \cap (P\times P)$ and (59) holds. Then, Lemma 6 enables us to obtain

$$i\left(A, B_{r_2} \cap (P\times P), P\times P\right) = 0. \quad (66)$$

Let $M_2 = \{(u,v) \in P\times P : (u,v) = \mu A(u,v), \mu \in [0,1]\}$. Then, we prove that M_2 is a bounded set in $P\times P$. If $(u,v) \in M_2$, then we have

$$\begin{aligned} u &= \mu A_1(u,v), \\ v &= \mu A_2(u,v), \text{ for } (u,v) \in P\times P. \end{aligned} \quad (67)$$

From Lemma 8, we have

$$u, v \in P_0. \quad (68)$$

Moreover, by (H5), we have

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \le \begin{pmatrix} \int_1^e G(t,s)\left(a_{11}u(s) + a_{12}v(s) + l_1\right)\frac{ds}{s} \\ \int_1^e G(t,s)\left(b_{11}u(s) + b_{12}v(s) + l_2\right)\frac{ds}{s} \end{pmatrix}. \quad (69)$$

Multiplying both sides of the above by $\phi(t)$, integrating over $[1,e]$, and using Lemma 4, we obtain

$$\begin{pmatrix} \int_1^e u(t)\phi(t)\frac{dt}{t} \\ \int_1^e v(t)\phi(t)\frac{dt}{t} \end{pmatrix} \le \begin{pmatrix} \kappa_2 \int_1^e \phi(t)\left(a_{11}u(t) + a_{12}v(t) + l_1\right)\frac{dt}{t} \\ \kappa_2 \int_1^e \phi(t)\left(b_{11}u(t) + b_{12}v(t) + l_2\right)\frac{dt}{t} \end{pmatrix}. \quad (70)$$

Consequently, we have

$$\begin{pmatrix} 1 - a_{11}\kappa_2 & -a_{12}\kappa_2 \\ -b_{11}\kappa_2 & 1 - b_{12}\kappa_2 \end{pmatrix} \begin{pmatrix} \int_1^e u(t)\phi(t)\frac{dt}{t} \\ \int_1^e v(t)\phi(t)\frac{dt}{t} \end{pmatrix} \le \begin{pmatrix} \frac{\kappa_2 l_1 \Gamma(\alpha)}{\Gamma(\alpha+2)} \\ \frac{\kappa_2 l_2 \Gamma(\alpha)}{\Gamma(\alpha+2)} \end{pmatrix}. \quad (71)$$

Solving this matrix inequality, we have

$$\begin{pmatrix} \int_1^e u(t)\phi(t)\frac{dt}{t} \\ \int_1^e v(t)\phi(t)\frac{dt}{t} \end{pmatrix} \le \frac{\begin{pmatrix} 1-b_{12}\kappa_2 & a_{12}\kappa_2 \\ b_{11}\kappa_2 & 1-a_{11}\kappa_2 \end{pmatrix}\begin{pmatrix} \kappa_2 l_1\Gamma(\alpha)/\Gamma(\alpha+2) \\ \kappa_2 l_2\Gamma(\alpha)/\Gamma(\alpha+2) \end{pmatrix}}{(1-a_{11}\kappa_2)(1-b_{12}\kappa_2)-a_{12}b_{11}\kappa_2^2}. \quad (72)$$

This implies that

$$\int_1^e u(t)\phi(t)\frac{dt}{t} \le \frac{(\kappa_2\Gamma(\alpha)/\Gamma(\alpha+2))[(1-b_{12}\kappa_2)l_1+a_{12}\kappa_2 l_2]}{(1-a_{11}\kappa_2)(1-b_{12}\kappa_2)-a_{12}b_{11}\kappa_2^2},$$
$$\int_1^e v(t)\phi(t)\frac{dt}{t} \le \frac{(\kappa_2\Gamma(\alpha)/\Gamma(\alpha+2))[b_{11}\kappa_2 l_1+(1-a_{11}\kappa_2)l_2]}{(1-a_{11}\kappa_2)(1-b_{12}\kappa_2)-a_{12}b_{11}\kappa_2^2}. \quad (73)$$

Note that $u, v \in P_0$, we have

$$\|u\| \le \frac{(\omega_0^{-1}\kappa_2\Gamma(\alpha)/\Gamma(\alpha+2))[(1-b_{12}\kappa_2)l_1+a_{12}\kappa_2 l_2]}{(1-a_{11}\kappa_2)(1-b_{12}\kappa_2)-a_{12}b_{11}\kappa_2^2},$$
$$\|v\| \le \frac{(\omega_0^{-1}\kappa_2\Gamma(\alpha)/\Gamma(\alpha+2))[b_{11}\kappa_2 l_1+(1-a_{11}\kappa_2)l_2]}{(1-a_{11}\kappa_2)(1-b_{12}\kappa_2)-a_{12}b_{11}\kappa_2^2}. \quad (74)$$

Taking $R_2 > (\omega_0^{-1}\kappa_2\Gamma(\alpha)/\Gamma(\alpha+2))[(1-b_{12}\kappa_2)l_1+a_{12}\kappa_2 l_2+b_{11}\kappa_2 l_1+(1-a_{11}\kappa_2)l_2]/(1-a_{11}\kappa_2)(1-b_{12}\kappa_2)-a_{12}b_{11}\kappa_2^2$ and $R_2 > r_2$ (r_2 is defined by (H4)), we have

$$(u,v) \ne \mu A(u,v), \quad \text{for } (u,v) \in \partial B_{R_2} \cap (P\times P), \mu \in [0,1]. \quad (75)$$

Then, Lemma 7 enables us to obtain

$$i\left(A, B_{R_2} \cap (P\times P), P\times P\right) = 1. \quad (76)$$

From (66) and (76), we have

$$\begin{aligned} & i\left(A, \left(B_{R_2}\backslash \overline{B}_{r_2}\right) \cap (P\times P), P\times P\right) \\ &= i\left(A, B_{R_2} \cap (P\times P), P\times P\right) - i\left(A, B_{r_2} \cap (P\times P), P\times P\right) \\ &= 1-0 = 1. \end{aligned} \quad (77)$$

Therefore, the operator A has at least one fixed point on $(B_{R_2}\backslash \overline{B}_{r_2}) \cap (P\times P)$. Equivalently, (1) has at least one positive solution. This completes the proof.

In (1), let $n=3$, $\alpha=2.5$, and $h(t)=\log t$, $t\in[1,e]$. Then, $\int_1^e h(t)(\log t)^{\alpha-1}(dt/t) = \int_1^e (\log t)^{\alpha}(dt/t) = (2/7) \in [0,1)$ and (H0) holds. Moreover, we can calculate $\mathscr{K}$, κ_1, and κ_2 as follows:

$$\begin{aligned} \mathscr{K} &= \frac{\alpha-1}{\Gamma(\alpha)}\left[1+\frac{\int_1^e h(t)(dt/t)}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right] = \frac{1.5}{\Gamma(2.5)}\left[1+\frac{\int_1^e (\log t)(dt/t)}{1-\int_1^e (\log t)^{2.5}(dt/t)}\right] \approx 1.92, \\ \kappa_1 &= \frac{\alpha^2\Gamma(\alpha)}{\Gamma(2\alpha+2)} + \frac{\Gamma(\alpha)}{2\Gamma(2\alpha)}\frac{\int_1^e h(t)(\log t)^{\alpha-1}(1-\log t)(dt/t)}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)} = \frac{(2.5)^2\Gamma(2.5)}{\Gamma(7)} + \frac{\Gamma(2.5)}{2\Gamma(5)}\frac{\Gamma(3.5)}{\Gamma(5.5)}\frac{7}{5} \approx 0.014, \\ \kappa_2 &= \frac{\alpha-1}{\Gamma(\alpha+2)}\left[1+\frac{\int_1^e h(t)(dt/t)}{1-\int_1^e h(t)(\log t)^{\alpha-1}(dt/t)}\right] = \frac{1.5}{\Gamma(4.5)}\left[1+\frac{\int_1^e (\log t)(dt/t)}{1-\int_1^e (\log t)^{2.5}(dt/t)}\right] \approx 0.22. \end{aligned} \quad (78)$$

□

Example 1. Let $f_1(t,u,v)=(u+v)^{\gamma_1}$, $f_2(t,u,v)=(u+v)^{\gamma_2}$, $p_1(v)=v^{1/3}$, $q_1(u)=u^4$, $p_2(v)=v^2$, and $q_2(u)=u$, for $(t,u,v)\in[1,e]\times\mathbb{R}^+\times\mathbb{R}^+$, where $\gamma_1>2$ and $\gamma_2>4$. Then, we have

(i) $\liminf_{v\longrightarrow+\infty}(f_1(t,u,v)/p_1(v)) = \liminf_{v\longrightarrow+\infty}((u+v)^{\gamma_1}/v^{1/3}) \ge \liminf_{v\longrightarrow+\infty}(v^{\gamma_1}/v^{1/3}) = +\infty$, for all $(t,u)\in[1,e]\times\mathbb{R}^+$

(ii) $\liminf_{u\longrightarrow+\infty}(f_2(t,u,v)/q_1(u)) = \liminf_{u\longrightarrow+\infty}((u+v)^{\gamma_2}/u^4) \ge \liminf_{u\longrightarrow+\infty}(u^{\gamma_2}/u^4) = +\infty$, for all $(t,v)\in[1,e]\times\mathbb{R}^+$

(iii) $\limsup_{u+v\longrightarrow0^+}(f_1(t,u,v)/p_2(v)) = \limsup_{u+v\longrightarrow0^+}((u+v)^{\gamma_1}/v^2) = 0$, for all $t\in[1,e]$

(iv) $\limsup_{u+v\longrightarrow0^+}(f_2(t,u,v)/q_2(u)) = \limsup_{u+v\longrightarrow0^+}((u+v)^{\gamma_2}/u) = 0$, for all $t\in[1,e]$

(v) $\liminf_{z\longrightarrow+\infty}(p_1(\mathscr{K}q_1(z))/z) = \liminf_{z\longrightarrow+\infty}(\sqrt[3]{\mathscr{K}}z^{4/3}/z) = +\infty$

(vi) $\limsup_{z\longrightarrow0^+}(p_2(\mathscr{K}q_2(z))/z) = \limsup_{z\longrightarrow0^+}(\mathscr{K}^2z^2/z) = 0$

Therefore, (H2)-(H3) hold.

Example 2. Let $a_{11}=0.05$, $a_{12}=0.6$, $b_{11}=0.4$, and $b_{12}=0.08$, then we calculate $a_{11}\kappa_2=0.011<1$, $b_{12}\kappa_2=0.0176<1$, and

$$\begin{vmatrix} 1-a_{11}\kappa_2 & -a_{12}\kappa_2 \\ -b_{11}\kappa_2 & 1-b_{12}\kappa_2 \end{vmatrix} = \begin{vmatrix} 0.989 & -0.132 \\ -0.088 & 0.9824 \end{vmatrix} \approx 0.96. \quad (79)$$

Let $f_1(t,u,v)=(a_{11}u+a_{12}v)^{\gamma_3}$, $f_2(t,u,v)=(b_{11}u+b_{12}v)^{\gamma_4}$, $p_3(v)=\sqrt{v}$, and $q_3(u)=u^{3/4}$, for $(t,u,v)\in[1,e]\times\mathbb{R}^+\times\mathbb{R}^+$, where $\gamma_3\in(0,(1/2))$ and $\gamma_4\in(0,(3/4))$. Then for all $t\in[1,e]$, we have

$$\begin{aligned}
&\liminf_{a_{11}u+a_{12}v\longrightarrow 0^+}\frac{f_1(t,u,v)}{p_3(v)}=\liminf_{a_{11}u+a_{12}v\longrightarrow 0^+}\frac{(a_{11}u+a_{12}v)^{\gamma_3}}{v^{1/2}}\geq\liminf_{a_{11}u+a_{12}v\longrightarrow 0^+}\frac{(a_{12}v)^{\gamma_3}}{v^{1/2}}=+\infty,\\
&\liminf_{b_{11}u+b_{12}v\longrightarrow 0^+}\frac{f_2(t,u,v)}{q_3(u)}=\liminf_{b_{11}u+b_{12}v\longrightarrow 0^+}\frac{(b_{11}u+b_{12}v)^{\gamma_4}}{u^{3/4}}\geq\liminf_{b_{11}u+b_{12}v\longrightarrow 0^+}\frac{(b_{11}u)^{\gamma_4}}{u^{3/4}}=+\infty,\\
&\limsup_{a_{11}u+a_{12}v\longrightarrow +\infty}\frac{f_1(t,u,v)}{a_{11}u+a_{12}v}=\limsup_{a_{11}u+a_{12}v\longrightarrow +\infty}\frac{(a_{11}u+a_{12}v)^{\gamma_3}}{a_{11}u+a_{12}v}=0,\\
&\limsup_{b_{11}u+b_{12}v\longrightarrow +\infty}\frac{f_2(t,u,v)}{b_{11}u+b_{12}v}=\limsup_{b_{11}u+b_{12}v\longrightarrow +\infty}\frac{(b_{11}u+b_{12}v)^{\gamma_4}}{b_{11}u+b_{12}v}=0,\\
&\liminf_{z\longrightarrow 0^+}\frac{p_3(\mathscr{K}q_3(z))}{z}=\liminf_{z\longrightarrow 0^+}\frac{\sqrt{\mathscr{K}}z^{3/8}}{z}=+\infty.
\end{aligned}\tag{80}$$

As a result, (H4)-(H5) hold.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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