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Research Article

Synchronization Control in Reaction-Diffusion Systems: Application to Lengyel-Epstein System

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Synchronization and control in high dimensional spatial-temporal systems have received increasing interest in recent years. In this paper, the problem of complete synchronization for reaction-diffusion systems is investigated. Linear and nonlinear synchronization control schemes have been proposed to exhibit synchronization between coupled reaction-diffusion systems. Synchronization behaviors of coupled Lengyel-Epstein systems are obtained to demonstrate the effectiveness and feasibility of the proposed control techniques.

1. Introduction

Synchronization of chaos is a phenomenon that may occur when two, or more, chaotic systems adjust a given property of their motion to a common behavior due to a coupling or to a forcing. This phenomenon has attracted the interest of many researchers from various fields due to its potential applications in physics, biology, chemistry, and engineering sciences since the pioneering work by Pecora and Carroll [1]. Various synchronization types have been presented, such as complete synchronization, phase synchronization, lag synchronization, anticipated synchronization, function projective synchronization, generalized synchronization, and Q-S synchronization.

Most of the research efforts have been devoted to the study of chaos control and chaos synchronization problems in low-dimensional nonlinear dynamical systems [2–10]. Synchronizing high dimensional systems in which state variables depend not only on time but also on the spatial position remains a challenge. These high dimensional systems are generally modelled in spatial-temporal domain by partial differential systems. Recently, the search for synchronization has moved to high dimensional nonlinear dynamical systems. Over the last years, some studies have investigated synchronization of spatially extended systems demonstrating spatiotemporal chaos such as the work presented in [11–32]. Synchronization dynamics of reaction-diffusion systems has been studied in [11, 12] using phase reduction theory. It has been shown that reaction-diffusion systems can exhibit synchronization in a similar way to low-dimensional oscillators. A general approach for synchronizing coupled partial differential equations with spatiotemporally chaotic dynamics by driving the response system only at a finite number of space points has been introduced in [13, 14]. Synchronization and control for spatially extended systems based on local spatially averaged coupling signals have been presented in [17]. The effect of asymmetric couplings in the synchronization of spatially extended chaotic systems has been investigated in [19]. The effect of time-delay autosynchronization on uniform oscillations in a general model described by the complex Ginzburg-Landau equation has been presented in [20]. Furthermore, generalized synchronization [21], complete-like synchronization [22], the backstepping synchronization approach [26], the graph-theoretic synchronization approach [27], pinning impulsive synchronization [30], and impulsive type synchronization strategy [31] for coupled reaction-diffusion systems have been introduced.

The main aim of the present paper is to study the problem of complete synchronization in coupled reaction-diffusion systems. Linear and nonlinear control schemes have been proposed to realize complete synchronization for partial differential systems. As a special case, we investigate complete synchronization behaviors of coupled Lengyel-Epstein systems.

2. Systems Description and Problem Formulation

Reaction-diffusion systems have shown important roles in modelling various spatiotemporal patterns that arise in chemical and biological systems [33, 34]. Reaction-diffusion systems can describe a wide class of rhythmic spatiotemporal patterns observed in chemical and biological systems, such as circulating pulses on a ring, oscillating spots, target waves, and rotating spirals. The most familiar way to study synchronization is to use a controller to make the output of the slave (response) system copy in some manner the master (drive) system one. In this case, we design the controller in which the difference of states of synchronized systems converges to zero. This phenomenon is called complete synchronization. Consider the master and the slave reaction-diffusion systems as

$$\textit{Master}\begin{cases}\dfrac{\partial u_1(x,t)}{\partial t}=\displaystyle\sum_{j=1}^{2}d_{1j}\Delta u_j+\sum_{j=1}^{2}a_{1j}u_j+f_1(u_1,u_2),\\ \dfrac{\partial u_2(x,t)}{\partial t}=\displaystyle\sum_{j=1}^{2}d_{2j}\Delta u_j+\sum_{j=1}^{2}a_{2j}u_j+f_2(u_1,u_2),\end{cases}\tag{1}$$

and

$$\textit{Slave}\begin{cases}\dfrac{\partial v_1(x,t)}{\partial t}=\displaystyle\sum_{j=1}^{2}d_{1j}\Delta v_j+\sum_{j=1}^{2}a_{1j}v_j+f_1(v_1,v_2)+\mathbf{U}_1,\\ \dfrac{\partial v_2(x,t)}{\partial t}=\displaystyle\sum_{j=1}^{2}d_{2j}\Delta v_j+\sum_{j=1}^{2}a_{2j}v_j+f_2(v_1,v_2)+\mathbf{U}_2,\end{cases}\tag{2}$$

where $(u_1(x,t),u_2(x,t))^T$ and $(v_1(x,t),v_2(x,t))^T$ are the corresponding states, $x\in\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, Δ is the Laplacian operator on Ω, $(d_{ij})\in\mathbb{R}^2$ are the diffusivity constants, $A=(a_{ij})\in\mathbb{R}^2$, f_1 and f_2 are nonlinear continuous functions, and $\mathbf{U}_1$ and $\mathbf{U}_2$ are controllers to be designed. We impose the homogeneous Neumann boundary conditions

$$\frac{\partial u_i}{\partial\eta}=\frac{\partial v_i}{\partial\eta}=0,\quad i=1,2,\ \textit{for all}\ x\in\partial\Omega.\tag{3}$$

where η is the unit outer normal to $\partial\Omega$. The aim of the synchronization process is to force the error between the master and slave systems, defined as

$$e_i=v_i-u_i,\quad i=1,2,\tag{4}$$

to zero. We assume that the diffusivity constants (d_{ij}) satisfy

$$\begin{aligned}d_{11},d_{22}&\geq 0,\\ d_{12}&=-d_{21},\end{aligned}\tag{5}$$

and the error system satisfies the homogeneous Neumann boundary condition

$$\frac{\partial e_1}{\partial\eta}=\frac{\partial e_2}{\partial\eta}=0,\quad \textit{for all}\ x\in\partial\Omega.\tag{6}$$

To realize complete synchronization between the master system given in (1) and the slave system given in (2), we discuss the asymptotical stable of zero solution of synchronization error system given in (4). That is, in the following sections, we find the controllers $\mathbf{U}_1$ and $\mathbf{U}_2$, in linear and nonlinear forms, such that the solution of the error system $e_i=v_i-u_i$ go to 0, $i=1,2$, as t goes to $+\infty$.

3. Synchronization via Nonlinear Controllers

In this section, we outline the issue of controlling the master-slave reaction-diffusion system given in (1) and (2) via nonlinear controllers. The time partial derivatives of the error system given in (4) can derived as

$$\begin{aligned}\frac{\partial e_1}{\partial t}&=\sum_{j=1}^{2}d_{1j}\Delta e_j+\sum_{j=1}^{2}a_{1j}e_j+f_1(v_1,v_2)-f_1(u_1,u_2)\\&\quad+\mathbf{U}_1,\\ \frac{\partial e_2}{\partial t}&=\sum_{j=1}^{2}d_{2j}\Delta e_j+\sum_{j=1}^{2}a_{2j}e_j+f_2(v_1,v_2)-f_2(u_1,u_2)\\&\quad+\mathbf{U}_2.\end{aligned}\tag{7}$$

That is,

$$\begin{aligned}\frac{\partial e_1}{\partial t}&=\sum_{j=1}^{2}d_{1j}\Delta e_j+\sum_{j=1}^{2}\left(a_{1j}-c_{1j}\right)e_j+R_1+\mathbf{U}_1,\\ \frac{\partial e_2}{\partial t}&=\sum_{j=1}^{2}d_{2j}\Delta e_j+\sum_{j=1}^{2}\left(a_{2j}-c_{2j}\right)e_j+R_2+\mathbf{U}_2,\end{aligned}\tag{8}$$

where $C=(c_{ij})_{2\times 2}$ is a control matrix to be determined later and

$$\begin{aligned}R_1&=\sum_{j=1}^{2}c_{1j}\left(v_j-u_j\right)+f_1(v_1,v_2)-f_1(u_1,u_2),\\ R_2&=\sum_{j=1}^{2}c_{2j}\left(v_j-u_j\right)+f_2(v_1,v_2)-f_2(u_1,u_2).\end{aligned}\tag{9}$$

Theorem 1. *If the control matrix C is chosen such that $A - C$ is a negative definite matrix, then the master-slave reaction-diffusion system given in (1) and (2) can be synchronized under the following nonlinear control law*

$$\mathbf{U}_i = -R_i, \quad i = 1, 2. \tag{10}$$

Proof. Substituting the control parameters given in (10) into (8) yields

$$\begin{aligned} \frac{\partial e_1}{\partial t} &= \sum_{j=1}^{2} d_{1j} \Delta e_j + \sum_{j=1}^{2} \left(a_{1j} - c_{1j}\right) e_j, \\ \frac{\partial e_2}{\partial t} &= \sum_{j=1}^{2} d_{2j} \Delta e_j + \sum_{j=1}^{2} \left(a_{2j} - c_{2j}\right) e_j. \end{aligned} \tag{11}$$

We may, now, construct our Lyapunov functional as

$$V = \frac{1}{2} \int_\Omega e^T e, \tag{12}$$

where $e = (e_1, e_2)^T$, then

$$\begin{aligned} \frac{\partial V}{\partial t} &= \int_\Omega \left(e_1 \frac{\partial e_1}{\partial t} + e_2 \frac{\partial e_2}{\partial t} \right) \\ &= \int_\Omega \left[e_1 \left(\sum_{j=1}^{2} d_{1j} \Delta e_j + \sum_{j=1}^{2} \left(a_{1j} - c_{1j}\right) e_j \right) \right. \\ &\quad \left. + e_2 \left(\sum_{j=1}^{2} d_{2j} \Delta e_j + \sum_{j=1}^{2} \left(a_{2j} - c_{2j}\right) e_j \right) \right] \\ &= \sum_{j=1}^{2} \int_\Omega d_{jj} e_j \left(\Delta e_j\right) + \int_\Omega d_{12} e_1 \Delta e_2 + \int_\Omega d_{21} e_2 \Delta e_1 \\ &\quad + \int_\Omega e^T (A - C) e. \end{aligned} \tag{13}$$

By using Green formula, we can get

$$\begin{aligned} \frac{\partial V}{\partial t} &= -\sum_{j=1}^{2} \int_\Omega d_{jj} \left(\nabla e_j\right)^2 \\ &\quad + \int_{\partial\Omega} \left(d_{12} \frac{\partial e_2}{\partial \eta} e_1 + d_{21} \frac{\partial e_1}{\partial \eta} e_2 \right) d\sigma \\ &\quad - \int_\Omega (d_{21} + d_{12}) \nabla e_1 \nabla e_2 + \int_\Omega e^T (A - C) e, \end{aligned} \tag{14}$$

where ∇ is the gradient vector, η is the is the unit outer normal to $\partial\Omega$, and σ is an auxiliary variable for integration. Then, using the assumption given in (6), the condition given in (5), and the fact that $A - C$ is a negative definite matrix, we obtain

$$\frac{\partial V}{\partial t} = -\sum_{j=1}^{2} \int_\Omega d_{jj} \left(\nabla e_j\right)^2 - \int_\Omega e^T (C - A) e < 0. \tag{15}$$

From Lyapunov stability theory, we can conclude that the zero solution of the error system (11) is globally asymptotically stable and therefore, the master system (1) and the slave system (2) are globally synchronized. □

4. Synchronization via Linear Controllers

In this section, we outline the issue of controlling the master-slave reaction-diffusion system given in (1) and (2) via linear controllers. In this case, we assume that

$$\begin{aligned} \left| f_1 (v_1, v_2) - f_1 (u_1, u_2) \right| &\le \alpha_1 \left| v_1 - u_1 \right| + \alpha_2 \left| v_2 - u_2 \right|, \\ \left| f_2 (v_1, v_2) - f_2 (u_1, u_2) \right| &\le \beta_1 \left| v_1 - u_1 \right| + \beta_2 \left| v_2 - u_2 \right|, \end{aligned} \tag{16}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants.

Theorem 2. *If there exists a control matrix $L = (l_{ij})_{2\times 2}$ such that $A - L$ is a definite negative matrix, then the master-slave reaction-diffusion system given in (1) and (2) can be synchronized under the following linear control law*

$$\begin{aligned} \mathbf{U}_1 &= -\sum_{j=1}^{2} l_{1j} e_j - \left(\alpha_1 + \frac{(\beta_1 + \alpha_2)^2}{4} \right) e_1, \\ \mathbf{U}_2 &= -\sum_{j=1}^{2} l_{2j} e_j - (\beta_2 + 1) e_2. \end{aligned} \tag{17}$$

Proof. Substituting (17) into the error system given in (7) yields

$$\begin{aligned} \frac{\partial e_1}{\partial t} &= \sum_{j=1}^{2} d_{1j} \Delta e_j + \sum_{j=1}^{2} \left(a_{1j} - l_{1j}\right) e_j + f_1 (v_1, v_2) \\ &\quad - f_1 (u_1, u_2) - \left(\alpha_1 + \frac{(\beta_1 + \alpha_2)^2}{4} \right) e_1, \\ \frac{\partial e_2}{\partial t} &= \sum_{j=1}^{2} d_{2j} \Delta e_j + \sum_{j=1}^{2} \left(a_{2j} - l_{2j}\right) e_j + f_2 (v_1, v_2) \\ &\quad - f_2 (u_1, u_2) - (\beta_2 + 1) e_2. \end{aligned} \tag{18}$$

Constructing a Lyapunov function in the form $V = (1/2) \int_\Omega e^T e$ gives

$$\begin{aligned} \frac{\partial V}{\partial t} &= \int_\Omega \left(e_1 \frac{\partial e_1}{\partial t} + e_2 \frac{\partial e_2}{\partial t} \right) = \sum_{j=1}^{2} \int_\Omega d_{1j} e_1 \Delta e_j + \int_\Omega e_1 \\ &\quad \cdot \sum_{j=1}^{2} \left(a_{1j} - l_{1j}\right) e_j + \int_\Omega e_1 \left[f_1 (v_1, v_2) - f_1 (u_1, u_2) \right] \\ &\quad - \int_\Omega \left(\alpha_1 + \frac{(\beta_1 + \alpha_2)^2}{4} \right) e_1^2 + \sum_{j=1}^{2} \int_\Omega d_{2j} e_2 \Delta e_j \\ &\quad + \int_\Omega e_2 \sum_{j=1}^{2} \left(a_{2j} - l_{2j}\right) e_j + \int_\Omega e_2 \left[f_2 (v_1, v_2) \right. \\ &\quad \left. - f_2 (u_1, u_2) \right] - \int_\Omega (\beta_2 + 1) e_2^2 = \sum_{j=1}^{2} \int_\Omega d_{jj} e_j \Delta e_j \\ &\quad + \int_\Omega \left(d_{12} e_1 \Delta e_2 + d_{21} e_2 \Delta e_1 \right) \end{aligned}$$

$$+\int_{\Omega}\left[e_1\sum_{j=1}^{2}\left(a_{1j}-l_{1j}\right)e_j+e_2\sum_{j=1}^{2}\left(a_{2j}-l_{2j}\right)e_j\right]$$
$$+\int_{\Omega}\left[e_1\left(f_1\left(v_1,v_2\right)-f_1\left(u_1,u_2\right)\right)\right.$$
$$\left.+e_2\left(f_2\left(v_1,v_2\right)-f_2\left(u_1,u_2\right)\right)\right]-\int_{\Omega}\left(\alpha_1\right.$$
$$\left.+\frac{\left(\beta_1+\alpha_2\right)^2}{4}\right)e_1^2-\int_{\Omega}\left(\beta_2+1\right)e_2^2$$
$$\leq\sum_{j=1}^{2}\int_{\Omega}d_{jj}e_j\Delta e_j+\int_{\Omega}\left(d_{12}e_1\Delta e_2+d_{21}e_2\Delta e_1\right)$$
$$+\int_{\Omega}\left[e_1\sum_{j=1}^{2}\left(a_{1j}-l_{1j}\right)e_j+e_2\sum_{j=1}^{2}\left(a_{2j}-l_{2j}\right)e_j\right]$$
$$+\int_{\Omega}\left[\left|e_1\right|\left|f_1\left(v_1,v_2\right)-f_1\left(u_1,u_2\right)\right|\right.$$
$$\left.+\left|e_2\right|\left|f_2\left(v_1,v_2\right)-f_2\left(u_1,u_2\right)\right|\right]-\int_{\Omega}\left(\alpha_1\right.$$
$$\left.+\frac{\left(\beta_1+\alpha_2\right)^2}{4}\right)e_1^2-\int_{\Omega}\left(\beta_2+1\right)e_2^2. \tag{19}$$

By using Green formula, we get

$$\frac{\partial V}{\partial t}\leq-\sum_{j=1}^{2}\int_{\Omega}d_{jj}\left(\nabla e_j\right)^2-\int_{\Omega}e^T\left(L-A\right)e$$
$$+\int_{\Omega}\left[\left|e_1\right|\left|f_1\left(v_1,v_2\right)-f_1\left(u_1,u_2\right)\right|\right.$$
$$\left.+\left|e_2\right|\left|f_2\left(v_1,v_2\right)-f_2\left(u_1,u_2\right)\right|\right]-\int_{\Omega}\left(\alpha_1\right.$$
$$\left.+\frac{\left(\beta_1+\alpha_2\right)^2}{4}\right)e_1^2-\int_{\Omega}\left(\beta_2+1\right)e_2^2, \tag{20}$$

and by using the conditions given in (16), we obtain

$$\frac{\partial V}{\partial t}\leq-\sum_{j=1}^{2}\int_{\Omega}d_{jj}\left(\nabla e_j\right)^2-\int_{\Omega}e^T\left(L-A\right)e$$
$$+\int_{\Omega}\left(\alpha_1e_1^2+\left(\alpha_2+\beta_1\right)\left|e_1\right|\left|e_2\right|+\beta_2e_2^2\right)$$
$$-\int_{\Omega}\left(\alpha_1+\frac{\left(\beta_1+\alpha_2\right)^2}{4}\right)e_1^2-\int_{\Omega}\left(\beta_2+1\right)e_2^2$$
$$=-\sum_{j=1}^{2}\int_{\Omega}d_{jj}\left(\nabla e_j\right)^2-\int_{\Omega}e^T\left(L-A\right)e$$
$$-\int_{\Omega}\left[\frac{\left(\beta_1+\alpha_2\right)^2}{4}e_1^2-\left(\beta_1+\alpha_2\right)\left|e_2\right|\left|e_1\right|+e_2^2\right]$$
$$=-\sum_{j=1}^{2}\int_{\Omega}d_{jj}\left(\nabla e_j\right)^2-\int_{\Omega}e^T\left(L-A\right)e$$
$$-\int_{\Omega}\left(\frac{\beta_1+\alpha_2}{2}\left|e_1\right|-\left|e_2\right|\right)^2<0. \tag{21}$$

Therefore, since $\partial V/\partial t<0$, we can conclude that the master system (1) and the slave system (2) are globally synchronized. □

5. Application and Numerical Simulation

In this section, numerical simulations are given to illustrate and validate the synchronization schemes derived in the previous sections. We take the Lengyel-Epstein system [35] as a special case of reaction-diffusion systems. Consider the following coupled master-slave systems:

$$\begin{aligned}\frac{\partial u_1(t,x)}{\partial t}&=\frac{\partial^2u_1}{\partial x^2}+5\gamma-u_1-\frac{4u_1u_2}{1+u_1^2},\\ \frac{\partial u_2(t,x)}{\partial t}&=\delta\left(d\frac{\partial^2u_2}{\partial x^2}+u_1-\frac{u_1u_2}{1+u_1^2}\right),\end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{\partial v_1(t,x)}{\partial t}&=\frac{\partial^2v_1}{\partial x^2}+5\gamma-v_1-\frac{4v_1v_2}{1+u_1^2}+\mathbf{U}_1,\\ \frac{\partial v_2(t,x)}{\partial t}&=\delta\left(d\frac{\partial^2v_2}{\partial x^2}+v_1-\frac{v_1v_2}{1+v_1^2}\right)+\mathbf{U}_2,\end{aligned} \tag{23}$$

where >0, $x\in(0,\theta)$, $(\delta,\gamma,\theta,d)=(9.7607,2.7034,13.03,1.75)$, and $(\mathbf{U}_1,\mathbf{U}_2)^T$ is the control law to be determined. The reaction-diffusion system given in (22) is called the Lengyel-Epstein system. When the initial conditions associated with system (22) are given by $(u_1(0,x),\ u_2(0,x))=(\theta+0.2\cos(5\pi x),1+\theta^2+0.6\cos(5\pi x))$ then the solutions u_1 and u_2 are shown in Figures 1 and 2. For the uncontrolled system (23) (i.e., $\mathbf{U}_1=\mathbf{U}_2=0$), if the initial conditions are given by $(v_1(0,x),v_2(0,x))=(\theta+0.2\cos(4\pi x),1+\theta^2+0.6\cos(4\pi x))$ then the solutions v_1 and v_2 are shown in Figures 3 and 4. The approximation and calculation of the solutions to the Lengyel-Epstein systems given in (22) and (23) are obtained using the Matlab function "pdepe".

Comparing with the master-slave reaction-diffusion systems given (1) and (2), the constants $(d_{ij})_{2\times2}$ and $A=(a_{ij})_{2\times2}$ can be given as

$$\left(d_{ij}\right)_{2\times2}=\begin{pmatrix}1&0\\0&\delta d\end{pmatrix}, \tag{24}$$

and

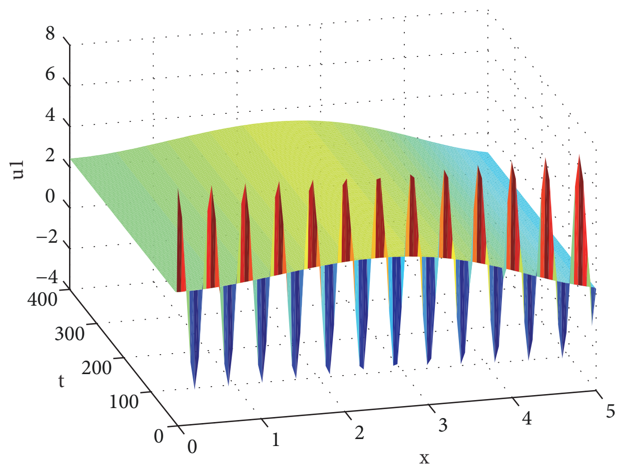

FIGURE 1: Dynamic behavior of solution u_1.

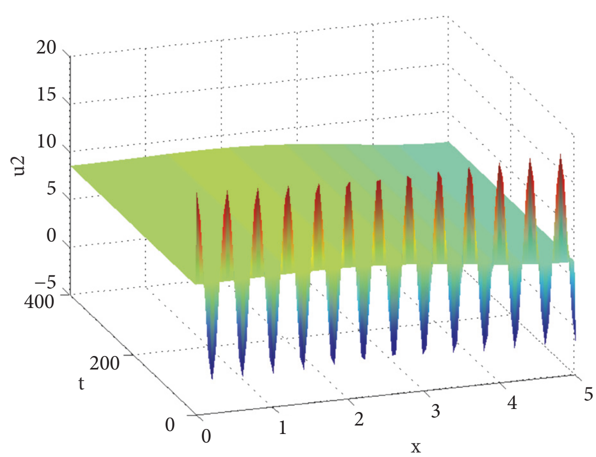

FIGURE 2: Dynamic behavior of solution u_2.

$$A = (a_{ij})_{2\times 2} = \begin{pmatrix} -1 & 0 \\ \delta & 0 \end{pmatrix}. \tag{25}$$

It is clear that our assumption (5) is satisfied. Also, the homogeneous Neumann boundary condition for systems (22) and (23) is described as

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = \frac{\partial v_1}{\partial x} = \frac{\partial v_2}{\partial x} = 0, \quad x = 0, \ \theta \text{ and } t > 0. \tag{26}$$

5.1. Case 1: Nonlinear Control. According to the control scheme proposed in Section 3, if we choose the control matrix C as

$$C = \begin{pmatrix} 0 & 0 \\ \delta & 2 \end{pmatrix}, \tag{27}$$

then the controllers $\mathbf{U}_1$ and $\mathbf{U}_2$ can be designed as

$$\mathbf{U}_1 = \frac{4v_1v_2}{1+v_1^2} - \frac{4u_1u_2}{1+u_1^2},$$

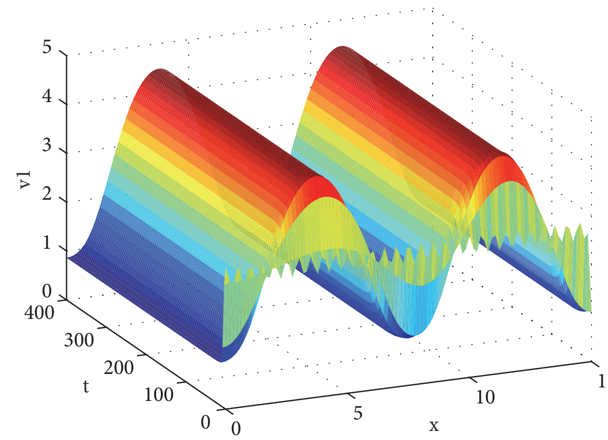

FIGURE 3: Dynamic behavior of solution v_1.

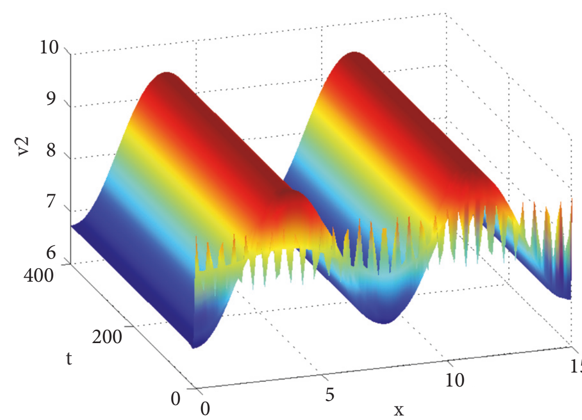

FIGURE 4: Dynamic behavior of solution v_2.

$$\mathbf{U}_2 = -\delta(v_1 - u_1) - 2(v_2 - u_2) + \frac{\delta v_1 v_2}{1+v_1^2} - \frac{\delta u_1 u_2}{1+u_1^2}, \tag{28}$$

and so, simply, we can show that $A - C$ is a negative definite matrix. Therefore, based on Theorem 1, systems (22) and (23) are globally synchronized. The time evolution of the error system states e_1 and e_2, in this case, is shown in Figures 5 and 6.

5.2. Case 2: Linear Control. First, the assumption given in (16) for controlling the master-slave reaction-diffusion system given in (1) and (2) via linear controllers is satisfied. One can easily verify that

$$\begin{aligned} |f_1(v_1,v_2) - f_1(u_1,u_2)| &\le |v_1 - u_1| + 4|v_2 - u_2|, \\ |f_2(v_1,v_2) - f_2(u_1,u_2)| &\le |v_1 - u_1| + \delta|v_2 - u_2|. \end{aligned} \tag{29}$$

According to the control scheme proposed in Section 4, if we choose the control matrix L as

$$L = \begin{pmatrix} 0 & 0 \\ \delta & 1 \end{pmatrix}, \tag{30}$$

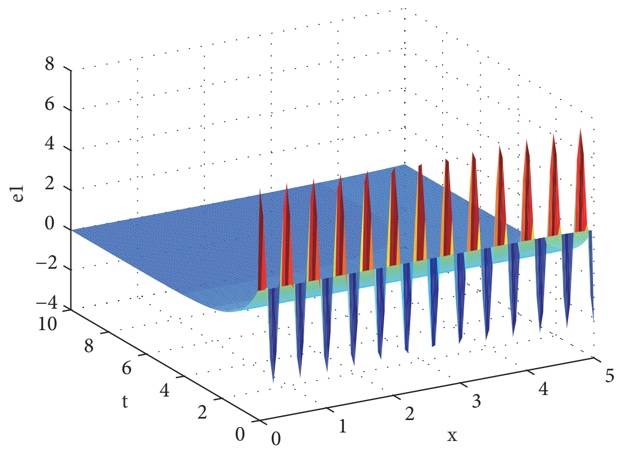

FIGURE 5: Time evolution of the nonlinear synchronization control error e_1.

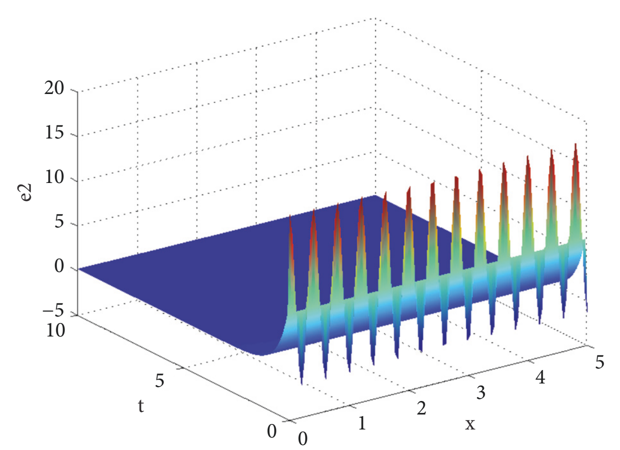

FIGURE 6: Time evolution of the nonlinear synchronization control error e_2.

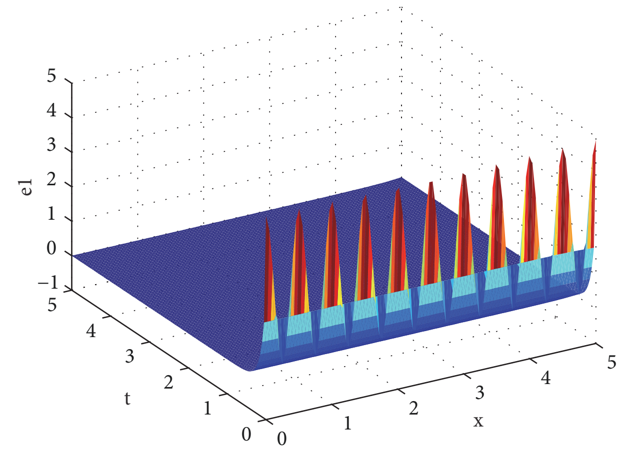

FIGURE 7: Time evolution of the linear synchronization control error e_1.

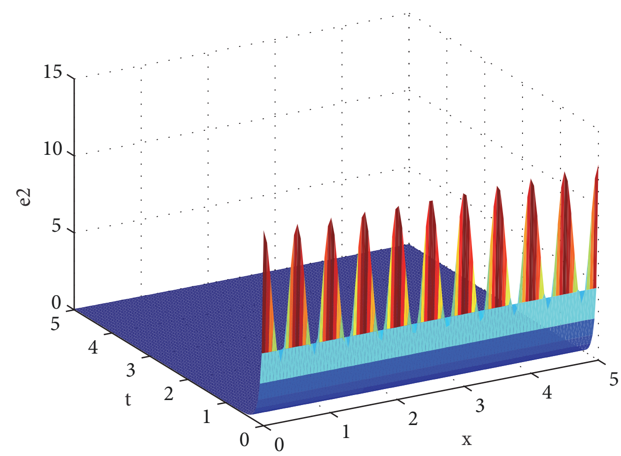

FIGURE 8: Time evolution of the linear synchronization control error e_2.

then the controllers $\mathbf{U}_1$ and $\mathbf{U}_2$ can be designed as

$$\begin{aligned} \mathbf{U}_1 &= -\frac{29}{4}\left(v_1 - u_1\right), \\ \mathbf{U}_2 &= -\delta\left(v_1 - u_1\right) - \left(\delta + 2\right)\left(v_2 - u_2\right), \end{aligned} \tag{31}$$

and so, simply, we can show that $A - L$ is a negative definite matrix. Therefore, based on Theorem 2, systems (22) and (23) are globally synchronized. The time evolution of the error system states e_1 and e_2, in this case, is shown in Figures 7 and 8.

As a result form the performed numerical simulations, we can observe that the addition of the designed linear and nonlinear controllers to the controlled Lengyel-Epstein system, given in (23), updates the coupled systems, given in (22) and (23), dynamics such that the systems states become synchronized. In both cases, the proposed control schemes stabilize the synchronization error states where the zero solution of the error system becomes globally asymptotically stable.

6. Conclusion

The study investigates the synchronization control for a class of reaction-diffusion systems. First, a spatial-time coupling protocol for the synchronization is suggested, then novel control methods, that include linear and nonlinear controllers, are proposed to realize complete synchronization between coupled reaction-diffusion systems. The synchronization results are derived based on Lyapunov stability theory and using the drive-response concept.

Suitable sufficient conditions for achieving synchronization of coupled Lengyel-Epstein systems via suitable linear and nonlinear controllers applied to the response system are derived. For this purpose, we design the controllers so that the zero solution of the error system becomes globally asymptotically stable. Numerical simulations consisting of displaying synchronization behaviors of coupled Lengyel-Epstein systems are given, using Matlab function “pdepe”, to verify the effectiveness of the proposed synchronization schemes. Comparing the numerical simulations shown in Figures 5, 6, 7, and 8, we can easily observe that the linear control scheme realizes synchronization faster than the nonlinear case. Also, the nonlinear control scheme requires the removal of nonlinear terms from the response system, which may increase the cost of the controllers. So, the cost of the controllers in the nonlinear case is more than the cost in the linear case.

The study confirms that the problem of complete synchronization in coupled high dimensional spatial-temporal systems can be realized using linear and nonlinear controllers. Also, we can easily see that the research results obtained in this paper can be extended to many other types of spatial-temporal systems with reaction-diffusion terms.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Physical Review Letters*, vol. 64, no. 8, pp. 821–824, 1990.

[2] H. Fujisaka and T. Yamada, "Stability theory of synchronized motion in coupled-oscillator systems," *Progress of Theoretical and Experimental Physics*, vol. 69, no. 1, pp. 32–47, 1983.

[3] E. Ott, C. Grebogi, and J. A. Yorke, "Controlling chaos," *Physical Review Letters*, vol. 64, no. 11, pp. 1196–1199, 1990.

[4] C. W. Wu and L. O. Chua, "A unified framework for synchronization and control of dynamical systems," *International Journal of Bifurcation and Chaos*, vol. 4, no. 4, pp. 979–998, 1994.

[5] H. M. Rodrigues, "Abstract methods for synchronization and applications," *Applicable Analysis: An International Journal*, vol. 62, no. 3-4, pp. 263–296, 1996.

[6] A. Acosta and P. García, "Synchronization of non-identical chaotic systems: an exponential dichotomies approach," *Journal of Physics A: Mathematical and General*, vol. 34, no. 43, pp. 9143–9151, 2001.

[7] G. Chen and X. Yu, *Chaos Control: Theory and Applications*, Springer-Verlag, Berlin, Germany, 2003.

[8] A. E. Hramov and A. A. Koronovskii, "An approach to chaotic synchronization," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 14, no. 3, pp. 603–610, 2004.

[9] M. A. Aziz-Alaoui, "Synchronization of chaos," *Encyclopedia of Mathematical Physics*, pp. 213–226, 2006.

[10] A. Ouannas and Z. Odibat, "Generalized synchronization of different dimensional chaotic dynamical systems in discrete time," *Nonlinear Dynamics*, vol. 81, no. 1-2, pp. 765–771, 2015.

[11] H. Nakao, T. Yanagita, and Y. Kawamura, "Phase-reduction approach to synchronization of spatiotemporal rhythms in reaction-diffusion systems," *Physical Review X*, vol. 4, no. 2, Article ID 021032, 2014.

[12] Y. Kawamura, S. Shirasaka, T. Yanagita, and H. Nakao, "Optimizing mutual synchronization of rhythmic spatiotemporal patterns in reaction-diffusion systems," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 96, no. 1, Article ID 012224, 2017.

[13] L. Kocarev, Z. Tasev, T. Stojanovski, and U. Parlitz, "Synchronizing spatiotemporal chaos," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 7, no. 4, pp. 635–643, 1997.

[14] L. Kocarev, Z. Tasev, and U. Parlitz, "Synchronizing spatiotemporal chaos of partial differential equations," *Physical Review Letters*, vol. 79, no. 1, pp. 51–54, 1997.

[15] H. G. Winful and L. Rahman, "Synchronized chaos and spatiotemporal chaos in arrays of coupled lasers," *Physical Review Letters*, vol. 65, no. 13, pp. 1575–1578, 1990.

[16] S. Boccaletti, J. Bragard, F. T. Arecchi, and H. Mancini, "Synchronization in nonidentical extended systems," *Physical Review Letters*, vol. 83, no. 3, pp. 536–539, 1999.

[17] L. Junge and U. Parlitz, "Synchronization and control of coupled Ginzburg-Landau equations using local coupling," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 61, no. 4, pp. 3736–3742, 2000.

[18] J. Bragard, F. T. Arecchi, and S. Boccaletti, "Characterization of synchronized spatiotemporal states in coupled nonidentical complex Ginzburg-Landau equations," *International Journal of Bifurcation and Chaos*, vol. 10, no. 10, pp. 2381–2389, 2000.

[19] J. Bragard, S. Boccaletti, and H. Mancini, "Asymmetric coupling effects in the synchronization of spatially extended chaotic systems," *Physical Review Letters*, vol. 91, no. 6, Article ID 064103, 2003.

[20] C. Beta and A. S. Mikhailov, "Controlling spatiotemporal chaos in oscillatory reaction-diffusion systems by time-delay autosynchronization," *Physica D: Nonlinear Phenomena*, vol. 199, no. 1-2, pp. 173–184, 2004.

[21] A. E. Hramov, A. A. Koronovskii, and P. V. Popov, "Generalized synchronization in coupled Ginzburg-Landau equations and mechanisms of its arising," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 72, no. 3, Article ID 037201, 2005.

[22] C. T. Zhou, "Synchronization in nonidentical complex Ginzburg-Landau equations," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 16, no. 1, Article ID 013124, 2006.

[23] P. García, A. Acosta, and H. Leiva, "Synchronization conditions for master-slave reaction diffusion systems," *EPL (Europhysics Letters)*, vol. 88, no. 6, Article ID 60006, 2009.

[24] B. Ambrosio and M. A. Aziz-Alaoui, "Synchronization and control of coupled reaction-diffusion systems of the FitzHugh-Nagumo type," *Computers & Mathematics with Applications*, vol. 64, no. 5, pp. 934–943, 2012.

[25] L. Wang and H. Zhao, "Synchronized stability in a reaction-diffusion neural network model," *Physics Letters A*, vol. 378, no. 48, pp. 3586–3599, 2014.

[26] K.-N. Wu, T. Tian, and L. Wang, "Synchronization for a class of coupled linear partial differential systems via boundary control," *Journal of The Franklin Institute*, vol. 353, no. 16, pp. 4062–4073, 2016.

[27] T. Chen, R. Wang, and B. Wu, "Synchronization of multi-group coupled systems on networks with reaction-diffusion terms based on the graph-theoretic approach," *Neurocomputing*, vol. 227, pp. 54–63, 2017.

[28] Z. Tu, N. Ding, L. Li, Y. Feng, L. Zou, and W. Zhang, "Adaptive synchronization of memristive neural networks with time-varying delays and reaction-diffusion term," *Applied Mathematics and Computation*, vol. 311, no. 15, pp. 118–128, 2017.

[29] S. Chen, C.-C. Lim, P. Shi, and Z. Lu, "Synchronization control for reaction-diffusion FitzHugh-Nagumo systems with spatial sampled-data," *Automatica*, vol. 93, pp. 352–362, 2018.

[30] H. Chen, P. Shi, and C.-C. Lim, "Pinning impulsive synchronization for stochastic reaction–diffusion dynamical networks with delay," *Neural Networks*, vol. 106, pp. 281–293, 2018.

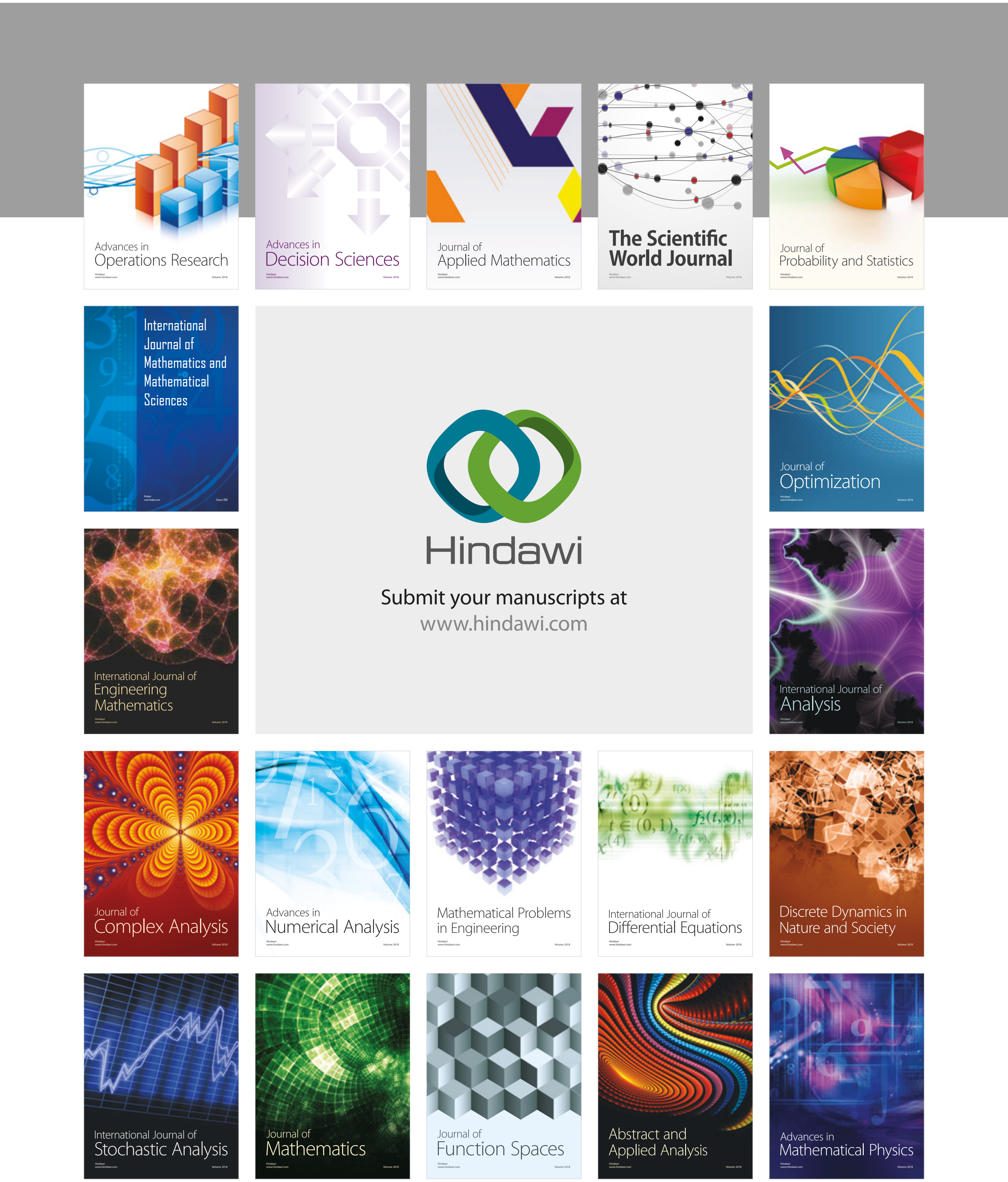
[31] L. Liu, W.-H. Chen, and X. Lu, "Impulsive H∞ synchronization for reaction–diffusion neural networks with mixed delays," *Neurocomputing*, vol. 272, no. 10, pp. 481–494, 2018.

[32] C. He and J. Li, "Hybrid adaptive synchronization strategy for linearly coupled reaction–diffusion neural networks with time-varying coupling strength," *Neurocomputing*, vol. 275, pp. 1769–1781, 2018.

[33] A. S. Mikhailov and K. Showalter, "Control of waves, patterns and turbulence in chemical systems," *Physics Reports*, vol. 425, no. 2-3, pp. 79–194, 2006.

[34] A. S. Mikhailov and G. Ertl, *Engineering of Chemical Complexity*, World Scientific Publishing Co., Singapore, 2nd edition, 2016.

[35] B. Lisena, "On the global dynamics of the Lengyel-Epstein system," *Applied Mathematics and Computation*, vol. 249, pp. 67–75, 2014.



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