Supplementary Information for: Effects of empathy on the evolution of fairness in group-structured populations

Yanling Zhang, Jian Liu and Aming Li

I. THE CALCULATION OF THE MEAN OFFER FOR THE EMPATHETIC STRATEGIES

When all players use the empathetic strategies, the mean offer p under weak selection is calculated as follows. We first discretize the continuous strategy $p \in [0, 1]$. Then, we calculate the mean frequencies of discrete multiple strategies according to the results on discrete strategies. Finally, when the number of discrete strategies tends to $+\infty$, we obtain the mean offer by calculating the weighted average of all discrete strategies whose weights are their frequencies.

A. The expression of discrete multiple strategies' frequencies

Assume that all players choose strategies from S discrete empathetic strategies $\{0, \frac{1}{S-1}, \cdots, 1\}$. Now, the original continuous problem is changed into a discrete one, and we focus on the mean frequencies of the above S strategies averaged over the stationary distribution. The mean frequency of strategy $k \in \{1, 2, \cdots, S\}$ can be calculated by the Mutation-Selection analysis [1]. Assume that the frequency of strategy k is denoted by x_k and the mean frequency of strategy k averaged over the stationary distribution under weak section is denoted by $\langle x_k \rangle_{\omega\to 0}$. The Mutation-Selection analysis has two steps: First, we calculate the expected change of x_k in a single-step update denoted by $\triangle x_k$ by comparing the increase and the decrease of x_k,

$$\triangle x_k = (1 - \frac{(S-1)u}{S})\frac{F_k}{F} + \frac{u}{S}(1 - \frac{F_k}{F}) - x_k, \tag{1}$$

where F_k and F are the total fitness of individuals using strategy k and of the population, respectively; Second, we obtain the expression of $\langle x_k \rangle_{\omega\to 0}$ by letting the mean $\triangle x_k$ averaged over the stationary distribution be zero and performing the perturbation theory in the limit $\omega \to 0$,

$$\langle x_k \rangle_{\omega\to 0} = \frac{1}{S} + \omega\frac{1-u}{Nu}\langle \Pi_k - x_k\Pi \rangle_0, \tag{2}$$

where $\langle\rangle_0$ denotes the average under neutral selection $\omega = 0$, Π_k and Π are the total payoffs of individuals using strategy k and of the population, respectively.

From Eq. (2), the calculation of $\langle x_k\rangle_{\omega\to 0}$ is based on the neutral selection $\omega = 0$, which implies there is no fitness differences between individuals. Assume that I_{ij} is the total number of games that individuals using strategy i play with individuals using strategy j (each game played by two individuals using strategy i is counted twice in computing I_{ii}). Since, all strategies in the neutral stationary states are equivalent, $\langle x_p I_{qr}\rangle_0 = \langle x_{p'} I_{q'r'}\rangle_0$ holds when a bijection operation from the set $\{1, 2, \cdots, S\}$ to $\{1, 2, \cdots, S\}$ satisfies $\pi((p, q, r)) = (p', q', r')$ [2, 3]. Specifically, we have

$$\begin{aligned} &\langle x_1 I_{11}\rangle_0 = \langle x_p I_{pp}\rangle_0, \langle x_1 I_{12}\rangle_0 = \langle x_p I_{pq}\rangle_0, \langle x_1 I_{21}\rangle_0 = \langle x_p I_{qp}\rangle_0, \\ &\langle x_1 I_{22}\rangle_0 = \langle x_p I_{qq}\rangle_0, \langle x_1 I_{23}\rangle_0 = \langle x_p I_{qr}\rangle_0, \end{aligned} \tag{3}$$

where $p \neq q \neq r \neq p$. Then, we write $\langle \Pi_k - x_k \Pi\rangle_0$ as

$$\begin{aligned} \langle \Pi_k - x_k \Pi\rangle_0 = \; & S(\langle x_1 I_{22}\rangle_0 - \langle x_1 I_{23}\rangle_0)(a_{kk} - \overline{a}_{**}) + S(\langle x_1 I_{21}\rangle_0 \\ & -\langle x_1 I_{23}\rangle_0)(\overline{a}_{k*} - \overline{a}_{*k}) + S^2\langle x_1 I_{23}\rangle_0(\overline{a}_{k*} - \overline{a}), \end{aligned} \tag{4}$$

where $\overline{a}_{**} = \frac{1}{S}\sum_{i=1}^{S} a_{ii}$, $\overline{a}_{k*} = \frac{1}{S}\sum_{i=1}^{S} a_{ki}$, $\overline{a}_{*k} = \frac{1}{S}\sum_{i=1}^{S} a_{ik}$, $\overline{a} = \frac{1}{S^2}\sum_{i=1}^{S}\sum_{j=1}^{S} a_{ij}$, and a_{ij} is the payoff of an individual with strategy i against an individual with strategy j. From Eqs. (2) and (4), we have the expression of $\langle x_k\rangle_{\omega\to 0}$ as

$$\langle x_k\rangle_{\omega\to 0} = \frac{1}{S} + \omega\frac{1-u}{Nu}(\Gamma_1(a_{kk} - \overline{a}_{**}) + \Gamma_2(\overline{a}_{k*} - \overline{a}_{*k}) + \Gamma_3(\overline{a}_{k*} - \overline{a})), \tag{5}$$

where $\Gamma_1 = S(\langle x_1 I_{22}\rangle_0 - \langle x_1 I_{23}\rangle_0)$, $\Gamma_2 = S(\langle x_1 I_{21}\rangle_0 - \langle x_1 I_{23}\rangle_0)$, and $\Gamma_3 = S^2\langle x_1 I_{23}\rangle_0$.

B. The concrete values of discrete multiple strategies' frequencies

The expression of $\langle x_k\rangle_{\omega\to 0}$ in Eq. (5) contains three unknown parameters Γ_1, Γ_2, and Γ_3, which are comprised of $\langle x_1 I_{22}\rangle_0$, $\langle x_1 I_{21}\rangle_0$, and $\langle x_1 I_{23}\rangle_0$. According to the definition of I_{22}, $\langle x_1 I_{22}\rangle_0$ (the mean $x_1 I_{22}$ averaged over the neutral stationary distribution) can be expressed by the probability under neutral selection assigned to the event that three randomly chosen (without replacement) players adopt given strategies and locations [2–4]. Furthermore, $\langle x_1 I_{21}\rangle_0$ and $\langle x_1 I_{23}\rangle_0$ can be expressed similarly. Assume that $Pr(s_1 = a, s_2 = b, h_1 \cdot h_2 = 1)$ is the probability that two randomly chosen (without replacement) players (say $1, 2$) adopt the strategies a, b ($s_1 = a, s_2 = b$) and they are located in the same group ($h_1 \cdot h_2 = 1$). Similarly, $Pr(s_1 = a, s_2 = b, s_3 = c, h_2 \cdot h_3 = 1)$ is the probability that three randomly chosen (without replacement) players (say $1, 2, 3$) adopt the strategies a, b, c ($s_1 = a, s_2 = b, s_3 = c$) and the latter two of them are located in the same group ($h_2 \cdot h_3 = 1$). Then, we have

$$\begin{aligned}
\langle x_1 I_{22}\rangle_0 &= (N-1)(N-2)Pr(s_1=1, s_2=2, s_3=2, h_2\cdot h_3=1),\\
\langle x_1 I_{21}\rangle_0 &= (N-1)Pr(s_1=1, s_2=2, h_1\cdot h_2=1) + (N-1)\\
&(N-2)Pr(s_1=1, s_2=2, s_3=1, h_2\cdot h_3=1),\\
\langle x_1 I_{23}\rangle_0 &= (N-1)(N-2)Pr(s_1=1, s_2=2, s_3=3, h_2\cdot h_3=1).
\end{aligned} \tag{6}$$

Under neutral selection, the above probabilities in Eq. (6) can be computed by combining the coalescence theory with the theory of random walks in spatial lattices [5]. The method has two steps: from the present (the time when multiple individuals are chosen) backwards to the time of their most recent common ancestor (MRCA), the coalescence theory is used to obtain the distribution about the number of migration events and the number of mutation events during the ancestral process; from the time of MRCA forwards to the present, the random walk is employed to trace the changing path of the strategy and the changing path of the location upon each lineage. Considering all possible states weighted by the corresponding probabilities, we finally accomplish the calculation of the probabilities in Eq. (6). Furthermore, we have the concrete values of Γ_1, Γ_2, and Γ_3 as

$$\begin{aligned}
\Gamma_1 =\ & (N-1)(N-2)/(3M)\textstyle\sum_{x=1}^{M}(-2\Phi_1\Psi_2 - \Phi_4\alpha_1 + 3\Psi_2),\\
\Gamma_2 =\ & (N-1)/(3M)\textstyle\sum_{x=1}^{M}(3\Psi_1 - 3\Psi_2 + (N-2)(-2\Phi_1\Psi_2\\
& -\Phi_4\alpha_1 + \Phi_2\Psi_2 + \Phi_3\Psi_1 + \Phi_5\alpha_1)),\\
\Gamma_3 =\ & (N-1)(N-2)/(3M)\textstyle\sum_{x=1}^{M}(3\Psi_1 - 3\Psi_2 + 2(2\Phi_1\Psi_2\\
& +\Phi_4\alpha_1 - \Phi_2\Psi_2 - \Phi_3\Psi_1 - \Phi_5\alpha_1)),
\end{aligned} \tag{7}$$

where the above Φ_i and Ψ_i omit $(f(x))$, $\Phi_1(f) = \frac{(1-u)(2-v(1-f))}{2+(N-2)u+\frac{2(N-2)(1-u)v}{3}(1-f)}$, $\Phi_2(f) = \frac{2-u-v(1-f)}{2+\frac{2(N-2)u}{3}+\frac{(N-2)(2-u)v}{3}(1-f)}$, $\Phi_3(f) = \frac{(1-u)(2-v(1-f))}{2+\frac{2(N-2)u}{3}+\frac{(N-2)(2-u)v}{3}(1-f)}$, $\Phi_4(f) = \frac{(1-u)(1-v(1-f))}{1+\frac{(N-2)u}{2}+\frac{(N-2)(1-u)v}{3}(1-f)}$, $\Phi_5(f) = \frac{(2-u)(1-v(1-f))}{2+\frac{2(N-2)u}{3}+\frac{(N-2)(2-u)v}{3}(1-f)}$, $\Psi_1(f) = \frac{1-v(1-f)}{1+(N-1)v(1-f)}$, $\Psi_2(f) = \frac{(1-u)(1-v(1-f))}{1+(N-1)u+(N-1)(1-u)v(1-f)}$, and $\alpha_1 = \frac{1-u}{1+(N-1)u}$. The omitted $f(x)$ describes the migration pattern. The expression of $f(x)$ is $f(x) = \frac{1}{M-1}\sum_{j=1}^{M-1}\cos\frac{2\pi jx}{M}$ for global migration and is $f(x) = \cos\frac{2\pi x}{M}$ for local migration. Substituting Eq. 7 into Eq. 5, we can obtain the concrete value of $\langle x_k\rangle_{\omega\to 0}$.

C. The mean offer for the continuous empathetic strategies

The definition of the mean offer p is $p = \lim_{S\to+\infty}\sum_{k=1}^{S}\frac{k-1}{S-1}\times\langle x_k\rangle_{\delta\to 0}$. According to Eq. (5), the mean offer p is

$$\begin{aligned}
p =\ & \lim_{S\to+\infty}\textstyle\sum_{k=1}^{S}\frac{k-1}{S-1}\frac{1}{S} + \omega\frac{1-u}{Nu}\lim_{S\to+\infty}\sum_{k=1}^{S}\frac{k-1}{S-1}(\Gamma_1(a_{kk}-\overline{a_{**}})\\
& +\Gamma_2(\overline{a_{k*}}-\overline{a_{*k}}) + \Gamma_3(\overline{a_{k*}}-\overline{a})).
\end{aligned} \tag{8}$$

After some easy calculations, we have

$$\begin{array}{l}
\lim\limits_{S\to+\infty} \sum_{k=1}^{S} \frac{k-1}{S-1}\frac{1}{S} = \int_0^1 xdx = \frac{1}{2}, \\
\lim\limits_{S\to+\infty} \sum_{k=1}^{S} \frac{k-1}{S-1}\frac{1}{S} a_{kk} = \int_0^1 xa(x,x)dx = 1/2, \\
\lim\limits_{S\to+\infty} \sum_{k=1}^{S} \frac{k-1}{S-1}\frac{1}{S} \overline{a_{**}} = \frac{1}{2}\int_0^1 a(x,x)dx = 1/2, \\
\lim\limits_{S\to+\infty} \sum_{k=1}^{S} \frac{k-1}{S-1}\frac{1}{S} \overline{a_{*k}} = \int_0^1 \int_0^y y^2 dxdy + \int_0^1 \int_y^1 y(1-x)dxdy = 7/24, \\
\lim\limits_{S\to+\infty} \sum_{k=1}^{S} \frac{k-1}{S-1}\frac{1}{S} \overline{a_{k*}} = \int_0^1 \int_0^y xydxdy + \int_0^1 \int_y^1 x(1-x)dxdy = 5/24, \\
\lim\limits_{S\to+\infty} \sum_{k=1}^{S} \frac{k-1}{S-1}\frac{1}{S} \overline{a} = \int_0^1 \int_0^y y/2dxdy + \int_0^1 \int_y^1 (1-x)/2dxdy = 1/4.
\end{array} \tag{9}$$

Substituting Eqs. (7) and (9) into Eq. (8), we have the mean offer p as

$$\begin{aligned}
p &= \frac{1}{2} - \omega\frac{1-u}{24Nu}(2\Gamma_2 + \Gamma_3) \\
&= \frac{1}{2} - \omega\frac{(1-u)(N-1)}{24Mu}\sum_{x=1}^{M}(\Psi_1(f(x) - \Psi_2(f(x)).
\end{aligned} \tag{10}$$

TABLE 1: For different levels of empathy α, the average values and the standard deviations of the mean offer p and the mean demand q. For each set of parameters, we perform 10 simulations generated by different random seeds. Each simulation runs 5×10^7 generations. Other parameters: $N = 50$, $M = 9$, $r = 1$, $u = 0.2$, $v = 0.1$, and $\omega = 1$.

α	0	0.1	0.3	0.5	0.7	0.9
Average value of p	0.43	0.43	0.42	0.41	0.41	0.40
Standard deviation	0.00034	0.00034	0.00037	0.00042	0.00019	0.00030
Average value of q	0.31	0.32	0.34	0.36	0.37	0.39
Standard deviation	0.00038	0.00039	0.00041	0.00023	0.00030	0.00030

TABLE 2: For different probability density distributions of initialization $f(x)$, the mean offer p and the mean demand q. For each set of parameters, we perform one simulation with 5×10^7 generations. Other parameters: $N = 50$, $M = 9$, $r = 1$, $u = 0.01$, $v = 0.1$, and $\omega = 1$.

	$f(x) = 1$	$f(x) = 2x$	$f(x) = 3x^2$	$f(x) = 4x^3$	$f(x) = 5x^4$
p	0.2268	0.2268	0.2268	0.2268	0.2268
q	0.1675	0.1675	0.1675	0.1675	0.1675

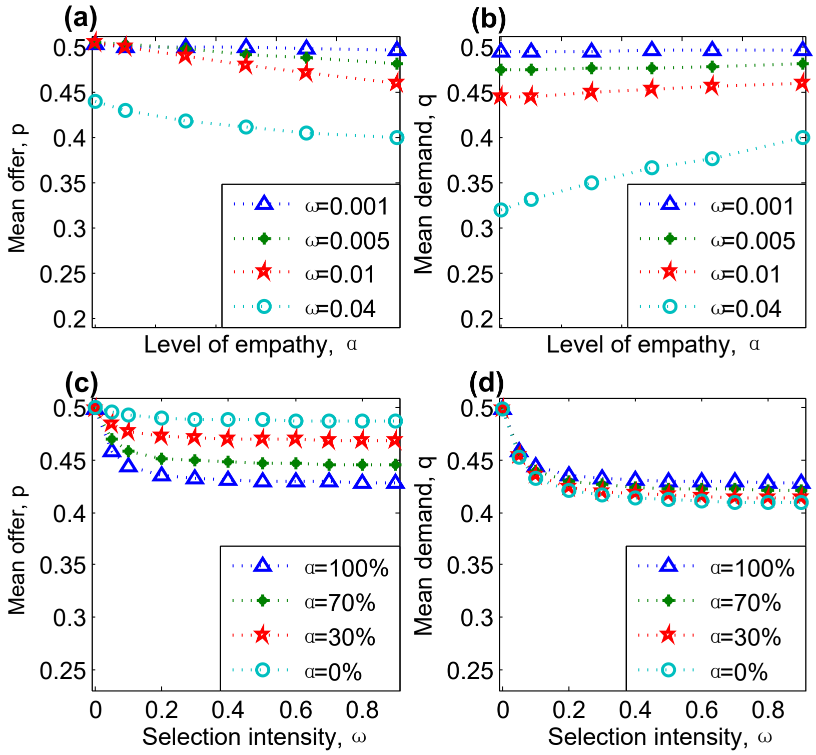

FIG. S1: The changing trends of the mean offer p and demand q with the level of empathy α or the intensity of selection ω. ((a), (b)) p and q change very little with α when ω is very small. ((c), (d)) When the mutation probability is high, the experimental behaviors cannot be observed. Parameters: $N = 50$, $M = 9$, $r = 1$, $v = 0.1$, ((a), (b)) $u = 0.1$, and ((c), (d)) $u = 0.4$.

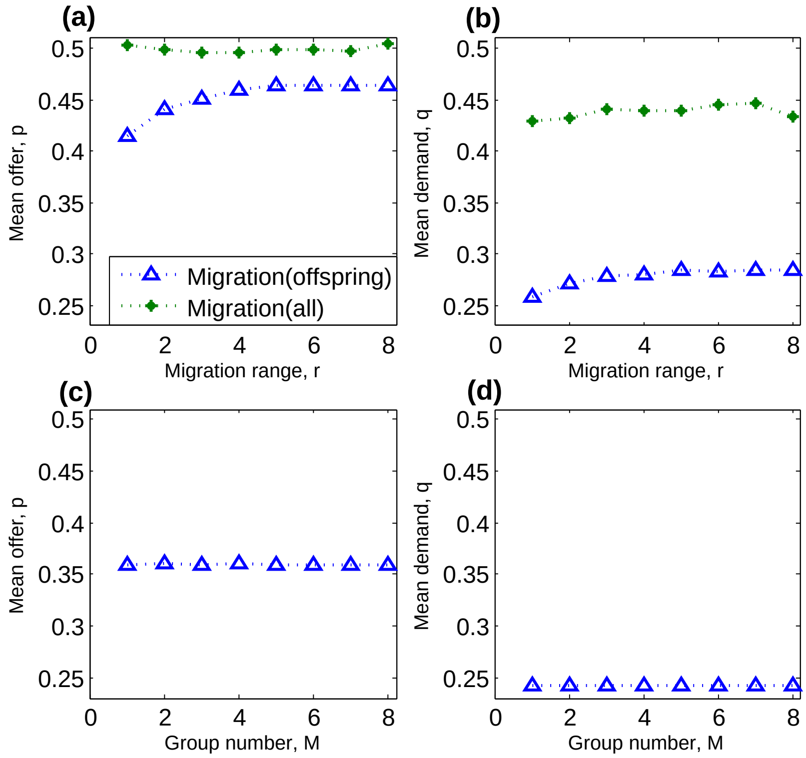

FIG. S2: The mean offer p and demand q for the smallest level of empathy $\alpha = 0$. ((a), (b)) When only the offspring migrates at a generation, p and q increase with small r but remain around constants when r is large; when all individuals migrate at a generation, p and q do not significantly change with r. ((c), (d)) When there is no migration (the migration probability $v = 0$), p and q do not change with the number of groups M. Parameters: $N = 50$, ((a), (b)) $\omega = 0.015$, $M = 16$, $v = 0.1$, ((c), (d)) $\omega = 0.01$, $v = 0$, $r = 1$.

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