SUPPLEMENTARY MATERIAL

APPENDIX A. PROOF OF THEOREM 1

Proof:

(1) Let

$$\mu = f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right)\right)\right),\quad \nu = g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\nu_{p(i)})\right)\right)\right)\right)\right)\right)$$

To prove $qROFAMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) = \langle \mu, \nu \rangle$, we need to prove

$$\left(\frac{1}{n!}\bigoplus_{p\in \boldsymbol{P}_n}\bigotimes_{i=1}^{n}\left(\Xi_{p(i)}^{\delta_i}\right)\right)^{\frac{1}{\sum_{i=1}^{n}\delta_i}} = \langle \mu,\ \nu\rangle$$

The proof process is as follow:

According to the power operation in Equation (4), we have

$$\Xi_{p(i)}^{\delta_i} = \left\langle f^{-1}\left(\delta_i f(\mu_{p(i)})\right),\ g^{-1}\left(\delta_i g(\nu_{p(i)})\right)\right\rangle$$

According to the product operation in Equation (2), we can obtain

$$\bigotimes_{i=1}^{n}\left(\Xi_{p(i)}^{\delta_i}\right) = \left\langle f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right),\ g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\nu_{p(i)})\right)\right)\right\rangle$$

According to the sum operation in Equation (1), we have

$$\bigoplus_{p\in \boldsymbol{P}_n}\bigotimes_{i=1}^{n}\left(\Xi_{p(i)}^{\delta_i}\right) = \left\langle g^{-1}\left(\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right),\ f^{-1}\left(\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\nu_{p(i)})\right)\right)\right)\right)\right\rangle$$

According to multiplication operation in Equation (3), we can obtain

$$\frac{1}{n!}\bigoplus_{p\in \boldsymbol{P}_n}\bigotimes_{i=1}^{n}\left(\Xi_{p(i)}^{\delta_i}\right) = \left\langle g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right),\ f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\nu_{p(i)})\right)\right)\right)\right)\right\rangle$$

The following expression is obtained according to the power operation in Equation (4)

$$\left(\frac{1}{n!}\bigoplus_{p\in \boldsymbol{P}_n}\bigotimes_{i=1}^{n}\left(\Xi_{p(i)}^{\delta_i}\right)\right)^{\frac{1}{\sum_{i=1}^{n}\delta_i}} = \left\langle f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right)\right)\right),\right.$$

$$\left. g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right\rangle$$

(2) To prove $qROFAMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n)$ is a qROFN, we need to prove $0 \le \mu \le 1$, $0 \le \nu \le 1$, and $0 \le \mu^q + \nu^q \le 1$. We firstly prove $0 \le \mu \le 1$ and $0 \le \nu \le 1$. The proof process is as follow:

1) According to the definition of a qROFN in Definition 1, we have $0 \le \mu_{p(i)} \le 1$. Because $f(t)$ is monotonically decreasing, we further have

$$\delta_i f(0) \geq \delta_i f(\mu_{p(i)}) \geq \delta_i f(1)$$

and

$$\left(\sum_{i=1}^{n}\delta_i\right) f(0) = \sum_{i=1}^{n}\left(\delta_i f(0)\right) \geq \sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right) \geq \sum_{i=1}^{n}\left(\delta_i f(1)\right) = \left(\sum_{i=1}^{n}\delta_i\right) f(1)$$

Since $f^{-1}(t)$ is monotonically decreasing, we can obtain

$$f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(0)\right) \leq f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right) \leq f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(1)\right)$$

Because $g(t)$ is monotonically increasing, we further have

$$g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(0)\right)\right) \leq g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right) \leq g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(1)\right)\right)$$

and

$$(n!)g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(0)\right)\right) = \sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(0)\right)\right) \leq \sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right) \leq$$
$$\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(1)\right)\right) = (n!)g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(1)\right)\right)$$

and

$$g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(0)\right)\right) \leq \frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right) \leq g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(1)\right)\right)$$

Since $g^{-1}(t)$ is monotonically increasing, we can obtain

$$f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(0)\right) = g^{-1}\left(g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(0)\right)\right)\right) \leq g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right) \leq$$
$$g^{-1}\left(g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(1)\right)\right)\right) = f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(1)\right)$$

Because $f(t)$ is monotonically decreasing, we further have

$$\left(\sum_{i=1}^{n}\delta_i\right) f(0) = f\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(0)\right)\right) \geq f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right)\right) \geq$$
$$f\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right) f(1)\right)\right) = \left(\sum_{i=1}^{n}\delta_i\right) f(1)$$

and

$$f(0) \geq \frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right)\right) \geq f(1)$$

Finally, since $f^{-1}(t)$ is monotonically decreasing, we can obtain

$$0 = f^{-1}\left(f(0)\right) \leq f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right)\right)\right) \leq f^{-1}\left(f(1)\right) = 1$$

That is $0 \le \mu \le 1$. Similarly, we can prove $0 \le \nu \le 1$.

2) We then prove $0 \le \mu^q + \nu^q \le 1$. The proof process is as follow:

Since $0 \le \mu \le 1$ and $0 \le \nu \le 1$, we have $0 \le \mu^q \le 1$ and $0 \le \nu^q \le 1$, and thus $0 \le \mu^q + \nu^q \le 2$.

According to the definition of a qROFN in Definition 1, we have $\mu_{p(i)}{}^q + \nu_{p(i)}{}^q \le 1$ and $\mu_{p(i)}{}^q \le 1 - \nu_{p(i)}{}^q$. Since $f(t)$ is monotonically decreasing, we further have

$$f\left(\mu_{p(i)}^q\right) \ge f\left(1-\nu_{p(i)}^q\right)$$

Because $f(1-t) = g(t)$, we have

$$f\left(\mu_{p(i)}^q\right) \ge g\left(\nu_{p(i)}^q\right)$$

and

$$\sum_{i=1}^n \left(\delta_i f(\mu_{p(i)}^q)\right) \ge \sum_{i=1}^n \left(\delta_i g\left(\nu_{p(i)}^q\right)\right)$$

Since $f^{-1}(t)$ is monotonically decreasing, we can obtain

$$f^{-1}\left(\sum_{i=1}^n \left(\delta_i f(\mu_{p(i)}^q)\right)\right) \le f^{-1}\left(\sum_{i=1}^n \left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)$$

Because $f^{-1}(t) = 1 - g^{-1}(t)$, we have

$$f^{-1}\left(\sum_{i=1}^n \left(\delta_i f(\mu_{p(i)}^q)\right)\right) \le 1 - g^{-1}\left(\sum_{i=1}^n \left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)$$

Since $g(t)$ is monotonically increasing, we further have

$$g\left(f^{-1}\left(\sum_{i=1}^n \left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right) \le g\left(1 - g^{-1}\left(\sum_{i=1}^n \left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)$$

Because $g(1-t) = f(t)$, we can obtain

$$g\left(f^{-1}\left(\sum_{i=1}^n \left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right) \le f\left(g^{-1}\left(\sum_{i=1}^n \left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)$$

and

$$\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^n \left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right) \le \frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^n \left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)$$

Since $g^{-1}(t)$ is monotonically increasing, we have

$$g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^n \left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right)\right) \le g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^n \left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)$$

Because $g^{-1}(t) = 1 - f^{-1}(t)$, we further have

$$g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^n \left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right)\right) \le 1 - f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^n \left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)$$

Since $f(t)$ is monotonically decreasing, we can obtain

$$\frac{1}{\sum_{i=1}^n \delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^n \left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right)\right)\right) \ge \frac{1}{\sum_{i=1}^n \delta_i} f\left(1 - f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^n \left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)$$

Because $f(1-t) = g(t)$, we can obtain

$$\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right)\right)\right) \geq \frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)$$

Since $f^{-1}(t)$ is monotonically decreasing, we have

$$f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right)\right)\right)\right) \leq f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)$$

Finally, because $f^{-1}(t) = 1 - g^{-1}(t)$, we can obtain

$$f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right)\right)\right)\right) \leq 1 - g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)$$

and

$$f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)}^q)\right)\right)\right)\right)\right)\right) + g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right) \leq 1$$

When $q = 1$, according to the above inequality, we have

$$f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right)\right)\right) + g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(\nu_{p(i)}\right)\right)\right)\right)\right)\right)\right) \leq 1$$

That is, $\mu + \nu \leq 1$.

Now we need to prove that the inequality also holds when $q = 2, 3, \ldots$. Let $m = 2, 3, \ldots$. The purpose is transformed into proof of $\mu^m + \nu^m \leq 1$.

According to $\mu + \nu \leq 1$ and the binomial theorem, we can obtain

$$(\mu+\nu)^m = \sum_{k=0}^{m}\left(C_m^k \mu^{m-k}\nu^k\right) = \mu^m + \nu^m + \sum_{k=1}^{m-1}\left(C_m^k \mu^{m-k}\nu^k\right) \leq 1$$

Because $\mu \geq 0$ and $\nu \geq 0$, we have

$$\sum_{k=1}^{m-1}\left(C_m^k \mu^{m-k}\nu^k\right) \geq 0$$

Therefore, we can obtain $\mu^m + \nu^m \leq 1$. Now it can be concluded that $\mu^q + \nu^q \leq 1$ for $q = 1, 2, 3, \ldots$.

Since we have proved $0 \leq \mu^q + \nu^q \leq 2$ and $\mu^q + \nu^q \leq 1$, we can obtain $0 \leq \mu^q + \nu^q \leq 1$.

APPENDIX B. PROOF OF THEOREM 2

Proof:

Since $\mu_i = \mu$ and $p(i)$ is a permutation of $(1, 2, \ldots, n)$, we have

$$\delta_i f(\mu_{p(i)}) = \delta_i f(\mu)$$

and

$$\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right) = \sum_{i=1}^{n}\left(\delta_i f(\mu)\right) = \left(\sum_{i=1}^{n}\delta_i\right)\left(f(\mu)\right)$$

Then we can obtain

$$f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)=f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(f(\mu)\right)\right)$$

and

$$g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)=g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(f(\mu)\right)\right)\right)$$

Further, we have

$$\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)=\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n}\left(g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(f(\mu)\right)\right)\right)\right)=g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(f(\mu)\right)\right)\right)$$

and

$$g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right)=g^{-1}\left(g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(f(\mu)\right)\right)\right)\right)=f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(f(\mu)\right)\right)$$

Finally, we can obtain

$$\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right)\right)=\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(f(\mu)\right)\right)\right)=f(\mu)$$

and

$$f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(i)})\right)\right)\right)\right)\right)\right)=f^{-1}\left(f(\mu)\right)=\mu$$

Similarly, we can prove

$$g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\nu_{p(i)})\right)\right)\right)\right)\right)\right)=\nu$$

Therefore, we can obtain $qROFAMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) = <\mu, \nu>$.

APPENDIX C. PROOF OF THEOREM 3

Proof:

According to Theorem 1, we have

$$qROFAMM^{\Delta}(\Xi_{1,1},\Xi_{1,2},...,\Xi_{1,n})=\langle\mu_{\mathrm{I}},\ \nu_{\mathrm{I}}\rangle \quad \text{and} \quad qROFAMM^{\Delta}(\Xi_{2,1},\Xi_{2,2},...,\Xi_{2,n})=\langle\mu_{\mathrm{II}},\ \nu_{\mathrm{II}}\rangle$$

where

$$\mu_{\mathrm{I}}=f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(1,i)})\right)\right)\right)\right)\right)\right),\quad \nu_{\mathrm{I}}=g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\nu_{p(1,i)})\right)\right)\right)\right)\right)\right)$$

$$\mu_{\text{II}} = f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(2,i)})\right)\right)\right)\right)\right)\right),\quad \nu_{\text{II}} = g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\nu_{p(2,i)})\right)\right)\right)\right)\right)\right)$$

and $0 \le \mu_{\text{I}} \le 1$ and $0 \le \mu_{\text{II}} \le 1$. Since $\mu_{1,i} \ge \mu_{2,i}$ for all $i = 1, 2,\ldots, n$, we have $\mu_{p(1,i)} \ge \mu_{p(2,i)}$. Because $f(x)$ is monotonically decreasing, we can obtain

$$\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(1,i)})\right) \le \sum_{i=1}^{n}\left(\delta_i f(\mu_{p(2,i)})\right)$$

Since $f^{-1}(x)$ is monotonically decreasing, we have

$$f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(1,i)})\right)\right) \ge f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(2,i)})\right)\right)$$

Because $g(x)$ is monotonically increasing, we can obtain

$$\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(1,i)})\right)\right)\right) \ge \frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(2,i)})\right)\right)\right)$$

Since $g^{-1}(x)$ is monotonically increasing, we have

$$g^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(1,i)})\right)\right)\right)\right) \ge g^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(2,i)})\right)\right)\right)\right)$$

Because $f(x)$ is monotonically decreasing, we can obtain

$$\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(1,i)})\right)\right)\right)\right)\right) \le \frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(2,i)})\right)\right)\right)\right)\right)$$

Finally, since $f^{-1}(x)$ is monotonically decreasing, we have

$$f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(1,i)})\right)\right)\right)\right)\right)\right) \ge f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\mu_{p(2,i)})\right)\right)\right)\right)\right)\right)$$

That is $\mu_{\text{I}} \ge \mu_{\text{II}}$. Similarly, we can prove $\nu_{\text{I}} \le \nu_{\text{II}}$. Since

$S(qROFAMM^{\Delta}(\Xi_{1,1}, \Xi_{1,2}, \ldots, \Xi_{1,n})) = \mu_{\text{I}}^q - \nu_{\text{I}}^q$ and $S(qROFAMM^{\Delta}(\Xi_{2,1}, \Xi_{2,2}, \ldots, \Xi_{2,n})) = \mu_{\text{II}}^q - \nu_{\text{II}}^q$

and $1 \ge \mu_{\text{I}} \ge \mu_{\text{II}} \ge 0$ and $0 \le \nu_{\text{I}} \le \nu_{\text{II}} \le 1$, we can obtain

$S(qROFAMM^{\Delta}(\Xi_{1,1}, \Xi_{1,2}, \ldots, \Xi_{1,n})) \ge S(qROFAMM^{\Delta}(\Xi_{2,1}, \Xi_{2,2}, \ldots, \Xi_{2,n}))$

and thus $qROFAMM^{\Delta}(\Xi_{1,1}, \Xi_{1,2}, \ldots, \Xi_{1,n}) \ge qROFAMM^{\Delta}(\Xi_{2,1}, \Xi_{2,2}, \ldots, \Xi_{2,n})$.

APPENDIX D. PROOF OF THEOREM 4

Proof:

According to Theorem 3, we have

$qROFAMM^{\Delta}(\Xi_{\text{LB}}, \Xi_{\text{LB}}, \ldots, \Xi_{\text{LB}}) \le qROFAMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) \le qROFAMM^{\Delta}(\Xi_{\text{UB}}, \Xi_{\text{UB}}, \ldots, \Xi_{\text{UB}})$

According to Theorem 2, we have

$qROFAMM^{\Delta}(\Xi_{\text{LB}}, \Xi_{\text{LB}}, \ldots, \Xi_{\text{LB}}) = \Xi_{\text{LB}}$ and $qROFAMM^{\Delta}(\Xi_{\text{UB}}, \Xi_{\text{UB}}, \ldots, \Xi_{\text{UB}}) = \Xi_{\text{UB}}$

Therefore, we can obtain $\Xi_{\text{LB}} \le qROFAMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) \le \Xi_{\text{UB}}$.

APPENDIX E. PROOF OF THEOREM 5

Proof:

(1) Let

$$\mu = f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right) f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right),$$

$$\nu = g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right) g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right)$$

To prove $qROFWAMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) = \langle\mu, \nu\rangle$, we need to prove

$$\left(\frac{1}{n!}\bigoplus_{p\in \boldsymbol{P}_n}\bigotimes_{i=1}^{n}\left((nw_{p(i)})\Xi_{p(i)}\right)^{\delta_i}\right)^{\frac{1}{\sum_{i=1}^{n}\delta_i}} = \langle \mu,\ \nu\rangle$$

The proof process is as follow:

According to the multiplication operation in Equation (3), we have

$$(nw_{p(i)})\Xi_{p(i)} = \left\langle g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right),\ f^{-1}\left((nw_{p(i)})f(\nu_{p(i)})\right)\right\rangle$$

According to the power operation in Equation (4), we can obtain

$$\left((nw_{p(i)})\Xi_{p(i)}\right)^{\delta_i} = \left\langle f^{-1}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right),\ g^{-1}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\nu_{p(i)})\right)\right)\right)\right\rangle$$

According to the product operation in Equation (2), we have

$$\bigotimes_{i=1}^{n}\left((nw_{p(i)})\Xi_{p(i)}\right)^{\delta_i} = \left\langle f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right),\ g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\nu_{p(i)})\right)\right)\right)\right)\right\rangle$$

According to the sum operation in Equation (1), we can obtain

$$\bigoplus_{p\in \boldsymbol{P}_n}\bigotimes_{i=1}^{n}\left((nw_{p(i)})\Xi_{p(i)}\right)^{\delta_i} = \left\langle g^{-1}\left(\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right)\right)\right),\right.$$
$$\left. f^{-1}\left(\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right\rangle$$

According to multiplication operation in Equation (3), we have

$$\frac{1}{n!}\bigoplus_{p\in \boldsymbol{P}_n}\bigotimes_{i=1}^{n}\left((nw_{p(i)})\Xi_{p(i)}\right)^{\delta_i} = \left\langle g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right)\right)\right),\right.$$
$$\left. f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right\rangle$$

The following expression is obtained according to the power operation in Equation (4)

$$\left(\frac{1}{n!}\bigoplus_{p\in \boldsymbol{P}_n}\bigotimes_{i=1}^{n}\left((nw_{p(i)})\Xi_{p(i)}\right)^{\delta_i}\right)^{\frac{1}{\sum_{i=1}^{n}\delta_i}} = \left\langle f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right) f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right),\right.$$
$$\left. g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right) g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right)\right\rangle$$

(2) To prove $qROFWAMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n)$ is a qROFN, we need to prove $0 \le \mu \le 1$, $0 \le \nu \le 1$, and $0 \le \mu^q + \nu^q \le 1$. We firstly prove $0 \le \mu \le 1$ and $0 \le \nu \le 1$. The proof process is as follow:

1) According to the definition of a qROFN in Definition 1, we have $0 \le \mu_{p(i)} \le 1$. Because $g(t)$ and $g^{-1}(t)$ are monotonically increasing, we further have

$$(nw_{p(i)})g(0) \le (nw_{p(i)})g(\mu_{p(i)}) \le (nw_{p(i)})g(1)$$

and

$$g^{-1}\left((nw_{p(i)})g(0)\right) \le g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right) \le g^{-1}\left((nw_{p(i)})g(1)\right)$$

Since $f(t)$ and $f^{-1}(t)$ are monotonically decreasing, we can obtain

$$\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(0)\right)\right)\right) \ge \sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right) \ge \sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(1)\right)\right)\right)$$

and

$$f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(0)\right)\right)\right)\right) \le f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right) \le f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(1)\right)\right)\right)\right)$$

Because $g(t)$ and $g^{-1}(t)$ are monotonically increasing, we have

$$g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)f(0)\right)\right) = \frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(0)\right)\right)\right)\right)\right) \le \frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right)\right) \le$$
$$\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(1)\right)\right)\right)\right)\right) = g\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)f(1)\right)\right)$$

and

$$f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)f(0)\right) \le g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right)\right)\right) \le f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)f(1)\right)$$

Finally, since $f(t)$ and $f^{-1}(t)$ are monotonically decreasing, we can obtain

$$f(0) = \left(1\Big/\sum_{i=1}^{n}\delta_i\right) f\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)f(0)\right)\right) \ge \left(1\Big/\sum_{i=1}^{n}\delta_i\right) f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right)\right)\right)\right) \ge$$
$$\left(1\Big/\sum_{i=1}^{n}\delta_i\right) f\left(f^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)f(1)\right)\right) = f(1)$$

and

$$0 = f^{-1}\left(f(0)\right) \le f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right) f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\mu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right) \le f^{-1}\left(f(1)\right) = 1$$

That is $0 \le \mu \le 1$. Similarly, we can prove $0 \le \nu \le 1$.

2) We then prove $0 \le \mu^q + \nu^q \le 1$. The proof process is as follow:

Since $0 \le \mu \le 1$ and $0 \le \nu \le 1$, we have $0 \le \mu^q \le 1$ and $0 \le \nu^q \le 1$, and thus $0 \le \mu^q + \nu^q \le 2$.

According to the definition of a qROFN in Definition 1, we have $\mu_{p(i)}{}^q + \nu_{p(i)}{}^q \le 1$ and $\mu_{p(i)}{}^q \le 1 - \nu_{p(i)}{}^q$. Because $g(t)$ is monotonically increasing and $g(1-t) = f(t)$, we further have

$$g\left(\mu_{p(i)}^q\right) \le g\left(1-\nu_{p(i)}^q\right) = f\left(\nu_{p(i)}^q\right)$$

and

$$(nw_{p(i)})g\left(\mu_{p(i)}^q\right) \le (nw_{p(i)})f\left(\nu_{p(i)}^q\right)$$

Because $g^{-1}(t)$ is monotonically increasing and $g^{-1}(t) = 1 - f^{-1}(t)$, we can obtain

$$g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right) \le g^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right) = 1 - f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)$$

Since $f(t)$ is monotonically decreasing and $f(1-t) = g(t)$, we have

$$f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right)\right) \ge f\left(1 - f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right) = g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)$$

and

$$\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right)\right)\right) \ge \sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)$$

Because $f^{-1}(t)$ is monotonically decreasing and $f^{-1}(t) = 1 - g^{-1}(t)$, we can obtain

$$f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right)\right)\right)\right) \le f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right) = 1 - g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)$$

Since $g(t)$ is monotonically increasing and $g(1-t) = f(t)$, we have

$$g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right)\right)\right)\right)\right) \le g\left(1 - g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right) =$$
$$f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)$$

and

$$\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right)\right)\right)\right)\right) \le \frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)$$

Because $g^{-1}(t)$ is monotonically increasing and $g^{-1}(t) = 1 - f^{-1}(t)$, we can obtain

$$g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right) \le g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right) =$$
$$1 - f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)$$

Since $f(t)$ is monotonically decreasing and $f(1-t) = g(t)$, we have

$$f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)\right) \ge f\left(1 - f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)\right) =$$
$$g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)\right)$$

and

$$\left(1\Big/\sum_{i=1}^{n}\delta_i\right) f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)\right) \ge$$
$$\left(1\Big/\sum_{i=1}^{n}\delta_i\right) g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)\right)$$

Finally, because $f^{-1}(t)$ is monotonically decreasing and $f^{-1}(t) = 1 - g^{-1}(t)$, we can obtain

$$f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right) f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)\right)\right) \le$$

$$f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)\right)\right)=$$
$$1-g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)\right)\right)$$

When q = 1, according to the above inequality, we have

$$f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n}g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\mu_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)+$$
$$g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\nu_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\le 1$$

That is, $\mu + \nu \le 1$.

Now we need to prove that the inequality also holds when q = 2, 3, …. Let m = 2, 3, …. The purpose is transformed into proof of $\mu^m + \nu^m \le 1$.

According to $\mu + \nu \le 1$ and the binomial theorem, we can obtain

$$\left(\mu+\nu\right)^m=\sum_{k=0}^{m}\left(C_m^k\mu^{m-k}\nu^k\right)=\mu^m+\nu^m+\sum_{k=1}^{m-1}\left(C_m^k\mu^{m-k}\nu^k\right)\le 1$$

Because $\mu \ge 0$ and $\nu \ge 0$, we have

$$\sum_{k=1}^{m-1}\left(C_m^k\mu^{m-k}\nu^k\right)\ge 0$$

Therefore, we can obtain $\mu^m + \nu^m \le 1$. Now it can be concluded that $\mu^q + \nu^q \le 1$ for q = 1, 2, 3, ….

Since we have proved $0 \le \mu^q + \nu^q \le 2$ and $\mu^q + \nu^q \le 1$, we can obtain $0 \le \mu^q + \nu^q \le 1$.

APPENDIX F. PROOF OF THEOREM 6

Proof:

(1) Let

$$\mu=g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i}g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right)\right)\right),\quad \nu=f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i}f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n}g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\nu_{p(i)})\right)\right)\right)\right)\right)\right)$$

To prove $qROFAGMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) = \langle\mu, \nu\rangle$, we need to prove

$$\frac{1}{\sum_{i=1}^{n}\delta_i}\left(\bigotimes_{p\in \boldsymbol{P}_n}\bigoplus_{i=1}^{n}\left(\delta_i\Xi_{p(i)}\right)\right)^{\frac{1}{n!}}=\langle\mu,\ \nu\rangle$$

The proof process is as follow:

According to the multiplication operation in Equation (3), we have

$$\delta_i\Xi_{p(i)}=\left\langle g^{-1}\left(\delta_i g(\mu_{p(i)})\right),\ f^{-1}\left(\delta_i f(\nu_{p(i)})\right)\right\rangle$$

According to the sum operation in Equation (1), we can obtain

$$\bigoplus_{i=1}^{n}\left(\delta_i\Xi_{p(i)}\right)=\left\langle g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right),\ f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\nu_{p(i)})\right)\right)\right\rangle$$

According to the product operation in Equation (2), we have

$$\bigotimes_{p\in\boldsymbol{P}_n}\bigoplus_{i=1}^{n}\left(\delta_i\Xi_{p(i)}\right)=\left\langle f^{-1}\left(\sum_{p\in\boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right),\ g^{-1}\left(\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\nu_{p(i)})\right)\right)\right)\right)\right\rangle$$

According to power operation in Equation (4), we can obtain

$$\left(\bigotimes_{p\in\boldsymbol{P}_n}\bigoplus_{i=1}^{n}\left(\delta_i\Xi_{p(i)}\right)\right)^{\frac{1}{n!}}=\left\langle f^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right),\ g^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\nu_{p(i)})\right)\right)\right)\right)\right\rangle$$

The following expression is obtained according to the multiplication operation in Equation (3)

$$\frac{1}{\sum_{i=1}^{n}\delta_i}\left(\bigotimes_{p\in\boldsymbol{P}_n}\bigoplus_{i=1}^{n}\left(\delta_i\Xi_{p(i)}\right)\right)^{\frac{1}{n!}}=\left\langle g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right)\right)\right),\right.$$
$$\left. f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right\rangle$$

(2) To prove $qROFAGMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n)$ is a qROFN, we need to prove $0 \le \mu \le 1$, $0 \le \nu \le 1$, and $0 \le \mu^q + \nu^q \le 1$. We firstly prove $0 \le \mu \le 1$ and $0 \le \nu \le 1$. The proof process is as follow:

1) According to the definition of a qROFN in Definition 1, we have $0 \le \mu_{p(i)} \le 1$. Because $g(t)$ is monotonically increasing, we further have

$$\delta_i g(0) \le \delta_i g(\mu_{p(i)}) \le \delta_i g(1)$$

and

$$\left(\sum_{i=1}^{n}\delta_i\right)g(0)=\sum_{i=1}^{n}\left(\delta_i g(0)\right)\le\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\le\sum_{i=1}^{n}\left(\delta_i g(1)\right)=\left(\sum_{i=1}^{n}\delta_i\right)g(1)$$

Since $g^{-1}(t)$ is monotonically increasing, we can obtain

$$g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right)\le g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\le g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)$$

Because $f(t)$ is monotonically decreasing, we further have

$$f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right)\right)\ge f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\ge f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)\right)$$

and

$$(n!)\,f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right)\right)=\sum_{p\in\boldsymbol{P}_n} f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right)\right)\ge\sum_{p\in\boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\ge$$
$$\sum_{p\in\boldsymbol{P}_n} f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)\right)=(n!)\,f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)\right)$$

and

$$f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right)\right)\ge\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\ge f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)\right)$$

Since $f^{-1}(t)$ is monotonically decreasing, we can obtain

$$g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right) = f^{-1}\left(f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right)\right)\right) \le f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right) \le$$

$$f^{-1}\left(f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)\right)\right) = g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)$$

Because $g(t)$ is monotonically increasing, we further have

$$\left(\sum_{i=1}^{n}\delta_i\right)g(0) = g\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right)\right) \le g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right)\right) \le$$

$$g\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)\right) = \left(\sum_{i=1}^{n}\delta_i\right)g(1)$$

and

$$g(0) \le \frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right)\right) \le g(1)$$

Finally, since $g^{-1}(t)$ is monotonically increasing, we can obtain

$$0 = g^{-1}\left(g(0)\right) \le g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right)\right)\right) \le g^{-1}\left(g(1)\right) = 1$$

That is $0 \le \mu \le 1$. Similarly, we can prove $0 \le \nu \le 1$.

2) We then prove $0 \le \mu^q + \nu^q \le 1$. The proof process is as follow:

Since $0 \le \mu \le 1$ and $0 \le \nu \le 1$, we have $0 \le \mu^q \le 1$ and $0 \le \nu^q \le 1$, and thus $0 \le \mu^q + \nu^q \le 2$.

According to the definition of a qROFN in Definition 1, we have $\mu_{p(i)}{}^q + \nu_{p(i)}{}^q \le 1$ and $\mu_{p(i)}{}^q \le 1 - \nu_{p(i)}{}^q$. Since $g(t)$ is monotonically increasing, we further have

$$g\left(\mu_{p(i)}^q\right) \le g\left(1-\nu_{p(i)}^q\right)$$

Because $g(1-t) = f(t)$, we have

$$g\left(\mu_{p(i)}^q\right) \le f\left(\nu_{p(i)}^q\right)$$

and

$$\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^q)\right) \le \sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^q\right)\right)$$

Since $g^{-1}(t)$ is monotonically increasing, we can obtain

$$g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^q)\right)\right) \le g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^q\right)\right)\right)$$

Because $g^{-1}(t) = 1 - f^{-1}(t)$, we have

$$g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^q)\right)\right) \le 1 - f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^q\right)\right)\right)$$

Since $f(t)$ is monotonically decreasing, we further have

$$f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^q)\right)\right)\right) \ge f\left(1 - f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^q\right)\right)\right)\right)$$

Because $f(1-t) = g(t)$, we can obtain

$$f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^{q})\right)\right)\right)\geq g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)$$

and

$$\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^{q})\right)\right)\right)\geq \frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)$$

Since $f^{-1}(t)$ is monotonically decreasing, we have

$$f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^{q})\right)\right)\right)\right)\leq f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)$$

Because $f^{-1}(t) = 1 - g^{-1}(t)$, we further have

$$f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^{q})\right)\right)\right)\right)\leq 1-g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)$$

Since $g(t)$ is monotonically increasing, we can obtain

$$\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^{q})\right)\right)\right)\right)\right)\leq \frac{1}{\sum_{i=1}^{n}\delta_i} g\left(1-g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)$$

Because $g(1-t) = f(t)$, we can obtain

$$\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^{q})\right)\right)\right)\right)\right)\leq \frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)$$

Since $g^{-1}(t)$ is monotonically increasing, we have

$$g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^{q})\right)\right)\right)\right)\right)\right)\leq g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)$$

Finally, because $g^{-1}(t) = 1 - f^{-1}(t)$, we can obtain

$$g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^{q})\right)\right)\right)\right)\right)\right)\leq 1-f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)$$

and

$$g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)}^{q})\right)\right)\right)\right)\right)\right)+f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}^{q}\right)\right)\right)\right)\right)\right)\right)\leq 1$$

When $q = 1$, according to the above inequality, we have

$$g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right)\right)\right)+f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(\nu_{p(i)}\right)\right)\right)\right)\right)\right)\right)\leq 1$$

That is, $\mu + \nu \leq 1$.

Now we need to prove that the inequality also holds when $q = 2, 3, \ldots$. Let $m = 2, 3, \ldots$. The purpose is transformed into proof of $\mu^m + \nu^m \le 1$.

According to $\mu + \nu \le 1$ and the binomial theorem, we can obtain

$$(\mu+\nu)^m = \sum_{k=0}^{m}\left(C_m^k \mu^{m-k}\nu^k\right) = \mu^m + \nu^m + \sum_{k=1}^{m-1}\left(C_m^k \mu^{m-k}\nu^k\right) \le 1$$

Because $\mu \ge 0$ and $\nu \ge 0$, we have

$$\sum_{k=1}^{m-1}\left(C_m^k \mu^{m-k}\nu^k\right) \ge 0$$

Therefore, we can obtain $\mu^m + \nu^m \le 1$. Now it can be concluded that $\mu^q + \nu^q \le 1$ for $q = 1, 2, 3, \ldots$.

Since we have proved $0 \le \mu^q + \nu^q \le 2$ and $\mu^q + \nu^q \le 1$, we can obtain $0 \le \mu^q + \nu^q \le 1$.

APPENDIX G. PROOF OF THEOREM 7

Proof:

Since $\mu_i = \mu$ and $p(i)$ is a permutation of $(1, 2, \ldots, n)$, we have

$$\delta_i g(\mu_{p(i)}) = \delta_i g(\mu)$$

and

$$\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right) = \sum_{i=1}^{n}\left(\delta_i g(\mu)\right) = \left(\sum_{i=1}^{n}\delta_i\right)\left(g(\mu)\right)$$

Then we can obtain

$$g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right) = g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(g(\mu)\right)\right)$$

and

$$f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right) = f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(g(\mu)\right)\right)\right)$$

Further, we have

$$\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right) = \frac{1}{n!}\sum_{p\in \boldsymbol{P}_n}\left(f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(g(\mu)\right)\right)\right)\right) = f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(g(\mu)\right)\right)\right)$$

and

$$f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right) = f^{-1}\left(f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(g(\mu)\right)\right)\right)\right) = g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(g(\mu)\right)\right)$$

Finally, we can obtain

$$\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right)\right) = \frac{1}{\sum_{i=1}^{n}\delta_i} g\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)\left(g(\mu)\right)\right)\right) = g(\mu)$$

and

$$g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(i)})\right)\right)\right)\right)\right)\right) = g^{-1}\left(g(\mu)\right) = \mu$$

Similarly, we can prove

$$f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\nu_{p(i)})\right)\right)\right)\right)\right)\right)=\nu$$

Therefore, we can obtain $qROFAGMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) = <\mu, \nu>$.

APPENDIX H. PROOF OF THEOREM 8

Proof:

According to Theorem 6, we have

$$qROFAGMM^{\Delta}(\Xi_{1,1},\Xi_{1,2},...,\Xi_{1,n})=\langle \mu_{\mathrm{I}},\ \nu_{\mathrm{I}}\rangle \quad \text{and} \quad qROFAGMM^{\Delta}(\Xi_{2,1},\Xi_{2,2},...,\Xi_{2,n})=\langle \mu_{\mathrm{II}},\ \nu_{\mathrm{II}}\rangle$$

where

$$\mu_{\mathrm{I}}=g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(1,i)})\right)\right)\right)\right)\right)\right),\quad \nu_{\mathrm{I}}=f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\nu_{p(1,i)})\right)\right)\right)\right)\right)\right)$$

$$\mu_{\mathrm{II}}=g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(2,i)})\right)\right)\right)\right)\right)\right),\quad \nu_{\mathrm{II}}=f^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f(\nu_{p(2,i)})\right)\right)\right)\right)\right)\right)$$

and $0 \le \mu_{\mathrm{I}} \le 1$ and $0 \le \mu_{\mathrm{II}} \le 1$. Since $\mu_{1,i} \ge \mu_{2,i}$ for all $i = 1, 2,\ldots, n$, we have $\mu_{p(1,i)} \ge \mu_{p(2,i)}$. Because $g(x)$ is monotonically increasing, we can obtain

$$\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(1,i)})\right)\ge \sum_{i=1}^{n}\left(\delta_i g(\mu_{p(2,i)})\right)$$

Since $g^{-1}(x)$ is monotonically increasing, we have

$$g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(1,i)})\right)\right)\ge g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(2,i)})\right)\right)$$

Because $f(x)$ is monotonically decreasing, we can obtain

$$\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(1,i)})\right)\right)\right)\le \frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(2,i)})\right)\right)\right)$$

Since $f^{-1}(x)$ is monotonically decreasing, we have

$$f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(1,i)})\right)\right)\right)\right)\ge f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(2,i)})\right)\right)\right)\right)$$

Because $g(x)$ is monotonically increasing, we can obtain

$$\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(1,i)})\right)\right)\right)\right)\right)\ge \frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(2,i)})\right)\right)\right)\right)\right)$$

Finally, since $g^{-1}(x)$ is monotonically increasing, we have

$$g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(1,i)})\right)\right)\right)\right)\right)\right) \geq g^{-1}\left(\frac{1}{\sum_{i=1}^{n}\delta_i} g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g(\mu_{p(2,i)})\right)\right)\right)\right)\right)\right)$$

That is $\mu_{\mathrm{I}} \geq \mu_{\mathrm{II}}$. Similarly, we can prove $\nu_{\mathrm{I}} \leq \nu_{\mathrm{II}}$. Since

$S(qROFAGMM^{\Delta}(\Xi_{1,1}, \Xi_{1,2}, \ldots, \Xi_{1,n})) = \mu_{\mathrm{I}}^q - \nu_{\mathrm{I}}^q$ and $S(qROFAGMM^{\Delta}(\Xi_{2,1}, \Xi_{2,2}, \ldots, \Xi_{2,n})) = \mu_{\mathrm{II}}^q - \nu_{\mathrm{II}}^q$

and $1 \geq \mu_{\mathrm{I}} \geq \mu_{\mathrm{II}} \geq 0$ and $0 \leq \nu_{\mathrm{I}} \leq \nu_{\mathrm{II}} \leq 1$, we can obtain

$S(qROFAGMM^{\Delta}(\Xi_{1,1}, \Xi_{1,2}, \ldots, \Xi_{1,n})) \geq S(qROFAGMM^{\Delta}(\Xi_{2,1}, \Xi_{2,2}, \ldots, \Xi_{2,n}))$

and thus $qROFAGMM^{\Delta}(\Xi_{1,1}, \Xi_{1,2}, \ldots, \Xi_{1,n}) \geq qROFAGMM^{\Delta}(\Xi_{2,1}, \Xi_{2,2}, \ldots, \Xi_{2,n})$.

APPENDIX I. PROOF OF THEOREM 9

Proof:

According to Theorem 8, we have

$qROFAGMM^{\Delta}(\Xi_{\mathrm{LB}}, \Xi_{\mathrm{LB}}, \ldots, \Xi_{\mathrm{LB}}) \leq qROFAGMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) \leq qROFAGMM^{\Delta}(\Xi_{\mathrm{UB}}, \Xi_{\mathrm{UB}}, \ldots, \Xi_{\mathrm{UB}})$

According to Theorem 7, we have

$qROFAGMM^{\Delta}(\Xi_{\mathrm{LB}}, \Xi_{\mathrm{LB}}, \ldots, \Xi_{\mathrm{LB}}) = \Xi_{\mathrm{LB}}$ and $qROFAGMM^{\Delta}(\Xi_{\mathrm{UB}}, \Xi_{\mathrm{UB}}, \ldots, \Xi_{\mathrm{UB}}) = \Xi_{\mathrm{UB}}$

Therefore, we can obtain $\Xi_{\mathrm{LB}} \leq qROFAGMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) \leq \Xi_{\mathrm{UB}}$.

APPENDIX J. PROOF OF THEOREM 10

Proof:

(1) Let

$$\mu = g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right) g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)}) f(\mu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right),$$

$$\nu = f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right) f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)}) g(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right)$$

To prove $qROFWAGMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n) = \langle\mu, \nu\rangle$, we need to prove

$$\frac{1}{\sum_{i=1}^{n}\delta_i}\left(\bigotimes_{p\in \boldsymbol{P}_n}\bigoplus_{i=1}^{n}\left(\delta_i \Xi_{p(i)}^{nw_{p(i)}}\right)\right)^{\frac{1}{n!}} = \langle\mu,\ \nu\rangle$$

The proof process is as follow:

According to the power operation in Equation (4), we have

$$\Xi_{p(i)}^{nw_{p(i)}} = \left\langle f^{-1}\left((nw_{p(i)}) f(\mu_{p(i)})\right),\ g^{-1}\left((nw_{p(i)}) g(\nu_{p(i)})\right)\right\rangle$$

According to the multiplication operation in Equation (3), we can obtain

$$\delta_i \Xi_{p(i)}^{nw_{p(i)}} = \left\langle g^{-1}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)}) f(\mu_{p(i)})\right)\right)\right),\ f^{-1}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)}) g(\nu_{p(i)})\right)\right)\right)\right\rangle$$

According to the sum operation in Equation (1), we have

$$\bigoplus_{i=1}^{n}\left(\delta_i \Xi_{p(i)}^{nw_{p(i)}}\right) = \left\langle g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)}) f(\mu_{p(i)})\right)\right)\right)\right),\ f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)}) g(\nu_{p(i)})\right)\right)\right)\right)\right\rangle$$

According to the product operation in Equation (2), we can obtain

$$\bigotimes_{p\in \boldsymbol{P}_n}\bigoplus_{i=1}^{n}\left(\delta_i \Xi_{p(i)}^{nw_{p(i)}}\right)=\left\langle f^{-1}\left(\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right)\right)\right)\right)\right)\right),\right.$$
$$\left. g^{-1}\left(\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right\rangle$$

According to power operation in Equation (4), we have

$$\left(\bigotimes_{p\in \boldsymbol{P}_n}\bigoplus_{i=1}^{n}\left(\delta_i \Xi_{p(i)}^{nw_{p(i)}}\right)\right)^{\frac{1}{n!}}=\left\langle f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right)\right)\right)\right)\right)\right),\right.$$
$$\left. g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right\rangle$$

The following expression is obtained according to the multiplication operation in Equation (3)

$$\frac{1}{\sum_{i=1}^{n}\delta_i}\left(\bigotimes_{p\in \boldsymbol{P}_n}\bigoplus_{i=1}^{n}\left(\delta_i \Xi_{p(i)}^{nw_{p(i)}}\right)\right)^{\frac{1}{n!}}=\left\langle g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right),\right.$$
$$\left. f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g(\nu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right)\right\rangle$$

(2) To prove $qROFWAGMM^{\Delta}(\Xi_1, \Xi_2, \ldots, \Xi_n)$ is a qROFN, we need to prove $0 \le \mu \le 1$, $0 \le \nu \le 1$, and $0 \le \mu^q + \nu^q \le 1$. We firstly prove $0 \le \mu \le 1$ and $0 \le \nu \le 1$. The proof process is as follow:

1) According to the definition of a qROFN in Definition 1, we have $0 \le \mu_{p(i)} \le 1$. Because $f(t)$ and $f^{-1}(t)$ are monotonically decreasing, we further have

$$(nw_{p(i)})f(0) \ge (nw_{p(i)})f(\mu_{p(i)}) \ge (nw_{p(i)})f(1)$$

and

$$f^{-1}\left((nw_{p(i)})f(0)\right) \le f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right) \le f^{-1}\left((nw_{p(i)})f(1)\right)$$

Since $g(t)$ and $g^{-1}(t)$ are monotonically increasing, we can obtain

$$\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(0)\right)\right)\right) \le \sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right)\right)\right) \le \sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(1)\right)\right)\right)$$

and

$$g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(0)\right)\right)\right)\right) \le g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right)\right)\right)\right) \le g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(1)\right)\right)\right)\right)$$

Because $f(t)$ and $f^{-1}(t)$ are monotonically decreasing, we have

$$f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right)\right)=\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(0)\right)\right)\right)\right)\right) \ge \frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right)\right)\right)\right)\right) \ge$$
$$\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(1)\right)\right)\right)\right)\right) = f\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)\right)$$

and

$$g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right) \le f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right)\right)\right)\right)\right)\right) \le g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)$$

Finally, since $g(t)$ and $g^{-1}(t)$ are monotonically increasing, we can obtain

$$g(0)=\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(0)\right)\right)\le\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\le$$

$$\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(g^{-1}\left(\left(\sum_{i=1}^{n}\delta_i\right)g(1)\right)\right)=g(1)$$

and

$$0=g^{-1}\left(g(0)\right)\le g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f(\mu_{p(i)})\right)\right)\right)\right)\right)\right)\right)\right)\le g^{-1}\left(g(1)\right)=1$$

That is $0\le\mu\le 1$. Similarly, we can prove $0\le\nu\le 1$.

2) We then prove $0\le\mu^q+\nu^q\le 1$. The proof process is as follow:

Since $0\le\mu\le 1$ and $0\le\nu\le 1$, we have $0\le\mu^q\le 1$ and $0\le\nu^q\le 1$, and thus $0\le\mu^q+\nu^q\le 2$.

According to the definition of a qROFN in Definition 1, we have $\mu_{p(i)}{}^q+\nu_{p(i)}{}^q\le 1$ and $\mu_{p(i)}{}^q\le 1-\nu_{p(i)}{}^q$. Because $f(t)$ is monotonically decreasing and $f(1-t)=g(t)$, we further have

$$f\left(\mu_{p(i)}^q\right)\ge f\left(1-\nu_{p(i)}^q\right)=g\left(\nu_{p(i)}^q\right)$$

and

$$(nw_{p(i)})f\left(\mu_{p(i)}^q\right)\ge(nw_{p(i)})g\left(\nu_{p(i)}^q\right)$$

Because $f^{-1}(t)$ is monotonically decreasing and $f^{-1}(t)=1-g^{-1}(t)$, we can obtain

$$f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\le f^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)=1-g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)$$

Since $g(t)$ is monotonically increasing and $g(1-t)=f(t)$, we have

$$g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\right)\le g\left(1-g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)=f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)$$

and

$$\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\right)\right)\le\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)$$

Because $g^{-1}(t)$ is monotonically increasing and $g^{-1}(t)=1-f^{-1}(t)$, we can obtain

$$g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\right)\right)\right)\le g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)=1-f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)$$

Since $f(t)$ is monotonically decreasing and $f(1-t)=g(t)$, we have

$$f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\right)\right)\right)\right)\ge f\left(1-f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)=$$

$$g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)$$

and

$$\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\right)\right)\right)\right)\ge\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n}g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)$$

Because $f^{-1}(t)$ is monotonically decreasing and $f^{-1}(t)=1-g^{-1}(t)$, we can obtain

$$f^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n}f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)\le f^{-1}\left(\frac{1}{n!}\sum_{p\in\boldsymbol{P}_n}g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)=$$

$$1-g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)$$

Since $g(t)$ is monotonically increasing and $g(1-t) = f(t)$, we have

$$g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)\right)\le g\left(1-g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)\right)=$$
$$f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)\right)$$

and

$$\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)\right)\le$$
$$\left(1\Big/\sum_{i=1}^{n}\delta_i\right)f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)\right)$$

Finally, because $g^{-1}(t)$ is monotonically increasing and $g^{-1}(t) = 1 - f^{-1}(t)$, we can obtain

$$g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)\right)\right)\le$$
$$g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)\right)\right)=$$
$$1-f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}^q\right)\right)\right)\right)\right)\right)\right)\right)\right)$$

When $q = 1$, according to the above inequality, we have

$$g^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)g\left(f^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} f\left(g^{-1}\left(\sum_{i=1}^{n}\left(\delta_i g\left(f^{-1}\left((nw_{p(i)})f\left(\mu_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)+$$
$$f^{-1}\left(\left(1\Big/\sum_{i=1}^{n}\delta_i\right)f\left(g^{-1}\left(\frac{1}{n!}\sum_{p\in \boldsymbol{P}_n} g\left(f^{-1}\left(\sum_{i=1}^{n}\left(\delta_i f\left(g^{-1}\left((nw_{p(i)})g\left(\nu_{p(i)}\right)\right)\right)\right)\right)\right)\right)\right)\right)\le 1$$

That is, $\mu + \nu \le 1$.

Now we need to prove that the inequality also holds when $q = 2, 3, \ldots$. Let $m = 2, 3, \ldots$. The purpose is transformed into proof of $\mu^m + \nu^m \le 1$.

According to $\mu + \nu \le 1$ and the binomial theorem, we can obtain

$$(\mu+\nu)^m=\sum_{k=0}^{m}\left(C_m^k\mu^{m-k}\nu^k\right)=\mu^m+\nu^m+\sum_{k=1}^{m-1}\left(C_m^k\mu^{m-k}\nu^k\right)\le 1$$

Because $\mu \ge 0$ and $\nu \ge 0$, we have

$$\sum_{k=1}^{m-1}\left(C_m^k\mu^{m-k}\nu^k\right)\ge 0$$

Therefore, we can obtain $\mu^m + \nu^m \le 1$. Now it can be concluded that $\mu^q + \nu^q \le 1$ for $q = 1, 2, 3, \ldots$.

Since we have proved $0 \le \mu^q + \nu^q \le 2$ and $\mu^q + \nu^q \le 1$, we can obtain $0 \le \mu^q + \nu^q \le 1$.