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Research Article

Investigations of Nonlinear Triopoly Models with Different Mechanisms

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This paper studies the dynamic characteristics of triopoly models that are constructed based on a 3-dimensional Cobb–Douglas utility function. The paper presents two parts. The first part introduces a competition among three rational firms on which their prices are isoelastic functions. The competition is described by a 3-dimensional discrete dynamical system. We examine the impact of rationality on the system's steady state point. Studying the stability/instability of this point, which is Nash equilibrium and is unique in those models, is illustrated. Numerically, we give some global analysis of Nash point and its stability. The second part deals with heterogeneous scenarios. It consists of two different models. In the first model, we assume that one competitor adopts the local monopolistic approximation mechanism (LMA) while the other opponents are rational. The second model assumes two heterogeneous players with LMA mechanism against one rational firm. Studies show that the stability of NE point of those models is not guaranteed. Furthermore, simulation shows that when firms behave rational with symmetric costs, the stability of NE point is achievable.

1. Introduction

Oligopolistic competition in economic market structure has got much attention recently. It is more complex and perfect competition when one compares it with the monopolistic one. When tackling this kind of competition, a wide range of different outcomes can arise, and therefore it is reasonable to use the theory of game. Game theory has been used heavily to model the behavior of such oligopolistic competitors.

Fully rational firms involved in oligopolistic competition are provided with cognitive and computational skills so that they can perfectly identify the demand curve of produced commodity, and therefore, the expectation of their competitors' production in the next period is achieved. Knowing such skills makes firms ready to solve a one period optimization problem. Recent works [1–6] have investigated the influence of reducing rationality in terms of computational and informational capabilities. Those works have yielded an important conclusion; that is, reducing such rationality leads to the appearance of complex dynamic characteristics of firms' behaviors. In [1], some triopolistic games have been introduced and studied using quasiconcave utility function. The complex behavior of a Cournot duopoly game has been investigated in [2]. Using heterogeneous expectations, a nonlinear duopoly game has been introduced and discussed in [4]. Askar et al. [5] have studied the dynamic characteristics of Cournot duopoly models based on unknown inverse demand function. With nonlinear demand function, whose inflection points do not exist, Askar has proposed and investigated the complex dynamic behavior of a Cournot duopoly game.

Puu [7] has constructed a duopolistic model on which he has used a unimodel reaction function accompanying Cobb–Douglas consumer's preferences. Studies carried out by Puu have shown that the firms' outputs are evolved through a chaotic scenario which in turn leads to chaos.

Since Puu's work, several studies have made by researchers to look into different decisional and adjustment mechanisms such as bounded rationality and LMA (local monopolistic approximation) mechanisms. The bounded rationality mechanism requires from the competitors some local knowledge in order to improve their outputs according to the variations in the marginal profit. Indeed, this mechanism makes the competitors unaware of any information or knowledge about the demand and cost functions. Only the competitors need to be aware of any change that occurs in market due to small changes in the produced quantities by estimating the marginal profit. For more details about this mechanism, readers are advised to refer to the literature [4, 8, 9]. The LMA mechanism was first introduced by Tuinstra [10]. Oligopolistic players adopting the LMA mechanism do not have any information or knowledge about the demand function of the market. Even though it is unknown for the competitors, they conjecture it in linear. Therefore, with local knowledge about the true demand curve and the current market state in terms of quantities produced and their prices, they estimate this linear function.

The literature has reported several works that have adopted such rationality and LMA mechanisms. For instance, Pecora and Sodini have analyzed a Cournot duopoly game whose demand function was isoelastic in continuous time periods [11]. In [12], the LMA and the gradient rule have been used to analyze the complex dynamic behavior of a duopoly model. A discrete duopolistic fishery model with two agents who adopt heterogeneous expectations has been investigated in [13]. Both the monopolistic approximation and the gradient approach have been used in [14] to study an evolutionary oligopoly competition. For more related works and simulation approaches, readers are advised to refer to [15, 16].

The current paper discusses the influences of some adjustment mechanisms on the stability of Nash equilibrium point. Here, we propose and investigate different types of triopoly games on which firms use bounded rationality and LMA mechanisms. Our obtained results show that the repeated triopoly game based on rationality mechanism converges at Nash point and implies more stability region. On the other two games where LMA mechanism is adopted by competed firms, the results show that the repeated games based on that mechanism or based on a mixed type of both rationality and LMA do not converge at Nash point due to the complicated behaviors of systems described those games and due to the negative quantities obtained which have no meaning in economic market.

Briefly, the current paper is described as follows. In Section 2, we introduce the Cobb–Douglas production function for three oligopolistic firms. After that, the first scenario of a rational game is constructed, and then an investigation on its complex dynamic characteristics is presented. After that, a heterogeneous game is introduced and studied in details. Finally, conclusion is provided in Section 3.

2. Model

In 1927, the first formulation of Cobb–Douglas function was described. In that time, Douglas sought for a functional form by which he could use to present the data he calculated for workers and capital. Economists today widely use this production function to study the relationship between the amount of two or more inputs and the amount of outputs that can be produced by those inputs. The current paper assumes that the market structure consists of three firms whose preferences are derived from Cobb–Douglas. It takes the following form:

$$U = \prod_{i=1}^{3} q_i^{\alpha_i}, \sum_{i=1}^{3} \alpha_i = 1. \quad (1)$$

Indeed, firms want to maximize their preferences due to a budget constraint $\sum_{i=1}^{3} p_i q_i = 1$, where p_i, $i = 1, 2, 3$ is a commodity price supplied by firm i. According to this constraint, the following maximization problem is constructed:

$$\begin{gathered} \operatorname{Max} \sum_{i=1}^{3} \alpha_i \log q_i \\ \sum_{i=1}^{3} p_i q_i = 1. \end{gathered} \quad (2)$$

Equation (2) has the following solution:

$$q_j = \frac{\alpha_j}{p_j}, \quad j = 1, 2, 3. \quad (3)$$

If we sum the above for all firms, we get

$$\begin{gathered} p(Q) = \frac{1}{Q}, \\ Q = \sum_{i=1}^{3} q_i. \end{gathered} \quad (4)$$

This kind of demand is called an isoelastic function. Now, we suppose that the firms' costs are

$$C_i(q_i) = c_i q_i, \quad i = 1, 2, 3, \quad (5)$$

where $c_i > 0$ is a constant marginal cost for each firm. Using (4) and (5), each firm has its own profit as follows:

$$\begin{gathered} \pi_1 = \frac{q_1}{q_1 + q_2 + q_3} - c_1 q_1, \\ \pi_2 = \frac{q_2}{q_1 + q_2 + q_3} - c_2 q_2, \\ \pi_3 = \frac{q_3}{q_1 + q_2 + q_3} - c_3 q_3. \end{gathered} \quad (6)$$

Game theory can be used to study the above scenario on which the firms are three oligopolists. The game feasible space will be constructed with all strategies $q_i > 0, i = 1, 2, 3$ and the payoff functions that are given in (6). Only one Nash equilibrium point for this game is given by

$$\text{NE} = (\bar{q}_1, \bar{q}_2, \bar{q}_3) = \left(\frac{2(c_2+c_3-c_1)}{(c_1+c_2+c_3)^2}, \frac{2(c_1+c_3-c_2)}{(c_1+c_2+c_3)^2}, \frac{2(c_1+c_2-c_3)}{(c_1+c_2+c_3)^2} \right). \quad (7)$$

It is positive under the conditions $c_2 + c_3 > c_1, c_1 + c_3 > c_2$ and $c_1 + c_2 > c_3$. In an oligopolistic competition, information that should be available for each player about its opponent is important and limited. The gradient mechanism which is important and intensively used in literature is a rule of thumb. It requires only local knowledge about the player's marginal profit. It depends on some thoughts each player should know about variations in the amount $q_{i,t+1} - q_{i,t}$ that in turn gives exact estimation of the marginal profit $\partial\pi_i/\partial q_{i,t}$. Firms in such competition are always seeking for a good estimation of the current marginal profit in order to see whether it increases or decreases its output level depending on the information given by the marginal profit in the previous time step. This is governed by a positive parameter called the speed of adjustment. The mechanism is described by the following discrete map:

$$q_{i,t+1} = q_{i,t} + k_i \phi_i, \quad i = 1, 2, 3, \quad (8)$$

where, $k_i > 0$ is the speed of adjustment and $\phi_i = \partial\pi_i/\partial q_{i,t}$. Here, we study two different scenarios: the first scenario assumes that the three oligopolistic firms adopt this mechanism, while in the other we suppose that one of the firms adopts the so-called Local Monopolistic Approximation (LMA) that is described later. Let us now construct the first secario. Using (6), the marginal profit of each firm is given by

$$\phi_i = \frac{Q - q_i}{Q^2} - c_i, \quad i = 1, 2, 3. \quad (9)$$

Substituting (9) in (8), the resulting oligopolistic game is presented by the following discrete dynamical system:

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + k\left[\frac{Q - q_1}{Q^2} - c_1\right], \\ q_{2,t+1} &= q_{2,t} + k\left[\frac{Q - q_2}{Q^2} - c_2\right], \\ q_{3,t+1} &= q_{3,t} + k\left[\frac{Q - q_3}{Q^2} - c_3\right]. \end{aligned} \quad (10)$$

2.1. Local Analysis. This subsection provides a discussion on the steady state of the game and the local stability of system (10). We investigate under what conditions should system (10) be stable and where complex dynamic can influence the stabilization of the steady states. The following proposition is given.

Proposition 1. The Nash equilibrium (7) is a steady state of the system (10) and is locally asymptotically stable provided that $k < (4/(c_1 + c_2 + c_3)^2)$.

Proof. System (10) at NE point has the following Jacobian matrix:

$$J = \begin{bmatrix} 1 - c_1 k(c_1+c_2+c_3) & \frac{-k}{4}(c_1+c_2+c_3)(3c_1-c_2-c_3) & \frac{-k}{4}(c_1+c_2+c_3)(3c_1-c_2-c_3) \\ \frac{-k}{4}(c_1+c_2+c_3)(3c_2-c_1-c_3) & 1 - c_2 k(c_1+c_2+c_3) & \frac{-k}{4}(c_1+c_2+c_3)(3c_2-c_1-c_3) \\ \frac{-k}{4}(c_1+c_2+c_3)(3c_3-c_1-c_2) & \frac{-k}{4}(c_1+c_2+c_3)(3c_3-c_1-c_2) & 1 - c_3 k(c_1+c_2+c_3) \end{bmatrix}. \quad (11)$$

When games described are by discrete dynamic systems, then studying the stability of the NE point of those systems depends on the eigenvalues of the Jacobian matrix. This makes us to impose the condition $|\lambda_i| < 1, i = 1, 2, 3$ that means that all the eigenvalues must be in the unit circle. This can be carried by recalling the following Jury conditions:

$$\begin{aligned} &\text{(i)}: 1 + a_1 + a_2 + a_3 > 0, \\ &\text{(ii)}: 1 - a_1 + a_2 - a_3 > 0, \\ &\text{(iii)}: 1 + a_2 - a_1 a_3 - a_3^2 > 0, \\ &\text{(iv)}: 1 - a_3^2 > 0, \end{aligned} \quad (12)$$

where $f(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ is the characteristic polynomial of the above Jacobian and

$$\begin{aligned} a_1 &= -(\lambda_1 + \lambda_2 + \lambda_3) = -\text{Trace}(J), \\ a_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ a_3 &= (-1)^3 \lambda_1\lambda_2\lambda_3 = -\text{Determinant}(J). \end{aligned} \quad (13)$$

From the above Jacobian and with simple calculations, we deduce

$$
\begin{aligned}
a_1 &= -(3-2\ell),\\
a_2 &= \left(1-\frac{1}{2}\ell\right)\left(3-\frac{5}{2}\ell\right),\\
a_3 &= -(1-\ell)\left(1-\frac{1}{2}\ell\right)^2,\\
\ell &= \frac{1}{2}k\left(c_1+c_2+c_3\right)^2.
\end{aligned}
\tag{14}
$$

This makes Jury conditions become

$$
\begin{aligned}
&\text{(i)}: \frac{1}{4}\ell^3 > 0,\\
&\text{(ii)}: \frac{1}{4}(2-\ell)(4-\ell)^2 > 0,\\
&\text{(iii)}: \frac{1}{16}\ell\left(128-208\ell+140\ell^2-49\ell^3+10\ell^4-\ell^5\right) > 0,\\
&\text{(iv)}: \frac{1}{16}\ell\left(8-5\ell+\ell^2\right)\left(8-8\ell+5\ell^2-\ell^3\right) > 0.
\end{aligned}
\tag{15}
$$

Simple calculations show that the first condition (i) is always fulfilled, the condition (ii) is fulfilled under $0<\ell<2$ (this is equivalent $k<(4/(c_1+c_2+c_3)^2)$), and the other two conditions hold providing that (ii) holds. The second condition (ii) becomes zero at $\ell=2$ or $\ell=4$ which means period-doubling bifurcation (flip bifurcation) may occurs. Furthermore, the condition (iv) can not be zero and hence Neimark–Sacker bifurcation does not exist for the system (10). In addition, one can easily get the eigenvalues as follows:

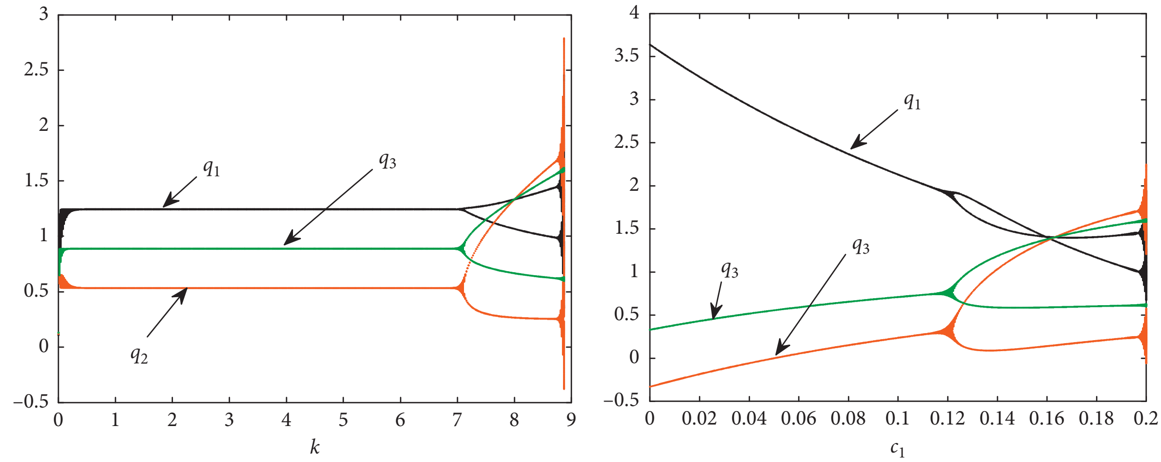
$$
\begin{aligned}
\lambda_1 &= 1-\frac{k}{2}\left(c_1+c_2+c_3\right)^2,\\
\lambda_{2,3} &= 1-\frac{k}{4}\left(c_1+c_2+c_3\right)^2.
\end{aligned}
\tag{16}
$$

Those eignvalues are real and $|\lambda_i|<1, i=1,2,3$ if $k<(4/(c_1+c_2+c_3)^2)$ which completes the proof. □

2.2. Simulation. In this section, we perform some numerical simulation to investigate the complex behavior of system (10). This includes the influences of the system's parameters on the stability of Nash point. We start our simulation by assuming the following parameter values: $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13)$ and $c_1=0.20, c_2=0.30$ and $c_3=0.25$. We assume different values for the firms' costs as we study first the asymmetric case. This makes Nash point equal $(1.2444, 0.53333, 0.88889)$. As shown in Figure 1(a) Nash point is asymptotically stable for any values for the parameter k till k approaches 7.11 on where birth of period 2-cycle arises. After that a period-doubling bifurcation (flip bifurcation) exists. Only flip bifurcation exists in this case as we have only two different real eigenvalues. Here, we should highlight on the values of the costs' parameters that must be selected in such a way that the conditions $c_2+c_3>c_1, c_1+c_3>c_2$ and $c_1+c_2>c_3$ are satisfied and at the same time, positivity of the quantities is preserved. Those costs parameters have a great impact on the system behavior as shown in Figure 1(b). Simulation shows that choosing values for the costs' parameters above 0.25 preserves positivity of quantities but does not guarantee stability of NE point. Figures 2(a) and 2(b) show the influence of c_2 and c_3 on the behavior of system (10). Now, we investigate more the influence of the parameter k on the stability of Nash point. It is confirmed by simulation that when increasing k above 7.11, different types of period cycles are obtained. For example, when $k=8.85$ and the other parameter values are fixed, a birth of stable period 2-cycle is emerged. The time series for this cycle is given in Figure 3(a); besides that, we give the phase portrait of it in Figure 3(b). This means that the system (10) jumps to these two cycles and around the stable Nash point during the period of competition. Fixing the quantity produced by the third firm to 0.13, the size and shape of the period 2-cycle basin of attraction are depicted in Figure 4. The red color refers to the basin of attraction of Nash point while the blue one denotes the basin of attraction of the period 2-cycle.

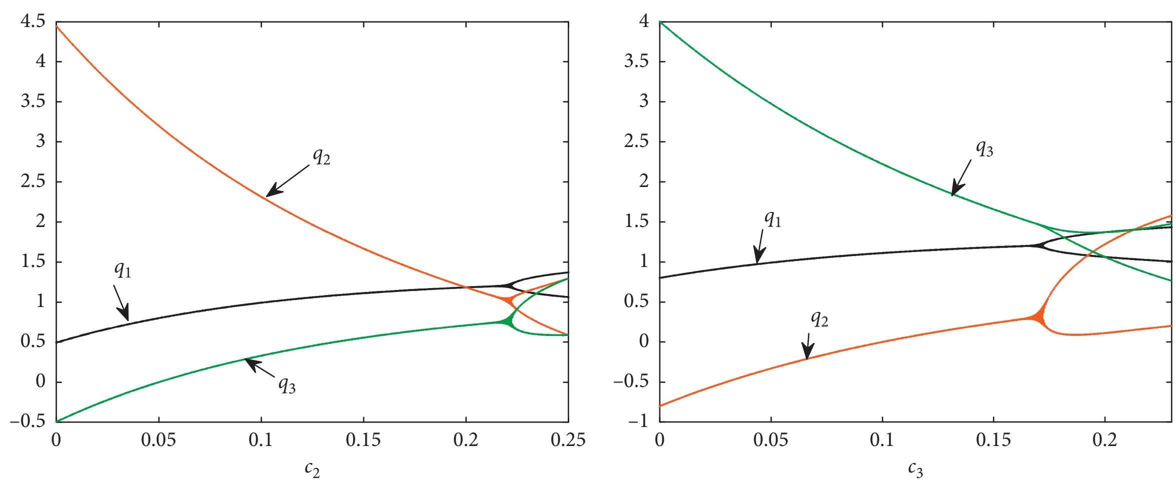
Increasing the parameter k slightly to the value of 9, we get a stable period 4-cycle. However, those cycles are stable but one can observe that there are negative quantities which have no economic meaning. Figure 5(a) shows the phase portrait of this cycle. Figure 5(b) shows the basin of attraction of this cycle. The basin of attraction given in Figure 5(b) seems more complicated than that of the period 2-cycle. The gray color is for the Nash point which in this case includes negative values while the other colors are for the basin of attraction of the period 4-cycle. Furthermore, we increase the parameter k to the value of 9.33, and hence, we get a stable period 8-cycle. Figures 6(a) and 6(b) show the phase portrait and the basin of attraction of this cycle, respectively. This figure contains fractal structure with different colors that are embedded with the colors of the basin of attraction of period 8-cycle. The white color is for nonconvergent points.

Now, we end this section by studying the symmetric case. This case is obtained when we set $c_1=c_2=c_3=c$. It makes Nash equilibrium point become $\mathrm{NE}=(2/9c, 2/9c, 2/9c)$. One can easily prove that this point is locally stable if $k<(4/9c^2)$. Numerical simulation shows that NE point is locally stable for values of k less than 11.22. Therefore, an increase in k more than that value makes the Nash point unstable via period-doubling bifurcation which is given in Figure 7(a) (the corresponding maximum Lyapunov exponent is plotted in Figure 7(b)). Moreover, the coexistence of period 2-cycle is detected at those parameters and for value of $k=12$. Figures 8(a) and 8(b) show this period with its time series. This makes us to investigate more to see whether there are more cycles.



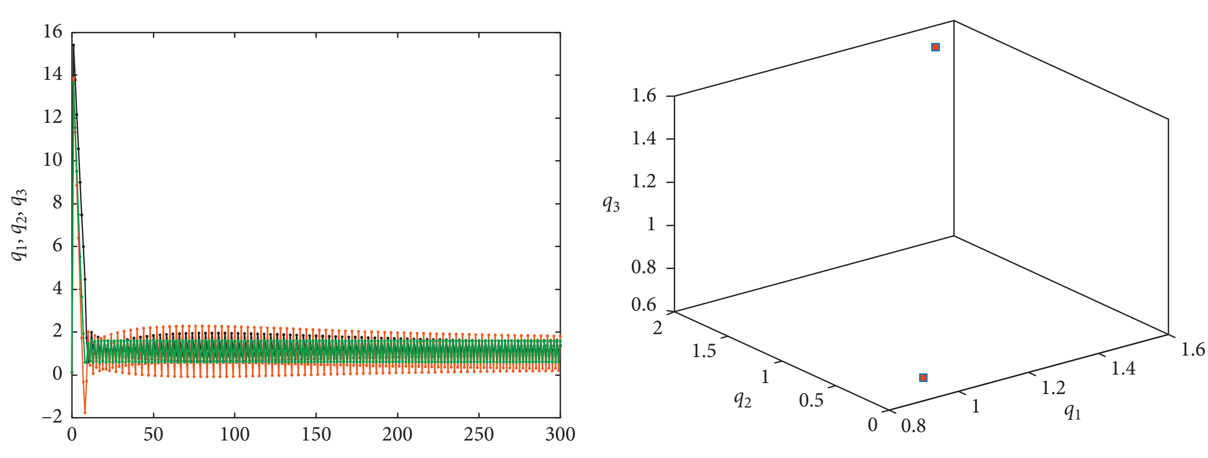
(a) (b)

Figure 1: (a) Bifurcation diagram of system (10) with respect to k at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), c_1 = 0.2, c_2 = 0.3, c_3 = 0.25$. (b) Bifurcation diagram of system (10) with respect to c_1 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 8.85, c_2 = 0.3, c_3 = 0.25$.



(a) (b)

Figure 2: (a) Bifurcation diagram of system (10) with respect to c_2 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 8.85, c_1 = 0.2, c_3 = 0.25$. (b) Bifurcation diagram of system (10) with respect to c_3 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 8.85, c_1 = 0.2, c_2 = 0.3$.



(a) (b)

Figure 3: (a) Time series of system (10) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 8.85, c_1 = 0.2, c_2 = 0.3, c_3 = 0.25$. (b) Phase space of the period 2-cycle of system (10) with respect to c_3 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 8.85, c_1 = 0.2, c_2 = 0.3$.

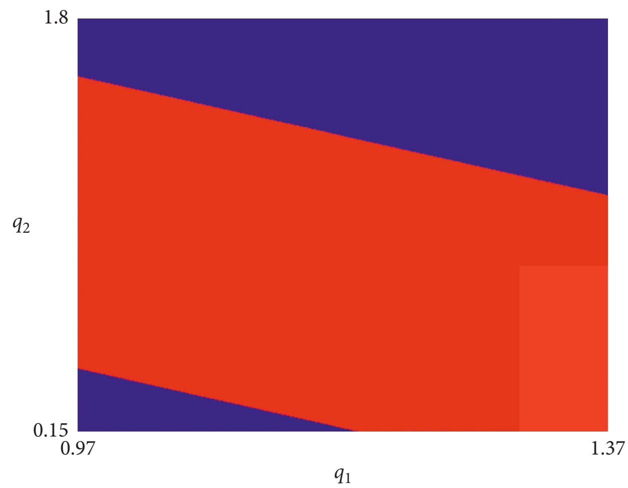
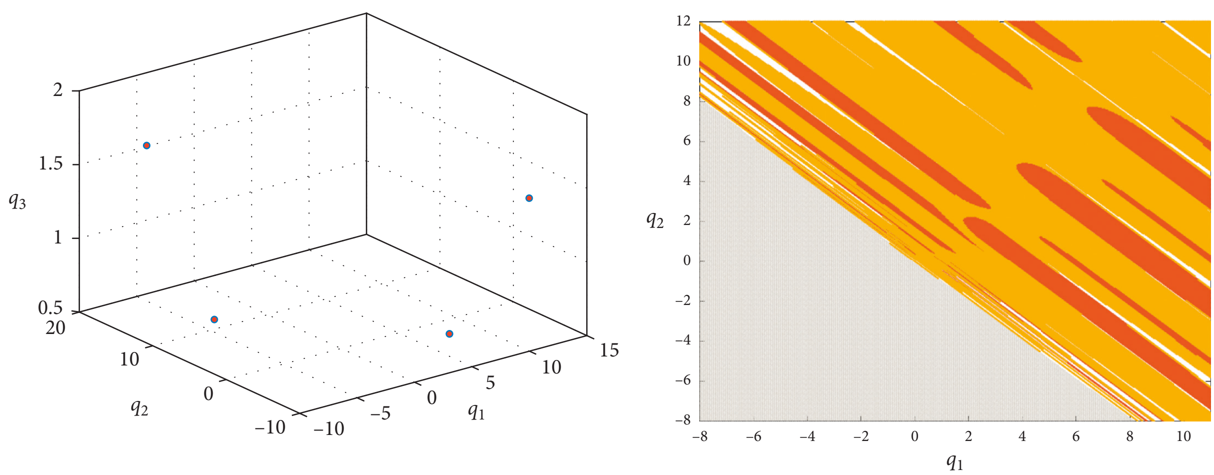


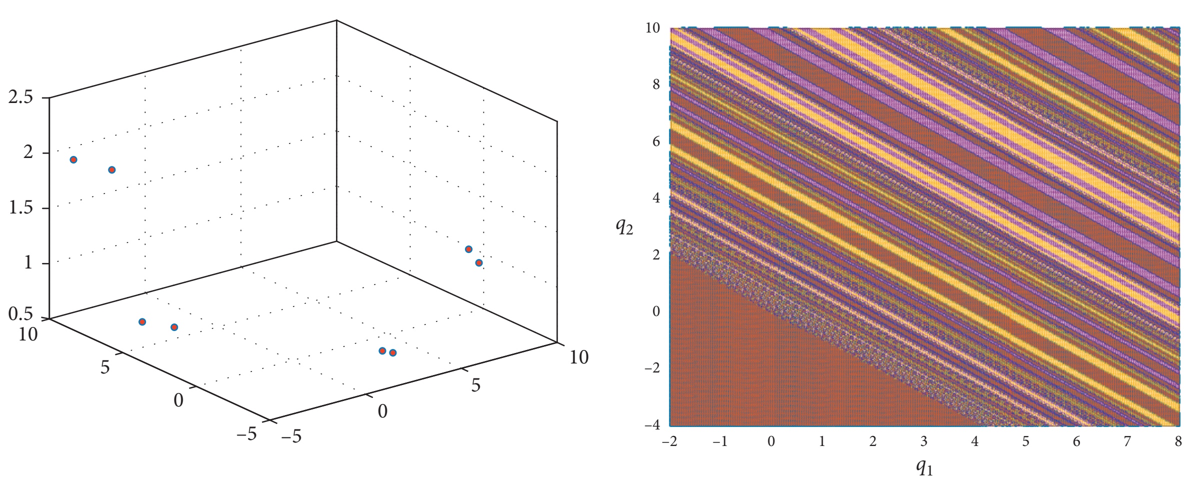
FIGURE 4: Basin of attraction of the period 2-cycle at $q_{0,3} = 0.13, k = 8.85, c_1 = 0.2, c_2 = 0.3, c_3 = 0.25$.



(a)

(b)

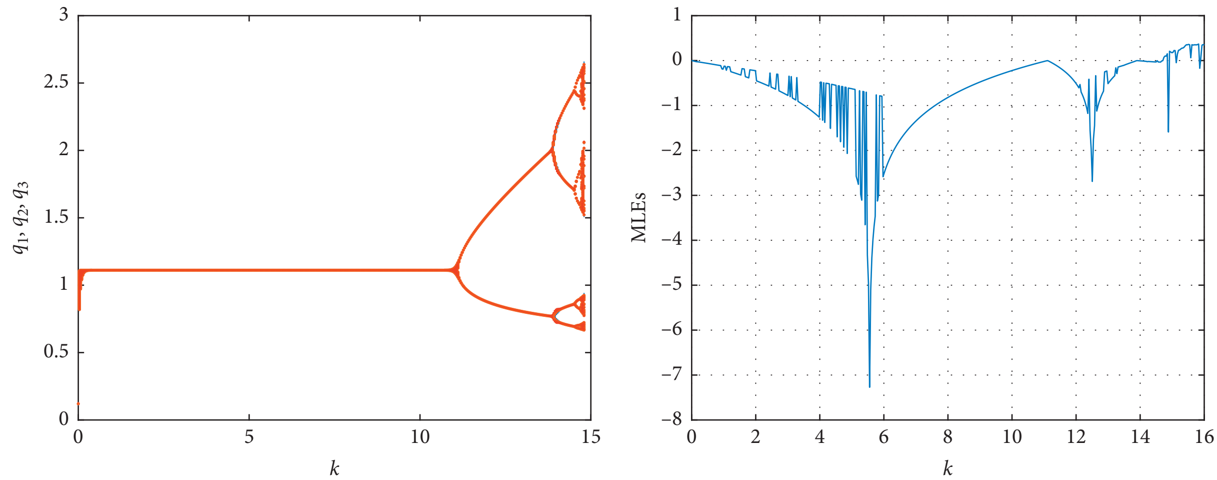
FIGURE 5: (a) Phase space of the period 4-cycle of system (10) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 9, c_1 = 0.2, c_2 = 0.3, c_3 = 0.25$. (b) Basin of attraction of the period 4-cycle at $q_{0,3} = 0.13, k = 9, c_1 = 0.2, c_2 = 0.3, c_3 = 0.25$.



(a)

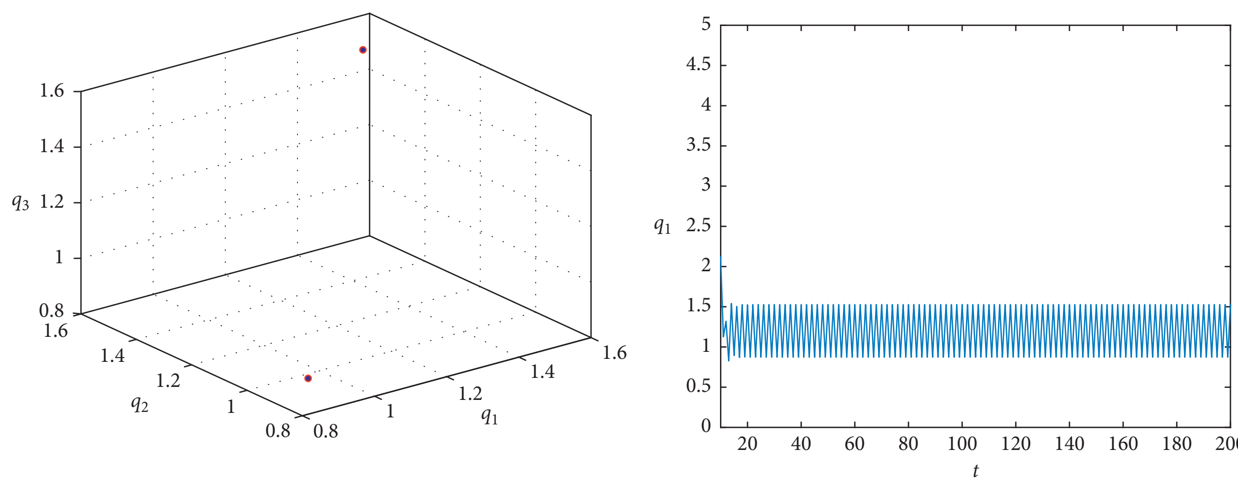
(b)

FIGURE 6: (a) Phase space of the period 8-cycle of system (10) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 9.33, c_1 = 0.2, c_2 = 0.3, c_3 = 0.25$. (b) Basin of attraction of the period 8-cycle at $q_{0,3} = 0.13, k = 9.33, c_1 = 0.2, c_2 = 0.3, c_3 = 0.25$.



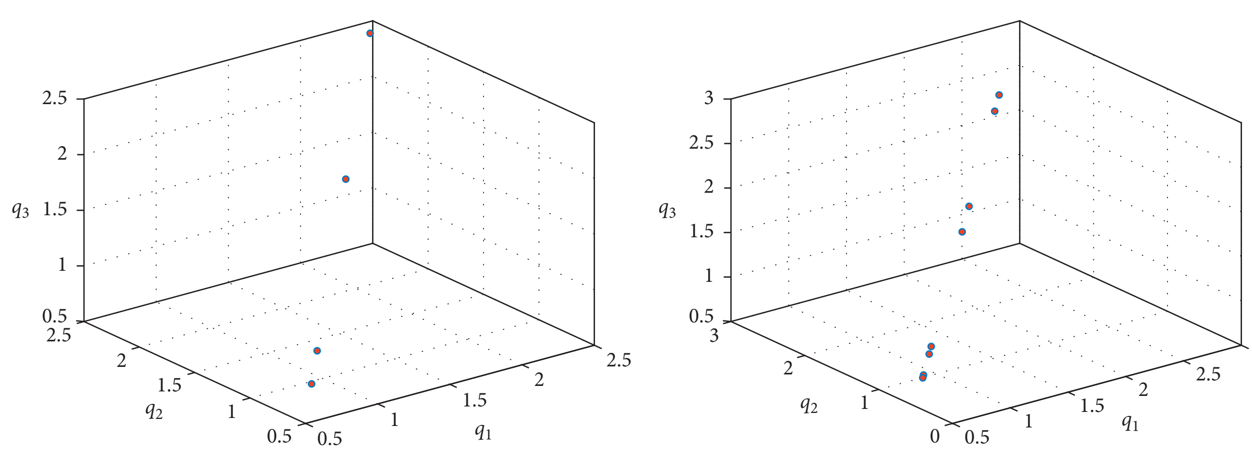
(a) (b)

FIGURE 7: (a) Bifurcation diagram of system (10) with respect to k at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), c = 0.2$. (b) Lyapunov exponents with respect to k at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), c = 0.2$.



(a) (b)

FIGURE 8: (a) Phase space of the period 2-cycle of system (10) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 12, c = 0.25$. (b) Time series for q_1 at $k = 12, c = 0.2$.



(a) (b)

FIGURE 9: (a) Phase space of the period 4-cycle of system (10) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 14.4, c = 0.25$. (b) Phase space of the period 8-cycle of system (10) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 14.6, c = 0.2$.

Figures 9(a) and 9(b) present the phase portrait of period 4-cyle and period 8-cyle at $k = 14.4$ and $k = 14.6$ respectively. We give the basin of attraction of period 8-cyle that is described by two colors in Figure 10. We conclude based on these obtained results that this case is better than the previous case and the region of stability of Nash point is bigger than that of the asymmetric case and positivity of quantities are guarantied.

2.3. Heterogeneous Effect. The competition described here includes heterogeneous competitors. We assume that two firms behave rational while the other adopts the LMA mechanism. In [10], the definition of the LMA mechanism has been introduced. It requires no global information about the demand function, yet it needs the players knowing market price p_t and the corresponding produced quantity Q_t. This can be called local information of the price function for (p_t, Q_t) only. Through some experiences on the market, the player may be able to estimate the price function for market values within a neighborhood of (p_t, Q_t) and then compute properly the following derivative:

$$\partial_{q_1} f\left(q_{1,t}, q_{2,t}\right) = f'\left(Q_t\right). \tag{17}$$

Scenario 1. We assume that the first oligopolist is adopting the LMA mechanism. It has been discussed that such mechanism can be calculated by the effect of small quantity and price variations which make the firm (the player) evaluates the price function at each time for the total supply Q_t. Therefore, equation (17) with the price of the first player $p_{1,t}$ gives the following price function:

$$p^e_{1,t+1} = p_{1,t} + p'\left(Q_t\right)\left[Q^e_{t+1} - Q_t\right], \tag{18}$$

where $Q^e_{t+1} = q_{1,t+1} + q^e_{2,t+1} + \cdots + q^e_{3,t+1}$ and $q^e_{2,t+1}, q^e_{3,t+1}$ are the expected outputs, i.e., the outputs which the first oligopolist expects from its opponents at time $t+1$. Now, we consider static expectations for the second and third oligopolists ($q^e_{2,t+1} = q_{2,t}, q^e_{3,t+1} = q_{3,t}$); then (18) takes the form

$$p^e_{1,t+1} = \frac{1}{\left(q_1 + q_2 + q_3\right)} - \frac{1}{\left(q_1 + q_2 + q_3\right)^2}\left[q_{1,t+1} - q_{1,t}\right], \tag{19}$$

where $p'\left(Q_t\right) = 1/Q^2$. Now, the first oligopolist chooses its next period strategy according to the following:

$$q_{1,t+1} = \arg\max_{q_{2,t+1}}\left[p^e_{1,t+1} q_{1,t+1} - c_1 q_{1,t+1}\right], \tag{20}$$

namely,

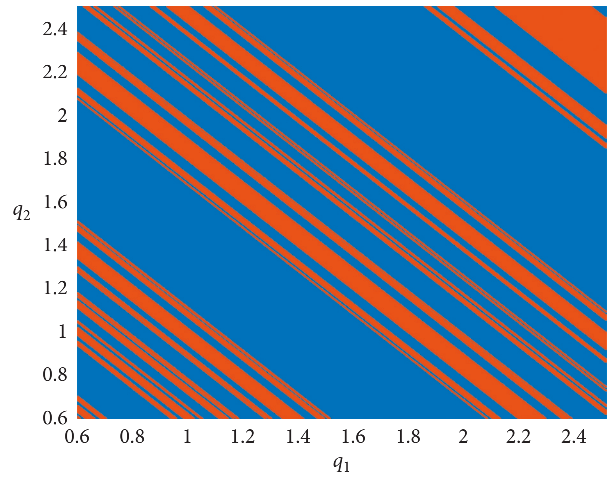


FIGURE 10: Basin of attraction of the period 8-cycle at $q_{0,3} = 0.13, k = 14.6, c = 0.2$.

$$\begin{aligned} q_{1,t+1} &= \frac{1}{2} q_{1,t} + \frac{Q}{2}\left(1 - c_1 Q\right), \\ Q &= q_1 + q_2 + q_3. \end{aligned} \tag{21}$$

Then the resulting oligopolistic game is now described by

$$\begin{aligned} q_{1,t+1} &= \frac{1}{2} q_{1,t} + \frac{Q}{2}\left[1 - c_1 Q\right], \\ q_{2,t+1} &= q_{2,t} + k\left[\frac{Q - q_2}{Q^2} - c_2\right], \\ q_{3,t+1} &= q_{3,t} + k\left[\frac{Q - q_3}{Q^2} - c_3\right]. \end{aligned} \tag{22}$$

The above system is a nonlinear discrete dynamic system which consists of one LMA player against two rational competitors.

2.3.1. Local Analysis. As previously done, we calculate here the steady state of system (22) and study its stability and the corresponding complex characteristics. Simple calculations yield the NE described in (7) as the steady state of system (22). Even though the heterogeneousness is carried out by the first oligopolist, the steady state of system (22) is the same steady state of system (10) where all the oligopolists adopt the bounded rationality mechanism. The following proposition is given.

Proposition 2. The Nash equilibrium (7) is a steady state of system (22) and it is locally asymptotically stable if the following conditions are satisfied.

$$
\begin{aligned}
&\frac{1}{16\bar{c}}\left[(3\bar{c}-8c_1)\bar{c}^4k^2+8(5\bar{c}-11c_1)\bar{c}^2k+16(c_1+3c_2+5c_3)\right]>0,\\
&\frac{-1}{64}\left[k^4\bar{c}^6(2c_1-\bar{c})+12k^3\bar{c}^4(2c_1-\bar{c})^2+8k\bar{c}^2(3c_1-3c_2-5c_3)(2c_1-\bar{c})+32((2c_1-\bar{c})(2c_2+c_3)-2c_1c_3)\right]>0,\\
&\frac{-1}{16\bar{c}}\left[k^2\bar{c}^4(8c_1-5\bar{c})+24k\bar{c}^2(\bar{c}-c_1)+16c_3-16c_2-48c_1\right]>0,\\
&\bar{c}=c_1+c_2+c_3.
\end{aligned}
\tag{23}
$$

System (22) at NE point has the following Jacobian matrix:

$$
\begin{bmatrix}
\dfrac{c_2+c_3-c_1}{c_1+c_2+c_3} & \dfrac{c_2+c_3-3c_1}{2(c_1+c_2+c_3)} & \dfrac{c_2+c_3-3c_1}{2(c_1+c_2+c_3)}\\
1-\dfrac{k}{4}(c_1+c_2+c_3)(3c_2-c_1-c_3) & -c_2k(c_1+c_2+c_3) & \dfrac{-k}{4}(c_1+c_2+c_3)(3c_2-c_1-c_3)\\
1-\dfrac{k}{4}(c_1+c_2+c_3)(3c_3-c_1-c_2) & \dfrac{-k}{4}(c_1+c_2+c_3)(3c_3-c_1-c_2) & -c_3k(c_1+c_2+c_3)
\end{bmatrix}.
\tag{24}
$$

For NE to be asymptotically stable, all the roots of the following characteristic equation must have magnitudes of eigenvalues less than one:

$$\lambda^3+\theta_1\lambda+\theta_2\lambda+\theta_3=0, \tag{25}$$

where

$$
\begin{aligned}
\theta_1&=\frac{k(c_2+c_3)(c_1+c_2+c_3)^2+(c_1-c_2+c_3)}{(c_1+c_2+c_3)},\\
\theta_2&=-\frac{\left[k(c_1+c_2+c_3)^2\left(k(c_1+c_2+c_3)^2(c_1-3c_2-3c_3)-12c_1+12c_2+12c_3\right)-48c_1+16c_2+16c_3\right]}{16(c_1+c_2+c_3)},\\
\theta_3&=\frac{k}{8}\left[k(c_1-c_2-c_3)(c_1+c_2+c_3)^3+2(c_1+c_2+c_3)(3c_1-c_2-c_3)\right].
\end{aligned}
\tag{26}
$$

This can be achieved if and only if the following Jury conditions are satisfied:

$$
\begin{aligned}
\ell_1&:=3+\theta_1-\theta_2-3\theta_3>0,\\
\ell_2&:=1-\theta_2+\theta_3(\theta_1-\theta_3)>0,\\
\ell_3&:=1-\theta_1+\theta_2-\theta_3>0.
\end{aligned}
\tag{27}
$$

Substituting (26) in (27) completes the proof. In order to get more insights about the above proposition, we perform some numerical simulations.

2.3.2. Simulation and Global Analysis. This simulation handles the complex characteristics of system (22). It is devoted to investigate the results obtained by Proposition 2 and to see whether NE is stable or not. We set the parameter values to $c_1=0.12, c_2=0.33$ and $c_3=0.25$. Figure 11(a) shows that the Nash point is locally stable whenever varying the parameter k until the system (22) starts bifurcating around Nash point and then high period cycles are formed. Comparing this case with the previous case, we see that the previous case is more stable in terms of the stability region with respect to the parameter k. This means that adopting LMA mechanism does not help the firm to be more stable against its rational competitors. Figure 11(a) shows the stability of Nash point when we increase k. We should highlight here that choosing the costs' parameters should satisfy the

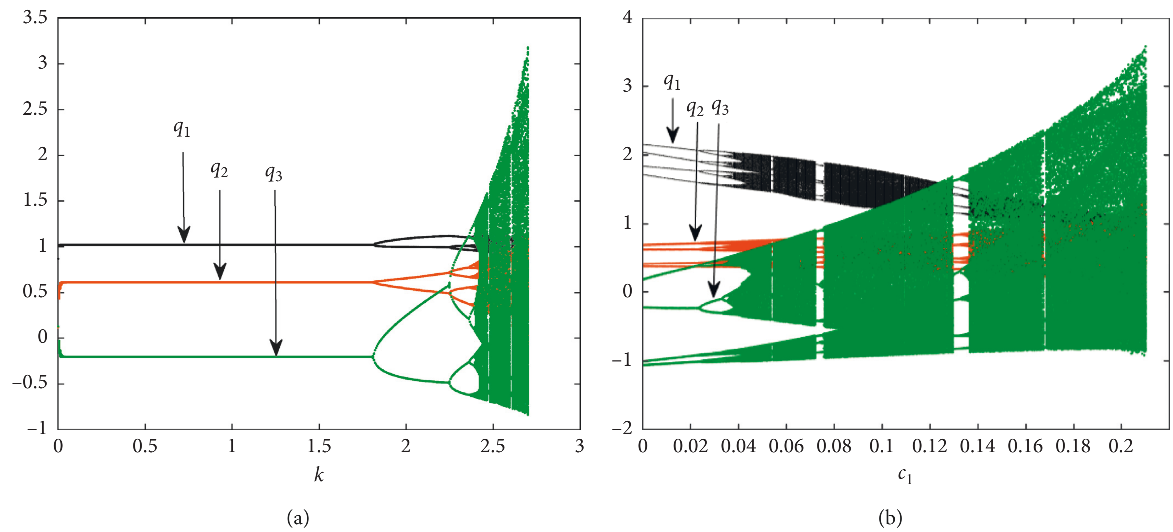


FIGURE 11: (a) Bifurcation diagram of system (22) with respect to k at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), c_1 = 0.2, c_2 = 0.4, c_3 = 0.8$. (b) Bifurcation diagram of system (22) with respect to c_1 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.7, c_2 = 0.4, c_3 = 0.8$.

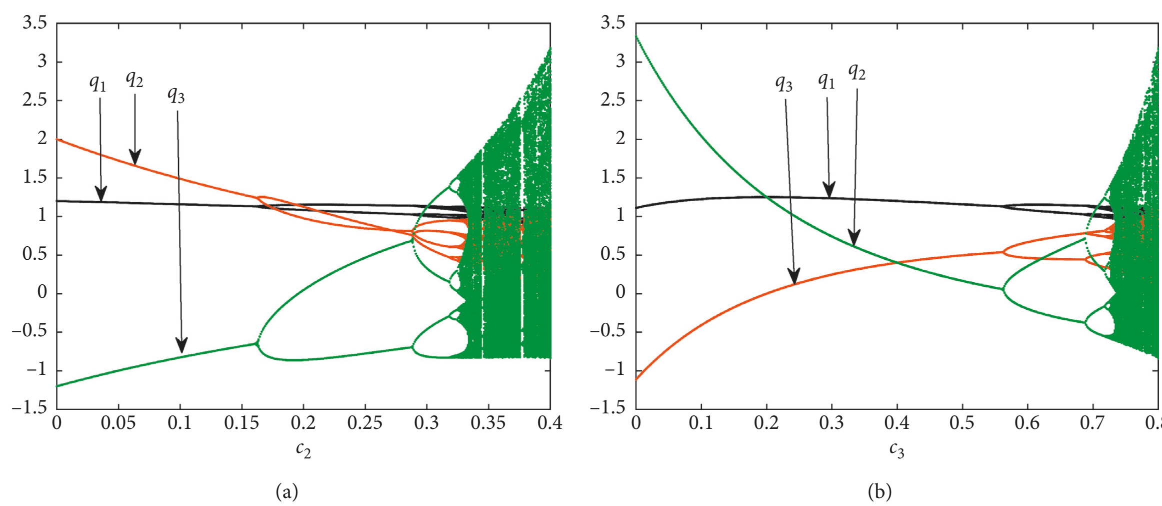
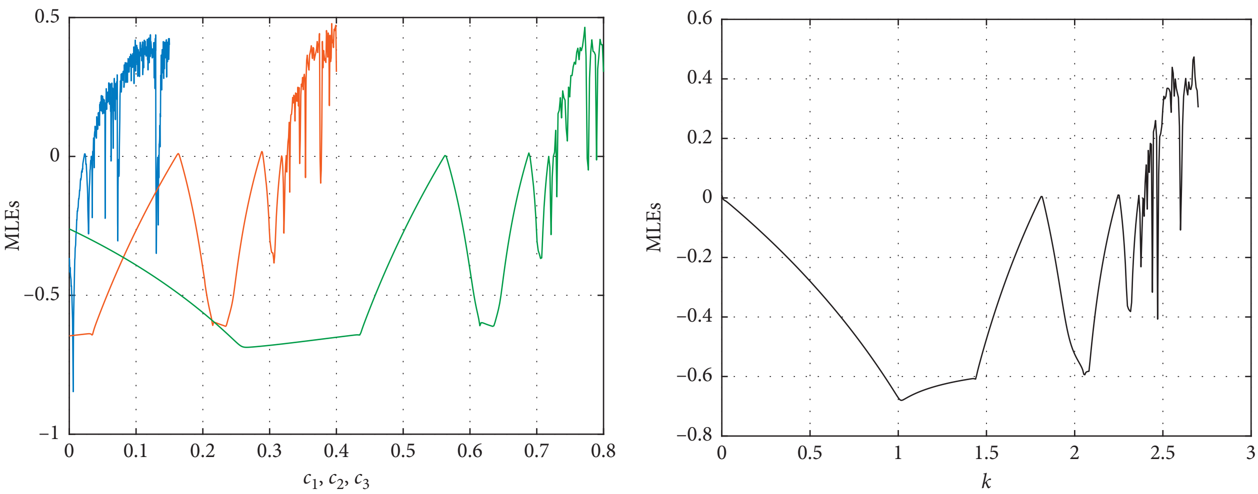


FIGURE 12: (a) Bifurcation diagram of system (22) with respect to c_2 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), c_1 = 0.2, k = 2.7, c_3 = 0.8$. (b) Bifurcation diagram of system (22) with respect to c_3 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.7, c_1 = 0.2, c_2 = 0.4$.

conditions, $c_2 + c_3 > c_1, c_1 + c_3 > c_2$ and $c_1 + c_2 > c_3$. Choosing very small values of cost parameters extends the region of stability of Nash point. Figures 11(b) and 12(a) and 12(b) show the influences of costs on the stability of Nash point. As one can see, only small values of costs should be selected; otherwise, chaotic behavior may arise. The corresponding Lyapunov exponent for those costs and k is given in Figures 13(a) and 13(b). Other interesting chaotic behaviors are given in Figures 14 and 15. Figure 14(a) gives the phase portrait of a chaotic attractor of system (22) at the parameters: $k = 2.7, c_1 = 0.2, c_2 = 0.4, c_3 = 0.8$. The time series of quantities at those parameters is shown in Figure 14(b). Figures 15(a) and 15(b) present different chaotic attractors of the system. Period 2-cycle and period 5-cycle are obtained in Figure 16. The basin of attraction of period 5-cycle is represented by many colors and is given in Figure 17. All these periods are unstable.

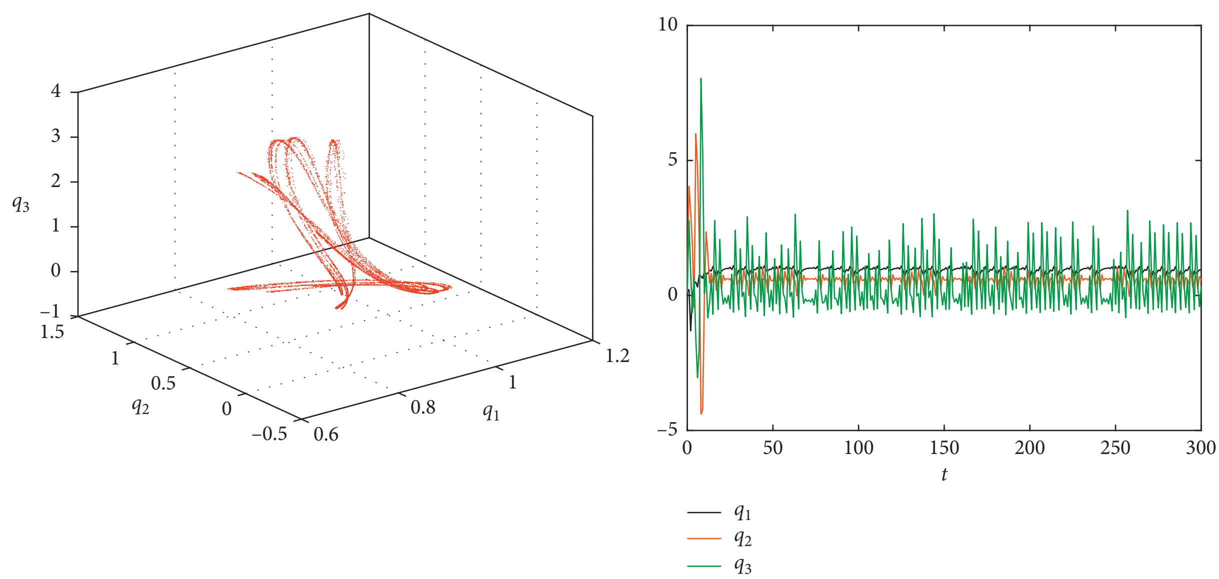
Scenario 2. In this scenario, we assume that two monopolists (firm 1 and firm 2) adopt the LMA mechanism while the other uses bounded rationality. This means that system (22) can be rewritten in the form



(a)

(b)

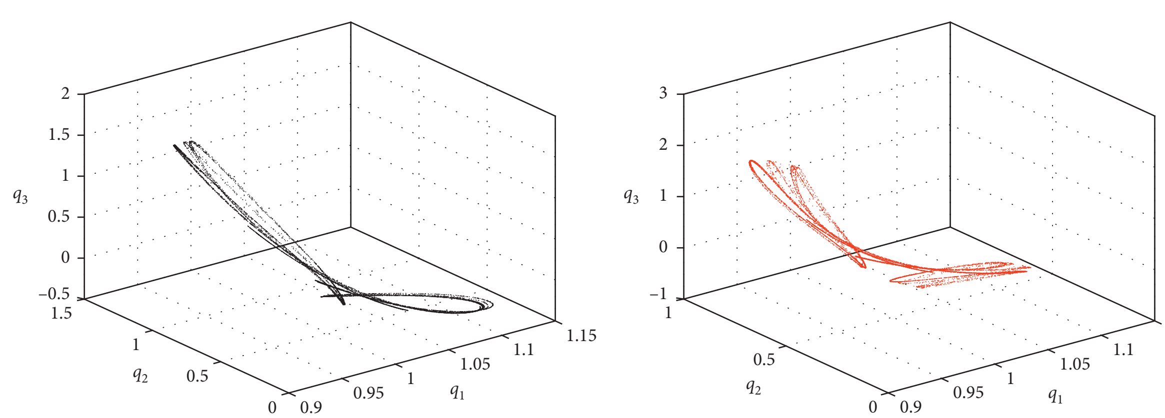
FIGURE 13: (a) Lyapunov exponents with respect to c_1, c_2, c_3 at $k = 2.7$. (b) Lyapunov exponents with respect to $k = 2.7, c_1 = 0.2, c_2 = 0.4, c_3 = 0.8$.



(a)

(b)

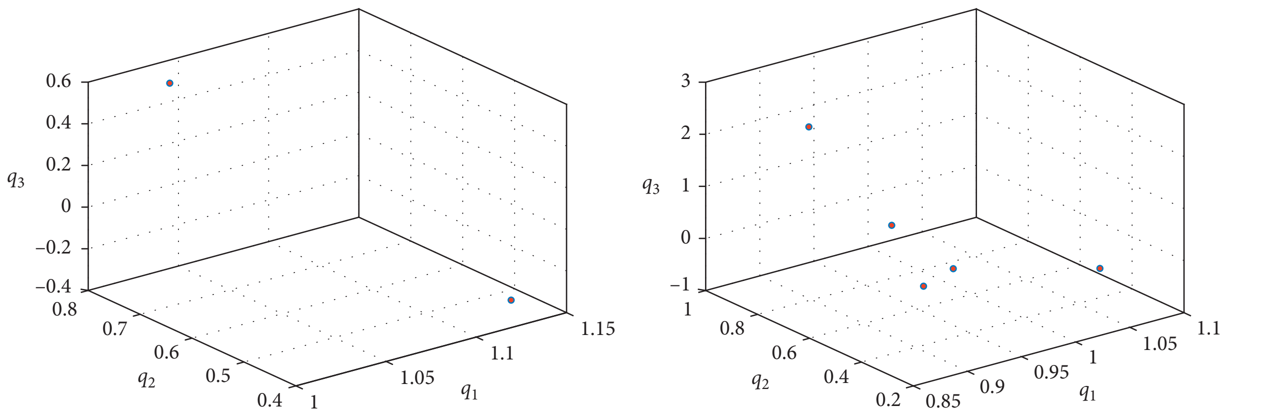
FIGURE 14: (a) Chaotic attractor of system (22) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.7, c_1 = 0.2, c_2 = 0.4, c_3 = 0.8$. (b) Time series for q_1, q_2, q_3 at $k = 2.7, c_1 = 0.2, c_2 = 0.4, c_3 = 0.8$.



(a)

(b)

FIGURE 15: (a) Chaotic attractor of system (22) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.55, c_1 = 0.2, c_2 = 0.5, c_3 = 0.7$. (b) Chaotic attractor of system (22) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.55, c_1 = 0.2, c_2 = 0.4, c_3 = 0.8$.



(a) (b)

Figure 16: (a) Phase space of the period 2-cycle of system (22) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.55, c_1 = 0.2, c_2 = 0.4, c_3 = 0.7$. (b) Phase space of the period 8-cycle of system (22) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 14.6, c = 0.2$.

$$
\begin{aligned}
q_{1,t+1} &= \frac{1}{2} q_{1,t} + \frac{Q}{2}\left[1 - c_1 Q\right], \\
q_{2,t+1} &= \frac{1}{2} q_{2,t} + \frac{Q}{2}\left[1 - c_2 Q\right], \\
q_{3,t+1} &= q_{3,t} + k\left[\frac{Q - q_3}{Q^2} - c_3\right].
\end{aligned}
\tag{28}
$$

Proposition 3. The Nash equilibrium (7) is a steady state of system (32).

Now, the stability of Nash point is as previously obtained, and we get Jacobian matrix as follows:

$$
\begin{bmatrix}
\dfrac{c_2 + c_3 - c_1}{c_1 + c_2 + c_3} & \dfrac{c_2 + c_3 - 3c_1}{2(c_1 + c_2 + c_3)} & \dfrac{c_2 + c_3 - 3c_1}{2(c_1 + c_2 + c_3)} \\
\dfrac{c_1 + c_3 - 3c_2}{2(c_1 + c_2 + c_3)} & \dfrac{c_1 + c_3 - c_2}{c_1 + c_2 + c_3} & \dfrac{c_1 + c_3 - 3c_2}{2(c_1 + c_2 + c_3)} \\
\dfrac{k}{4}(c_1 + c_2 + c_3)(c_1 + c_2 - 3c_3) & \dfrac{k}{4}(c_1 + c_2 + c_3)(c_1 + c_2 - 3c_3) & 1 - c_3 k (c_1 + c_2 + c_3)
\end{bmatrix},
\tag{29}
$$

whose characteristic equation is given by

$$
\lambda^3 + \Lambda_1 \lambda + \Lambda_2 \lambda + \Lambda_3 = 0, \tag{30}
$$

where

$$
\begin{aligned}
\Lambda_1 &= \frac{c_3 k (c_1 + c_2 + c_3)^2 - (c_1 + c_2 + 3c_3)}{c_1 + c_2 + c_3}, \\
\Lambda_2 &= \frac{k(c_1 + c_2 - 5c_3)(c_1 + c_2 + c_3)^2 - (c_1 + c_2 - 11c_3)}{4(c_1 + c_2 + c_3)}, \\
\Lambda_3 &= -\frac{k(c_1 + c_2 - 3c_3)(c_1 + c_2 + c_3)^2 - 2(c_1 + c_2 - 3c_3)}{8(c_1 + c_2 + c_3)}.
\end{aligned}
\tag{31}
$$

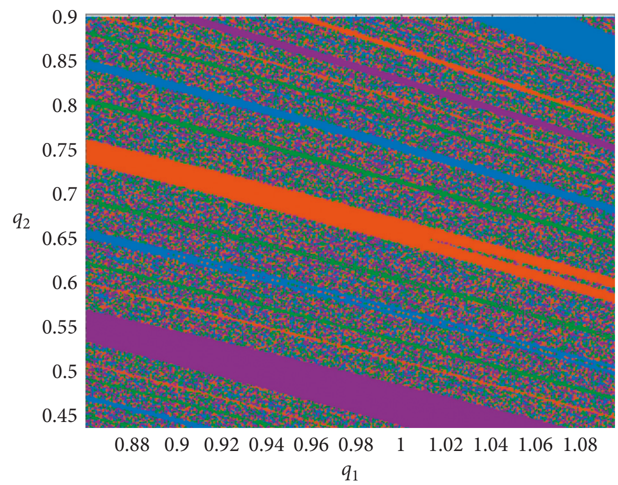


Figure 17: Basin of attraction of the period 5-cycle at $q_{0,3} = 0.13, k = 2.6, c_1 = 0.2, c_2 = 0.4, c_3 = 0.8$.

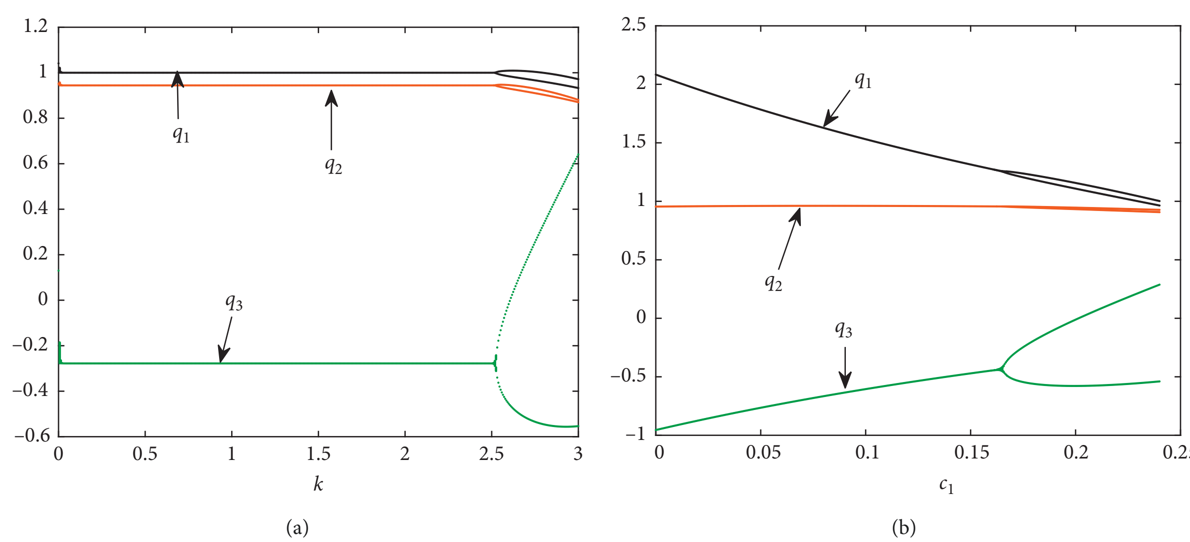


Figure 18: (a) Bifurcation diagram of system (32) with respect to k at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), c_1 = 0.24, c_2 = 0.26, c_3 = 0.785$. (b) Bifurcation diagram of system (32) with respect to c_1 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.77, c_2 = 0.26, c_3 = 0.785$.

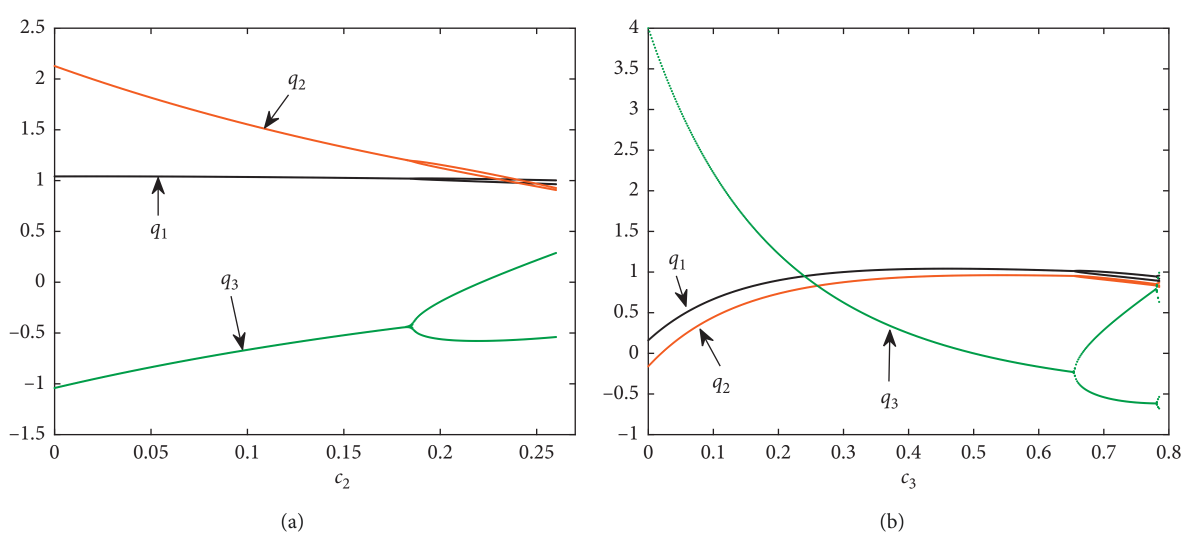


Figure 19: (a) Bifurcation diagram of system (32) with respect to c_2 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), c_1 = 0.24, k = 2.77, c_3 = 0.785$. (b) Bifurcation diagram of system (32) with respect to c_3 at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.77, c_1 = 0.24, c_2 = 0.26$.

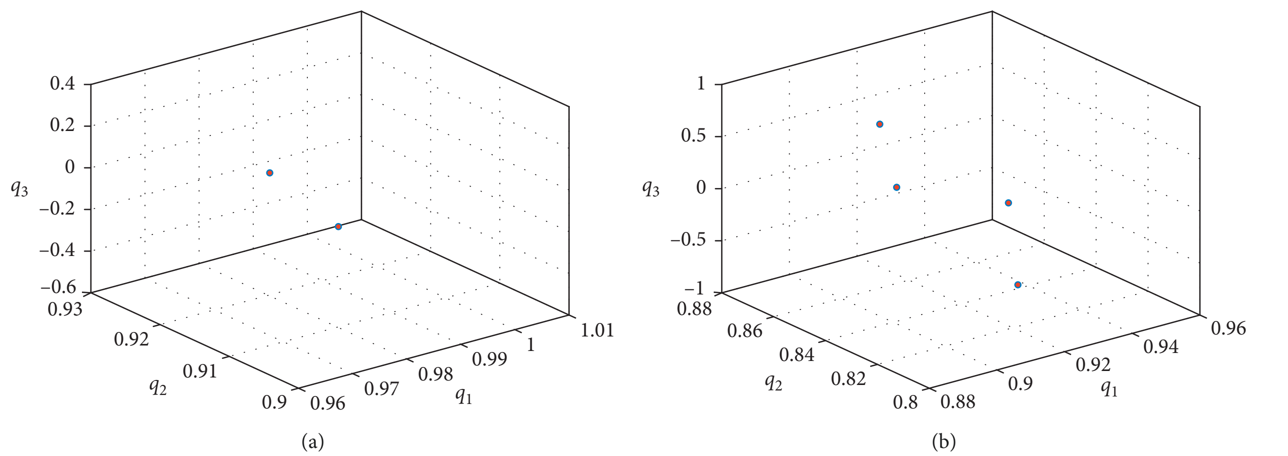


FIGURE 20: (a) Phase space of the period 2-cycle of system (32) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.77, c_1 = 0.24, c_2 = 0.26, c_3 = 0.785$. (b) Phase space of the period 4-cycle of system (32) at $(q_{0,1}, q_{0,2}, q_{0,3}) = (0.11, 0.12, 0.13), k = 2.77, c_1 = 0.24, c_2 = 0.26, c_3 = 0.785$.

Using Jury conditions and (31) we get

$$
\begin{aligned}
\ell_1 &= \frac{k(c_1 + c_2 + 9c_3)(c_1 + c_2 + c_3)^2 + 4(3c_1 + 3c_2 - c_3)}{c_1 + c_2 + c_3},\\
\ell_2 &= -\frac{\left[k(c_1 + c_2 + c_3)^2\left(k(c_1 + c_2 + 5c_3)(c_1 + c_2 - 3c_3)(c_1 + c_2 + c_3)^2 + 4(c_1 + c_2)^2 - 4c_3(14c_1 + 14c_2 - c_3)\right) - 4(5c_1 + 5c_2 + c_3)(3c_1 + 3c_2 - c_3)\right]}{64(c_1 + c_2 + c_3)^2},\\
\ell_3 &= \frac{3\left[k(c_1 + c_2 - 7c_3)(c_1 + c_2 + c_3)^2 + 4(c_1 + c_2 + 5c_3)\right]}{8(c_1 + c_2 + c_3)}.
\end{aligned}
\tag{32}
$$

Therefore, the Nash point system (32) is asymptotically stable if $\ell_i > 0, i = 1, 2, 3$. The eigenvalues and Jury conditions take complicated forms, and then some simulations are carried out to investigate the conditions (32). We observe that when $c_i = c, i = 1, 2, 3$, both ℓ_1 and ℓ_2 are positive while $\ell_3 > 0$ provided that $kc^2 < 0.62$, and hence, Nash point is asymptotically stable. On the other hand, when we take different values of costs, i.e., $c_1 = 0.24, c_2 = 0.26$, and $c_3 = 0.785$, bifurcated behaviors of the system appear and then the local stability of Nash point does not exist. Figure 18(a) shows different bifurcated behaviors with respect to the parameter k. It seems that all firms get unstable due to the bad influences of those cost parameters and the negative quantities that appear which are nonsense in any economic context. Figures 18(b) and 19 give the influence of costs on the stability of Nash point. As one can see, those costs affect the system's behavior even if they have taken small values. This is also clear from Figures 20(a) and 20(b) where the period 2-cycle and period 4-cycle appear.

3. Conclusion

The current paper has investigated an oligopolistic game that consists of three competitors. Different complicated dynamic routes have been raised due to the adoption of two different adjustment mechanisms, bounded rationality and the LMA mechanism. The demand function used to build this game and its corresponding dynamical systems has been derived from Cobb–Douglas production function. The obtained results have shown that the stability of Nash equilibrium loses its stability due to the appearance of bifurcated behaviors of those discussed systems in the manuscript. We have concluded from the obtained results that bounded rationality mechanism when it is adopted by firms has given better stability for the Nash point in comparison with the results of stability given by the LMA mechanism. Our obtained results extend results existed in literature. Furthermore, we have detected several fractal structures which require more analysis and investigations that will be addressed in future research works. The limitation of the current work lies in the application of only two types of mechanisms; the bounded rationality and LMA mechanism. Other types of mechanisms should be considered in comparison, and this might include expected cooperation among firms.

Data Availability

The availability of data is carried out by the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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