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Research Article

The Extremal Permanental Sum for a Quasi-Tree Graph

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Let G be a graph and $A(G)$ the adjacency matrix of G. The permanent of matrix $(xI - A(G))$ is called the permanental polynomial of G. The permanental sum of G is the sum of the absolute values of the coefficients of permanental polynomial of G. Computing the permanental sum is #p-complete. In this note, we prove the maximum value and the minimum value of permanental sum of quasi-tree graphs. And the corresponding extremal graphs are also determined. Furthermore,we also determine the graphs with the minimum permanental sum among quasi-tree graphs of order n and size m, where $n - 1 \le m \le 2n - 3$.

1. Introduction

The *permanent* of $n \times n$ matrix $M = (b_{ij})$ $(i, j = 1, 2, \dots, n)$ is defined as

$$\operatorname{per}(M) = \sum_{\sigma} \prod_{i=1}^{n} b_{i\sigma(i)}, \quad (1)$$

where the sum is taken over all permutations σ of $\{1, 2, \dots, n\}$.

Let G be a graph with n vertices and let $A(G)$ be its adjacency matrix. The *permanental polynomial* of G is defined as

$$\pi(G, x) = \operatorname{per}(xI - A(G)) = \sum_{k=0}^{n} b_k(G)\, x^{n-k}, \quad (2)$$

where I is the unit matrix of order n. Basic theory of permanental polynomials is well studied recently in [1–3] and the references therein. Kasum et al.[4] and Merris et al. [5] gave the coefficients of the permanental polynomial of G, i.e.,

$$b_k(G) = (-1)^k \sum_{H} 2^{c(H)}, \quad 0 \le k \le n, \quad (3)$$

where the sum is taken over all Sachs subgraphs H of G on k vertices and $c(H)$ is the number of cycles in H. Recall that a *Sachs subgraph* is a graph in which each component is a single edge or a cycle.

The *permanental sum* of graph G, denoted by $PS(G)$, can be defined as the summation of all absolute values of coefficients of permanental polynomial of G, i.e.,

$$PS(G) = \sum_{i=0}^{n} |b_i(G)| = \sum_{i=0}^{n} \sum_{H} 2^{c(H)}. \quad (4)$$

Thus, $PS(G) = 1$ if G is an empty graph. Wu and So [6] have shown that computing permanental sum of a graph is #P-complete.

The permanental sum of a graph was first considered by Tong [7]. In [8], Xie et al. captured a labile fullerene $C_{50}(D_{5h})$. Tong computed all 271 fullerenes in C_{50}. In his study, Tong found that the permanental sum of $C_{50}(D_{5h})$ achieves the minimum among all 271 fullerenes in C_{50}. He pointed that the permanental sum would be closely related to stability of molecular graphs. Recently, the permanental sum of a graph has received much attention. Li et al. [9] determined the extremal hexagonal chains with respect to permanental sum. Li and Wei [10] proved the lower and upper bounds for the permanental sum of an octagonal chain. Wu and Lai [11] systematically introduced the properties of permanental sum of a graph.

A connected graph G is called a *quasi-tree* graph, if there exists a vertex u^* in G such that $G - u^*$ is a tree. Let G be a quasi-tree graph with n vertices and m edges. Then $n-1 \leq m \leq 2n-3$, and the degree of u^* in G equals $m-n+2$. Denote $\mathcal{G}_n = \{G : G$ is a quasi-tree graph of order $n\}$, and $\mathcal{G}_{n,m} = \{G : G$ is a quasi-tree graph of order n and size $m\}$. As an important class of graphs, quasi-tree graphs have been widely studied. For the background and some known results about quasi-tree graphs, we refer the reader to [12–15].

The purpose of this note is to investigate the properties of permanental sum of quasi-tree graphs. The note is organized as follows. In the next section, we review some previous results that will be needed in the sequel. In Section 3, we discuss the permanental sum of quasi-tree graphs.

2. Some Preliminary

In this note, we only consider finite, undirected, and simple graph. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v. The graph that arises from G by deleting a vertex $u \in V(G)$ or an edge $uv \in E(G)$ will be denoted by $G-u$ or $G-uv$. Let $G+H$ denote the union of two vertex disjoint graphs G and H. For any positive integer l, lG denotes the union of l disjoint copies of G. The path, cycle, and star of order n are denoted by P_n, C_n and $K_{1,n-1}$, respectively.

Two edges of G are said to be *independent* if they are not adjacent in G. A k*-matching* of G is a set of k mutually independent edges. For an integer $k \geq 0$, let $m(G,k)$ denote the number of k-matchings of a graph G. The *Hosoya index* $Z(G)$ of a graph G is defined to be the total number of matchings of G, that is,

$$Z(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} m(G,k), \tag{5}$$

where n is the number of the vertices of the graph G. Some results on Hosoya indices were studied in [3, 16–18].

For $n \geq 2$, let $F(n) = F(n-1) + F(n-2)$ denote the sequence of Fibonacci numbers, in particular, $F(0) = 0$ and $F(1) = 1$.

Lemma 1 (see [10]). *Let $G = T_1 \cup T_2 \cup \ldots \cup T_t$ be a forest with order $n \geq 2$ and $t \geq 2$, where T_i is a tree with n_i vertices, $i = 1, 2, \ldots, t$. Then $Z(G) \leq \prod_{i=1}^{t} F(n_i + 1)$ with equality if and only if $T_i \cong P_{n_i}$. Moreover $Z(G) \leq F(m_1+1)F(m_2+1)$, where $m_1 + m_2 = n$ with equality if and only if $G \cong P_{m_1} \cup P_{m_2}$.*

Let E_k be the empty graph of order k. Denote $\vee$ the graph joint of two graphs, and $\cup$ the disjoint union of two graphs. The graphs $F_n^m = (K_{1,m-n+1} \cup E_{2n-m+3}) \vee E_1$ and $H_n^{n+2} = (C_3 \cup E_{n-4}) \vee E_1$ are shown in Figure 1[19].

Lemma 2 (see [19]). *Let $\mathcal{B}_{n,m}$ be the set consisting of all graphs of order n and size m. For $G \in \mathcal{B}_{n,m}$ with $n-1 \leq m \leq 2n-3$,*

$$Z(F_n^m) \leq Z(G). \tag{6}$$

Equality holds if and only if $G = F_n^m$, or H_n^{n+2} when $m = n+2$, where Graphs F_n^m and H_n^{n+2} see Figure 1.

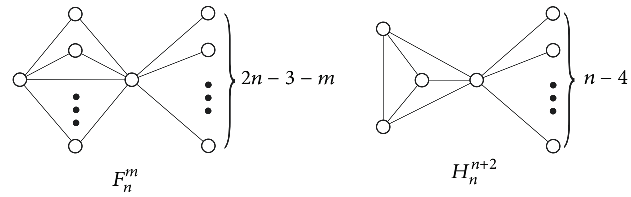

Figure 1: Graphs F_n^m and H_n^{n+2} in Lemma 2.

By the definitions of $\mathcal{B}_{n,m}$ and $\mathcal{G}_{n,m}$, we obtain that $\mathcal{G}_{n,m} \subset \mathcal{B}_{n,m}$. By Lemma 2, we have the following.

Corollary 3. *Let $G \in \mathcal{G}_{n,m}$ be a quasi-tree graph with $n-1 \leq m \leq 2n-3$. Then*

$$Z(F_n^m) \leq Z(G), \tag{7}$$

where the equality holds if and only if $G = F_n^m$.

Lemma 4 (see [11]). *Letting T be a tree with order $n \geq 1$, then $n \leq PS(T) \leq F(n+1)$, the first equality holds if and only if $T \cong K_{1,n-1}$, and the second equality holds if and only if $T \cong P_n$.*

Lemma 5 (see [11]). *Let P_n be a path with n vertices. Then*

$$PS(P_n) = \begin{cases} 1 & if\ n = 0, \\ 1 & if\ n = 1, \\ F(n+1) & if\ n \geq 2. \end{cases} \tag{8}$$

Lemma 6 (see [11]). *The permanental sum of a graph satisfies the following identities:*

(i) *Let G and H be two connected graphs. Then*

$$PS(G \cup H) = PS(G)\,PS(H). \tag{9}$$

(ii) *Let $e = uv$ be an edge of a graph G and $\mathcal{C}(e)$ the set of cycles containing e. Then*

$$PS(G) = PS(G-e) + PS(G-v-u) + 2\sum_{C_k \in \mathcal{C}(e)} PS(G - V(C_k)). \tag{10}$$

(iii) *Let v be a vertex of a graph G and $\mathcal{C}(v)$ the set of cycles containing v. Then*

$$PS(G) = PS(G-v) + \sum_{u \in N_G(v)} PS(G-v-u) + 2\sum_{C_k \in \mathcal{C}(v)} PS(G - V(C_k)). \tag{11}$$

By Lemma 6, we have the following.

Corollary 7. *Let G be a graph and e an edge of G. Then $PS(G-e) < PS(G)$.*

3. Main Results

In this section, we will investigate the properties of permanental sum of a quasi-tree graph.

Theorem 8. *Let $G \in \mathcal{G}_{n,m}$ be a quasi-tree graph with $n-1 \le m \le 2n-3$. Then*

$$m^2 - mn + m + n \le PS(G), \quad (12)$$

where the equality holds if and only if $G \cong F_n^m$.

Proof. By the definition of permanental sum of a graph, it can be known that $PS(G) = Z(G) + 2w(G)$, where $w(G)$ denotes the number of all Sachs graphs containing cycles of G. Checking G, we know that G has exactly $\binom{d_G(u^*)}{2}$ cycles and $d_G(u^*) = m - n + 2$. Thus $\binom{d_G(u^*)}{2} = w(F_n^m) \le w(G)$. By Corollary 3, we have $PS(F_n^m) \le PS(G)$ with equality if and only if $G \cong F_n^m$. By Lemma 6, we obtain that

$$\begin{aligned} PS(F_n^m) &= PS(K_{1,m-n+1}) \\ &\quad + (2n-3-m)\,PS(K_{1,m-n+1}) \\ &\quad + (m-n+1)\,PS(K_{1,m-n}) + 1 \\ &\quad + 2\binom{m-n+2}{2} = m^2 - mn + m + n. \end{aligned} \quad (13)$$

This completes the proof. □

Theorem 9. *Let $G \in \mathcal{G}_n$. Then*

$$\begin{aligned} PS(G) &\le F(n) + \sum_{i=1}^{n-1} F(i)\,F(n-i) \\ &\quad + 2\sum_{r=1}^{n-2}\sum_{j_r=1}^{n-r-1} F(j_r)\,F(n-r-j_r), \end{aligned} \quad (14)$$

where the equality holds if and only if $G \cong G^*$.

Proof. Let $G \in \mathcal{G}_n$, and let $u \in N_G(u^*)$ and $\mathcal{C}_G(u^*) = \{C : C$ is a cycle containing u^* in $G \in \mathcal{G}_n\}$. Suppose that $G \in \mathcal{G}_n$ has the maximum permanental sum. We will characterize the structure of G. By (iii) of Lemma 6, it can be known that if $G \in \mathcal{G}_n$ has the maximum permanental sum, then $PS(G-u^*)$, $\sum_{u^* \in N_G(v)} PS(G-u^*-u)$, and $\sum_{C_k \in \mathcal{C}(u^*)} PS(G-V(C_k))$ must attain maximum value. From the definition of a quasi-tree graph, we know that $G-u^*$ is a tree. By Lemma 4, $G-u^*$ has the maximum permanental sum when $G-u^*$ is isomorphic to path P_{n-1}. Since a path has exactly two vertices of degree 1. Thus there exist exactly two vertices, say u' and u'', such that $G-u^*-u'$ and $G-u^*-u''$ are paths. Set $u \in N_G(u^*)-\{u', u''\}$. By Lemma 1, only when $G-u^*-u$ has two components, each of which is a path, does $\sum_{u \in N_G(u^*)-\{u',u''\}} PS(G-u^*-u)$ attain the maximum value. Similarly, by Lemma 1 and Lemma 6, $\sum_{C_k \in \mathcal{C}(u^*)} PS(G-V(C_k))$ attains the maximum value if and only if $G-V(C_k)$ has two components, each of which is a path, and G has the largest number of cycles in $\mathcal{G}_n$. Combining arguments above and Corollary 7, G must be isomorphic to G^*(see Figure 2). Let the number of 3-cycles, 4-cycles,..., n-cycles in G^* be $j_1, j_2, \ldots, j_r, \ldots, j_{n-3}, j_{n-2}$, respectively. By Lemma 6, we obtain that

$$\begin{aligned} PS(G^*) &= PS(P_{n-1}) + \sum_{i=1}^{n-1} PS(P_{i-1})\,PS(P_{n-1-i}) \\ &\quad + 2\Bigg(\sum_{j_1=1}^{n-2} PS(P_{j_1-1})\,PS(P_{n-2-j_1}) \\ &\quad + \sum_{j_2=1}^{n-3} PS(P_{j_2-1})\,PS(P_{n-3-j_2}) \\ &\quad + \sum_{j_3=1}^{n-4} PS(P_{j_3-1})\,PS(P_{n-4-j_3}) + \cdots \\ &\quad + \sum_{j_{n-3}=1}^{2} PS(P_{j_{n-3}-1})\,PS(P_{n-(n-2)-j_{n-3}}) \\ &\quad + PS(P_{j_{n-2}-1})\,PS(P_{n-(n-1)-j_{n-2}})\Bigg) = PS(P_{n-1}) \\ &\quad + \sum_{i=1}^{n-1} PS(P_{i-1})\,PS(P_{n-1-i}) \\ &\quad + 2\sum_{r=1}^{n-2}\sum_{j_r=1}^{n-r-1} PS(P_{j_r-1})\,PS(P_{n-r-1-j_r}) = F(n) \\ &\quad + \sum_{i=1}^{n-1} F(i)\,F(n-i) + 2\sum_{r=1}^{n-2}\sum_{j_r=1}^{n-r-1} F(j_r)\,F(n-r-j_r). \end{aligned} \quad (15)$$

□

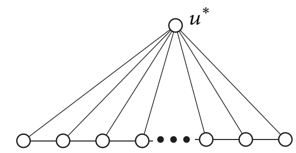
G^*

FIGURE 2: Graph G^*.

By Theorems 8 and 9, we obtain the following result.

Theorem 10. *Let $G \in \mathcal{G}_n$. Then*

$$\begin{aligned} n &\le PS(G) \\ &\le F(n) + \sum_{i=1}^{n-1} F(i)\,F(n-i) \\ &\quad + 2\sum_{r=1}^{n-2}\sum_{j_r=1}^{n-r-1} F(j_r)\,F(n-r-j_r), \end{aligned} \quad (16)$$

The first equality holds if and only if $G \cong K_{1,n-1}$*, and the second equality holds if and only if* $G \cong G^*$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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