Hindawi
Complexity
Volume 2019, Article ID 4510586, 12 pages
https://doi.org/10.1155/2019/4510586

Research Article

Optimal Utilization of Ports' Free-of-Charge Times in One Distribution Center and Multiple Ports Inventory Systems

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Received 1 November 2018; Accepted 22 January 2019; Published 28 February 2019

Academic Editor: Honglei Xu

In this paper, we consider a distribution system consisting of one distribution center (DC), a set of ports, and a set of retailers, in which the product is distributed to the retailers from the DC through the ports by the water transport, and study inventory management for the distribution system with considering the effect of the free storage periods provided by the ports. Inventory management for the distribution system is to determine the order intervals of the DC and the retailers while minimizing the inventory ordering and holding costs. Focusing on stationary and integer-ratio policies, we formulate this inventory management problem as an optimization problem with a convex objective function and a set of integer-ratio constraints and present $O(N \log N)$ time algorithm to solve the relaxed problem (relaxing the integer-ratio constraints) to optimality, where N is the number of the retailers. We prove that the relaxed problem provides a lower bound on average cost for all the feasible policies (containing dynamic policies) for this inventory management problem. By using the optimal solution of the relaxed problem, we build a stationary integer-ratio policy (a power-of-two policy) for this inventory management problem and prove that the power-of-two policy can approximate the optimal inventory policy to 83% accuracy.

1. Introduction

With the growth of international trade and regional economic, from January 2017 to June 2018, the word seaborne trade increased by 4%, and total volumes reached 10.7 billion tons [1]. In China, from January 2018 to August 2018, the cargo volumes of the domestic water transport reached 4.48 billion tons and increased by 3.3% over the same period last year (http://xxgk.mot.gov.cn/jigou/zhghs/201809/t20180914_3087694.html (in Chinese)). The above data shows that more and more firms distribute their product by the water transport. Thus, in this paper, we consider a distribution system consisting of one distribution center (DC), a set of retailers, and a set of ports, in which the product is distributed to the retailers from the DC through the ports by the water transport, and study inventory management for the distribution system.

In practice, when the cargo arrives at the port, the port normally allows the cargo (in-transit inventory) to stay in the ports for free for a certain time period [2, 3]. For example, the free storage times at the major container ports are from 3 to 9 days in Europe, from 3 to 5 days in Asia, and about 10 days in Egypt (http://www.cma-cgm.com/ebusiness/tariffs/demurrage-detention). To take advantage of these free storage periods, the distributors should take into account the free storage periods to better coordinate their DC-retailer inventory replenishment activities to minimize the two-echelon inventory costs. Therefore, we study inventory management for the distribution system with considering the free storage periods and explore the impact of the free storage periods provided by the ports on the inventory policies for the distribution system.

Inventory management for the distribution system with one DC multiretailer or the one warehouse multiretailer (OWMR) system has been extensively studied, and we refer the readers to Roundy [4], Muckstadt and Roundy [5], Levi et al. [6], and Chu and Shen [7] for the related research. The research associated with the free storage period in the framework of supply chain is few. Dekker et al. [8] and Pourakbar et al. [9] consider a floating stock distribution strategy in the intermodal transport for the fast moving customer goods supply chain, in which the stocks are deployed at the intermodal terminals in advance of customer demands within the free storage period provided by the terminals. They analyze four different distribution strategies on a conceptual model and a container shipping scheduling problem and show that the floating stock strategy may lead to lower storage costs and a shorter ordering lead time. Furthermore, they use a real case study to support their findings. van Asperen and Dekker [10] discuss application of the floating stock in the evaluation of port-of-entry choices. Additionally, the research on the storage pricing for the container terminals also considers the effect of the free storage period provided by the terminals or ports, in which the pricing schedules associated with the free storage period are always assumed [2, 3, 11, 12].

In this paper, we study the inventory problem for the one DC multiretailer distribution system with considering the effect of the free storage periods provided by the ports. The objective is to determine the order intervals of the DC and the retailers while minimizing the inventory ordering and holding costs. We focus on stationary and integer-ratio policies and formulate this problem as a nonlinear optimization problem. We first solve a relaxed problem of the nonlinear optimization problem and prove that the optimal solution of the relaxed problem provides a lower bound for all the feasible policies (stationary and dynamic policies) for this inventory problem. Then we build a stationary integer-ratio policy (a power-of-two policy) based on the optimal solution of the relaxed problem and also discuss the gap between the power-of-two policy and the optimal policy for this inventory problem. Note that some results of this paper were presented in the 2017 2nd International Conference on Mechanical Control and Automation [13].

The remainder of this paper is organized as follows. We formulate the inventory management problem and give the solution approach for the optimization problem in Section 2. In Section 3, we prove that the optimal solution of the relaxed problem provides a lower bound on average cost for all feasible policies for the inventory problem, and, in Section 4, we build a power-of-two policy for the inventory problem. We give a numerical example in Section 5 and conclude this paper in Section 6.

2. Model Formulation and Solution Approach

We consider a distribution system with one DC, a set of ports, and a set of retailers, which is shown in Figure 1. The factory/supplier supplies one kind of product, and the DC orders from the single factory/supplier and replenishes the retailers through the ports by the water transport. For the distribution system based on ports, we make the following assumptions:

(i) The distribution system is a centralized system. That is to say, the decisions for inventory replenishment for the DC and the retailers are made centrally.

(ii) The demand at each retailer is deterministic.

Table 1: The order intervals and quantities in the example.

Facility	DC	1st retailer	2nd retailer
Order interval	1	1/2	2
Order quantity	3,1,3,1,⋯	1/2	2

(iii) There are no limits on the capacities of the factory/supplier and the DC.

(iv) In the inventory replenishment, no shortages are allowed.

(v) The leading times for replenishing inventories for the DC and the retailers are deterministic. Without loss of generality, we assume that the leading times are zero. Note that the model we formulated in this paper can be extended to the case that the leading times are not zero easily [4, 5].

(vi) We only consider the free storage periods provided by the ports associated with the retailers and ignore the free storage period provided by the port associated with the DC. That is to say, we only consider the free storage periods for the inbound cargo [2, 3, 11, 12].

In the inventory management, the inventory policies contain stationary policies and dynamic policies. The stationary policies mean that the order intervals and the order quantities do not change over time, and the dynamic policies mean that the order intervals and the order quantities change over time. Inventory management for this distribution system is to determine the optimal inventory policies (the optimal order intervals) for the DC and the retailers while minimizing the long-run average system-wide inventory ordering and holding costs over an infinite time horizon, which we call *the primal problem*. It is known that the optimal inventory policy for the primal problem is unknown [4, 14]. The optimal inventory policy for the primal problem might be very complicated, and we even do not know the optimal inventory policy is stationary or dynamic. Thus, we focus on stationary and integer-ratio inventory policies for the primal problem, and the reason is that stationary integer-ratio inventory policies are more practical in production planning and scheduling [5].

For this distribution system, the integer-ratio policies mean that, for each retailer, the ratio of the order interval at the DC to that at the retailer or the ratio of the order interval at the retailer to that at the DC is an integer [4, 5, 15]. For example, suppose that there is a distribution system with two retailers, and the demand rate at each retailer is 1. Let the order intervals at the DC, the 1st retailer, and the 2nd retailer be 1, 1/2, and 2, respectively. We know that this policy satisfies the integer-ratio constraint and is an integer-ratio policy. Under this integer-ratio policy, the optimal order quantities at the DC and the retailers are shown in Table 1.

From Table 1, we see that the order quantities at the 1st retailer and 2nd retailer are stationary, but the order quantity at the DC is not stationary. Thus this policy is not a stationary integer-ratio policy for this simple distribution system. In this paper, we study the stationary integer-ratio policies for

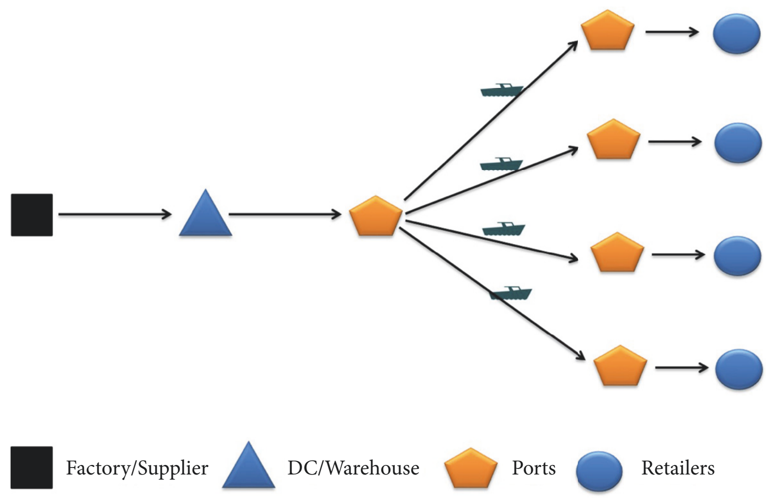

FIGURE 1: A distribution system with one DC, a set of ports, and a set of retailers.

the one DC and multiple ports distribution system and use the stationary integer-ratio policies (power-of-two policies) to approximate the optimal policy for the distribution system as well.

In order to formulate this inventory management problem, we first introduce the following notation:

(i) R: the set of retailers, where $R = \{1, 2, \cdots, N\}$

(ii) K_0: the fixed ordering cost at the DC

(iii) h_0: the inventory holding cost rate at the DC

(iv) K_i: the fixed ordering cost at the retailer i, $i \in R$

(v) h_i: the inventory holding cost rate at the retailer i, $i \in R$

(vi) λ_i: the constant demand rate at the retailer i, $i \in R$

(vii) t_i: the free storage period provided by the port associated with retailer i, $i \in R$

Note that, for ease of exposition, we assume $h_0 \le h_i, \forall i \in R$; i.e., the holding cost rate at warehouse is no more than those at retailers, which is practically reasonable [5, 15].

Let $T = \{T_0, T_1, \cdots, T_N\}$ be a feasible stationary integer-ratio inventory policy for this distribution system, where T_0 is the order interval at the DC and T_i is that at retailer $i, i \in R$. Let $C(T)$ denote the average system-wide inventory holding and ordering cost under policy T and $c_i(T_0, T_i)$ denote the average inventory holding and ordering cost for retailer i under T_0 and T_i, $i \in R$. Then we have

$$C(T) = \frac{K_0}{T_0} + \sum_{i \in R} c_i(T_0, T_i), \quad (1)$$

where K_0/T_0 is average ordering cost at the DC.

Next we show how to calculate $c_i(T_0, T_i)$, $i \in R$. For each $i \in R$, we consider two cases:

(i) $T_0 \ge T_i$. In this case, we know that the order frequency at the DC is less than that at retailer i. That is to say, retailer i should order at least once from the DC before the DC places an order next time from the factory, and the DC needs to hold inventory to serve the demand at retailer i [4, 5, 16]. For example, the demand rate at retailer i is 1, the order interval at retailer i is 1, and the order interval at the DC is 2. Within the order interval at the DC, retailer i places order twice. Obviously, the DC needs to hold 1 inventory to serve retailer i before he places an order next time. Then we have

$$c_i(T_0, T_i) = \frac{K_i}{T_i} + \frac{1}{2}\lambda_i h_i \frac{(T_i - t_i)^+ (T_i - t_i)^+}{T_i} + \frac{1}{2}\lambda_i h_0 (T_0 - T_i), \quad (2)$$

where K_i/T_i is the average ordering cost at retailer i, $(1/2)\lambda_i h_i((T_i - t_i)^+ (T_i - t_i)^+ / T_i)$ is the average holding cost at retailer i, in which we consider the holding cost is zero within the free storage period t_i, and $(1/2)\lambda_i h_0(T_0 - T_i)$ is the average holding cost at the DC for serving demand at retailer i.

(ii) $T_0 < T_i$. In this case, we know that the order frequency at the DC is higher than that at retailer i. That is to say, retailer i will not order again from the DC before the DC places an order next time from the factory, and the DC does not need to hold any inventory to serve the demand at retailer i [4, 5, 16]. For example, the demand rate at retailer i is 1, the order interval at retailer i is 2, and the order interval at the DC is 1. When retailer i places an order from the DC, the DC also places an order from the factory. Obviously, there is not any inventory to be carried at the DC to serve retailer i. Then we have

$$c_i(T_0, T_i) = \frac{K_i}{T_i} + \frac{1}{2}\lambda_i h_i \frac{(T_i - t_i)^+ (T_i - t_i)^+}{T_i}. \quad (3)$$

Based on the above analysis, we formulate $c_i(T_0, T_i),\ i \in R$, as follows:

$$c_i(T_0, T_i) = \frac{K_i}{T_i} + \frac{1}{2}\lambda_i h_i \frac{(T_i - t_i)^+ (T_i - t_i)^+}{T_i} + \frac{1}{2}\lambda_i h_0 \left[\max(T_0, T_i) - T_i\right]. \tag{4}$$

Focusing on stationary integer-ratio policies, we formulate inventory management for the distribution system based on ports as the following optimization problem:

$$\mathcal{Q}\text{: min}\quad \left(\frac{K_0}{T_0} + \sum_{i\in R}\frac{K_i}{T_i} + \frac{1}{2}\sum_{i\in R}\lambda_i h_i \frac{(T_i - t_i)^+ (T_i - t_i)^+}{T_i} + \frac{1}{2}\sum_{i\in R}\lambda_i h_0 \left[\max(T_0, T_i) - T_i\right]\right) \tag{5}$$

$$\text{s.t.}\quad \frac{T_0}{T_i},\ \text{or}\ \frac{T_i}{T_0} \in Z^+,\quad i = 1, \ldots, N, \tag{6}$$

$$T_i > 0,\quad i = 0, 1, \ldots, N, \tag{7}$$

where the first term of the objective function of the model $\mathcal{Q}$ is the average ordering cost at the DC, the second term is average ordering cost at the retailers, the third term is the average holding cost at the retailers, and the last term is average holding cost at the DC. Constraints (6) are the integer-ratio restrictions, and constraints (7) describe that the order intervals for the DC and the retailers are positive. Note that, for each retailer $i \in R$, the holding cost within the free storage period t_i is zero, and we conclude that the optimal order interval for retailer i is greater than t_i. Thus, we have the following lemma.

Lemma 1. *Let T_i^*, $i = 1, \cdots, N$, denote the optimal order intervals for the retailers. Then $T_i^* \geq t_i, \forall i = 1, \cdots, N$.*

Proof. We prove it by contradiction. Let T_0 denote the order interval for the DC, and suppose that there exists $i \in R$ such that $T_i^* = t_i' < t_i$. Then we have the following:

(i) If $t_i' < t_i \leq T_0$, $\max(T_0, t_i') - t_i' = T_0 - t_i' \geq T_0 - t_i = \max(T_0, t_i) - t_i$.

(ii) If $t_i' \leq T_0 < t_i$, $\max(T_0, t_i') - t_i' = T_0 - t_i' \geq \max(T_0, t_i) - t_i = 0$.

(iii) If $T_0 < t_i' < t_i$, $\max(T_0, t_i') - t_i' = \max(T_0, t_i) - t_i = 0$.

It follows directly that

$$\frac{K_i}{t_i'} + \frac{1}{2}\sum_{i\in R}\lambda_i h_0 \left[\max(T_0, t_i') - t_i'\right] > \frac{K_i}{t_i} + \frac{1}{2}\sum_{i\in R}\lambda_i h_0 \left[\max(T_0, t_i) - t_i\right]. \tag{8}$$

Since the cost associated with all the other retailers is unchanged, $T_i^* = t_i$ gives another solution which is better than the one with $T_i^* = t_i' < t_i$. □

By Lemma 1, $\mathcal{Q}$ can be rewritten as

$$\mathcal{Q}\text{: min}\quad \left(\frac{K_0}{T_0} + \sum_{i\in R}\frac{K_i}{T_i} + \frac{1}{2}\sum_{i\in R}\lambda_i h_i \frac{(T_i - t_i)^2}{T_i} + \frac{1}{2} \cdot \sum_{i\in R}\lambda_i h_0 \left[\max(T_0, T_i) - T_i\right]\right) \tag{9}$$

$$\text{s.t.}\quad (6), (7).$$

Obviously, it is very hard for us to directly solve the model $\mathcal{Q}$ for the integer-ratio constraints (6). Thus, we first relax the integer-ratio constraints (6) and solve the corresponding relaxed problem. Then we build stationary integer-ratio policies for the primal problem using the optimal solution of the relaxed problem. By relaxing the integer-ratio constraints (6), we obtain the following relaxed problem:

$$\min_{T_i > 0, i\in R\cup\{0\}} \left(\frac{K_0}{T_0} + \sum_{i\in R}\frac{K_i}{T_i} + \frac{1}{2}\sum_{i\in R}\lambda_i h_i \frac{(T_i - t_i)^2}{T_i} + \frac{1}{2}\sum_{i\in R}\lambda_i h_0 \left[\max(T_0, T_i) - T_i\right]\right). \tag{10}$$

Note that the relaxed problem (10) is a convex optimization problem. Next we show how to solve the relaxed problem (10). We first give the following theorem.

Theorem 2. *Let T_0^* and T_i^* denote the optimal solution to (10), $i \in R$, and we have*

(1) *$T_i^* > T_0^*$ if and only if $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i}$, and this set of retailers is denoted by G.*

(2) *$T_i^* < T_0^*$ if and only if $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)}$, and this set of retailers is denoted by L.*

(3) *$T_i^* = T_0^*$ if and only if $\sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i} < T_i^* = T_0^* < \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)}$, and this set of retailers is denoted by E.*

Proof. Let $g_i(T_i) = K_i/T_i + (1/2)\lambda_i h_i((T_i - t_i)^2/T_i)$, and $l_i(T_i) = K_i/T_i + (1/2)\lambda_i h_i((T_i - t_i)^2/T_i) + (1/2)\lambda_i h_0(T_0 - T_i)$, $i \in R$, and we have $g''(T_i) > 0$ for $T_i > 0$, and $l_i''(T_i) > 0$ for $T_i > 0$, $i \in R$. Thus, $g_i(T_i)$ and $l_i(T_i)$ are both strictly convex function, $i \in R$.

(1) $\Longrightarrow$: Suppose that $T_0^* < T_i^*$, and we have $T_i^* \in \arg\min_{T_i > 0} g_i(T_i)$. Since $g_i(T_i)$ is a strictly convex function, we know that T_i^* is the unique minimizer of $g_i(T_i)$ over $T_i > 0$ [17], and then

$$g_i'(T_i^*) = -\frac{K_i}{(T_i^*)^2} + \frac{1}{2}\lambda_i h_i \left(1 - \frac{t_i^2}{(T_i^*)^2}\right) = 0$$

$$\Longrightarrow T_i^* = \sqrt{\frac{\lambda_i h_i t_i^2 + 2K_i}{\lambda_i h_i}} \geq t_i. \quad (11)$$

Therefore, if $T_0^* < T_i^*$, we always have $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i} > T_0^*$.

$\Longleftarrow$: Suppose that $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i}$. We next show that $T_i^* > T_0^*$. If $T_i^* \leq T_0^*$, then we have $l_i(T_i) \leq l_i(T_i^*)$ for $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i} < T_i \leq \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)}$. This implies that any value of T_i satisfying $\sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i} < T_i \leq \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)}$ can give a better objective function value and contradicts the optimality of T_i^*.

Therefore, if $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i}$, we always have $T_i^* > T_0^*$.

(2) $\Longrightarrow$: Suppose that $T_0^* > T_i^*$, and we have $T_i^* \in \arg\min_{T_i>0} l_i(T_i)$. Since $l_i(T_i)$ is a strictly convex function, we know that T_i^* is the unique minimizer of $l_i(T_i)$ over $T_i > 0$ [17], and then

$$f_i'(T_i^*) = -\frac{K_i}{(T_i^*)^2} + \frac{1}{2}\lambda_i h_i \left(1 - \frac{t_i^2}{(T_i^*)^2}\right) - \frac{1}{2}\lambda_i h_0 = 0$$

$$\Longrightarrow T_i^* = \sqrt{\frac{\lambda_i h_i t_i^2 + 2K_i}{\lambda_i (h_i - h_0)}} \geq t_i. \quad (12)$$

Therefore, if $T_i^* < T_0^*$, we always have $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)} < T_0^*$.

$\Longleftarrow$: Suppose that $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)}$. We next show that $T_i^* < T_0^*$. If $T_i^* \geq T_0^*$, then we have $g_i(T_i) \leq g_i(T_i^*)$ for $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i} < T_i \leq \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)}$. This implies that any value of T_i satisfying $\sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i} < T_i \leq \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)}$ can give a better objective function value and hence contradicts the optimality of T_i^*.

Therefore, if $T_i^* = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)}$, we always have $T_i^* < T_0^*$.

(3) According to (1) and (2), we can easily establish that $T_0^* = T_i^*$ if and only if $\sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i} < T_0^* = T_i^* < \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/(\lambda_i (h_i - h_0))}$. □

From Theorem 2, we know that no matter how much t_i is, $\forall i$, the optimal distribution strategy, will always force the inventories to stay at each port associated with retailer i for some time longer than t_i. Based on Theorem 2 and by using of the ideal of Roundy [4], we introduce the following algorithm to solve the relaxed problem (10) to optimality. For any retailer $i \in R$, let us define

$$T_i^G = \sqrt{\frac{\lambda_i h_i t_i^2 + 2K_i}{\lambda_i h_i}}$$

$$\text{and } T_i^L = \sqrt{\frac{\lambda_i h_i t_i^2 + 2K_i}{\lambda_i (h_i - h_0)}}. \quad (13)$$

We depict the details of the algorithm as follows.

Step 1. Let $Z^* = +\infty$. Partition the real line by T_i^G, T_i^L for all $i \in R$. Note that $T_i^G \leq T_i^L$ for any i.

Step 2. Suppose T_0^* falls in a particular interval, say $[a, b]$ (choosing the intervals from left to right). We can use Theorem 2 to determine the sets G, E, and L, depending on whether T_i^G, T_i^L fall to the left or right of the interval $[a, b]$.

More specifically, $i \in G$ and $T_i = T_i^G$ if $a \leq b \leq T_i^G$, $i \in E$ if $T_i^G \leq a \leq b \leq T_i^L$, $i \in L$, and $T_i = T_i^L$ if $T_i^L \leq a \leq b$.

After determining the retailers in the sets G and L, we need to solve the following problem to find optimal value of T_0

$$\frac{K_0}{T_0} + \sum_{i\in E} \frac{K_i}{T_0} + \frac{1}{2}\sum_{i\in E} \lambda_i h_i \frac{(T_0 - t_i)^2}{T_0} + \frac{1}{2}\sum_{i\in L} \lambda_i h_0 T_0. \quad (14)$$

According to the first-order condition, we have

$$-\frac{K_0}{T_0^2} - \sum_{i\in E} \frac{K_i}{T_0^2} + \frac{1}{2}\sum_{i\in E} \lambda_i h_i \left(1 - \frac{t_i^2}{T_0^{*2}}\right) + \frac{1}{2}\sum_{i\in L} \lambda_i h_0 = 0, \quad (15)$$

and hence

$$-2K_0 - 2\sum_{i\in E} K_i + \sum_{i\in E} \lambda_i h_i T_0^2 - \sum_{i\in E} \lambda_i h_i t_i^2 + \sum_{i\in L} \lambda_i h_0 T_0^2 = 0$$

$$\overline{T}_0 = \sqrt{\frac{2K_0 + 2\sum_{i\in E} K_i + \sum_{i\in E} \lambda_i h_i t_i^2}{\sum_{i\in E} \lambda_i h_i + \sum_{i\in L} \lambda_i h_0}}. \quad (16)$$

If $\overline{T_0} \in [a, b]$, then set $T_0 = T_i = \overline{T_0}$ for any $i \in E$, calculate the value of the cost Z using (6), and let $Z^* := Z$ if $Z^* > Z$. Note that, for any $i \in E$, we have $T_0 = T_i \geq t_i$ as $t_i \leq T_i^G \leq a$. Otherwise, move to the next interval (i.e., our guess that T_0^* is in $[a, b]$ is wrong).

Step 3. Go to Step 2 till it reaches the last interval. The value of T_0, T_i corresponding to Z^* is the optimal reorder interval of the warehouse and retailer i, respectively.

We get at most $2N + 1$ intervals along the line in Step 1. Note that as long as T_0^* falls within any interval, we have enough information to determine for all $i \in R$, whether $i \in G, E, L$. We also note that, by construction of the intervals, none of the values in $T_i^G, T_i^L, \forall i \in R$, will fall in the interval (a, b). Hence $G \cup E \cup L = R$. Step 1 requires a sorting operation for $O(N)$ values, which requires $O(N\log N)$ comparisons. The number of operations in Steps 2 and 3 can be performed in $O(N)$ operations, which is dominated by the number of operations in Step 1. Thus we have the following.

Theorem 3. *The computational complexity of the algorithm to solve the relaxed problem (10) is $O(N\log N)$, where N is the number of the retailers.*

3. Lower Bound Theorem

Obviously the optimal solution of the relaxed problem (10) provides a lower bound for $\mathcal{Q}$ (a lower bound for the stationary integer policies), but not for the primal problem (the optimal inventory policy for the primal problem may be dynamic). Thus we want to know whether the optimal solution of the relaxed problem of $\mathcal{Q}$ also provides a lower bound for the primal problem.

In this section, we give a Theorem 4, which is primarily designed to demonstrate that the optimal solution of the relaxed problem of $\mathcal{Q}$ also provides a lower bound for the primal problem. That is to say, the stationary policy obtained by the optimal solution of the relaxed problem of $\mathcal{Q}$ provides a lower bound for all the feasible stationary and dynamic policies for the primal problem.

Theorem 4 (lower bound theorem). *Let Z^* be the optimal objective function value of the relaxed problem (10). Then Z^* is a lower bound on the average cost for any feasible inventory policy (possibly dynamic) for the primal problem.*

Proof. We prove Theorem 4 using the similar spirits of Roundy [4]. Let $T^* = (T_0^*, T_1^*, T_2^* \cdots, T_N^*)$ denote the optimal solution to (10) and Z^* be the corresponding optimal objective value for (10), where T^* can be obtained by the algorithm proposed in Section 2. Note that T^* is the optimal relaxed order intervals for the DC and retailers. From Theorem 2, we see that the retailers naturally fall into three groups $G = \{i : T_0^* < T_i^G = T_i^*\}$, $E = \{i : T_i^G \le T_0^* = T_i^* \le T_i^L\}$, and $L = \{i : T_i^* = T_i^L < T_0^*\}$, where $T_i^G = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i h_i}$ and $T_i^L = \sqrt{(\lambda_i h_i t_i^2 + 2K_i)/\lambda_i (h_i - h_0)}$, $\forall i \in \mathcal{R}$.

For each policy $T = \{T_0, T_1, T_2, \cdots, T_n\}$, we say that the policy T *preserves the order* of T^* if $T_i \ge T$ whenever $i \in G$, $T_i = T$ whenever $i \in E$, and $T_i \le T$ whenever $i \in L$. For each order-preserving policy T, (10) can be rewritten as

$$\min_{T_0 \ge t_i, \forall i \in E, T_i \ge t_i, \forall i \in E^c} \left(\left(\frac{\overline{K}_0}{T_0} + \frac{1}{2}\lambda_0 \overline{H}_0 T_0 \right) + \sum_{i \in E^c} \left(\frac{\overline{K}_i}{T_i} + \frac{1}{2}\lambda_i H_i T_i \right) - M \right) \tag{17}$$

where $E^c \equiv G \cup L$, $\overline{K}_0 \equiv K_0 + \sum_{i\in E}(K_i + (1/2)\lambda_i h_i t_i^2)$, $\lambda_0 \equiv \sum_{i\in E\cup L}\lambda_i$, $\overline{H}_0 \equiv (1/\lambda_0)(\sum_{i\in E}\lambda_i h_i + \sum_{i\in L}\lambda_i h_0)$, $\overline{K}_i \equiv K_i + (1/2)\lambda_i h_i t_i^2$, $H_i \equiv h_i$ for all $i \in G$, $H_i \equiv (h_i - h_0)$ for all $i \in L$, and $M \equiv \sum_{i\in\mathcal{R}}\lambda_i h_i t_i$. From the definitions of T^* and Z^* and the formulation of (17), we observe that the minimum relaxed average cost Z^* is

$$Z^* = \overline{M}_0 + \sum_{i\in E^c} M_i - M, \tag{18}$$

where $\overline{M}_0 \equiv (2\overline{K}_0\lambda_0\overline{H}_0)^{1/2}$, and $M_i \equiv (2\overline{K}_i\lambda_i H_i)^{1/2}$ for all $i \in E^c$.

For each retailer $i \in E$, we define $H_i = 2(K_i + (1/2)\lambda_i h_i t_i^2)/\lambda_0 T_0^{*2}$, and, for the warehouse, we define $H_0 = 2K_0/\lambda_0 T_0^{*2}$. We also define $\overline{\lambda}_i, i \in \mathcal{R}$, as follows:

$$\overline{\lambda}_i = \begin{cases} \lambda_i, & i \in E, \\ \lambda_0, & i \in E^c. \end{cases} \tag{19}$$

Then we have

(1) $\overline{H}_0 = \sum_{i\in E\cup 0} H_i$;

(2) $\lambda_0 H_0 = \sum_{i\in\mathcal{R}}(\lambda_i h_i - \overline{\lambda}_i H_i)$;

(3) $\lambda_i(h_i - h_0) \le \overline{\lambda}_i H_i \le \lambda_i h_i, \ \forall i \in \mathcal{R}$.

We prove (1), (2), and (3) as follows. From the definitions of $H_i, \ \forall i \in \mathcal{R}$, we have

$$\begin{aligned}\overline{H}_0 &= \frac{2\overline{K}_0}{\lambda_0 T^{*2}} = \frac{2K_0 + 2\sum_{i\in E}\left(K_i + (1/2)\lambda_i h_i t_i^2\right)}{\lambda_0 T^{*2}} \\ &= \frac{2K_0}{\lambda_0 T^{*2}} + \sum_{i\in E}\frac{2\left(K_i + (1/2)\lambda_i h_i t_i^2\right)}{\lambda_0 T^{*2}} = \sum_{i\in E\cup 0} H_i,\end{aligned} \tag{20}$$

and

$$\begin{aligned}\lambda_0 H_0 &= \lambda_0 \overline{H}_0 - \sum_{i\in E}\lambda_0 H_i \\ &= \left(\sum_{i\in E}\lambda_i h_i + \sum_{i\in L}\lambda_i h_0\right) - \sum_{i\in E}\lambda_0 H_i \\ &= \sum_{i\in E}(\lambda_i h_i - \lambda_0 H_i) + \sum_{i\in L}\lambda_i h_0 \\ &= \sum_{i\in E}(\lambda_i h_i - \lambda_0 H_i) + \sum_{i\in E^c}(\lambda_i h_i - \lambda_i H_i) \\ &= \sum_{i\in\mathcal{R}}\left(\lambda_i h_i - \overline{\lambda}_i H_i\right).\end{aligned} \tag{21}$$

For each $i \in \mathcal{R}$, if $i \in G$, then $H_i = h_i$ and $\overline{\lambda}_i = \lambda_i$, and we have $\lambda_i(h_i - h_0) < \overline{\lambda}_i H_i = \lambda_i h_i$; if $i \in L$, then $H_i = h_i - h_0$ and $\overline{\lambda}_i = \lambda_i$, and we have $\lambda_i(h_i - h_0) = \overline{\lambda}_i H_i < \lambda_i h_i$; if $i \in E$, then $\overline{\lambda}_i = \lambda_0$ and $H_i = 2(K_i + (1/2)\lambda_i h_i t_i^2)/\lambda_0 T_0^{*2}$, and since $T_i^G \le T_0^* = T_i^* \le T_i^L$, we have $\lambda_i(h_i - h_0) \le \overline{\lambda}_i H_i \le \lambda_i h_i$. Therefore, we complete the proof for (1), (2), and (3).

Since $T_0^{*2} = 2\overline{K}_0/\lambda_0\overline{H}_0 = 2K_0/\lambda_0 H_0 = 2(K_i + (1/2)\lambda_i h_i t_i^2)/\lambda_0 H_i, \forall i \in E$, we also have

$$\begin{aligned}\overline{M}_0 &= \left(2\overline{K}_0\lambda_0\overline{H}_0\right)^{1/2} = \frac{2\overline{K}_0}{T_0^*} \\ &= \frac{2K_0}{T_0^*} + \sum_{i\in E}\frac{2\left(K_i + (1/2)\lambda_i h_i t_i^2\right)}{T_0^*} \\ &= \left(2K_0\lambda_0 H_0\right)^{1/2} + \sum_{i\in E}\sqrt{2\left(K_i + \frac{1}{2}\lambda_i h_i t_i^2\right)\lambda_0 H_i} \\ &= \sum_{i\in E\cup 0} M_i,\end{aligned} \tag{22}$$

where M_0 is the minimum value of $K_0/x + (1/2)\lambda_0 H_0 x$ and M_i is the minimum value of $(K_i + (1/2)\lambda_i h_i t_i^2)/x + (1/2)\lambda_0 H_i x$ for all $x \ge t_i, i \in E$. Then we can rewrite (18) as

$$Z^* = \sum_{i\in\mathscr{R}\cup 0} M_i - M, \tag{23}$$

where M_0 is the minimum value of $K_0/x + (1/2)\lambda_0 H_0 x$ for all $x \ge t_i \ \forall i \in E$ and M_i is the minimum value of $(K_i + (1/2)\lambda_i h_i t_i^2)/x + (1/2)\overline{\lambda}_i H_i x$ for all $x \ge t_i, i \in \mathscr{R}$.

We are ready to prove Z^* is a lower bound on average cost of all the feasible inventory policies for the primal problem. Let $T' = \{T_0', T_1', T_2', \cdots, T_N'\}$ be an arbitrary inventory policy over the infinite horizon for the primal problem and $Z(t')$ be the average cost incurred in the interval $[0, t')$ by the policy. For the feasible policy T', if there are some order intervals $T_i' < t_i, i \in \mathscr{R}$, we can find a better feasible policy by letting $T_i' = t_i$. From Lemma 1, we note that the cost for the new feasible policy is less than T'. Thus, for the policy T', without loss of generality, we assume that $T_i' \ge t_i, \forall i \in \mathscr{R}$. It suffices to show that $Z^* \le Z(t')$, for every $t' > 0$, and we also assume t' is large enough for the problem.

Let m_i be the number of orders placed by the retailer i in $[0, t']$, I_i^t be the inventory at retailer i at time t, and I_{i0}^t be the inventory at the DC destined for retailer $i, i \in \mathscr{R}$. We note that I_i^t is zero if t is in the initial time interval $[0, t_i]$ in each order interval for each retailer $i \in \mathscr{R}$. Thus, the total holding cost for the policy T' in the interval $[0, t']$ is

$$\sum_{i\in\mathscr{R}}\int_0^{t'}\left(h_i I_i^t + h_0 I_{i0}^t\right)dt. \tag{24}$$

We next show that

$$\begin{aligned}&\sum_{i\in\mathscr{R}}\int_0^{t'}\left(h_i I_i^t + h_0 I_{i0}^t\right)dt \\ &\quad\ge \sum_{i\in\mathscr{R}}\int_0^{t'}\left(H_i I_i^t + \left(\frac{\lambda_i h_i}{\overline{\lambda}_i} - H_i\right)I_{i0}^t\right)dt.\end{aligned} \tag{25}$$

In order to prove (25), for each $i \in \mathscr{R}$, we should prove $h_i I_i^t + h_0 I_{i0}^t \ge H_i I_i^t + (\lambda_i h_i/\overline{\lambda}_i - H_i)I_{i0}^t$. There are three cases to consider.

Case 1 ($i \in G$). We have $H_i = h_i$ and $\overline{\lambda}_i = \lambda_i$, and thus $h_i I_i^t + h_0 I_{i0}^t \ge H_i I_i^t + (\lambda_i h_i/\overline{\lambda}_i - H_i)I_{i0}^t$.

Case 2 ($i \in L$). We have $H_i = h_i - h_0$ and $\overline{\lambda}_i = \lambda_i$, and thus $h_i I_i^t + h_0 I_{i0}^t \ge H_i I_i^t + (\lambda_i h_i/\overline{\lambda}_i - H_i)I_{i0}^t$.

Case 3 ($i \in E$). We have $\lambda_i(h_i - h_0) \le \overline{\lambda}_i H_i \le \lambda_i h_i$ and $\overline{\lambda}_i = \lambda_0$, and then $H_i/h_i \le \lambda_i/\overline{\lambda}_i \le 1$ and $H_i - \lambda_i h_i/\lambda_0 + h_0 \ge H_i - \lambda_i h_i/\lambda_0 + \lambda_i h_0/\lambda_0 \ge 0$. Since $h_i I_i^t + h_0 I_{i0}^t - H_i I_i^t - (\lambda_i h_i/\overline{\lambda}_i - H_i)I_{i0}^t = (h_i - H_i)I_i^t + (H_i - \lambda_i h_i/\lambda_0 + h_0)I_{i0}^t \ge 0$, then $h_i I_i^t + h_0 I_{i0}^t \ge H_i I_i^t + (\lambda_i h_i/\overline{\lambda}_i - H_i)I_{i0}^t$.

Let $I_0^t = (1/H_0)\sum_{i\in\mathscr{R}}(\lambda_i h_i/\overline{\lambda}_i - H_i)I_{i0}^t$ be the average inventory at the DC at time t, and then we have

$$\begin{aligned}&\sum_{i\in\mathscr{R}}\int_0^{t'}\left(h_i I_i^t + h_0 I_{i0}^t\right)dt \\ &\quad\ge \sum_{i\in\mathscr{R}}\int_0^{t'}\left(H_i I_i^t + \left(\frac{\lambda_i h_i}{\overline{\lambda}_i} - H_i\right)I_{i0}^t\right)dt \\ &\quad= \sum_{i\in\mathscr{R}}\int_0^{t'} H_i I_i^t dt + \int_0^{t'} H_0 I_0^t dt.\end{aligned} \tag{26}$$

Note that the ith term in the sum of the first term on the right hand of (26) can be thought of the total holding cost incurred in the interval $[0, t']$ in a *single-item lot-size* problem in which m_i orders are placed in $[0, t']$, the demand rate per unit time is $\overline{\lambda}_i$, the per unit inventory holding cost per unit time is H_i, the setup cost is K_i, and the free inventory storage time is t_i in each order interval, and the second term on the right hand of (26) can be thought of the total holding cost incurred in $[0, t']$ in a *single-item lot-size* problem in which m_0 orders are placed in $[0, t']$, the demand rate per unit time is λ_0, the per unit inventory holding cost per unit time is H_0, and the setup cost is K_0. For the ith term in the sum of the first term on the right hand of (26), the inventory policy with minimum cost for this problem is every $t'/m_i (t'/m_i \ge t_i)$ unit time orders $\overline{\lambda}_i t'/m_i$ units, and the resulting total holding cost is $m_i\overline{\lambda}_i H_i(t'/m_i - t_i)^2/2$. For the second term on the right hand of (26), the inventory policy with minimum cost for this problem is every t'/m_0 unit time orders $\lambda_0 t'/m_0$ units, and the resulting total holding cost is $m_0\lambda_0 H_0(t'/m_0)^2/2$. Thus we have

$$\begin{aligned}Z\left(T', t'\right)t' &\ge \sum_{i\in\mathscr{R}}\left(K_i m_i + \int_0^{t'} H_i I_i^t dt\right) \\ &\quad + \left(K_0 m_0 + \int_0^{t'} H_0 I_0^t dt\right) \\ &\ge \sum_{i\in\mathscr{R}}\left(K_i m_i + \frac{m_i\overline{\lambda}_i H_i\left(t'/m_i - t_i\right)^2}{2}\right) \\ &\quad + \left(K_0 m_0 + \frac{m_0\lambda_0 H_0\left(t'/m_0\right)^2}{2}\right).\end{aligned} \tag{27}$$

In what follows, we prove

$$\frac{1}{t'}\sum_{i\in\mathcal{R}}\left(K_i m_i + \frac{m_i\overline{\lambda}_i H_i\left(t'/m_i - t_i\right)^2}{2}\right) + \frac{1}{t'}\left(K_0 m_0 + \frac{m_0\lambda_0 H_0\left(t'/m_0\right)^2}{2}\right) \geq Z^*. \tag{28}$$

There are three cases to consider.

Case 1 ($i \in G$). Then $H_i = h_i$ and $\overline{\lambda}_i = \lambda_i$, and we have

$$\frac{1}{t'}\left(K_i m_i + \frac{m_i\overline{\lambda}_i H_i\left(t'/m_i - t_i\right)^2}{2}\right) = \frac{K_i m_i}{t'} + \frac{\overline{\lambda}_i H_i t'}{2m_i} + \frac{m_i\lambda_i h_i t_i^2}{2t'} - \lambda_i h_i t_i. \tag{29}$$

Case 2 ($i \in L$). Then $H_i = h_i - h_0$ and $\overline{\lambda}_i = \lambda_i$, and we have

$$\begin{aligned}\frac{1}{t'}\left(K_i m_i + \frac{m_i\overline{\lambda}_i H_i\left(t'/m_i - t_i\right)^2}{2}\right) &= \frac{K_i m_i}{t'} + \frac{\overline{\lambda}_i H_i t'}{2m_i} + \frac{m_i\lambda_i h_i t_i^2}{2t'} - \lambda_i h_i t_i \\ &\quad + \lambda_i h_0 t_i\left(1 - \frac{m_i t_i}{2t'}\right) \\ &\geq \frac{K_i m_i}{t'} + \frac{\overline{\lambda}_i H_i t'}{2m_i} + \frac{m_i\lambda_i h_i t_i^2}{2t'} - \lambda_i h_i t_i.\end{aligned} \tag{30}$$

Case 3 ($i \in E$). Then $\lambda_i(h_i - h_0) \leq \overline{\lambda}_i H_i \leq \lambda_i h_i$ and $\overline{\lambda}_i = \lambda_0$, and we have

$$\begin{aligned}\frac{1}{t'}\left(K_i m_i + \frac{m_i\overline{\lambda}_i H_i\left(t'/m_i - t_i\right)^2}{2}\right) &= \frac{K_i m_i}{t'} + \frac{\overline{\lambda}_i H_i t'}{2m_i} + \frac{m_i\overline{\lambda}_i H_i t_i^2}{2t'} - \overline{\lambda}_i H_i t_i t' \\ &= \frac{K_i m_i}{t'} + \frac{\overline{\lambda}_i H_i t'}{2m_i} + \frac{m_i\lambda_i h_i t_i^2}{2t'} - \lambda_i h_i t_i \\ &\quad + t' t_i\left(\lambda_i h_i - \overline{\lambda}_i H_i\right) + \frac{1}{2} m_i t_i^2\left(\overline{\lambda}_i H_i - \lambda_i h_i\right) \\ &= \frac{K_i m_i}{t'} + \frac{\overline{\lambda}_i H_i t'}{2m_i} + \frac{m_i\lambda_i h_i t_i^2}{2t'} - \lambda_i h_i t_i \\ &\quad + \left(\lambda_i h_i t_i - \overline{\lambda}_i H_i t_i\right)\left(t' - \frac{1}{2} m_i t_i\right) \\ &\geq \frac{K_i m_i}{t'} + \frac{\overline{\lambda}_i H_i t'}{2m_i} + \frac{m_i\lambda_i h_i t_i^2}{2t'} - \lambda_i h_i t_i.\end{aligned} \tag{31}$$

Thus we have

$$\frac{1}{t'}\sum_{i\in\mathcal{R}}\left(K_i m_i + \frac{m_i\overline{\lambda}_i H_i\left(t'/m_i - t_i\right)^2}{2}\right) \geq \sum_{i\in\mathcal{R}}\left(\frac{m_i\left(K_i + (1/2)\lambda_i h_i t_i^2\right)}{t'} + \frac{1}{2}\overline{\lambda}_i H_i\frac{t'}{m_i} - \lambda_i h_i t_i\right), \tag{32}$$

and then

$$\begin{aligned}Z\left(T', t'\right) &\geq \frac{1}{t'}\sum_{i\in\mathcal{R}}\left(K_i m_i + \frac{m_i\overline{\lambda}_i H_i\left(t'/m_i - t_i\right)^2}{2}\right) \\ &\quad + \left(\frac{K_0 m_0}{t'} + \frac{1}{2}\lambda_0 H_0\frac{t'}{m_0}\right) \\ &\geq \sum_{i\in\mathcal{R}}\left(\frac{m_i\left(K_i + (1/2)\lambda_i h_i t_i^2\right)}{t'} + \frac{1}{2}\overline{\lambda}_i H_i\frac{t'}{m_i}\right) \\ &\quad + \left(\frac{K_0 m_0}{t'} + \frac{1}{2}\lambda_0 H_0\frac{t'}{m_0}\right) - \sum_{i\in\mathcal{R}}\lambda_i h_i t_i \geq Z^*.\end{aligned} \tag{33}$$

□

4. Power-of-Two Policies

In this section, we make use of the optimal solution of the relaxed problem (10) to build a stationary integer-ratio, i.e., a power-of-two policy, for the model $\mathcal{Q}$. The power-of-two policy is a power-of-two multiple of the base planning period, which is an easy-to-use policy in practice. Muckstadt and Roundy [5] and Muckstadt and Sapra [15] discuss the advantages of the power-of-two policy in detail, especially on the inventory management and production planning and scheduling.

Let $T^* = \{T_0^*, T_1^*, \cdots, T_N^*\}$ be the optimal solution for the relaxed problem (10) and T_L be the base planning period, such as a day or a week. We use T^* to build a power-of-two policy, denoted by $\overline{T} = \{\overline{T}_0, \overline{T}_1, \cdots, \overline{T}_N\}$, in the following: For each $i \in R$,

(i) if $i \in E^c$, let $\overline{T}_i = 2^{l_i}T_L$, where l_i is a positive integer, such that $(1/\sqrt{2})T_i^* \leq 2^{l_i}T_L \leq \sqrt{2}T_i^*$;

(ii) if $i \in E \cup \{0\}$, let $\overline{T}_i = 2^{l_i}T_L$, where l_i is a positive integer, such that $(1/\sqrt{2})T_0^* \leq 2^{l_i}T_L \leq \sqrt{2}T_0^*$.

Obviously, $\overline{T} = \{\overline{T}_0, \overline{T}_1, \cdots, \overline{T}_N\}$ is a feasible stationary integer-ratio policy for the model $\mathcal{Q}$. Next we explore the gap between the power-of-two policy $\overline{T}$ and the optimal solution of the model $\mathcal{Q}$.

In practice, the fixed cost in the water transport is relatively high [18, 19], and then for each retailer $i \in R$, we assume that $t_i \leq \sqrt{2K_i/\lambda_i h_i}$, which means the free storage period t_i is less than the optimal ordering interval for the EOQ model with the fixed ordering cost K_i, the holding cost rate h_i, and the demand rate λ_i. With this assumption, we give the following lemma.

Lemma 5. *Let $T^* = (T_0^*, T_1^*, T_2^* \cdots, T_N^*)$ be the optimal solution of the relaxed problem (10) and Z^* be the corresponding optimal objective function value. For each retailer $i \in R$, if $t_i \leq \sqrt{2K_i/\lambda_i h_i}$, then we have*

(1) $T_i^* \geq \sqrt{2} t_i$;

(2) $Z^* \geq (\sqrt{2}-1)M$, *where* $M \equiv \sum_{i\in R} \lambda_i h_i t_i$.

Proof. (1) For each $i \in R$, if $t_i \leq \sqrt{2K_i/\lambda_i h_i}$, we consider the following three cases.

Case 1. If $i \in G$, then

$$T_i^* = \sqrt{\frac{\lambda_i h_i t_i^2 + 2K_i}{\lambda_i h_i}} = \sqrt{t_i^2 + \frac{2K_i}{\lambda_i h_i}} \geq \sqrt{2} t_i. \quad (34)$$

Case 2. If $i \in L$, then

$$T_i^* = \sqrt{\frac{\lambda_i h_i t_i^2 + 2K_i}{\lambda_i (h_i - h0)}} > \sqrt{\frac{\lambda_i h_i t_i^2 + 2K_i}{\lambda_i h_i}} \geq \sqrt{2} t_i. \quad (35)$$

Case 3. If $i \in E$, then

$$T_i^* \geq \sqrt{\frac{\lambda_i h_i t_i^2 + 2K_i}{\lambda_i h_i}} \geq \sqrt{2} t_i. \quad (36)$$

Thus, for each $i \in R$, if $t_i \leq \sqrt{2K_i/\lambda_i h_i}$, we have $T_i^* \geq \sqrt{2} t_i$.

(2) Since $T^* = (T_0^*, T_1^*, T_2^* \cdots, T_n^*)$ is optimal for the relaxed problem (10), then

$$\begin{aligned} Z^* &= \frac{K_0}{T_0^*} + \sum_{i\in R} \frac{K_i}{T_i^*} + \frac{1}{2} \sum_{i\in R} \lambda_i h_i \frac{(T_i^* - t_i)}{T_i^*} \\ &\quad + \frac{1}{2} \sum_{i\in G} \lambda_i h_0 (T_0^* - T_i^*) \\ &= \frac{K_0}{T_0^*} + \frac{1}{2} \sum_{i\in G} \lambda_i h_0 (T_0^* - T_i^*) \\ &\quad + \sum_{i\in R} \frac{K_i + (1/2)\lambda_i h_i t_i^2}{T_i^*} + \frac{1}{2} \sum_{i\in R} \lambda_i h_i T_i^* + \sum_{i\in R} \lambda_i h_i t_i \\ &> \sum_{i\in R} \frac{K_i + (1/2)\lambda_i h_i t_i^2}{T_i^*} + \frac{1}{2} \sum_{i\in R} \lambda_i h_i T_i^* + \sum_{i\in R} \lambda_i h_i t_i. \end{aligned} \quad (37)$$

Since $(K_i + (1/2)\lambda_i h_i t_i^2)/T_i^* = (1/2) \sum_{i\in R} \lambda_i h_i T_i^*$, $i \in R$, and $T_i^* \geq \sqrt{2} t_i$, then

$$\begin{aligned} Z^* &> \sum_{i\in R} \frac{K_i + (1/2)\lambda_i h_i t_i^2}{T_i^*} + \frac{1}{2} \sum_{i\in R} \lambda_i h_i T_i^* + \sum_{i\in R} \lambda_i h_i t_i \\ &= \sum_{i\in R} \lambda_i h_i T_i^* - \sum_{i\in R} \lambda_i h_i t_i \geq (\sqrt{2}-1) \sum_{i\in R} \lambda_i h_i t_i \\ &= (\sqrt{2}-1) M. \end{aligned} \quad (38)$$

□

Based on Lemma 5, we show that the power-of-two policy $\overline{T} = \{\overline{T}_0, \overline{T}_1, \cdots, \overline{T}_N\}$ we build above can approximate the optimal inventory policy of the primal problem to 83% accuracy. We give the following theorem.

Theorem 6. *Let C^* denote the optimal objective value of the primal problem, Z^* denote the optimal objective value of the relaxed problem (10), and $Z(\overline{T})$ denote the objective value of the model $\mathbb{Q}$ under the power-of-two policy $\overline{T}$. For each retailer $i \in R$, if $t_i \leq \sqrt{2K_i/\lambda_i h_i}$, we have $Z^* \leq C^* \leq Z(\overline{T}) \leq 1.20 Z^* \leq 1.20 C^*$; i.e., $Z(\overline{T})$ approximates C^* to 83% accuracy.*

Proof. Obviously, we have $Z^* \leq C^* \leq Z(\overline{T})$. In order to prove Theorem 6, we only need to prove $Z(\overline{T}) \leq 1.20 Z^*$. Let $T^* = (T_0^*, T_1^*, T_2^* \cdots, T_N^*)$ be the optimal solution of the relaxed problem (10). By (17), we have

$$Z(\overline{T}) = \left(\frac{\overline{K}_0}{\overline{T}_0} + \frac{1}{2}\lambda_0 \overline{H}_0 \overline{T}_0\right) + \sum_{i\in E^c} \left(\frac{\overline{K}_i}{\overline{T}_i} + \frac{1}{2}\lambda_i H_i \overline{T}_i\right) - M. \quad (39)$$

Since $\overline{K}_0/T_0^* = (1/2)\lambda_0 \overline{H}_0 T_0^* = \overline{M}_0/2$ and $\overline{K}_i/T_i^* = (1/2)\lambda_i H_i T_i^* = M_i/2, i \in E^c$, we can rewrite $Z(\overline{T})$ as

$$Z(\overline{T}) = \frac{\overline{M}_0}{2}\left(\frac{T_0^*}{\overline{T}_0} + \frac{\overline{T}_0}{T_0^*}\right) + \sum_{i\in E^c} \frac{M_i}{2}\left(\frac{T_i^*}{\overline{T}_i} + \frac{\overline{T}_i}{T_i^*}\right) - M. \quad (40)$$

Note that $(1/\sqrt{2})T_i^* \leq \overline{T}_i \leq \sqrt{2} T_i^*, i \in E^c$, and $(1/\sqrt{2})T_0^* \leq \overline{T}_i \leq \sqrt{2} T_0^*, i \in E \cup 0$, and then we have

$$\begin{aligned} Z(\overline{T}) &= \frac{\overline{M}_0}{2}\left(\frac{T_0^*}{\overline{T}_0} + \frac{\overline{T}_0}{T_0^*}\right) + \sum_{i\in E^c} \frac{M_i}{2}\left(\frac{T_i^*}{\overline{T}_i} + \frac{\overline{T}_i}{T_i^*}\right) \\ &\quad - M \\ &\leq \frac{\overline{M}_0}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) + \sum_{i\in E^c} \frac{M_i}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) \\ &\quad - M \\ &= \frac{1}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right)\left(\overline{M}_0 + \sum_{i\in E^c} M_i - M\right) \\ &\quad + \frac{1}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) M - M \\ &= \frac{1}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) Z^* + \frac{3 - 2\sqrt{2}}{2\sqrt{2}} M. \end{aligned} \quad (41)$$

Table 2: The values for t_i.

Parameters	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
Day	3	11	4	18	6	6	12	15	13	10
Year	3/365	11/365	4/365	18/365	6/365	6/365	12/365	15/365	13/365	10/365

Table 3: The values for K_i, h_i, and λ_i.

Parameters	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
Values	254	859	595	371	363	301	415	871	160	550
Parameters	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}
Values	30	22	30	26	13	23	14	16	24	19
Parameters	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
Values	603	592	284	310	925	1264	423	1032	1186	525

From Lemma 5, we know that, for each $i \in R$, if $t_i \leq \sqrt{2K_i/\lambda_i h_i}$, then we have $Z^* \geq (\sqrt{2}-1)M$. Since $Z^* \geq (\sqrt{2}-1)M$, we have

$$\begin{aligned} Z(\overline{T}) &\leq \frac{1}{2}\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)Z^* + \frac{3-2\sqrt{2}}{2\sqrt{2}}M \\ &\leq \frac{1}{2}\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)Z^* \\ &\quad + \frac{3-2\sqrt{2}}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}-1}\right)Z^* \approx 1.20Z^*. \end{aligned} \tag{42}$$

Finally, we have

$$Z^* \leq C^* \leq Z(\overline{T}) \leq 1.20Z^* \leq 1.20C^*. \tag{43}$$

□

5. Numerical Example

In this section, we give a numerical example to show how to calculate $c_i(T_0, T_i)$, $i \in R$, how to solve the relaxed problem (10), and how to build the power-of-two policy via the optimal solution of the relaxed problem (10). We suppose that there is a distribution system consisting of one DC, ten ports, and ten retailers. For the DC, we set the fixed ordering cost, K_0, as 500 and set the holding cost rate per unit per year, h_0, as 10. For the ports, we randomly generate the free time period, t_i, in $[1, 20]$, and the values for t_i are shown in Table 2.

For the retailers, we randomly generate the fixed ordering cost, K_i, in $[100, 1000]$, the inventory holding cost rate per unit per year, h_i, in $(10, 30]$, and the demand rate per year, λ_i, in $[100, 1500]$, and those values are shown in Table 3.

Let T_0 be the order interval at the DC and T_i be the order at the retailer i, $i \in R$. We first calculate $c_i(T_0, T_i)$ as follows:

$$c_1(T_0, T_1) = \frac{254}{T_1} + \frac{1}{2} \times 603 \times 30 \times \frac{(T_1-3)^2}{T_1} + \frac{1}{2} \times 603 \times 10 \times [\max(T_0, T_1) - T_1],$$

$$c_2(T_0, T_2) = \frac{859}{T_2} + \frac{1}{2} \times 592 \times 22 \times \frac{(T_2-11)^2}{T_2} + \frac{1}{2} \times 592 \times 10 \times [\max(T_0, T_2) - T_2],$$

$$c_3(T_0, T_3) = \frac{595}{T_3} + \frac{1}{2} \times 284 \times 30 \times \frac{(T_3-4)^2}{T_3} + \frac{1}{2} \times 284 \times 10 \times [\max(T_0, T_3) - T_3],$$

$$c_4(T_0, T_4) = \frac{371}{T_4} + \frac{1}{2} \times 310 \times 26 \times \frac{(T_4-18)^2}{T_4} + \frac{1}{2} \times 310 \times 10 \times [\max(T_0, T_4) - T_4],$$

$$c_5(T_0, T_5) = \frac{363}{T_5} + \frac{1}{2} \times 925 \times 13 \times \frac{(T_5-6)^2}{T_5} + \frac{1}{2} \times 925 \times 10 \times [\max(T_0, T_5) - T_5],$$

$$c_6(T_0, T_6) = \frac{301}{T_6} + \frac{1}{2} \times 1264 \times 23 \times \frac{(T_6-6)^2}{T_6} + \frac{1}{2} \times 1264 \times 10 \times [\max(T_0, T_6) - T_6],$$

$$c_7(T_0, T_7) = \frac{415}{T_7} + \frac{1}{2} \times 423 \times 14 \times \frac{(T_7-12)^2}{T_7} + \frac{1}{2} \times 423 \times 10 \times [\max(T_0, T_7) - T_7],$$

$$c_8(T_0, T_8) = \frac{871}{T_8} + \frac{1}{2} \times 1032 \times 16 \times \frac{(T_8-15)^2}{T_8} + \frac{1}{2} \times 1032 \times 10 \times [\max(T_0, T_8) - T_8],$$

$$c_9(T_0, T_9) = \frac{160}{T_9} + \frac{1}{2} \times 1186 \times 24 \times \frac{(T_9-13)^2}{T_9}$$

Table 4: The optimal solution for the relaxed problem.

Order interval	T_0^*	T_1^*	T_2^*	T_3^*	T_4^*	T_5^*	T_6^*	T_7^*	T_8^*	T_9^*	T_{10}^*
Year	0.189	0.189	0.364	0.374	0.307	0.246	0.189	0.376	0.327	0.146	0.333
Day	69.16	69.16	133.02	136.47	112.20	89.89	69.16	137.17	119.50	53.45	121.62

Table 5: A power-of-two policy for the primal problem.

Order interval	$\overline{T}_0$	$\overline{T}_1$	$\overline{T}_2$	$\overline{T}_3$	$\overline{T}_4$	$\overline{T}_5$	$\overline{T}_6$	$\overline{T}_7$	$\overline{T}_8$	$\overline{T}_9$	$\overline{T}_{10}$
Year	0.25	0.25	0.5	0.5	0.25	0.25	0.25	0.5	0.25	0.125	0.25
Day	91.25	91.25	182.5	182.5	91.25	91.25	91.25	182.5	91.25	45.625	91.25

$$
\begin{aligned}
&+\frac{1}{2}\times 1186\times 10 \\
&\times\left[\max\left(T_0,T_9\right)-T_9\right], \\
c_{10}\left(T_0,T_{10}\right) &= \frac{550}{T_{10}}+\frac{1}{2}\times 525\times 19\times\frac{\left(T_{10}-10\right)^2}{T_{10}} \\
&+\frac{1}{2}\times 525\times 10 \\
&\times\left[\max\left(T_0,T_{10}\right)-T_{10}\right].
\end{aligned}
\tag{44}
$$

By the above $c_i(T_0,T_i)$, $i\in R$, we get the following relaxed problem (10):

$$
\begin{aligned}
\min_{T_i>0,i\in R\cup\{0\}} &\frac{500}{T_0}+\left(\frac{254}{T_1}+\frac{859}{T_2}+\frac{595}{T_3}+\frac{371}{T_4}+\frac{363}{T_5}\right. \\
&\left.+\frac{301}{T_6}+\frac{415}{T_7}+\frac{871}{T_8}+\frac{160}{T_9}+\frac{550}{T_{10}}\right) \\
&+\left[\frac{9045\left(T_1-3\right)^2}{T_1}+\frac{6512\left(T_2-11\right)^2}{T_2}\right. \\
&+\frac{4260\left(T_3-4\right)^2}{T_3}+\frac{4030\left(T_4-18\right)^2}{T_4} \\
&+\frac{12025\left(T_5-6\right)^2}{2T_5}+\frac{14536\left(T_6-6\right)^2}{T_6} \\
&+\frac{2961\left(T_7-12\right)^2}{T_7}+\frac{8256\left(T_8-15\right)^2}{T_8} \\
&\left.+\frac{14232\left(T_9-13\right)^2}{T_9}+\frac{9975\left(T_{10}-10\right)^2}{2T_{10}}\right] \\
&+\left[3015\max\left(T_0,T_1\right)+2960\max\left(T_0,T_2\right)\right. \\
&+1420\max\left(T_0,T_3\right)+1550\max\left(T_0,T_4\right) \\
&+4625\max\left(T_0,T_5\right)+6320\max\left(T_0,T_6\right) \\
&+2115\max\left(T_0,T_7\right)+5160\max\left(T_0,T_8\right) \\
&\left.+5930\max\left(T_0,T_9\right)+2625\max\left(T_0,T_{10}\right)\right] \\
&-\left[3015T_1+2960T_2+1420T_3+1550T_4+4625T_5\right. \\
&+6320T_6+2115T_7+5160T_8+5930T_9 \\
&\left.+2625T_{10}\right].
\end{aligned}
\tag{45}
$$

We solve the above relaxed problem (10) by the algorithm proposed in Section 2 and obtain the optimal T_i, $i\in R\cup\{0\}$, in Table 4.

Note that the optimal order quantity at each facility (the DC and the retailers) equals the optimal order interval multiplied by the demand rate. By the method proposed in Section 4, we build a power-of-two policy for the primal problem via the optimal solution for the relaxed problem (10) as shown in Table 5.

In this example, the optimal objective function value for the relaxed problem is 34073.143, and the objective function value for the primal problem under the power-of-two policy is 35391.838. Then we know that the gap between the power-of-two policy built in above and the optimal policy for the primal problem is 3.87% (100 × (35391.838 − 34073.143)/34073.143). Thus we conclude that the power-of-two policies may perform much better in practice.

Additionally, in this example, we also solve the inventory management problem without considering the effect of the free time periods provided by the ports and comparing the objective function values for the primal problem with and without considering the effect of the free time periods; we obtain that the system-wide cost for this distribution system reduces 9.16% by making use of the free time periods provided by the ports. That is to say, making use of the free time periods provided by the ports can significantly reduce the system-wide cost for the distribution systems that distribute their product by the water transport.

6. Conclusions

In this paper, we study inventory management for the distribution system consisting with one DC, a set of ports, and a set of retailers. The DC orders the product from a single factory/supplier and replenishes the retailers through the ports by the water transport. Since the ports always allow the cargo arriving in the ports to stay for free for a period of time, which we call the free storage period, we study inventory management for this distribution system with integrating the impact of the free storage periods provided by the ports. Focusing on stationary and integer-ratio policies, we formulate this inventory management problem as a nonlinear optimization model with a convex objective function and a set of integer-ratio constraints. We first solve the relaxed problem by relaxing the integer-ratio constraints in $O(N \log N)$ time and then build a stationary integer-ratio policy (a power-of-two policy) for this inventory management problem by using the optimal solution of the relaxed problem. More importantly, we prove that the optimal solution of the relaxed problem provides a lower bound on the average cost of any feasible policies (possibly dynamic policies) for this inventory management problem and that the power-of-two policy we build can approximate the optimal inventory policy for this inventory management problem to 83% accuracy. Finally, we give an example to show how to apply the models and algorithms proposed in this paper in practice, and we also obtain some management insights from the example.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this article.

Acknowledgments

This research is supported by Humanity and Social Science Youth Foundation of Ministry of Education of China (no. 14YJC630078).

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